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Integral stochastic ordering of the multivariate normal mean-variance and the skew-normal scale-shape mixture models

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Abstract In this paper, we introduce integral stochastic ordering of two most important classes of distributions that are commonly used to fit data possessing high values of skewness and (or) kurtosis. The first one is based on the selection distributions started by the univariate skew-normal distribution. A broad, flexible and newest class in this area is the scale and shape mixture of multivariate skew-normal distributions. The second one is the general class of Normal Mean-Variance Mixture distributions. We then derive necessary and sufficient conditions for comparing the random vectors from these two classes of distributions. The integral orders considered here are the usual, concordance, supermodular, convex, increasing convex and directionally convex stochastic orders. Moreover, for bivariate random vectors, in the sense of stop-loss and bivariate concordance stochastic orders, the dependence strength of random portfolios is characterized in terms of order of correlations.

Keywords Integral order, Skew-normal, Scale-shape mixture, Mean-variance mixture

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1. Introduction

There are numerous situations in which the assumption of normality is not supported by the data. Several examples of such cases have been provided by Genton [21]. Some families of near normal distributions, which include the normal distribution and to some extent some of its desirable properties, have played a crucial role in the analysis of data arising from different fields. A decisive point in the development of such distributions is the paper of Azzalini [5].

A substantial amount of work on multivariate skew distributions has resulted from the proposal of the multivariate skew-normal (SN) distribution [6, 8]. In recent years, work on multivariate SN distribution, in both theoretical and applied studies, has increased substantially.

In this regard, Arellano-Valle et al. [1] introduced a broad and flexible class of multivariate distributions obtained by both scale and shape mixtures of multivariate skew-normal distributions. This family of multivariate distributions unifies and extends many existing models in the literature such as scale mixtures of skew-normal distributions [15] and shape mixtures of skew-normal distributions [2].

Another extension of multivariate normal distribution is the mean-variance mixtures of multivariate normal distribution. An important case of Normal mean-variance mixtures is Generalized Hyperbolic (GH) distributions

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introduced by Barndorff-Nielsen and Blaesild [10]. Recently, GH distributions have been used by many authors to fit financial data; see, for example, [42, 14, 16].

In a variety of applications, researchers are often interested in comparing treatment groups on the basis of several, potentially dependent outcomes. The statistical problem of interest is then to compare multivariate distributions of the outcomes from the control and treatment groups. The theory of stochastic orders provides an useful theoretical foundation for such comparisons as it facilitates comparing random variables and vectors. The books by Shaked and Shanthikumar [44, 45] and Müller and Stoyan [35] provide an elaborate discussion on various stochastic orders and their applications to diverse problems. Belzunce [13] has reviewed some of the main stochastic orders in the literature and showed relationships between them. He also pointed out some applications of stochastic orders in several fields such as reliability theory, risk theory, epidemiology, ecology and biology.

Some results on stochastic orderings of multivariate normal distributions can be found in [11], [32] and [43]. Müller [33] hss provided a general treatments to the so-called integral stochastic orders and subsequently Müller [34] has discussed necessary and sufficient conditions for many important examples of integral stochastic orders for multivariate normal distributions. Landsman and Tsanakas [29] introduced the convex and concordance orders of bivariate elliptical distributions. Davidov and Peddada [17] obtained necessary and sufficient conditions for the usual stochastic ordering of multivariate elliptical random vectors. Pan et al. [38] studied convex and increasing convex orderings of multivariate elliptical random vectors.

Stochastic ordering for the univariate SN distribution and its extensions have been discussed recently, which include some reliability orderings for the generalized SN distribution [23], ordering of univariate SN distribution and general skew-symmetric distributions [9] and characterizations of likelihood ratio order and usual stochastic order for the univariate skew-symmetric distributions [24]. Jamali et al. [25] extended some orderings in [34] for the multivariate SN distributions.

In the present paper, we introduce integral stochastic ordering of the general families of multivariate SN scale-shape mixture and normal mean-variance mixture models. We derive necessary and sufficient conditions for various types of integral orders such as usual stochastic order, convex order, increasing convex order, directionally convex order, concordance order and supermodular order. Furthermore, the effect of correlation in the bivariate case is examined. It is shown that for two bivariate random vectors belonging to *Frechet Space* [19], the ordering of their correlation coefficients is equivalent to their concordance order. Moreover, it is shown that, in the stop-loss order sense, riskiness of a portfolio of two risks with the considered distributions increases in terms of the correlation coefficient. This is a stronger version of a result obtained by Dhaene and Goovaerts [19] and in the more general family by Landsman and Tsanakas [29] regarding the skewness.

The remainder of this paper is organized as follows. In Section 2, we introduce briefly the concepts of integral stochastic order, multivariate SN scale-shape mixtures and multivariate normal mean-variance mixtures. Section 3 describes the univariate stochastic orders which form the basis for all the results established subsequently. In Section 4, the bivariate concordance order is characterized in terms of the order of correlations of portfolio risks. Section 5 provides the results of integral orderings as well as necessary and sufficient conditions for characterizations various orders. Section 6 concludes with a short discussion and some possible directions for future research.

2. Preliminaries

In this section, we first present a brief overview of integral stochastic orders and the families of distributions that we are in focus here.

2.1. Integral stochastic orders

Many of the stochastic orders that are in common use are defined as follows. Let $(S, \mathbb{A})$ be some measure space, and let $\mathbb{F}$ be some class of measurable functions $f : S \to \mathbb{R}$. Then, a relation $\preceq_{\mathbb{F}}$ is defined on the set of all probability measures on $(S, \mathbb{A})$ by

$$P \preceq_{\mathbb{F}} Q \qquad if \qquad \int f dP \leq \int f dQ \quad for \;\; all \;\; f \in \mathbb{F},$$

when the involved integrals exist. It is often helpful to know that it is sufficient to check $\int f dP \leq \int f dQ$ for all $f \in \mathbb{F}$ which are sufficiently smooth. A general study of this type of order has been given by Müller [33]. We wish to mention some important examples of integral orders. In the following, if $f : \mathbb{R}^d \to \mathbb{R}$ is twice continuously differentiable, then we write as usual

$$\nabla_f(\mathbf{x}) = \left[\frac{\partial}{\partial x_i} f(\mathbf{x})\right]_{i=1}^{d} \quad , \quad H_f(\mathbf{x}) = \left[\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x})\right]_{i,j=1}^{d}$$

for the gradient vector and the Hessian matrix of f, respectively.

Definition 1
([34, 18]) Given two d-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$ and twice differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ in $\mathbb{F}$,

(i) (Usual order) $\mathbf{X} \preceq_{st} \mathbf{Y}$ if, and only if, $E(f(\mathbf{X})) \leq E(f(\mathbf{Y}))$ holds for all $f \in \mathbb{F}$ satisfying $\nabla_f(\mathbf{x}) \geq \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^d$ (for all increasing functions f);

(ii) (Convex order) $\mathbf{X} \preceq_{cx} \mathbf{Y}$ if, and only if, $E(f(\mathbf{X})) \leq E(f(\mathbf{Y}))$ holds for all $f \in \mathbb{F}$ that $H_f(\mathbf{x})$ is positive semi-definite for all $\mathbf{x} \in \mathbb{R}^d$ (for all convex functions f);

(iii) (Supermodular order) $\mathbf{X} \preceq_{sm} \mathbf{Y}$ if, and only if, $E(f(\mathbf{X})) \leq E(f(\mathbf{Y}))$ holds for all $f \in \mathbb{F}$ satisfying $\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and all $1 \leq i < j \leq d$ (for all suprmodular functions f);

(iv) (Increasing convex order) $\mathbf{X} \preceq_{icx} \mathbf{Y}$ if, and only if, $E(f(\mathbf{X})) \leq E(f(\mathbf{Y}))$ holds for all $f \in \mathbb{F}$ that $\nabla_f(\mathbf{x}) \geq \mathbf{0}$ and $H_f(\mathbf{x})$ is positive semi-definite for all $\mathbf{x} \in \mathbb{R}^d$ (for all increasing-convex functions f);

(v) (Directionally convex order) $\mathbf{X} \preceq_{dcx} \mathbf{Y}$ if, and only if, $E(f(\mathbf{X})) \leq E(f(\mathbf{Y}))$ holds for all $f \in \mathbb{F}$ satisfying $\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and all $1 \leq i, j \leq d$ (for all directionally-convex functions f).

The functions in the above definition are generally defined in terms of difference operators. The difference operator Δ_i^{ϵ}, $1 \leq i \leq d$, $\epsilon > 0$, for a function $f : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\Delta_i^{\epsilon} f(\mathbf{x}) = f(\mathbf{x} + \epsilon \mathbf{e}_i) - f(\mathbf{x}),$$

where $\mathbf{e}_i$ is the i-th unit basis vector of $\mathbb{R}^d$. The function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be Δ-monotone if, for any subset $\{i_1, ..., i_k\} \subseteq \{1, ..., d\}$ and every $\epsilon_i > 0$, $i = 1, ..., k$,

$$\Delta_{i_1}^{\epsilon_{i_1}} ... \Delta_{i_k}^{\epsilon_{i_k}} f(\mathbf{x}) \geq 0 \quad for \;\; all \;\; \mathbf{x} \in \mathbb{R}^d. \tag{1}$$

The stochastic order generated by Δ-monotone functions is called upper orthant order, since it can be defined alternatively through a comparison of upper orthants [41]. Now, we recall the definitions of upper orthant and concordance orders which are studied for comparison of dependence structures.

