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Nonsmooth Vector Optimization Problem Involving Second-Order Semipseudo, Semiquasi Cone-Convex Functions

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Abstract Recently, Suneja et al. [26] introduced new classes of second-order cone-(η, ξ)-convex functions along with their generalizations and used them to prove second-order Karush–Kuhn–Tucker (KKT) type optimality conditions and duality results for the vector optimization problem involving first-order differentiable and second-order directionally differentiable functions. In this paper, we move one step ahead and study a nonsmooth vector optimization problem wherein the functions involved are first and second-order directionally differentiable. We introduce new classes of nonsmooth second-order cone-semipseudoconvex and nonsmooth second-order cone-semiquasiconvex functions in terms of second-order directional derivatives. Second-order KKT type sufficient optimality conditions and duality results for the same problem are proved using these functions.

Keywords Vector optimization, Cones, second-order cone-semipseudoconvexity (semiquasiconvexity), Second-order Optimality, Duality

AMS 2010 subject classifications 90C29, 90C46, 90C25, 90C26

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1. Introduction

Second-order optimality conditions have been widely studied for past many years because they refine first-order by second-order information which is very useful for recognizing efficient solutions. These conditions have important applications in sensitivity analysis and optimal algorithms, for example penalty methods [20, 24].

Various types of second-order (cone) convex functions like second-order (F, ρ) convex [2], second-order (F, α, ρ, d) convex [3], second-order cone-convex [25] and recently many others like second-order univexities, second-order hybrid $(\Phi, \rho, \eta, \zeta, \theta)$-invexity [29, 30, 31] along with their weaker notions have been defined for twice differentiable functions and used to study second-order duality results for multiobjective and vector optimization problems. Mangasarian [21] first formulated the second-order dual involving second-order derivatives for nonlinear programming problem and established second-order duality results under certain inclusion conditions. By introducing two additional parameters, Hanson [15] formulated a second-order dual similar to that of Mangasarian [21] and established duality results under the assumption of second-order type I invexity. Mishra [22] deduced second-order duality results involving second-order derivatives for multiobjective programming problem using classes of second order pseudo-type I, second-order quasi-type I and related functions. Recently, Jayswal and Jha [19] and Dubey et al. [8, 9, 10] have studied various second-order symmetric dual programs under the assumptions of second-order F-convexity, G_f-bonvexity/G_f-pseudobonvexity and (G, α_f)-bonvexity/(G, α_f)-pseudobonvexity involving second-order derivatives.

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In the absence of second-order derivatives, Ivanov [17] defined second-order (type I) invexity for first-order differentiable and second-order directionally differentiable functions. He used them to prove necessary and sufficient optimality conditions for nonlinear programming problem. In 2019, using limiting second-order subdifferentials, Feng and Li [12] obtained second-order Fritz-John optimality conditions for (strict) local minimizer of nonlinear programming problem with $C^{1,1}$ functions. Feng and Li [11, 13], Tuyen et al. [28], Ivanov [18], Xiao et al. [32] studied multiobjective/vector optimization problems with inequality constraints as well as ones with both inequality and equality constraints involving $C^{1,1}$ and locally Lipschitz functions. They obtained second-order necessary and sufficient KKT optimality conditions for different kinds of efficiency using various second-order constraint qualifications and regularity conditions in terms of second-order symmetric subdifferential, second-order upper generalized directional derivatives and second-order tangent sets. Using the idea of cones, Suneja et al. [26] extended the functions introduced by Ivanov [17] to second-order cone-(η, ξ)-convex and its weaker notions and used them to derive second-order KKT type optimality and duality results for vector optimization problem over cones involving first-order differentiable and second-order directionally differentiable vector valued functions.

The present paper is motivated by the works of Ivanov [17] and Suneja et al. [26]. In this paper, we have considered nondifferentiable functions and extended the class of second-order cone-(η, ξ)-convex functions and their weaker notions [26] for first and second-order directionally differentiable functions. Nonsmooth second-order cone-convex, nonsmooth second-order cone-(strictly) semipseudoconvex and nonsmooth second-order cone-semiquasiconvex functions have been introduced. Interrelations among these functions have been discussed and illustrated by examples. Using these functions, second-order KKT type sufficient optimality conditions for nonsmooth vector optimization problem over cones have been proved. Since first-order differentiable functions are also first-order directionally differentiable, so the results obtained by us can be applied to a wider class of functions as compared to Suneja et al. [26]. Also, second-order Wolfe type and Mond-Weir type duals are formulated and duality results are established. The results are well supported by various examples.

2. Notations and Definitions

Let $K \subseteq \mathbb{R}^m$ be a closed convex pointed $(K \cap (-K) = \{0\})$ cone with non-empty interior $(intK \neq \emptyset)$. We denote $K \setminus \{0\}$ by K_0. The positive dual cone K^+ and strict positive dual cone K^{s+} are defined as follows:

$$K^+ := \{y \in \mathbb{R}^m : z^T y \geqslant 0 \ \ \forall\, z \in K\}$$

and

$$K^{s+} := \{y \in \mathbb{R}^m : z^T y > 0 \ \ \forall\, z \in K_0\}.$$

Since the cone under consideration is closed and convex, by bipolar theorem $K = (K^+)^+$. In this case,

$$x \in K \iff \lambda^T x \geqslant 0, \ \ \forall\, \lambda \in K^+.$$

As given by Flores–Baźan et al. [14], we have

$x \in intK \iff \lambda^T x > 0 \ \ \forall\, \lambda \in K^+ \setminus \{0\}$.

Let $S \subseteq \mathbb{R}^n$ be a non-empty open subset and $f = (f_1, f_2, \ldots, f_m)^T : S \to \mathbb{R}^m$ be a vector valued function. We recall the definitions of first and second-order directionally differentiable functions which are weaker notions as compared to that of differentiability and twice differentiability respectively.

Definition 2.1
The first-order directional derivative of f_i at $x \in S$ in the direction $d \in \mathbb{R}^n$ is defined as an element of $\mathbb{R}$ given by

$$f_i'(x, d) := \lim_{t \to 0^+} \frac{(f_i(x + td) - f_i(x))}{t}.$$

If $f_i'(x, d)$ exists and is finite, then function f_i is called first-order directionally differentiable at x in the direction d. The function f_i is said to be first-order directionally differentiable on S if the derivative $f_i'(x, d)$ exists finitely for each $x \in S$ and direction $d \in \mathbb{R}^n$.

Definition 2.2
[7] Suppose f_i is first-order directionally differentiable at $x \in S$ in the direction $d \in \mathbb{R}^n$. The second-order directional derivative of f_i at x in the direction d is defined as an element of $\mathbb{R}$ given by

$$f_i''(x,d) := \lim_{t \to 0^+} \frac{2(f_i(x+td) - f_i(x) - tf_i'(x,d))}{t^2}.$$

If $f_i''(x,d)$ exists and is finite, then function f_i is called second-order directionally differentiable at x in the direction d. The function f_i is said to be second-order directionally differentiable on S if it is first-order directionally differentiable on S and the derivative $f_i''(x,d)$ exists finitely for each $x \in S$ and direction $d \in \mathbb{R}^n$.