Definition 2 (i) (Upper orthant order) $\mathbf{X} \preceq_{uo} \mathbf{Y}$ if, and only if, $P(\mathbf{X} \geq \mathbf{t}) \leq P(\mathbf{Y} \geq \mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^d$;

(ii) (Concordance order) $\mathbf{X} \preceq_{conc} \mathbf{Y}$ if, and only if, $P(\mathbf{X} \geq \mathbf{t}) \leq P(\mathbf{Y} \geq \mathbf{t})$ and $P(\mathbf{X} \leq \mathbf{t}) \leq P(\mathbf{Y} \leq \mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^d$.

2.2. Some general families of distributions

The following notations will be used throughout this paper: $\Phi(\cdot)$ denotes the cumulative distribution function (cdf) of the univariate standard normal distribution, $\phi_n(\cdot\,; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes the probability density function (pdf) of $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (the n-variate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$).

2.2.1. Normal mean-variance mixture distributions Consider the d-dimensional random vector $\mathbf{X}$ that can be expressed as

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \boldsymbol{\lambda} W + \sqrt{W}\mathbf{Z}, \tag{2}$$

where $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathbb{R}^d$, $\mathbf{Z} \sim N_d(\mathbf{0}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ as a positive definite matrix of order d, and W is an independent non-negative random variable with cdf $H(.;\boldsymbol{\eta})$ and support $\mathbb{S}_H$, which is indexed by the parameter vector $\boldsymbol{\eta}$. Then, the random vector $\mathbf{X}$ is said to have a normal mean-variance mixture (NMVM) distribution, denoted by $\mathbf{X} \sim NMVM_d(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, h)$. The pdf of $\mathbf{X}$ can be expressed as

$$f(\mathbf{x}) = \int_{\mathbb{S}_H} w^{-\frac{d}{2}} \phi_d\left((\mathbf{x} - \boldsymbol{\mu}) w^{-\frac{1}{2}} - \boldsymbol{\lambda} w^{\frac{1}{2}}; \mathbf{0}, \boldsymbol{\Sigma}\right) dH(w; \boldsymbol{\eta}). \tag{3}$$

The mean vector and the covariance matrix of $\mathbf{X}$ are given by

$$E(\mathbf{X}) = \boldsymbol{\mu} + E(W)\boldsymbol{\lambda}, \quad Cov(\mathbf{X}) = Var(W)\boldsymbol{\lambda}\boldsymbol{\lambda}^T + E(W)\boldsymbol{\Sigma}. \tag{4}$$

In the following lemma, we extend Slepian's inequality [46] for NMVM distributions. Let two d-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$ be distributed as

$$\mathbf{X} \sim NMVM_d(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, h), \qquad \mathbf{Y} \sim NMVM_d(\boldsymbol{\mu}', \boldsymbol{\Sigma}', \boldsymbol{\lambda}', h) \tag{5}$$

and $\boldsymbol{\sigma} = Diag(\boldsymbol{\Sigma})^{\frac{1}{2}}$ and $\boldsymbol{\sigma}' = Diag(\boldsymbol{\Sigma}')^{\frac{1}{2}}$.

Lemma 1
Let $\mathbf{X}$ and $\mathbf{Y}$ be the random vectors as given in (5). If $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$, $\boldsymbol{\lambda} \leq \boldsymbol{\lambda}'$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}'$, $\sigma_{ij} \leq \sigma'_{ij}$ for all $1 \leq i, j \leq d$, then

$$P(\mathbf{X} \geq \mathbf{t}) \leq P(\mathbf{Y} \geq \mathbf{t}) \quad for \;\; all \; \mathbf{t} \in \mathbb{R}^d.$$

If $\mu_i < \mu'_i$ or $\lambda_i < \lambda'_i$ for some i or $\sigma_{ij} < \sigma'_{ij}$ for some (i, j), then the above inequality is in fact strict.

Proof
From (2), we have, for all $w \in \mathbb{S}_H$,

$$\begin{aligned} P(\mathbf{X} \geq \mathbf{t} | W = w) &= P(\mathbf{Z} \geq w^{-\frac{1}{2}}(\mathbf{t} - \boldsymbol{\mu}) - w^{\frac{1}{2}}\boldsymbol{\lambda}), \\ P(\mathbf{Y} \geq \mathbf{t} | W = w) &= P(\mathbf{Z}' \geq w^{-\frac{1}{2}}(\mathbf{t} - \boldsymbol{\mu}') - w^{\frac{1}{2}}\boldsymbol{\lambda}'), \end{aligned}$$

where $\mathbf{Z} \sim N_d(\mathbf{0}, \boldsymbol{\Sigma})$ and $\mathbf{Z}' \sim N_d(\mathbf{0}, \boldsymbol{\Sigma}')$. Considering $\boldsymbol{\sigma} = \boldsymbol{\sigma}'$ and $\sigma_{ij} \leq \sigma'_{ij}$, and using Slepian's inequality, we have $P(\mathbf{Z} \geq \mathbf{u}) \leq P(\mathbf{Z}' \geq \mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^d$, and the inequality is strict if $\sigma_{ij} < \sigma'_{ij}$ for some (i, j). It follows from $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$ and $\boldsymbol{\lambda} \leq \boldsymbol{\lambda}'$ that

$$\begin{aligned} P(\mathbf{X} \geq \mathbf{t}) &= \int_{\mathbb{S}_H} P(\mathbf{Z} \geq w^{-\frac{1}{2}}(\mathbf{t} - \boldsymbol{\mu}) - w^{\frac{1}{2}}\boldsymbol{\lambda}) dH(w; \eta), \\ &\leq \int_{\mathbb{S}_H} P(\mathbf{Z}' \geq w^{-\frac{1}{2}}(\mathbf{t} - \boldsymbol{\mu}') - w^{\frac{1}{2}}\boldsymbol{\lambda}') dH(w; \eta) \\ &= P(\mathbf{Y} \geq \mathbf{t}) \end{aligned}$$

and the inequality is strict if $\mu_i < \mu'_i$, $\lambda_i < \lambda'_i$ or $\sigma_{ij} < \sigma'_{ij}$, for some index. □

The following lemma provides sufficient conditions for the integral stochastic ordering of the random vectors from NMVM distributions.

Lemma 2
Let $\mathbf{X}$ and $\mathbf{Y}$ be the random vectors as given in (5) and $\phi_p(.|w) = \phi_d(.; \boldsymbol{\mu}_p + w\boldsymbol{\lambda}_p, w\boldsymbol{\Sigma}_p)$, $0 \leq p \leq 1$, where

$$\begin{aligned} \boldsymbol{\mu}_p &= p\boldsymbol{\mu} + (1-p)\boldsymbol{\mu}', \\ \boldsymbol{\lambda}_p &= p\boldsymbol{\lambda} + (1-p)\boldsymbol{\lambda}', \\ \boldsymbol{\Sigma}_p &= p\boldsymbol{\Sigma} + (1-p)\boldsymbol{\Sigma}'. \end{aligned} \tag{6}$$

Further, let $f : \mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable and, for any $w \in \mathbb{S}_H$, satisfy the following conditions:

(i) $\lim_{x_j \to \pm\infty} f(\mathbf{x})\phi_p(\mathbf{x}|w) = 0$, for all $1 \leq j \leq d$,
(ii) $\lim_{x_j \to \pm\infty} f(\mathbf{x})\frac{\partial}{\partial x_i}\phi_p(\mathbf{x}|w) = 0$, for all $1 \leq i, j \leq d$,
(iii) $\lim_{x_j \to \pm\infty} \phi_p(\mathbf{x}|w)\frac{\partial}{\partial x_i} f(\mathbf{x}) = 0$, for all $1 \leq i, j \leq d$.

If the following conditions

$$\begin{aligned} &\sum_{i=1}^d (\mu_i' - \mu_i)\frac{\partial}{\partial x_i} f(\mathbf{x}) \geq 0, \\ &\sum_{i=1}^d (\lambda_i' - \lambda_i)\frac{\partial}{\partial x_i} f(\mathbf{x}) \geq 0, \\ &\sum_{i=1}^d \sum_{j=1}^d (\sigma_{ij}' - \sigma_{ij})\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) \geq 0 \end{aligned} \tag{7}$$

hold, then $E(f(\mathbf{X})) \leq E(f(\mathbf{Y}))$.

Proof
From (2) and considering conditions (i)-(iii), then using Theorem 2 of Müller [34] and applying double expectation formula, we obtain

$$\begin{aligned} E(f(\mathbf{Y})) - E(f(\mathbf{X})) = \int_{\mathbb{S}_H}\int_0^1\int_{\mathbb{R}^d} &\Big((\boldsymbol{\mu}' - \boldsymbol{\mu})^T \nabla_f(\mathbf{x}) + w\,(\boldsymbol{\lambda}' - \boldsymbol{\lambda})^T \nabla_f(\mathbf{x}) \\ &+ \frac{w}{2} Trace\left[(\boldsymbol{\Sigma}' - \boldsymbol{\Sigma})\, H_f(\mathbf{x})\right]\Big)\, \phi_p(\mathbf{x}|w) d\mathbf{x} dp dH(w; \boldsymbol{\eta}). \end{aligned}$$

Then, $Trace\left[(\boldsymbol{\Sigma}' - \boldsymbol{\Sigma})\, H_f(\mathbf{x})\right] = \sum_{i=1}^d \sum_{j=1}^d (\sigma_{ij}' - \sigma_{ij})\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x})$ and the conditions in (7) implies the negativity of $E(f(\mathbf{X}) - f(\mathbf{Y}))$. □

Remark 1
Let the mixing variable W in (2) follow the generalized inverse Gaussian (GIG) distribution with pdf

$$h(w; \boldsymbol{\eta}) = \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} w^{\lambda - 1} e^{-\frac{1}{2}\left(\chi w^{-1} + \psi w\right)}, \quad w > 0,$$

where $\chi > 0, \psi \geq 0$ if $\lambda < 0$, $\chi > 0, \psi > 0$ if $\lambda = 0$, $\chi \geq 0, \psi > 0$ if $\lambda > 0$ and $K_\lambda(.)$ being the Bessel function of the third kind with index λ. Then, the random vector $\mathbf{X}$ in (2) follows the d-variate GH distribution. Some important cases of GH are the Normal Inverse Gaussian distribution when $\lambda = -0.5$, the hyperbolic distribution when $\lambda = 1$, variance gamma distribution when $\lambda > 0$ and $\psi = 0$, and the skewed-t distribution with $\lambda < 0$ and $\chi = 0$ [31]. Some other examples of NMVM distributions are generalized hyperbolic skew-slash [4] when $W \sim Beta(\nu, 1)$, normal-mean-variance Birnbaum-Saunders [39] and normal-mean-variance Lindley [36].

2.2.2. Scale-shape mixture of multivariate SN distributions The multivariate SN distribution was originally introduced by Azzalini and Dalla-Valle [8]. Following Azzalini and Capitanio [6], the pdf of a d-variate SN distribution, with location vector $\boldsymbol{\mu}$, dispersion matrix $\boldsymbol{\Sigma}$ and shape/skewness vector $\boldsymbol{\alpha}$, can be written as

$$f(\mathbf{x}) = 2\phi_d(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})\Phi(\boldsymbol{\alpha}^T \boldsymbol{\sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})) \quad for \;\; all \;\; \mathbf{x} \in \mathbb{R}^d, \tag{8}$$

and it is denoted by $\mathbf{X} \sim SN_d(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\alpha})$.

Now, we consider the multivariate scale-shape mixtures of skew-normal (SSMSN) distributions proposed recently by Arellano-Valle et al. [1].

Let $\boldsymbol{\tau} = (\tau_1, \tau_2)$ be an arbitrary bivariate random vector with a joint cdf $H(\boldsymbol{\tau}; \boldsymbol{\eta})$ and support $\mathbb{S}_H$. Then, a d-dimensional random vector $\mathbf{Y}$ is said to follow the SSMSN distribution, if conditionally on $\boldsymbol{\eta}$, it takes on the form

$$\mathbf{Y}|\boldsymbol{\tau} \sim SN_d(\boldsymbol{\mu}, a_1(\tau_1)\boldsymbol{\Sigma}, a_2(\boldsymbol{\tau})\boldsymbol{\alpha}), \tag{9}$$

where $a_1(\tau_1)$ is a positive scale (weight) function and $a_2(\tau_1, \tau_2)$ is a real-valued shape function which is not symmetric about zero. Without loss of generality, we suppose that $a_2(\boldsymbol{\tau})$ is a positive function.

Suppose $Z_1 \sim N(0,1)$ and $\mathbf{Z}_2 \sim N_d(\mathbf{0}, \mathbf{\Sigma})$, and assume that Z_1 and $\mathbf{Z}_2$ are independent. Then, the SSMSN distribution in (9) has the following selection representation:

$$\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + \sqrt{a_1(\tau_1)}\mathbf{Z}_2 | \left(Z_1 \leq a_2(\boldsymbol{\tau})\boldsymbol{\alpha}^T\boldsymbol{\sigma}^{-1}\mathbf{Z}_2\right). \tag{10}$$

Now, by using the multivariate SN characteristic function [28], the characteristic function of SSMSN distribution is given by

$$\Psi(\mathbf{t}) = 2\int_{\mathbb{S}_H} \exp(i\boldsymbol{\mu}^T\mathbf{t} - \frac{a_1(\tau_1)}{2}\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}) \left\{1 + i\delta\left(\sqrt{a_1(\tau_1)}\boldsymbol{\lambda}_{\boldsymbol{\tau}}^T\mathbf{t}\right)\right\} dH(\boldsymbol{\tau};\boldsymbol{\eta}), \quad \mathbf{t} \in \mathbb{R}^d, \tag{11}$$

where

$$\boldsymbol{\lambda}_{\boldsymbol{\tau}} = \frac{a_2(\boldsymbol{\tau})}{\sqrt{1 + a_2^2(\boldsymbol{\tau})\boldsymbol{\alpha}^T\overline{\mathbf{\Sigma}}\boldsymbol{\alpha}}}\boldsymbol{\sigma}\overline{\mathbf{\Sigma}}\,\boldsymbol{\alpha}, \quad \delta(u) = \sqrt{2/\pi}\int_0^u \exp\left(z^2/2\right)dz, \tag{12}$$

with $\overline{\mathbf{\Sigma}} = \boldsymbol{\sigma}^{-1}\mathbf{\Sigma}\boldsymbol{\sigma}^{-1}$. The skewness parameter $\boldsymbol{\alpha}$ in (10) can also be written in terms of $\boldsymbol{\lambda}_{\boldsymbol{\tau}}$ in (12) as

$$\boldsymbol{\alpha} = \left(1 - \boldsymbol{\lambda}_{\boldsymbol{\tau}}^T\mathbf{\Sigma}^{-1}\boldsymbol{\lambda}_{\boldsymbol{\tau}}\right)^{-1/2}\boldsymbol{\sigma}\mathbf{\Sigma}^{-1}\boldsymbol{\lambda}_{\boldsymbol{\tau}}. \tag{13}$$

Let $\mathbf{X}$ and $\mathbf{Y}$ be two d-dimensional random vectors from SSMSN distributions as follows:

$$\begin{aligned} \mathbf{X}|\boldsymbol{\tau} &\sim SN_d(\boldsymbol{\mu}, a_1(\tau_1)\mathbf{\Sigma}, a_2(\boldsymbol{\tau})\boldsymbol{\alpha}), \\ \mathbf{Y}|\boldsymbol{\tau} &\sim SN_d(\boldsymbol{\mu}', a_1(\tau_1)\mathbf{\Sigma}', a_2(\boldsymbol{\tau})\boldsymbol{\alpha}'). \end{aligned} \tag{14}$$

The mean vector and the covariance matrix of (14) are given by

$$\begin{aligned} E(\mathbf{Y}) &= \boldsymbol{\mu} + \sqrt{\tfrac{2}{\pi}}E\left[a_1^{1/2}(\tau_1)\boldsymbol{\lambda}_{\boldsymbol{\tau}}\right], \\ Cov(\mathbf{Y}) &= E(a_1(\tau_1))\mathbf{\Sigma} - \tfrac{2}{\pi}E\left[a_1^{1/2}(\tau_1)\boldsymbol{\lambda}_{\boldsymbol{\tau}}\right]E\left[a_1^{1/2}(\tau_1)\boldsymbol{\lambda}_{\boldsymbol{\tau}}\right]^T. \end{aligned} \tag{15}$$

In the following lemma, we establish Slepian's inequality for SSMSN distributions.

Lemma 3
Let $\mathbf{X}$ and $\mathbf{Y}$ be the random vectors as given in (14). If $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$, $\boldsymbol{\lambda}_{\boldsymbol{\tau}} \leq \boldsymbol{\lambda}'_{\boldsymbol{\tau}}$ for all $\boldsymbol{\tau} \in \mathbb{S}_H$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}'$, $\sigma_{ij} \leq \sigma'_{ij}$, for all $1 \leq i,j \leq d$, then

$$P(\mathbf{X} \geq \mathbf{t}) \leq P(\mathbf{Y} \geq \mathbf{t}) \quad for \;\; all \; \mathbf{t} \in \mathbb{R}^d.$$

If $\mu_i < \mu'_i$ or $\lambda_{\boldsymbol{\tau},i} < \lambda'_{\boldsymbol{\tau},i}$ for some i or $\sigma_{ij} < \sigma'_{ij}$ for some (i,j), then the above inequality is indeed strict.