Remark 2.1
f is said to be first-order directionally differentiable at $x \in S$ in the direction $d \in \mathbb{R}^n$ if each f_i is first-order directionally differentiable at x in the direction d. The first-order directional derivative of f at x in the direction d is defined to be the vector:

$$(f_1'(x,d), f_2'(x,d), \ldots, f_m'(x,d))^T.$$

Remark 2.2
Suppose f is first-order directionally differentiable at $x \in S$ in the direction $d \in \mathbb{R}^n$. f is said to be second-order directionally differentiable at x in the direction d if each f_i is second-order directionally differentiable at x in the direction d. The second-order directional derivative of f at x in the direction d is defined to be the vector:

$$(f_1''(x,d), f_2''(x,d), \ldots, f_m''(x,d))^T.$$

Next, we introduce new classes of nonsmooth second-order cone-convex, nonsmooth second-order cone-semipseudoconvex and nonsmooth second-order cone-semiquasiconvex functions that will be used to study second-order KKT type optimality conditions and duality results for nonsmooth vector optimization problem. Let $\bar{x} \in S$ where S is a non-empty open subset of $\mathbb{R}^n$, $K \subseteq \mathbb{R}^m$ be a closed convex pointed cone with $intK \neq \emptyset$ and $f : S \to \mathbb{R}^m$ be first and second-order directionally differentiable vector valued function.

Definition 2.3
f is said to be nonsmooth second-order K-convex at $\bar{x}$, if there exists a real valued function $\omega : S \times S \to [0,\infty)$ such that for all $x \in S$

$$f(x) - f(\bar{x}) - f'(\bar{x}, x-\bar{x}) - \omega(x,\bar{x}) f''(\bar{x}, x-\bar{x}) \in K.$$

Remark 2.3
Suppose f is first-order differentiable at $\bar{x}$. Then, $f'(\bar{x}, x-\bar{x}) = \nabla f(\bar{x})(x-\bar{x})$ where $\nabla f(\bar{x}) = [\nabla f_1(\bar{x}), \nabla f_2(\bar{x}), \ldots, \nabla f_m(\bar{x})]^T$ is the $m \times n$ Jacobian matrix of f at $\bar{x}$ and for each $i = 1, 2, \ldots, m, \nabla f_i(\bar{x}) = \left(\frac{\partial f_i}{\partial x_1}(\bar{x}), \frac{\partial f_i}{\partial x_2}(\bar{x}), \ldots, \frac{\partial f_i}{\partial x_n}(\bar{x})\right)^T$ is the $n \times 1$ Gradient vector of f_i at $\bar{x}$. If $\omega(.,.) \equiv 1$, then nonsmooth second-order K-convex becomes second-order K-(η,ξ)-convex with $\eta(x,\bar{x}) \equiv \xi(x,\bar{x}) \equiv x - \bar{x}$ defined by Suneja et al. [26]. Further, if $m = 1, K = \mathbb{R}_+$, then nonsmooth second-order K-convex becomes second-order invex defined by Ivanov [17].

Definition 2.4
f is said to be nonsmooth second-order K-semipseudoconvex at $\bar{x}$, if there exists a real valued function $\omega : S \times S \to [0,\infty)$ such that for all $x \in S$

$$-[f'(\bar{x}, x-\bar{x}) + \omega(x,\bar{x}) f''(\bar{x}, x-\bar{x})] \notin intK \Longrightarrow -[f(x) - f(\bar{x})] \notin intK.$$

Remark 2.4
Clearly, every nonsmooth second-order K-convex function with respect to $\omega(.,.)$ is nonsmooth second-order K-semipseudoconvex with respect to same $\omega(.,.)$ but the converse is not true as can be seen from the following example.

Example 2.1
Let $S=(-1,1)\subseteq\mathbb{R}$. Define $f=(f_1,f_2):S\longrightarrow\mathbb{R}^2$ as

$$f_1(x)=\frac{1}{\mid x\mid+1},\, f_2(x)=\begin{cases}\dfrac{x}{x^2+1}, & x\geqslant 0\\ x^2, & x<0\end{cases}.$$

Let $\bar{x}=0$, then

$$f'(0,x)=\begin{cases}(-x,x), & x\geqslant 0\\ (x,0), & x<0\end{cases} \text{ and } f''(0,x)=\begin{cases}(2x^2,0), & x\geqslant 0\\ (2x^2,2x^2), & x<0\end{cases}.$$

Let $K=\{(x_1,x_2)^T\in\mathbb{R}^2: x_2\leqslant 0, x_1\leqslant -x_2\}$ and $\omega:S\times S\longrightarrow[0,\infty)$ be defined as

$$\omega(x,\bar{x})=\frac{1-x}{4(1+x)(1+x^2)}+\bar{x}^2.$$

Now, f is nonsmooth second-order K-semipseudoconvex at $\bar{x}=0$ with respect to $\omega(,.,)$ as

$$intK\ni -[f(x)-f(0)]=$$
$$\begin{cases}\left(\dfrac{x}{x+1},\dfrac{-x}{x^2+1}\right), & x\geqslant 0\\ \left(\dfrac{-x}{1-x},-x^2\right), & x<0.\end{cases}$$

This shows that $x\in\{x:0<x<1\}\cup\{x:-1<x<\frac{1-\sqrt{5}}{2}\}$

$$\Longrightarrow intK\ni -[f'(0,x)+\omega(x,0)f''(0,x)]=$$
$$\begin{cases}\left(x-\dfrac{(1-x)x^2}{2(1+x)(1+x^2)},-x\right), & x\geqslant 0\\ \left(-x-\dfrac{(1-x)x^2}{2(1+x)(1+x^2)},\dfrac{-(1-x)x^2}{2(1+x)(1+x^2)}\right), & x<0.\end{cases}$$

However, f is not nonsmooth second-order K-convex at $\bar{x}$ with respect to $\omega(.,.)$ as for $x=\frac{1}{2}$

$$f(x)-f(0)-f'(0,x)-\omega(x,0)f''(0,x)=\left(\frac{2}{15},\frac{-1}{10}\right)\notin K.$$

Definition 2.5
f is said to be nonsmooth second-order K-semiquasiconvex at $\bar{x}$, if there exists a real valued function $\omega:S\times S\rightarrow[0,\infty)$ such that for all $x\in S$

$$[f(x)-f(\bar{x})]\notin intK\Longrightarrow -[f'(\bar{x},x-\bar{x})+\omega(x,\bar{x})f''(\bar{x},x-\bar{x})]\in K.$$

Definition 2.6
f is said to be nonsmooth second-order K-strictly semipseudoconvex at $\bar{x}$, if there exists a real valued function $\omega:S\times S\rightarrow[0,\infty)$ such that for all $x\in S$

$$-[f(x)-f(\bar{x})]\in K_0\Longrightarrow -[f'(\bar{x},x-\bar{x})+\omega(x,\bar{x})f''(\bar{x},x-\bar{x})]\in intK.$$

Remark 2.5
We glance at few important reductions of the new classes defined above.