Proof
Let

$$\mathbf{Z}_{\boldsymbol{\tau}} \sim N_{d+1}\left(\mathbf{0}, \begin{pmatrix} \mathbf{\Sigma} & \boldsymbol{\lambda}_{\boldsymbol{\tau}} \\ \boldsymbol{\lambda}_{\boldsymbol{\tau}}^T & 1 \end{pmatrix}\right), \quad \mathbf{Z}'_{\boldsymbol{\tau}} \sim N_{d+1}\left(\mathbf{0}, \begin{pmatrix} \mathbf{\Sigma}' & \boldsymbol{\lambda}'_{\boldsymbol{\tau}} \\ \boldsymbol{\lambda}'^T_{\boldsymbol{\tau}} & 1 \end{pmatrix}\right).$$

Then, from (10), we have

$$\begin{aligned} P(\mathbf{X} \geq \mathbf{t}) &= 2\int_{\mathbb{S}_H} P(\mathbf{Z}_{\boldsymbol{\tau}} \geq a_1(\tau_1)^{-1/2}(\mathbf{t} - \boldsymbol{\mu}))dH(\boldsymbol{\tau};\boldsymbol{\eta}), \\ P(\mathbf{Y} \geq \mathbf{t}) &= 2\int_{\mathbb{S}_H} P(\mathbf{Z}'_{\boldsymbol{\tau}} \geq a_1(\tau_1)^{-1/2}(\mathbf{t} - \boldsymbol{\mu}'))dH(\boldsymbol{\tau};\boldsymbol{\eta}), \end{aligned}$$

and the result then follows from Slepian's inequality. □

Now, in the following lemma, we provide sufficient conditions for stochastic comparisons of SSMSN distributions.

Lemma 4
Let $\mathbf{X}$ and $\mathbf{Y}$ be the random vectors as given in (14), and

$$\begin{aligned}\boldsymbol{\mu}_p &= p\boldsymbol{\mu} + (1-p)\boldsymbol{\mu}',\\ \boldsymbol{\lambda}_p &= p\boldsymbol{\lambda}_{\boldsymbol{\tau}} + (1-p)\boldsymbol{\lambda}'_{\boldsymbol{\tau}},\\ \boldsymbol{\Sigma}_p &= p\boldsymbol{\Sigma} + (1-p)\boldsymbol{\Sigma}',\end{aligned} \tag{16}$$

for $0 \le p \le 1$. Further, let $f : \mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable and for all $\boldsymbol{\tau} \in \mathbb{S}_H$, $k = 1, 2$ satisfy the following conditions:

(i) $\lim_{x_j \to \pm\infty} f(\mathbf{x})\varphi_k(\mathbf{x}; p|\boldsymbol{\tau}) = 0$ for all $1 \le j \le d$, $k = 1, 2$,
(ii) $\lim_{x_j \to \pm\infty} f(\mathbf{x})\frac{\partial}{\partial x_i}\varphi_1(\mathbf{x}; p|\boldsymbol{\tau}) = 0$ for all $1 \le i, j \le d$,
(iii) $\lim_{x_j \to \pm\infty} \varphi_1(\mathbf{x}; p|\boldsymbol{\tau})\frac{\partial}{\partial x_i} f(\mathbf{x}) = 0$ for all $1 \le i, j \le d$,

where $\varphi_1(\mathbf{x}; p|\boldsymbol{\tau})$ is the pdf of $SN_d(\boldsymbol{\mu}_p, a_1(\tau_1)\boldsymbol{\Sigma}_p, a_2(\boldsymbol{\tau})\boldsymbol{\alpha}_p)$, $\boldsymbol{\alpha}_p$ is obtained by using the parameters in (16) in Equation (13), and $\varphi_2(.; p|\boldsymbol{\tau}) = \phi_d(.; \boldsymbol{\mu}_p, a_1(\tau_1)[\boldsymbol{\Sigma}_p - \boldsymbol{\lambda}_p\boldsymbol{\lambda}_p^T])$. If the following conditions

$$\begin{aligned}&\textstyle\sum_{i=1}^d (\mu'_i - \mu_i)\frac{\partial}{\partial x_i} f(\mathbf{x}) \ge 0,\\ &\textstyle\sum_{i=1}^d (\lambda'_{\boldsymbol{\tau},i} - \lambda_{\boldsymbol{\tau},i})\frac{\partial}{\partial x_i} f(\mathbf{x}) \ge 0,\\ &\textstyle\sum_{i=1}^d \sum_{j=1}^d (\sigma'_{ij} - \sigma_{ij})\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) \ge 0\end{aligned} \tag{17}$$

hold, then $E(f(\mathbf{X})) \le E(f(\mathbf{Y}))$.

Proof
Suppose $\mathbf{Z}_p$, given $\boldsymbol{\tau}$, has the pdf $\varphi_1(.; p|\boldsymbol{\tau})$. Then,

$$\begin{aligned}E(f(\mathbf{Y}) - f(\mathbf{X})) &= \textstyle\int_0^1 \int_{\mathbb{S}_H} \frac{\partial}{\partial p} E(f(\mathbf{Z}_p)|\boldsymbol{\tau}) dH(\boldsymbol{\tau}; \boldsymbol{\eta}) dp\\ &= \textstyle\int_0^1 \int_{\mathbb{S}_H} \int_{\mathbb{R}^d} f(\mathbf{z}) \frac{\partial}{\partial p}\varphi_1(\mathbf{z}; p|\boldsymbol{\tau}) d\mathbf{z} dH(\boldsymbol{\tau}; \boldsymbol{\eta}) dp.\end{aligned} \tag{18}$$

Considering (11), we can write the pdf of $\mathbf{Z}_p$, given $\boldsymbol{\tau}$, in terms of its characteristic function, $\Psi(.; p|\boldsymbol{\eta})$, as follows:

$$\begin{aligned}\varphi_1(\mathbf{z}; p|\boldsymbol{\tau}) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(-i\mathbf{t}^T\mathbf{z})\Psi(\mathbf{t}; p|\boldsymbol{\tau}) d\mathbf{t}\\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp\left\{-i\mathbf{t}^T(\mathbf{z} - \boldsymbol{\mu}_p) - \frac{a_1(\tau_1)}{2}\mathbf{t}^T\boldsymbol{\Sigma}_p\mathbf{t}\right\}\left\{1 + i\delta\left(\sqrt{a_1(\tau_1)}\boldsymbol{\lambda}_p^T\mathbf{t}\right)\right\} d\mathbf{t}.\end{aligned}$$

Upon differentiating the above expression with respect to p and then integrating over $\mathbb{R}^d$, we obtain

$$\begin{aligned}\frac{\partial}{\partial p}\varphi_1(\mathbf{z}; p|\boldsymbol{\tau}) &= \frac{a_1(\tau_1)}{2}\sum_{i=1}^d \sum_{j=1}^d (\sigma'_{ij} - \sigma_{ij}) \frac{\partial^2}{\partial z_i \partial z_j}\varphi_1(\mathbf{z}; p|\boldsymbol{\tau})\\ &- \sum_{i=1}^d ({\mu'}_i - \mu_i)\frac{\partial}{\partial z_i}\varphi_1(\mathbf{z}; p|\boldsymbol{\tau})\\ &- \frac{2a_1(\tau_1)^{1/2}}{\sqrt{2\pi}}\sum_{i=1}^n (\lambda'_i - \lambda_i)\frac{\partial}{\partial z_i}\varphi_2(\mathbf{z}; p|\boldsymbol{\tau}).\end{aligned}$$

Using the expression above in (18), and integrating by part and using the conditions (1)-(3), we derive the following:

$$\begin{aligned} E(f(\mathbf{Y})-f(\mathbf{X})) &= \int_0^1 \int_{\mathbb{S}_H} \int_{\mathbb{R}^n} (\boldsymbol{\mu}'-\boldsymbol{\mu})^T \nabla_f(\mathbf{z})\, \varphi_1(\mathbf{z};p|\boldsymbol{\tau}) d\mathbf{z} dH(\boldsymbol{\tau};\boldsymbol{\eta})dp \\ &+ \int_0^1 \int_{\mathbb{S}_H} \int_{\mathbb{R}^n} \frac{a_1(\tau_1)}{2} Trace\left[\left(\boldsymbol{\Sigma}'-\boldsymbol{\Sigma}\right) H_f(\mathbf{z})\right] \varphi_1(\mathbf{z};p|\boldsymbol{\tau})\, d\mathbf{z} dH(\boldsymbol{\tau};\boldsymbol{\eta})dp \\ &+ \int_0^1 \int_{\mathbb{S}_H} \int_{\mathbb{R}^n} \sqrt{\frac{2a_1(\tau_1)}{\pi}} \left(\boldsymbol{\lambda}'_{\boldsymbol{\tau}}-\boldsymbol{\lambda}_{\boldsymbol{\tau}}\right)^T \nabla_f(\mathbf{z})\, \varphi_2(\mathbf{z};p|\boldsymbol{\tau})\, d\mathbf{z} dH(\boldsymbol{\tau};\boldsymbol{\eta})dp. \end{aligned}$$

Now, by using the conditions in (17), we obtain the required result. □

Remark 2
Observe that the conditions (i)-(iii) in Lemmas 2 and 4 are weak regularity conditions, which assure the existence of all occurring integrals. They are always fulfilled if the function f, together with its first derivatives, fulfils a polynomial growth condition at infinity [34].

Remark 3
The family of SSMSN distributions is quite large, and contains several subfamilies of asymmetric distributions discussed considerably in the literature due to some desirable properties. Some well-known SSMSN subfamilies are as follows:

1. Scale mixture of SN distribution [15] when $a_2(\boldsymbol{\tau})=1$. For $a_1(\tau)=\tau^{-1}$, we have the multivariate skew-t [7] when $\tau \sim Gamma(\nu/2,\nu/2)$ and skew-Cauchy when $\nu=1$, the multivariate skew-slash [47] when $\tau \sim Beta(\nu,1)$, the skew-contaminated-normal [30] when τ has a discrete distribution with support $\mathbb{S}_H=\{\gamma,1\}$, finite mixture of SN distributions [15] when τ has a discrete distribution with support $\mathbb{S}_H=\{\tau_1,...,\tau_k\}$;
2. Shape mixture of SN distribution when $a_1=1$ and $a_2(\boldsymbol{\tau})=s(\tau)$. The multivariate skew-generalized-normal [3] when $s(\tau)=\tau\sim N(1,a)$, the multivariate skew-normal-Cauchy [27] when $s(\tau)=|\tau|$ and $\tau\sim N(0,1)$;
3. If $a_2(\tau)=\sqrt{a_1(\tau)}$, then two examples are the multivariate skew-t-normal and multivariate skew-slash-normal distributions, when $\tau\sim Gamma(\nu/2,\nu/2)$ and $\tau\sim Beta(\nu,1)$, respectively [20]. If $a_1(\tau_1)=\tau_1^{-1}$ and $a_2(\boldsymbol{\tau})=\tau_2^{-1/2}$, where τ_i are independent and $\tau_i\sim Gamma(\nu_i/2,\nu_i/2)$, $i=1,2$, we have the skew-t-t distribution [26].

3. Univariate stochastic orders

Some results on the univariate stochastic ordering of random variables with NMVM and SSMSN distributions are briefly presented here. We will use them later on for the proofs of necessity parts.

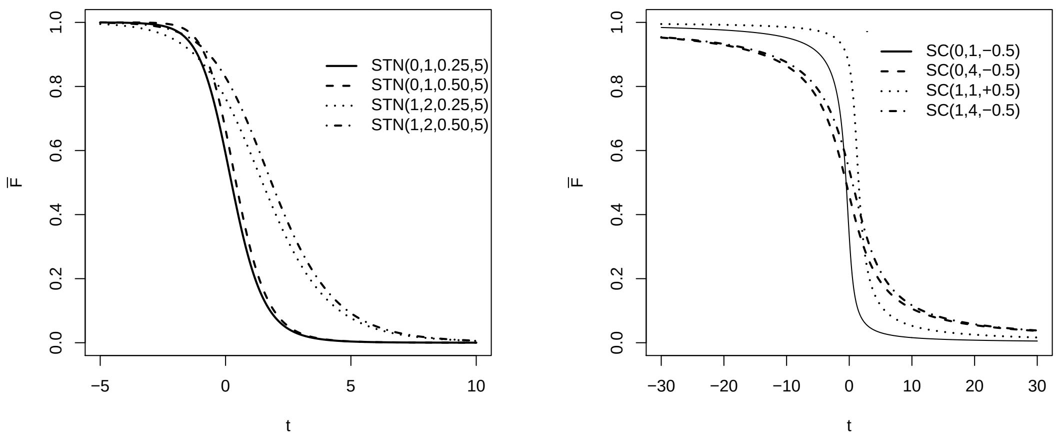
Let X_1 and X_2 be random variables with pdfs f_1 and f_2, cdfs F_1 and F_2, and survival functions $\overline{F}_1=1-F_1$ and $\overline{F}_2=1-F_2$, respectively. An equivalent condition in Definition 1 for the usual stochastic order $X_1\preceq_{st}X_2$ is that $\overline{F}_1(t)\le\overline{F}_2(t)$ for all $t\in\mathbb{R}$. The following Lemma provides necessary and sufficient conditions for the comparison of the general class of univariate SSMSN distribution [26].

Lemma 5
Let X_1 and X_2 be two SSMSN random variables given by

$$X_i|\boldsymbol{\tau}\sim SN_1(\mu_i,a_1(\tau_1)\sigma_i^2,a_2(\boldsymbol{\tau})\alpha_i),\quad i=1,2. \tag{19}$$

Then, $X_1\preceq_{st}X_2$ if and only if $\mu_1\le\mu_2$, $\sigma_1=\sigma_2$ and $\alpha_1\le\alpha_2$ $(\lambda_{\boldsymbol{\tau},1}\le\lambda_{\boldsymbol{\tau},2})$.

Proof
The sufficiency of $\mu_1\le\mu_2$, $\sigma_1=\sigma_2$ and $\alpha_1\le\alpha_2$ $(\lambda_{\boldsymbol{\tau},1}\le\lambda_{\boldsymbol{\tau},2})$ follows from Slepian's inequality in Lemma 3.



(i) (ii)

Figure 1. Plots of survival functions of (i) skew-t-normal distribution and (ii) skew-Cauchy distribution.

On the other hand, $X_1 \preceq_{st} X_2$ can only hold if

$$\begin{array}{l} \lim_{t\to+\infty}(f_2(t) - f_1(t)) = \int_{\mathbb{S}_H} \lim_{t\to+\infty} D(t;\boldsymbol{\tau})dH(\boldsymbol{\tau};\boldsymbol{\eta}) \geq 0, \\ \lim_{t\to-\infty}(f_2(t) - f_1(t)) = \int_{\mathbb{S}_H} \lim_{t\to-\infty} D(t;\boldsymbol{\tau})dH(\boldsymbol{\tau};\boldsymbol{\eta}) \leq 0, \end{array} \tag{20}$$

where $D(t;\boldsymbol{\tau}) = f_2(t|\boldsymbol{\tau}) - f_1(t|\boldsymbol{\tau})$. With $g\,(t;\boldsymbol{\tau}) = \frac{f_2(t|\boldsymbol{\tau})}{f_1(t|\boldsymbol{\tau})}$, from (8), we then get

$$\begin{aligned} g\,(t;\boldsymbol{\tau}) &= \frac{\sigma_1}{\sigma_2}\exp\left[\frac{1}{a_1(\boldsymbol{\tau})}\left[\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_2^2}\right)\frac{t^2}{2}+\left(\frac{\mu_2}{\sigma_2^2}-\frac{\mu_1}{\sigma_1^2}\right)t+\left(\frac{\mu_2^2}{\sigma_2^2}-\frac{\mu_1^2}{\sigma_1^2}\right)\right]\right] \\ &\times \frac{\Phi\left(\frac{a_2(\boldsymbol{\tau})\alpha_2}{\sigma_2\sqrt{a_1(\tau_1)}}\,(t-\mu_2)\right)}{\Phi\left(\frac{a_2(\boldsymbol{\tau})\alpha_1}{\sigma_1\sqrt{a_1(\tau_1)}}\,(t-\mu_1)\right)}. \end{aligned}$$