1. If $\omega(.,.)\equiv 0$, then nonsmooth second-order K-(strictly) semipseudoconvex function becomes (strictly) pseudoconvex with respect to K and nonsmooth second-order K-semiquasiconvex function becomes quasiconvex with respect to K defined by Aggarwal [1].

2. Suppose f is first-order differentiable and $\omega(.,.) \equiv 1$. Then, nonsmooth second-order K-(strictly) semipseudoconvex becomes second-order K-(η,ξ)-(strictly) pseudoconvex function and nonsmooth second-order K-semiquasiconvex becomes second-order K-(η,ξ)-quasiconvex function with $\eta(x,\bar{x}) \equiv \xi(x,\bar{x}) \equiv x-\bar{x}$ defined by Suneja et al. [26].

Remark 2.6
Every nonsmooth second-order K-strictly semipseudoconvex function with respect to $\omega(.,.)$ is nonsmooth second-order K-semipseudoconvex with respect to same $\omega(.,.)$. However, the converse is not true as illustrated by the following example.

Example 2.2
Let $S = (-8, 8) \subseteq \mathbb{R}$. Define $f = (f_1, f_2)^T : S \longrightarrow \mathbb{R}^2$ as

$$f_1(x) = \begin{cases} 0, & x \geq 0 \\ x^2, & x < 0 \end{cases} \text{ and } f_2(x) = x^2.$$

Let $\bar{x} = 0$. Then,

$$f'(0,x) = (0,0)^T \text{ and } f''(0,x) = \begin{cases} (0, 2x^2)^T, & x \geq 0 \\ (2x^2, 2x^2)^T, & x < 0 \end{cases}.$$

Let $K = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0, x_2 \leq x_1\}$ and $\omega : S \times S \longrightarrow [0, \infty)$ be a constant real valued function with $\omega(.,.) \equiv 1$. Now, f is nonsmooth second-order K-semipseudoconvex at $\bar{x} = 0$ with respect to $\omega(,.,)$ as

$$intK \ni -[f(x) - f(0)] = \begin{cases} \left(0, -x^2\right)^T, & x \geq 0 \\ \left(-x^2, -x^2\right)^T, & x < 0. \end{cases}$$

This shows that

$$x > 0 \Longrightarrow intK \ni -[f'(0,x) + \omega(x,0) f''(0,x)] = \begin{cases} \left(0, -2x^2\right)^T, & x \geq 0 \\ \left(-2x^2, -2x^2\right)^T, & x < 0. \end{cases}$$

However, f is not nonsmooth second-order K-strictly semipseudoconvex at $\bar{x} = 0$ with respect to $\omega(.,.)$ as for $x < 0$,

$$K_0 \ni -[f(x) - f(0)] = \begin{cases} \left(0, -x^2\right)^T, & x \geq 0 \\ \left(-x^2, -x^2\right)^T, & x < 0 \end{cases}$$

but

$$-[f'(0,x) + \omega(x,0) f''(0,x)] = \begin{cases} \left(0, -2x^2\right)^T, & x \geq 0 \\ \left(-2x^2, -2x^2\right)^T, & x < 0 \end{cases} \notin intK.$$

3. Second-Order Optimality Conditions

We consider the following nonsmooth vector optimization problem:

$$\begin{aligned} &K\text{-Minimize} \quad f(x) \\ &\text{subject to} \quad -g(x) \in Q, \end{aligned} \tag{VOP}$$

where $f = (f_1, f_2, \dots, f_m)^T : S \to \mathbb{R}^m$, $g = (g_1, g_2, \dots, g_p)^T : S \to \mathbb{R}^p$ are first and second-order directionally differentiable on S, S is non-empty open subset of $\mathbb{R}^n$, K and Q are closed convex pointed cones with non-empty interiors in $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively. $S_0 = \{x \in S : -g(x) \in Q\}$ denotes the set of all feasible solutions of (VOP).

Definition 3.1
Let $\bar{x} \in S_0$. Then, $\bar{x}$ is called a

(i) weak minimum of (VOP) if for all $x \in S_0, f(\bar{x}) - f(x) \notin intK$;

(ii) minimum of (VOP) if for all $x \in S_0, f(\bar{x}) - f(x) \notin K_0$;

(iii) strong minimum of (VOP) if for all $x \in S_0, f(x) - f(\bar{x}) \in K$.

Next, we prove second-order KKT type sufficient optimality conditions for (VOP) using second-order cone-convexity.

Theorem 1
Let f be nonsmooth second-order K-convex and g be nonsmooth second-order Q-convex at $\bar{x} \in S_0$ with respect to same $\omega : S \times S \longrightarrow [0, \infty)$. Suppose there exist $\lambda \in K^+ \setminus \{0\}, \mu \in Q^+$ such that for all $x \in S_0$,

$$\lambda^T [f'(\bar{x}, x - \bar{x}) + \omega(x, \bar{x}) f''(\bar{x}, x - \bar{x})] + \mu^T [g'(\bar{x}, x - \bar{x}) + \omega(x, \bar{x}) g''(\bar{x}, x - \bar{x})] \geqslant 0, \quad (1)$$

$$\mu^T g(\bar{x}) \geqslant 0. \quad (2)$$

Then, $\bar{x}$ is a weak minimum of (VOP).

Proof
Let if possible $\bar{x}$ be not a weak minimum of (VOP). Then, there exists $\hat{x} \in S_0$ such that

$$f(\bar{x}) - f(\hat{x}) \in intK.$$

Using $\lambda \in K^+ \setminus \{0\}$, we get

$$\lambda^T [f(\bar{x}) - f(\hat{x})] > 0. \quad (3)$$

As f is nonsmooth second-order K-convex at $\bar{x}$ with respect to $\omega(.,.)$ and $\lambda \in K^+ \setminus \{0\}$, we get

$$\lambda^T [f(\hat{x}) - f(\bar{x}) - f'(\bar{x}, \hat{x} - \bar{x}) - \omega(\hat{x}, \bar{x}) f''(\bar{x}, \hat{x} - \bar{x})] \geqslant 0. \quad (4)$$

Adding (3) and (4), we get

$$-\lambda^T [f'(\bar{x}, \hat{x} - \bar{x}) + \omega(\hat{x}, \bar{x}) f''(\bar{x}, \hat{x} - \bar{x})] > 0.$$

Using (1), we obtain

$$\mu^T [g'(\bar{x}, \hat{x} - \bar{x}) + \omega(\hat{x}, \bar{x}) g''(\bar{x}, \hat{x} - \bar{x})] > 0. \quad (5)$$

Since g is nonsmooth second-order Q-convex at $\bar{x}$ with respect to $\omega(.,.)$ and $\mu \in Q^+$, therefore

$$\mu^T [g(\hat{x}) - g(\bar{x}) - g'(\bar{x}, \hat{x} - \bar{x}) - \omega(\hat{x}, \bar{x}) g''(\bar{x}, \hat{x} - \bar{x})] \geqslant 0. \quad (6)$$

Adding (5) and (6), we get $\mu^T [g(\hat{x}) - g(\bar{x})] > 0$. Using (2), we get $\mu^T g(\hat{x}) > 0$ which is a contradiction to $\hat{x} \in S_0$. Thus, $\bar{x}$ is a weak minimum of (VOP). □

Following second-order KKT type sufficient optimality conditions for minimum and strong minimum of (VOP) can be proved on the similar lines.