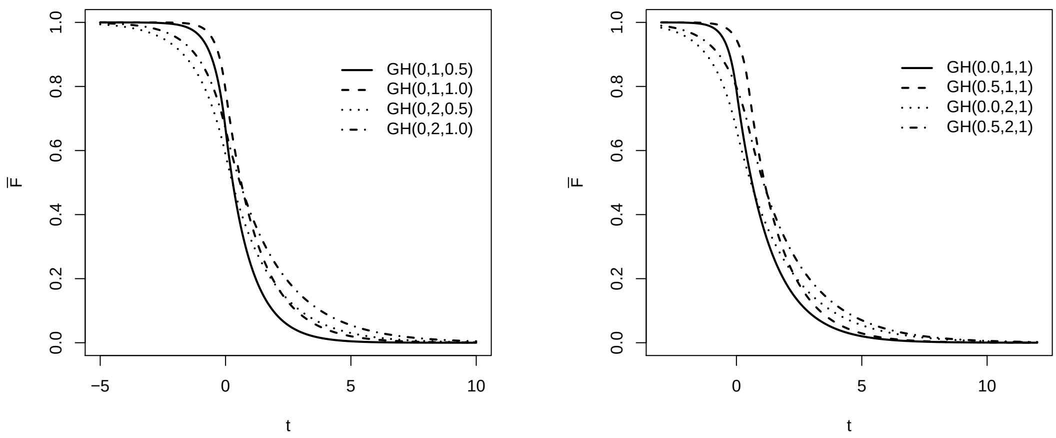
Suppose $\alpha_1\alpha_2 \geq 0$ and conditions $\mu_1 \leq \mu_2$, $\sigma_1 = \sigma_2$ and $\alpha_1 \leq \alpha_2$ $(\lambda_{\boldsymbol{\tau},1} \leq \lambda_{\boldsymbol{\tau},2})$ hold. Otherwise, for $\boldsymbol{\tau} \in \mathbb{S}_H$ we have $\lim_{t\to-\infty} g(t;\boldsymbol{\tau}) = +\infty$ implying $D(t;\boldsymbol{\tau}) > 0$ for sufficiently large negative t, or $\lim_{t\to+\infty} g(t;\boldsymbol{\tau}) = 0$ implying $D(t;\boldsymbol{\tau}) < 0$ for sufficiently large positive t. Then, the necessary conditions in (20) will not be established. Similarly, suppose $\alpha_1\alpha_2 < 0$, it would follow $\alpha_1 \leq 0 \leq \alpha_2$, $\mu_1 \leq \mu_2$ and $\sigma_1 = \sigma_2$. □

To illustrate the result of Lemma 5, the survival functions for two special cases of univariate SSMSN distributions are plotted in Figure 1: (i) the skew-t-normal distributions [22] denoted by $STN(\mu,\sigma,\alpha,\nu)$, and (ii) the skew-Cauchy distribution [12] denoted by $SC(\mu,\sigma,\alpha)$. Distributions whose parameters satisfy the conditions of Lemma 5 are stochastically ordered. Otherwise, their survival functions intersect each other and are therefore not stochastically ordered.

We now explore the conditions for the usual stochastic ordering in the univariate NMVM distributions. Let X_1 and X_2 be two random variables from NMVM distribution in (2) as follows:

$$X_i \sim NMVM_1(\mu_i,\sigma_i^2,\lambda_i,h) \ \ for \ \ i=1,2. \tag{21}$$

Considering the conditions $\mu_1 \leq \mu_2$, $\sigma_1 = \sigma_2$ and $\lambda_1 \leq \lambda_2$ and then using Slepian's inequality in Lemma 1, we conclude that $X_1 \preceq_{st} X_2$. Unfortunately, we are not able to show that these conditions are necessary for $X_1 \preceq_{st} X_2$, and so we are unable to provide necessary and sufficient conditions for the usual stochastic order of univariate NMVM distributions. However, in the special cases when $\mu_1 = \mu_2$ and $\lambda_1 = \lambda_2$, we can characterize the usual stochastic order.



(i) (ii)

Figure 2. Plots of survival functions of univariate GH distributions with $W \sim Exp(1)$ and (i) $\mu_1 = \mu_2 = 0$, (ii) $\lambda_1 = \lambda_2 = 1$.

Lemma 6
Let X_1 and X_2 be the random variables as given in (21).

(i) If $\mu_1 = \mu_2$, then $X_1 \preceq_{st} X_2$ if and only if $\sigma_1 = \sigma_2$ and $\lambda_1 \leq \lambda_2$;
(ii) If $\lambda_1 = \lambda_2$, then $X_1 \preceq_{st} X_2$ if and only if $\sigma_1 = \sigma_2$ and $\mu_1 \leq \mu_2$.

Proof
The sufficiency parts of (i) and (ii) are obvious. Now, let $X_1 \preceq_{st} X_2$. Then, we must have

$$\begin{aligned} \lim_{t\to+\infty}(f_2(t) - f_1(t)) = \int_{\mathbb{S}_H} \lim_{t\to+\infty} D(t;w)dH(w;\boldsymbol{\eta}) \geq 0, \\ \lim_{t\to-\infty}(f_2(t) - f_1(t)) = \int_{\mathbb{S}_H} \lim_{t\to-\infty} D(t;w)dH(w;\boldsymbol{\eta}) \leq 0, \end{aligned} \tag{22}$$

where $D(t;w) = \phi(t;\mu_2 + w\lambda_2, w\sigma_2^2) - \phi(t;\mu_1 + w\lambda_1, w\sigma_1^2)$. Let

$$\begin{aligned} g(t;w) &= \frac{\phi(t;\mu_2 + w\lambda_2, w\sigma_2^2)}{\phi(t;\mu_1 + w\lambda_1, w\sigma_1^2)} \\ &= \frac{\sigma_1}{\sigma_2} \exp\left\{ w^{-1}\left[\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)\frac{t^2}{2} + \left(\frac{\mu_2 + w\lambda_2}{\sigma_2^2} - \frac{\mu_1 + w\lambda_1}{\sigma_1^2}\right)t\right]\right\} \\ &\times \exp\left\{ w^{-1}\frac{(\mu_2 + w\lambda_2)^2}{\sigma_2^2} - w^{-1}\frac{(\mu_1 + w\lambda_1)^2}{\sigma_1^2}\right\}. \end{aligned}$$

In both cases (i)-(ii), we claim $\sigma_1 = \sigma_2$. If not, $\sigma_1 < \sigma_2$ implies $g(t;w)$ goes to infinity as $t \to -\infty$ and $\sigma_1 > \sigma_2$ implies $g(t;w)$ goes to 0 as $t \to +\infty$. So, for all $w \in \mathbb{S}_H$, $D(t;w)$ will be positive for large enough positive t and will be negative for large enough negative t. This violates the necessary conditions in (22). In a similar way, we can show that $\lambda_1 \leq \lambda_2$ if $\mu_1 = \mu_2$ and $\mu_1 \leq \mu_2$ if $\lambda_1 = \lambda_2$. □

To illustrate the result of Lemma 6, we consider the special case of univariate NMVM distributions with $W \sim Exp(1)$, which is a univariate generalized hyperbolic distribution (hyperbolic distribution) introduced by [10], denoted by $GH(\mu, \sigma, \lambda)$. Plots of the survival functions of two cases of GH distribution with common locations (i) and GH distribution with common skewness parameters (ii) are shown in Figure 2. Distributions whose parameters satisfy the conditions of Lemma 6 are stochastically ordered, and are not ordered otherwise.

We now consider the increasing convex ordering in the univariate case. An equivalent condition for the increasing convex order $X_1 \preceq_{icx} X_2$ is that [35]

$$E[(X_1-t)_+] \le E[(X_2-t)_+] \text{ for all } t \in \mathbb{R}.$$

In economics literature, the increasing convex order is commonly referred to as stop-loss order. It corresponds to the notion of second order stochastic dominance [40], while in insurance terms it is interpreted as a comparison of the stop-loss premiums of risks for any given retention. In the univariate case studied here, $\preceq_{sl}$ is used to denote the stop-loss order.

We consider the univariate SSMSN distributions in (19) with $a_2(\boldsymbol{\tau}) = 1$, which belongs to scale mixtures of SN distributions introduced by Branco and Dey [15]. In this case, the parameter $\boldsymbol{\lambda_\tau}$ in (12) does not depend on $\boldsymbol{\tau}$ and so we remove it from the index. In the following lemma, we derive conditions for increasing convex order of random variables from univariate SSMSN distributions with common locations.

Lemma 7
Let the random variables X_1 and X_2 distributed as

$$X_i|\tau \sim SN_1(\mu_i, a_1(\tau)\sigma_i^2, \alpha_i), \quad i = 1, 2. \tag{23}$$

If $\mu_1 = \mu_2$, then $X_1 \preceq_{icx} X_2$ if and only if $\lambda_1 \le \lambda_2$ and $\sigma_1 \le \sigma_2$, where $\lambda_i = \frac{\sigma_i \alpha_i}{\sqrt{1+\alpha_i^2}}$.

Proof
For the if part, let $f : \mathbb{R} \to \mathbb{R}$ be an increasing convex function. Then, conditions $\mu_1 = \mu_2$, $\lambda_1 \le \lambda_2$ and $\sigma_1 \le \sigma_2$ provides the conditions in (17) in Lemma 4, and so we have $X_1 \preceq_{icx} X_2$. To show the only if part, let us assume that $X_1 \preceq_{icx} X_2$. Then, $E(X_1) \le E(X_2)$, and so using (15) and considering $\mu_1 = \mu_2$, we have $\lambda_1 \le \lambda_2$. Let us assume that $\sigma_1 > \sigma_2$. Then, $\lim_{t\to+\infty} f_2(t;\tau)/f_1(t;\tau) = 0$ for all $\tau \in \mathbb{S}_H$ (see the proof of Lemma 5), and therefore

$$E(X_1-t)_+ = \int_t^{+\infty} \overline{F}_1(x|\tau)dx > \int_t^{+\infty} \overline{F}_2(x|\tau)dx = E(X_2-t)_+$$

for sufficiently large t, which is a contradiction to $X_1 \preceq_{icx} X_2$. □

We now get the the same result in the above lemma for the univariate NMVM distributions.

Lemma 8
Let the random variables X_1 and X_2 be distributed as

$$X_i \sim NMVM_1(\mu_i, \sigma_i^2, \lambda_i, h), \quad i = 1, 2. \tag{24}$$

If $\mu_1 = \mu_2$, then $X_1 \preceq_{icx} X_2$ if and only if $\lambda_1 \le \lambda_2$ and $\sigma_1 \le \sigma_2$.

Proof
The proof is similar to that of Lemma 7 and is therefore not presented here for the sake of brevity. □

Remark 4
Results of usual stochastic order in Lemma 5 can be applied for the univariate version of the distributions in Remark 3. The stop-loss order in Lemma 7 can be used to order the random variables from the univariate distributions in Case 1 of Remark 3. Also, the usual stochastic order in Lemma 6 and stop-loss order in Lemma 8 can be used to order all of the univariate versions of distributions in Remark 1.

4. Concordance order and portfolio risk

In this section, it is shown that for bivariate SMSSN and NMVM distributed random vectors, the concordance ordering is equivalent to the ordering of correlation coefficients. We note that a similar result has been proved for the normal [35] and elliptical [29] families of distributions.

Consider risks $X_1 \stackrel{d}{=} Y_1$ and $X_2 \stackrel{d}{=} Y_2$ with probability distributions F_1 and F_2, respectively, where $\stackrel{d}{=}$ denotes the equality in distribution. The random vectors (X_1, X_2) and (Y_1, Y_2) are then different only in the way that their elements depend on each other. The *Frechet Space* $\Re_2(F_1, F_2)$ is defined as the space of two-dimensional random vectors with fixed marginals F_1 and F_2 [19]. Concordance order can also be understood via the following result [19, 35].

Lemma 9
Consider the random vectors $(X_1, X_2), (Y_1, Y_2) \in \Re_2(F_1, F_2)$. Then, $(X_1, X_2) \preceq_{conc} (Y_1, Y_2)$ if and only if

$$Cov(h_1(X_1), h_2(X_2)) \leq Cov(h_1(Y_1), h_2(Y_2))$$

for all increasing functions h_1 and h_2, when the involved covariances exist.

Firstly, suppose $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ belong to $\Re_2(F_1, F_2)$ and

$$\mathbf{X}|\tau \sim SN_2(\boldsymbol{\mu}_{\mathbf{X}}, a_1(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}, \boldsymbol{\alpha}_{\mathbf{X}}), \qquad \mathbf{Y}|\tau \sim SN_2(\boldsymbol{\mu}_{\mathbf{Y}}, a_1(\tau)\boldsymbol{\Sigma}_{\mathbf{Y}}, \boldsymbol{\alpha}_{\mathbf{Y}}). \tag{25}$$

To deal with cases when the covariance matrix does not exist, a generalized correlation coefficient for bivariate vectors in (25) with

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} \sigma_{\mathbf{X},11} & \sigma_{\mathbf{X},12} \\ \sigma_{\mathbf{X},21} & \sigma_{\mathbf{X},22} \end{bmatrix}, \qquad \boldsymbol{\Sigma}_{\mathbf{Y}} = \begin{bmatrix} \sigma_{\mathbf{Y},11} & \sigma_{\mathbf{Y},12} \\ \sigma_{\mathbf{Y},21} & \sigma_{\mathbf{Y},22} \end{bmatrix},$$

is defined as

$$\rho_{\mathbf{X}} = \frac{\sigma_{\mathbf{X},12}}{\sqrt{\sigma_{\mathbf{X},11}\sigma_{\mathbf{X},22}}}, \qquad \rho_{\mathbf{Y}} = \frac{\sigma_{\mathbf{Y},12}}{\sqrt{\sigma_{\mathbf{Y},11}\sigma_{\mathbf{Y},22}}}. \tag{26}$$

Of course, the generalized correlation coefficients coincide with the usual (Pearson) correlation coefficient if covariance matrices exist. From Lemma 3, we immediately have the following results.

Corollary 1
Consider the bivariate SSMSN random vectors $\mathbf{X}$ and $\mathbf{Y}$ in (25). Then, $\mathbf{X} \preceq_{conc} \mathbf{Y}$ if and only if $\rho_{\mathbf{X}} \leq \rho_{\mathbf{Y}}$.

Corollary 2
Consider the bivariate NMVM random vectors $\mathbf{X}, \mathbf{Y} \in \Re_2(F_1, F_2)$ as

$$\mathbf{X} \sim NMVM_2(\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\Sigma}_{\mathbf{X}}, \boldsymbol{\lambda}_{\mathbf{X}}, h), \qquad \mathbf{Y} \sim NMVM_2(\boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\Sigma}_{\mathbf{Y}}, \boldsymbol{\lambda}_{\mathbf{Y}}, h). \tag{27}$$

Then, $\mathbf{X} \preceq_{conc} \mathbf{Y}$ if and only if $\rho_{\mathbf{X}} \leq \rho_{\mathbf{Y}}$.

It is apparent from Lemma 9 that concordance order is invariant under monotone transformations of the random variables considered. This implies that concordance order relates only to the copulas of the random vectors. Copulas are joint distributions with uniform marginals, which summarize the dependence structure of random vectors [37]. It is noted that, in the two-dimensional case, concordance order is equivalent to the supermodular order [35]. Hence, in higher dimensions, the supermodular order can be viewed as a generalization of the concordance order.

Dhaene and Goovaerts [19] showed that a portfolio consisting of two positive random variables becomes more risky in the stop-loss order sense when the two risks become more concordant. Landsman and Tsanakas [29] proved this result for the elliptical family. A stronger version can be obtained for SSMSN and NMVM distributed risks, as established below.

Proposition 1
Consider $\mathbf{X}, \mathbf{Y} \in \Re_2(F_1, F_2)$, where $\mathbf{X}$ and $\mathbf{Y}$ are random vectors for both cases (25) and (27). Then, $X_1 + X_2 \preceq_{icx} Y_1 + Y_2$ if and only if $\rho_{\mathbf{X}} \leq \rho_{\mathbf{Y}}$.

Proof
By Lemma 7 and (11), and Lemma 8 and (2), respectively, for the cases (25) and (27), we have $X_1 + X_2 \preceq_{icx} Y_1 + Y_2$ if and only if $\sigma_{\mathbf{X},11} + \sigma_{\mathbf{X},22} + 2\sigma_{\mathbf{X},12} \leq \sigma_{\mathbf{Y},11} + \sigma_{\mathbf{Y},22} + 2\sigma_{\mathbf{X},12}$. Since $\sigma_{\mathbf{X},11} = \sigma_{\mathbf{Y},11}$ and $\sigma_{\mathbf{X},22} = \sigma_{\mathbf{Y},22}$, it becomes equivalent to $\rho_{\mathbf{X}} \leq \rho_{\mathbf{Y}}$, as required. □

Remark 5
By Proposition 1, we can order the portfolio risks in the sense of stop-loss order, by ordering the correlations when the risks jointly follow bivariate versions of the distribution families discussed in Remark 1 and Case 1 of Remark 3.

5. Multivariate stochastic orders

Now, we focus our attention on the integral stochastic orderings of the multivariate case.

Corollary 3
Let the random vectors $\mathbf{X}$ and $\mathbf{Y}$ be as given in (5) and (14).

(i) If $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$, $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}'$ and $\boldsymbol{\lambda}_{\boldsymbol{\tau}} \leq \boldsymbol{\lambda}'_{\boldsymbol{\tau}}$ for all $\boldsymbol{\tau} \in \mathbb{S}_H(\boldsymbol{\lambda} \leq \boldsymbol{\lambda}')$, then $\mathbf{X} \preceq_{st} \mathbf{Y}$;
(ii) If $\boldsymbol{\mu} = \boldsymbol{\mu}'$, $\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}$ is positive semi-definite and $\boldsymbol{\lambda}_{\boldsymbol{\tau}} = \boldsymbol{\lambda}'_{\boldsymbol{\tau}}$ for all $\boldsymbol{\tau} \in \mathbb{S}_H(\boldsymbol{\lambda} = \boldsymbol{\lambda}')$, then $\mathbf{X} \preceq_{cx} \mathbf{Y}$;
(iii) If $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$, $\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}$ is positive semi-definite and $\boldsymbol{\lambda}_{\boldsymbol{\tau}} \leq \boldsymbol{\lambda}'_{\boldsymbol{\tau}}$ for all $\boldsymbol{\tau} \in \mathbb{S}_H(\boldsymbol{\lambda} \leq \boldsymbol{\lambda}')$, then $\mathbf{X} \preceq_{icx} \mathbf{Y}$;
(iv) If $\boldsymbol{\mu} = \boldsymbol{\mu}'$, $\sigma_{ij} \leq \sigma'_{ij}$ and $\boldsymbol{\lambda}_{\boldsymbol{\tau}} = \boldsymbol{\lambda}'_{\boldsymbol{\tau}}$ for all $\boldsymbol{\tau} \in \mathbb{S}_H(\boldsymbol{\lambda} = \boldsymbol{\lambda}')$, then $\mathbf{X} \preceq_{dcx} \mathbf{Y}$ and $\mathbf{X} \preceq_{sm} \mathbf{Y}$.