Theorem 2
Let f be nonsmooth second-order K-convex and g be nonsmooth second-order Q-convex at $\bar{x} \in S_0$ with respect to same $\omega(.,.) : S \times S \longrightarrow [0, \infty)$. Suppose there exist $\lambda \in K^{s+}, \mu \in Q^+$ such that for all $x \in S_0$, (1) and (2) hold. Then, $\bar{x}$ is a minimum of (VOP).

Theorem 3
Let f be nonsmooth second-order K-convex and g be nonsmooth second-order Q-convex at $\bar{x} \in S_0$ with respect to same $\omega(.,.) : S \times S \longrightarrow [0, \infty)$. Suppose there exists $\mu \in Q^+$ such that for all $x \in S_0$, (1) and (2) hold and (1) holds for all $\lambda \in K^+$. Then, $\bar{x}$ is a strong minimum of (VOP).

In the next theorem, we obtain second-order KKT type sufficient optimality conditions under the weaker assumption of nonsmooth second-order cone-semipseudoconvexity and nonsmooth second-order cone-semiquasiconvexity.

Theorem 4
Let f be nonsmooth second-order K-semipseudoconvex and g be nonsmooth second-order Q-semiquasiconvex at $\bar{x} \in S_0$ with respect to same $\omega(.,.) : S \times S \longrightarrow [0, \infty)$. If there exist $\lambda \in K^+ \setminus \{0\}, \mu \in Q^+$ such that for all $x \in S_0$, (1) and (2) hold, then $\bar{x}$ is a weak minimum of (VOP).

Proof
For all $x \in S_0, \mu^T g(x) \leqslant 0$. Using (2), we can write

$$\mu^T g(x) - \mu^T g(\bar{x}) \leqslant 0, \quad \forall x \in S_0.$$

If $\mu \neq 0$, then

$$g(x) - g(\bar{x}) \notin intQ, \quad \forall x \in S_0.$$

Since g is nonsmooth second-order Q-semiquasiconvex at $\bar{x}$ with respect to $\omega(.,.)$ and $\mu \in Q^+ \setminus \{0\}$, therefore

$$-\mu^T g^{'}(\bar{x}, x - \bar{x}) - \omega(x, \bar{x}) \mu^T g^{''}(\bar{x}, x - \bar{x})] \geqslant 0, \quad \forall x \in S_0.$$

Above inequality also holds for $\mu = 0$. From (1), we get

$$\lambda^T f^{'}(\bar{x}, x - \bar{x}) + \omega(x, \bar{x}) \lambda^T f^{''}(\bar{x}, x - \bar{x}) \geqslant 0, \quad \forall x \in S_0.$$

This implies for all $x \in S_0$,

$$-[f'(\bar{x}, x - \bar{x}) + \omega(x, \bar{x}) f''(\bar{x}, x - \bar{x})] \notin intK.$$

As f is nonsmooth second-order K-semipseudoconvex at $\bar{x}$ with respect to $\omega(.,.)$, we have

$$-(f(x) - f(\bar{x})) \notin intK \; \forall \, x \in S_0.$$

Thus, $\bar{x}$ is a weak minimum of (VOP). □

We give an example to illustrate Theorem 4.

Example 3.1
Let $S = (-1, 2) \subseteq \mathbb{R}, K = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \geqslant 0, x_2 \geqslant x_1\}$ and $Q = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \geqslant 0, x_1 \geqslant x_2\}$. Define $f = (f_1, f_2)^T : S \longrightarrow \mathbb{R}^2$ and $g = (g_1, g_2)^T : S \longrightarrow \mathbb{R}^2$ as

$$f_1(x) = \begin{cases} \dfrac{x}{x^2+1}, & x \geqslant 0 \\ x^3, & x < 0 \end{cases}, f_2(x) = \sin \mid x \mid + x^2, g_1(x) = - \mid x \mid - x^2 - 1 \text{ and } g_2(x) = - \mid x \mid .$$

The feasible set of corresponding problem (VOP) is $S_0 = (-1, 2)$. Let $\bar{x} = 0$.
Then,

$$f'(0, x) = \begin{cases} (x, x), & x \geqslant 0 \\ (0, -x), & x < 0 \end{cases} \text{ and } f''(0, x) = (0, 2x^2).$$

$$g'(0, x) = \begin{cases} (-x, -x), & x \geqslant 0 \\ (x, x), & x < 0 \end{cases} \text{ and } g''(0, x) = (-2x^2, 0).$$

Let $\omega : S \times S \longrightarrow [0,\infty)$ be defined as

$$\omega(x,\bar{x}) = \begin{cases} \dfrac{1}{4|x|} + \bar{x}^2, & x \neq 0 \\ \dfrac{1}{\bar{x}^2+1}, & x = 0. \end{cases}$$

Now, f is nonsmooth second-order K-semipseudoconvex at $\bar{x}=0$ with respect to $\omega(.,.)$ as

$$-[f'(0,x)+\omega(x,0)f''(0,x)] = \begin{cases} \left(-x, \frac{-3x}{2}\right), & x \geqslant 0 \\ \left(0, \frac{3x}{2}\right), & x < 0 \end{cases} \notin intK$$

$\Longrightarrow x \in (-1,2)$ and for all such x,

$$-[f(x)-f(\bar{x})] = \begin{cases} \left(\frac{-x}{x^2+1}, -\sin x - x^2\right), & x \geqslant 0 \\ \left(-x^3, \sin x - x^2\right), & x < 0 \end{cases} \notin intK.$$

(see Fig.1, Fig.2). Also, g is nonsmooth second-order Q-semiquasiconvex at $\bar{x}=0$ with respect to $\omega(.,.)$ as

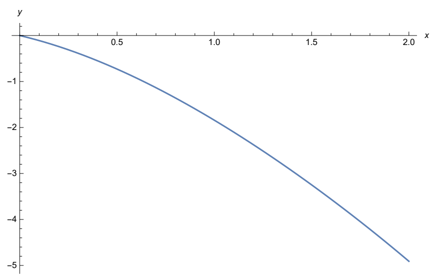

Figure 1. Graph of $-\sin x - x^2$

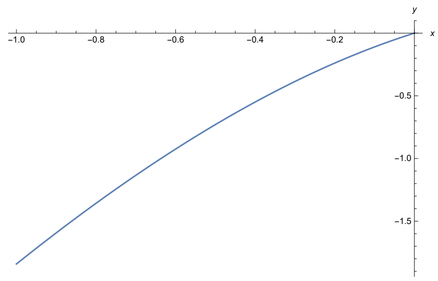

Figure 2. Graph of $\sin x - x^2$

$$[g(x)-g(0)] = \begin{cases} (-x-x^2, -x), & x \geqslant 0 \\ (x-x^2, x), & x < 0 \end{cases} \notin intQ$$

$$\Longrightarrow x \in (-1,2)$$

$$\Longrightarrow -[g'(0,x)+\omega(x,0)g''(0,x)] = \begin{cases} (x+2\omega(x,0)x^2, x), & x \geqslant 0 \\ (-x+2\omega(x,0)x^2, -x), & x < 0 \end{cases} \in Q.$$

Here,

$$K^+ = \{(x_1,x_2) : x_2 \geqslant 0, x_1 \geqslant -x_2\} \text{ and } Q^+ = \{(x_1,x_2) : x_1 \geqslant 0, x_1 \geqslant -x_2\}.$$

For $\lambda = (-1,1) \in K^+ \setminus \{0\}$ and $\mu = (0, \frac{1}{4}) \in Q^+$, following conditions hold for all $x \in S_0$:

$$\lambda^T[f'(0,x)+\omega(x,0)f''(0,x)] + \mu^T[g'(0,x)+\omega(x,0)g''(0,x)] = \begin{cases} \frac{x}{4}, & x \geqslant 0 \\ \frac{-5x}{4}, & x < 0 \end{cases} \geqslant 0,$$

$$\mu^T g(\bar{x}) = 0 \geqslant 0.$$

Thus, by Theorem 4, $\bar{x}=0$ is a weak minimum of (VOP).

Theorem 5
Let f be nonsmooth second-order K-strictly semipseudoconvex and g be nonsmooth second-order Q-semiquasiconvex at $\bar{x} \in S_0$ with respect to same $\omega(.,.) : S \times S \longrightarrow [0,\infty)$. If there exist $\lambda \in K^+ \setminus \{0\}, \mu \in Q^+$ such that for all $x \in S_0$, (1) and (2) hold ,then $\bar{x}$ is a minimum of (VOP).

Proof
Let if possible $\bar{x}$ be not a minimum of (VOP), then there exists $\hat{x} \in S_0$ such that

$$f(\bar{x}) - f(\hat{x}) \in K_0.$$

Since f is nonsmooth second-order K-strictly semipseudoconvex at $\bar{x}$ with respect to $\omega(,.,)$, therefore

$$-[f'(\bar{x}, \hat{x} - \bar{x}) + \omega(\hat{x}, \bar{x}) f''(\bar{x}, \hat{x} - \bar{x})] \in intK.$$

As $\lambda \in K^+ \setminus \{0\}$, we get

$$\lambda^T [f'(\bar{x}, \hat{x} - \bar{x}) + \omega(\hat{x}, \bar{x}) f''(\bar{x}, \hat{x} - \bar{x})] < 0. \tag{7}$$

Using (2) and the fact that $\hat{x} \in S_0$, we get

$$\mu^T [g(\hat{x}) - g(\bar{x})] \leqslant 0.$$

If $\mu \neq 0$, then

$$g(\hat{x}) - g(\bar{x}) \notin intQ.$$

Again g is nonsmooth second-order Q-semiquasiconvex at $\bar{x}$ with respect to $\omega(,.,)$ and $\mu \in Q^+$, we get

$$\mu^T [g'(\bar{x}, \hat{x} - \bar{x}) + \omega(\hat{x}, \bar{x}) g''(\bar{x}, \hat{x} - \bar{x})] \leqslant 0. \tag{8}$$

If $\mu = 0$, still above inequality holds. Adding (7) and (8), we get

$$\lambda^T f'(\bar{x}, \hat{x} - \bar{x}) + \mu^T g'(\bar{x}, \hat{x} - \bar{x}) + \omega(\hat{x}, \bar{x}) [\lambda^T f''(\bar{x}, \hat{x} - \bar{x}) + \mu^T g''(\bar{x}, \hat{x} - \bar{x})] < 0$$

which is contradiction to (1). Thus, $\bar{x}$ is a minimum of (VOP). □

4. Second-Order Duality

Aggarwal [1] associated a first-order dual in terms of first-order directional derivatives with (VOP) and proved duality results under the assumption of pseudoconvexity and quasiconvexity with respect to cone. Suneja et al. [26] formulated a second-order dual involving first-order derivatives and second-order directional derivatives for (VOP) and established duality results using second-order (η, ξ)-cone-convexity and its weaker notions.

In this section, we formulate second-order Wolfe type and Mond-Weir type duals for (VOP) in terms of first and second-order directional derivatives and prove duality results using nonsmooth second-order cone-convexity and its weaker notions. We begin with following second-order Wolfe type dual (WD).

Let $k \in intK$ be any arbitrary fixed vector.

$$K\text{-Maximize} \quad f(u) + \mu^T g(u) k \tag{WD}$$

$$\text{subject to} \quad \lambda^T f'(u, x - u) + \mu^T g'(u, x - u)$$

$$+ \xi[\lambda^T f''(u, x - u) + \mu^T g''(u, x - u)] \geqslant 0 \ \ \forall\, x \in S_0, \tag{9}$$

$$\lambda^T k = 1, \tag{10}$$

$\lambda \in K^+ \setminus \{0\}, \mu \in Q^+, u \in S, \xi \in \mathbb{R}_+$. In general, ξ can be regarded as a function.
Let D_0 be the feasible set of (WD).

Definition 4.1
A point $(\bar{u}, \bar{\lambda}, \bar{\mu}, \bar{\xi}) \in D_0$ is called weakly efficient solution (weak maximum) of (WD) if for all $(u, \lambda, \mu, \xi) \in D_0$, $f(u) + \mu^T g(u) k - f(\bar{u}) - \bar{\mu}^T g(\bar{u}) k \notin intK$.

Theorem 6 (Weak Duality)
Let $\bar{x} \in S_0$ and $(u, \lambda, \mu, \xi) \in D_0$. Assume that f is nonsmooth second-order K-convex and g is nonsmooth second-order Q-convex at u with respect to $\xi(,.,)$. Then, $f(u) + \mu^T g(u) k - f(\bar{x}) \notin intK$.