Again, we note that when $a_2(\boldsymbol{\tau}) = 1$, the index $\boldsymbol{\tau}$ in $\boldsymbol{\lambda}_{\boldsymbol{\tau}}$ will be removed.

we now consider the necessary and sufficient conditions for the characterization of the integral order of the SSMSN and NMVM families of distributions. To begin with, we consider the usual stochastic order.

Proposition 2
Suppose $\mathbf{X}$ and $\mathbf{Y}$ are SSMSN distributed random vectors as given in (14). Then, $\mathbf{X} \preceq_{st} \mathbf{Y}$ if and only if $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$, $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}'$ and $\boldsymbol{\lambda}_{\boldsymbol{\tau}} \leq \boldsymbol{\lambda}'_{\boldsymbol{\tau}}$ for all $\boldsymbol{\tau} \in \mathbb{S}_H$.

Proof
The if part follows immediately from Corollary 3. Conversely, $\mathbf{X} \preceq_{st} \mathbf{Y}$ implies $X_i \preceq_{st} Y_i$ and $X_i + X_j \preceq_{st} Y_i + Y_j$ for all $1 \leq i, j \leq d$. Then, we can deduce from Lemma 5 that we must have $\mu_i \leq \mu'_i$, $\sigma_{ii} = \sigma'_{ii}$, $\lambda_{\boldsymbol{\tau},i} \leq \lambda'_{\boldsymbol{\tau},i}$ and then $\sigma_{ij} = \sigma'_{ij}$ for $i \neq j$. □

Proposition 3
Suppose $\mathbf{X}$ and $\mathbf{Y}$ are the NMVM distributed random vectors as given in (5).

(i) If $\boldsymbol{\mu} = \boldsymbol{\mu}'$, then $\mathbf{X} \preceq_{st} \mathbf{Y}$ if and only if $\boldsymbol{\lambda} \leq \boldsymbol{\lambda}'$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}'$;
(ii) If $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, then $\mathbf{X} \preceq_{st} \mathbf{Y}$ if and only if $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}'$.

Proof
The result can be proved along the lines of Proposition 2, but we refrain from the presenting proof here for the sake of brevity. □

In what follows, we consider the random vectors in (14) with $a_2(\boldsymbol{\tau}) = 1$ and use $\boldsymbol{\lambda}$ instead of $\boldsymbol{\lambda}_{\boldsymbol{\tau}}$. Using (12), we have

$$\boldsymbol{\lambda} = \left(1 + \boldsymbol{\alpha}^T \overline{\boldsymbol{\Sigma}} \boldsymbol{\alpha}\right)^{-1/2} \boldsymbol{\sigma} \overline{\boldsymbol{\Sigma}} \boldsymbol{\alpha}.$$

So, we consider the following random vectors:

$$\mathbf{X}|\boldsymbol{\tau} \sim SN_d(\boldsymbol{\mu}, a_1(\tau_1)\boldsymbol{\Sigma}, \boldsymbol{\alpha}), \qquad \mathbf{Y}|\boldsymbol{\tau} \sim SN_d(\boldsymbol{\mu}', a_1(\tau_1)\boldsymbol{\Sigma}', \boldsymbol{\alpha}'). \tag{28}$$

It is evident from Definition 1 and (1) that every Δ-monotone function is supermodular, and so it is clear that the supermodular order is stronger than the upper orthant order. Also, the super modular order is stronger than the concordance order. In the following proposition, we characterize the supermodular ordering of random vectors in (2) and (28) by comparing their covariances.

Proposition 4
Suppose the random vectors $\mathbf{X}$ and $\mathbf{Y}$ are as given in both cases (2) and (28). Then, $\mathbf{X} \preceq_{sm} \mathbf{Y}$ if and only if $\mathbf{X}$ and $\mathbf{Y}$ have the same marginals ($X_i \overset{d}{=} Y_i$,$i = 1, ..., d$) and $\sigma_{ij} \leq \sigma'_{ij}$ for any $1 \leq i \neq j \leq d$.

Proof
Suppose $\mathbf{X} \preceq_{sm} \mathbf{Y}$. It can only hold if the random vectors have the same marginals. Then, using Lemmas 1 and 3, we can conclude that $\sigma_{ij} \leq \sigma'_{ij}$. Then, Corollary 3 yields the converse, and hence the result. □

The convex order and its extensions, like directionally convex and increasing convex orders, have assumed great interest, recently. Müller [34] derived necessary and sufficient conditions for these orders in the case of multivariate normal distribution and Pan et al. [38] derived the result in the case of multivariate elliptical distributions. Here, we now establish necessary and sufficient conditions for the case of more general NMVM and SSMSN families.

Proposition 5 (i) If $\boldsymbol{\mu} = \boldsymbol{\mu}'$, then $\mathbf{X} \preceq_{cx} \mathbf{Y}$ if and only if $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ and $\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}$ is positive semi-definite;
(ii) If $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, then $\mathbf{X} \preceq_{cx} \mathbf{Y}$ if and only if $\boldsymbol{\mu} = \boldsymbol{\mu}'$ and $\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}$ is positive semi-definite.

Proof
The sufficiency parts of both cases (i) and (ii) immediately follow from the stated conditions and Corollary 3. To show the converse, let us suppose $\mathbf{X} \preceq_{cx} \mathbf{Y}$. Hence, $E(\mathbf{X}) = E(\mathbf{X})$, using (4) and (15), for part (i) $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ and for part (ii) $\boldsymbol{\mu} = \boldsymbol{\mu}'$ follow. Now, we claim that $\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}$ should be positive semi-definite. Since otherwise, there exists some $\mathbf{a} \in \mathbb{R}^d$ such that $\mathbf{a}^T (\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}) \mathbf{a} < 0$. Let $f(\mathbf{X}) = \left(\mathbf{a}^T \mathbf{X}\right)^2$, which is convex. Again, for both cases (i) and (ii), using (4) and (15), respectively, we can get the contradiction $E(f(\mathbf{X})) - E(f(\mathbf{Y})) > 0$. □

Proposition 6 (i) If $\boldsymbol{\mu} = \boldsymbol{\mu}'$, then $\mathbf{X} \preceq_{dcx} \mathbf{Y}$ if and only if $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ and $\sigma_{ij} \leq \sigma'_{ij}$ for all $1 \leq i \neq j \leq d$;
(ii) If $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, then $\mathbf{X} \preceq_{dcx} \mathbf{Y}$ if and only if $\boldsymbol{\mu} = \boldsymbol{\mu}'$ and $\sigma_{ij} \leq \sigma'_{ij}$ for all $1 \leq i \neq j \leq d$.

Proof
The sufficiency parts of (i) and (ii) follow from the stated conditions and Corollary 3. Conversely, let us assume $\mathbf{X} \preceq_{dcx} \mathbf{Y}$. It follows $E(\mathbf{X}) = E(\mathbf{X})$, and then using (4) and (15) for cases (i) and (ii), respectively, we conclude $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ and $\boldsymbol{\mu} = \boldsymbol{\mu}'$. Consider the directionally convex function $f(\mathbf{X}) = x_i x_j$. Then, from $E(X_i X_j) \leq E(Y_i Y_j)$, relations (4) and (15) for cases (i) and (ii), respectively, it follows that $\sigma_{ij} \leq \sigma'_{ij}$. □

Remark 6
All the ordering results in Section 5 can be used to compare random vectors from the multivariate distributions in Remark 1 with common location vectors or common skewness vectors. Characterization of usual stochastic order in Proposition 2 can be applied in the general form of multivariate families in Remark 3, and the two orders coincide in these families. The result of supermodular order in Proposition 4, convex order in Proposition 5 and directionally convex order in Proposition 6 can all be used for comparing all the multivariate distributions in Case 1 of Remark 3 with common location vectors or common skewness vectors.

6. Concluding remarks

By considering random vectors from the multivariate NMVM and SSMSN distributions, we have established necessary and sufficient conditions for comparing the vectors using integral orders. We have derived necessary and sufficient conditions for the usual order in the case of SSMSN families in general. By considering the random vectors with common locations or with common skewness vectors, we have characterized the integral orders in the NMVM and SSMSN families. For stochastic orderings here, we use the integral orders of the usual stochastic, concordance, supermodular, convex, increasing convex and directionally convex orders. We show that for bivariate random vectors, the riskiness and dependence strength of random portfolios, in the sense of stop-loss and bivariate concordance stochastic orders, respectively, can be simply characterized in terms of the order of correlations. It will be of great interest to extend the results established here to some other general families such as unified skew-normal, unified skew-elliptical and selection distributions. We are currently working on this problem and hope to report the findings in a future paper.

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