Proof
Let if possible $f(u)+\mu^T g(u)k - f(\bar{x}) \in intK$. Then,

$$\lambda^T[f(u)-f(\bar{x})]+\mu^T g(u) > 0. \tag{11}$$

As f is nonsmooth second-order K-convex at u with respect to $\xi(.,.)$ and $\lambda \in K^+ \setminus \{0\}$, we get

$$\lambda^T[f(\bar{x})-f(u)-f'(u,\bar{x}-u)-\xi(\bar{x},u)f''(u,\bar{x}-u)] \geqslant 0. \tag{12}$$

Adding (11) and (12), we get

$$\mu^T g(u) - \lambda^T[f'(u,\bar{x}-u)+\xi(\bar{x},u)f''(u,\bar{x}-u)] > 0.$$

Using (9), we get

$$\mu^T[g(u)+g'(u,\bar{x}-u)+\xi(\bar{x},u)g''(u,\bar{x}-u)] > 0. \tag{13}$$

Again g is nonsmooth second-order Q-convex at u with respect to $\xi(.,.)$ and $\mu \in Q^+$, therefore

$$\mu^T[g(\bar{x})-g(u)-g'(u,\bar{x}-u)-\xi(\bar{x},u)g''(u,\bar{x}-u)] \geqslant 0. \tag{14}$$

Adding (13) and (14), we get $\mu^T g(\bar{x}) > 0$ which is a contradiction to $\bar{x} \in S_0$. Hence $f(u)+\mu^T g(u)k - f(\bar{x}) \notin intK$. □

To prove Strong Duality result, we use the KKT type necessary optimality conditions derived by Aggarwal [1] under the following regularity condition.

Definition 4.2
The function g is said to satisfy the regularity condition at $\bar{x} \in S$ if

$$g'(\bar{x}; S-\bar{x}) + \{\alpha g(\bar{x}) \mid \alpha \geqslant 0\} + Q = \mathbb{R}^p. \tag{15}$$

Theorem 7
[1] Let $\bar{x}$ be a weak minimum of (VOP). If $f'(\bar{x}, x-\bar{x})$ is K-subconvexlike, $g'(\bar{x}, x-\bar{x})$ is Q-subconvexlike on S and the regularity condition (15) holds at $\bar{x}$, then there exist $\lambda \in K^+ \setminus \{0\}, \mu \in Q^+$ such that

$$\lambda^T f'(\bar{x}, x-\bar{x}) + \mu^T g'(\bar{x}, x-\bar{x}) \geqslant 0 \; \forall \, x \in S, \tag{16}$$

$$\mu^T g(\bar{x}) = 0. \tag{17}$$

Theorem 8 (Strong Duality)
Let $\bar{x}$ be a weak minimum of (VOP). Assume that $f'(\bar{x}, x-\bar{x})$ is K-subconvexlike, $g'(\bar{x}, x-\bar{x})$ is Q-subconvexlike on S and the regularity condition (15) holds at $\bar{x}$. Then, there exist $\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is feasible for the dual problem (WD) and the objective function values of (VOP) and (WD) are equal. Moreover, if the conditions of Weak Duality Theorem 6 hold for all $(u, \lambda, \mu, \xi) \in D_0$, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is a weak maximum of (WD).

Proof
Since $\bar{x}$ is a weak minimum of (VOP), by Theorem 7 there exist $\lambda \in K^+ \setminus \{0\}, \mu \in Q^+$ such that (16) and (17) are satisfied. Since $\lambda \in K^+ \setminus \{0\}$ and $k \in intK$, therefore $\lambda^T k > 0$. Set $\bar{\lambda} = \frac{\lambda}{\lambda^T k} \in K^+ \setminus \{0\}, \bar{\mu} = \frac{\mu}{\lambda^T k} \in Q^+$, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is feasible for the dual problem (WD) and objective function values of (VOP) and (WD) are equal. Let if possible $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ be not a weak maximum of (WD), then there exists $(u, \lambda, \mu, \xi) \in D_0$ such that $f(u)+\mu^T g(u)k - f(\bar{x}) \in intK$ which is a contradiction to Weak Duality Theorem 6. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is a weak maximum of (WD). □

Strong Duality result in literature has mainly been proved by taking the parameter ξ (usually denoted by p) associated with the second-order derivative as zero (for instance [2, 3, 4, 16, 23, 26, 27, 33]). However, we shall next prove the Strong Duality result in which the variable ξ may not be equal to zero and hence we will be having the Strong Duality result for the non-trivial case.

Theorem 9 (Non-trivial Strong Duality)
Let $\bar{x}$ be a weak minimum of (VOP). Assume that $f'(\bar{x}, x-\bar{x})$ is K-subconvexlike, $g'(\bar{x}, x-\bar{x})$ is Q-subconvexlike on S and the regularity condition (15) holds at $\bar{x}$. If $f''(\bar{x}, x-\bar{x}) \in K$ and $g''(\bar{x}, x-\bar{x}) \in Q$ for all $x \in S_0$, then there exist $\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is feasible for the dual problem (WD) for all $\bar{\xi} \in \mathbb{R}_+$ and the objective function values of (VOP) and (WD) are equal. Moreover, if the conditions of Weak Duality Theorem 6 hold for all $(u, \lambda, \mu, \xi) \in D_0$, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a weak maximum of (WD).

Proof
Since $\bar{x}$ is a weak minimum of (VOP), by Theorem 7 there exist $\lambda \in K^+ \setminus \{0\}, \mu \in Q^+$ such that (16) and (17) are satisfied. Using $f''(\bar{x}, x-\bar{x}) \in K$ and $g''(\bar{x}, x-\bar{x}) \in Q$, we get

$$\lambda^T f'(\bar{x}, x-\bar{x}) + \mu^T g'(\bar{x}, x-\bar{x}) \\ + \bar{\xi}[\lambda^T f''(\bar{x}, x-\bar{x}) + \mu^T g''(\bar{x}, x-\bar{x})] \geqslant 0 \quad \forall x \in S_0, \bar{\xi} \in \mathbb{R}_+.$$

Since $\lambda \in K^+ \setminus \{0\}, k \in intK$, therefore $\lambda^T k > 0$. Set $\bar{\lambda} = \frac{\lambda}{\lambda^T k} \in K^+ \setminus \{0\}, \bar{\mu} = \frac{\mu}{\lambda^T k} \in Q^+$, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is feasible for the dual problem (WD) for all $\bar{\xi} \in \mathbb{R}_+$ and objective function values of (VOP) and (WD) are equal. Let if possible $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ be not a weak maximum of (WD), then there exists $(u, \lambda, \mu, \xi) \in D_0$ such that $f(u) + \mu^T g(u)k - f(\bar{x}) \in intK$ which is a contradiction to Weak Duality Theorem 6. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a weak maximum of (WD). □

Following is an example to illustrate Theorem 6.

Example 4.1
Let $S = (-0.5, 2) \subseteq \mathbb{R}, K = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geqslant 0, x_1 \geqslant x_2\}$ and $Q = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leqslant 0, x_2 \leqslant x_1\}$. Define $f = (f_1, f_2)^T : S \longrightarrow \mathbb{R}^2$ and $g = (g_1, g_2)^T : S \longrightarrow \mathbb{R}^2$ as

$$f_1(x) = \sin \mid x \mid + x^2 \text{ and } f_2(x) = \begin{cases} \dfrac{x}{x+1}, & x \geqslant 0 \\ x^2 + \frac{x^3}{6}, & x < 0 \end{cases}.$$

$$g_1(x) = - \mid x \mid \text{ and } g_2(x) = \begin{cases} \sin x, & x \geqslant 0 \\ \cos x - 1, & x < 0 \end{cases}.$$

The feasible set of corresponding problem (VOP) is $S_0 = [0, 2)$ and let $u = 0$. Now,

$$f'(0, x) = \begin{cases} (x, x)^T, & x \geqslant 0 \\ (-x, 0)^T, & x < 0 \end{cases} \text{ and } f''(0, x) = \begin{cases} (2x^2, -2x^2)^T, & x \geqslant 0 \\ (2x^2, 2x^2)^T, & x < 0 \end{cases}.$$

$$g'(0, x) = \begin{cases} (-x, x)^T, & x \geqslant 0 \\ (x, 0)^T, & x < 0 \end{cases} \text{ and } g''(0, x) = \begin{cases} (0, 0)^T, & x \geqslant 0 \\ (0, -x^2)^T, & x < 0 \end{cases}.$$

Let $\xi : S \times S \longrightarrow [0, \infty)$ be defined as $\xi(x, u) = \frac{1}{4} + u^2 x^2$. Then, f is nonsmooth second-order K-convex at $u = 0$ with respect to $\xi(.,.)$ as for all $x \in S$

$$K \ni f(x) - f(0) - f'(0, x) - \xi(x, 0) f''(0, x) = \\ \begin{cases} \left(\sin x + \frac{x^2}{2} - x, \frac{x}{x+1} - x + \frac{x^2}{2}\right)^T, & x \geqslant 0 \\ \left(-\sin x + x + \frac{x^2}{2}, \frac{x^2}{2} + \frac{x^3}{6}\right)^T, & x < 0 \end{cases}$$

[see Figure 3, Figure 4, Figure 5].

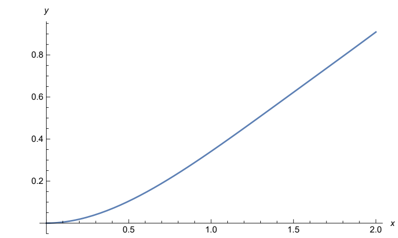

Figure 3. Graph of $\sin x + \frac{x^2}{2} - x$

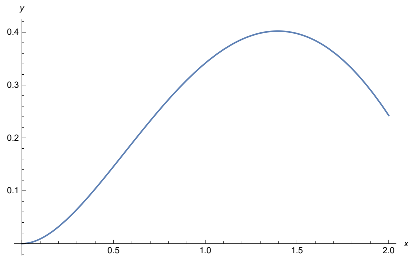

Figure 4. Graph of $\sin x - \frac{x}{x+1}$

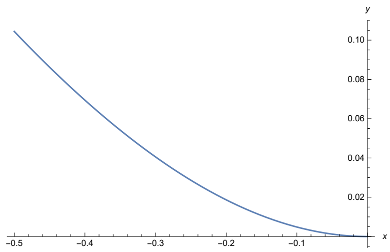

Figure 5. Graph of $-\sin x + x + \frac{x^2}{2}$

Also, g is nonsmooth second-order Q-convex at $u = 0$ with respect to $\xi(.,.)$ as for all $x \in S$

$$Q \ni g(x) - g(0) - g'(0,x) - \xi(x,0)g''(0,x) = \begin{cases} (0, \sin x - x)^T, & x \geqslant 0 \\ (0, \cos x - 1 + \frac{x^2}{4})^T, & x < 0 \end{cases}$$

[see Figure 6, Figure 7]. Here,

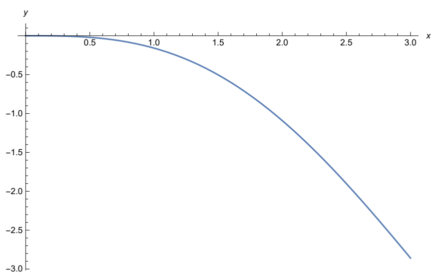

Figure 6. Graph of $\sin x - x$

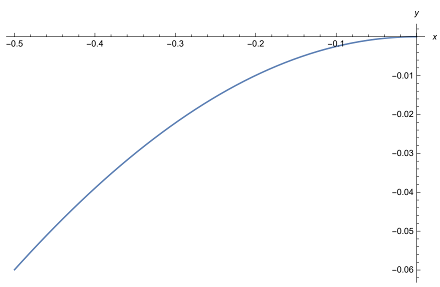

Figure 7. Graph of $\cos x - 1 + \frac{x^2}{4}$

$$K^+ = \{(x_1,x_2)^T \in \mathbb{R}^2 : x_1 \geqslant -x_2 \geqslant 0\} \text{ and } Q^+ = \{(x_1,x_2)^T \in \mathbb{R}^2 : 0 \leqslant x_1 \leqslant -x_2\}.$$

For $\lambda = (1,0)^T \in K^+ \setminus \{0\}, \mu = (0,-1)^T \in Q^+, k = (1, \frac{1}{2})^T \in intK$ and for all $x \in S_0$, following conditions hold:

(i) $\lambda^T f'(0,x) + \mu^T g'(0,x) + \xi(x,0)[\lambda^T f''(0,x) + \mu^T g''(0,x)] = \begin{cases} \frac{x^2}{2}, & x \geqslant 0 \\ -x + \frac{3x^2}{4}, & x < 0 \end{cases} \geqslant 0;$

(ii) $\lambda^T k = 1.$

Thus, $(u = 0, \lambda = (1,0)^T, \mu = (0,-1)^T, \xi = \frac{1}{4})$ is a dual feasible point. Moreover, for all $\bar{x} \in S_0$

$$f(u) + \mu^T g(u)k - f(\bar{x}) = \Big(-\sin\bar{x} - \bar{x}^2, \frac{-\bar{x}}{\bar{x}+1}\Big)^T \notin intK.$$

Hence Weak Duality Theorem 6 holds for all feasible point $\bar{x}$ of (VOP) and the dual feasible point $(u = 0, \lambda = (1,0)^T, \mu = (0,-1)^T, \xi = \frac{1}{4})$.

Next, we associate following second-order Mond-Weir type dual with (VOP) and establish duality results using nonsmooth second-order cone-semipseudoconvexity and nonsmooth second-order cone-semiquasiconvexity.

$$\begin{aligned} &K\text{-Maximize} \quad f(u) && \text{(MD)}\\ &\text{subject to} \quad \lambda^T f'(u, x-u) + \mu^T g'(u, x-u) \\ &\qquad + \xi[\lambda^T f''(u, x-u) + \mu^T g''(u, x-u)] \geqslant 0, \quad \forall x \in S_0 && (18)\\ &\qquad \mu^T g(u) \geqslant 0, && (19) \end{aligned}$$

$\lambda \in K^+ \setminus \{0\}, \mu \in Q^+, u \in S, \xi \in \mathbb{R}_+$. In general, ξ can be regarded as a function.
Let D_1 be the feasible set of (MD).

Definition 4.3
A point $(\bar{u}, \bar{\lambda}, \bar{\mu}, \bar{\xi}) \in D_1$ is called weakly efficient solution (weak maximum) of (MD) if for all $(u, \lambda, \mu, \xi) \in D_1$, $f(u) - f(\bar{u}) \notin intK$.

Theorem 10 (Weak Duality)
Let $\bar{x} \in S_0$ and $(u, \lambda, \mu, \xi) \in D_1$. Assume f is nonsmooth second-order K-semipseudoconvex and g is nonsmooth second-order Q-semiquasiconvex at u with respect to $\xi(,.,)$. Then $f(u) - f(\bar{x}) \notin intK$.

Proof
The proof follows on the lines of Theorem 4. □

Theorem 11 (Strong Duality)
Let $\bar{x}$ be a weak minimum of (VOP). Assume $f'(\bar{x}, x-\bar{x})$ is K-subconvexlike, $g'(\bar{x}, x-\bar{x})$ is Q-subconvexlike on S and the regularity condition (15) holds at $\bar{x}$. Then, there exist $\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is feasible for the dual problem (MD) and the objective function values of (VOP) and (MD) are equal. Moreover, if the conditions of Weak Duality Theorem 10 hold for all $(u, \lambda, \mu, \xi) \in D_1$, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is a weak maximum of (MD).

Proof
Since $\bar{x}$ is a weak minimum of (VOP), by Theorem 7 there exist $\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+$ such that (16) and (17) are satisfied. Then, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is feasible for the dual problem (MD) and objective function values of (VOP) and (MD) are equal. Let if possible $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ be not a weak maximum of (MD), then there exists $(u, \lambda, \mu, \xi) \in D_1$ such that $f(u) - f(\bar{x}) \in intK$ which is a contradiction to Weak Duality Theorem 10. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is a weak maximum of (MD). □

Next, we have the Strong Duality result in which the variable ξ may not be equal to zero.

Theorem 12 (Non-trivial Strong Duality)
Let $\bar{x}$ be a weak minimum of (VOP). Assume $f'(\bar{x}, x-\bar{x})$ is K-subconvexlike, $g'(\bar{x}, x-\bar{x})$ is Q-subconvexlike on S and the regularity condition (15) holds at $\bar{x}$. If $f''(\bar{x}, x-\bar{x}) \in K$ and $g''(\bar{x}, x-\bar{x}) \in Q$ for all $x \in S_0$, then there exist $\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is feasible for the dual problem (MD) for all $\bar{\xi} \in \mathbb{R}_+$ and the objective function values of (VOP) and (MD) are equal. Moreover, if the conditions of Weak Duality Theorem 10 hold for all $(u, \lambda, \mu, \xi) \in D_1$, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a weak maximum of (MD).

Proof
Since $\bar{x}$ is a weak minimum of (VOP), by Theorem 7 there exist $\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+$ such that (16) and (17) are satisfied. Using $f''(\bar{x}, x-\bar{x}) \in K$ and $g''(\bar{x}, x-\bar{x}) \in Q$, we get

$$\bar{\lambda}^T f'(\bar{x}, x-\bar{x}) + \bar{\mu}^T g'(\bar{x}, x-\bar{x}) \\ + \bar{\xi}[\bar{\lambda}^T f''(\bar{x}, x-\bar{x}) + \bar{\mu}^T g''(\bar{x}, x-\bar{x})] \geqslant 0 \quad \forall x \in S_0, \bar{\xi} \in \mathbb{R}_+.$$

Then, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is feasible for the dual problem (MD) for all $\bar{\xi} \in \mathbb{R}_+$ and objective function values of (VOP) and (MD) are equal. Let if possible $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ be not a weak maximum of (MD), then there exists $(u, \lambda, \mu, \xi) \in D_1$ such that $f(u) - f(\bar{x}) \in intK$ which is a contradiction to Weak Duality Theorem 10. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a weak maximum of (MD). □

We conclude this section with an example in which we find a feasible solution of (MD) given a weak minimum of (VOP) using Theorem 12.

Example 4.2
Let $S = (-4, 4) \subseteq \mathbb{R}, K = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \leqslant 0, x_2 \geqslant 0\}$ and $Q = \{x_1 \in \mathbb{R} : x_1 \geqslant 0\}$. Define $f = (f_1, f_2)^T : S \longrightarrow \mathbb{R}^2$ and $g : S \longrightarrow \mathbb{R}$ as

$$f_1(x) = \begin{cases} \dfrac{x}{x^2+1}, & x \geqslant 0 \\ x^3, & x < 0 \end{cases}, f_2(x) = \sin \mid x \mid + x^2 \text{ and } g(x) = \begin{cases} -1, & x \geqslant 0 \\ -2x-1, & x < 0 \end{cases}.$$

The feasible set of corresponding problem (VOP) is $S_0 = [\frac{-1}{2}, 4)$ and let $u = 0$. Clearly, u is a weak minimum of (VOP) as

$$f(u) - f(x) = \begin{cases} \left(\dfrac{-x}{x^2+1}, -\sin x - x^2\right)^T, & x \geqslant 0 \\ \left(-x^3, \sin x - x^2\right)^T, & x < 0 \end{cases} \notin intK \text{ for all } x \in S_0$$ [see Figure 8, Figure 9]. Now,

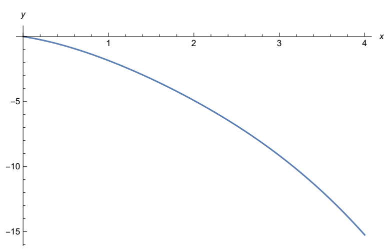

Figure 8. Graph of $-\sin x - x^2$

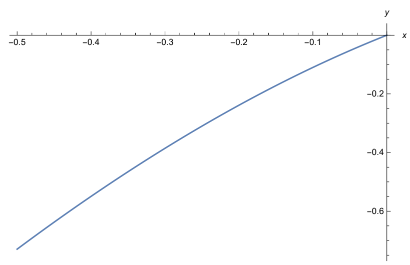

Figure 9. Graph of $\sin x - x^2$

$$f'(0, x) = \begin{cases} (x, x)^T, & x \geqslant 0 \\ (0, -x)^T, & x < 0 \end{cases} \text{ and } f''(0, x) = (0, 2x^2).$$

$$g'(0, x) = \begin{cases} 0, & x \geqslant 0 \\ -2x, & x < 0 \end{cases} \text{ and } g''(0, x) = 0.$$

Since $f'(0; S) + intK = \{(a, b) \in \mathbb{R}^2 : a < 4, b > -4, a < b\}$ and $g'(0; S) +$ $intQ = \{c \in \mathbb{R} : c > 0\}$ are convex sets, therefore by Proposition 6.4 [5], $f'(0, x)$ and $g'(0, x)$ are K-subconvexlike and Q-subconvexlike respectively on S. Also, $g'(0; S) + \{\alpha g(0) : \alpha \geqslant 0\} + Q = \mathbb{R}$ implies that regularity condition (15) holds at $u = 0$. Since $f''(0, x) \in K$ and $g''(0, x) \in Q$ for all $x \in S_0$, therefore $(u, \lambda, \mu, \xi) = (0, (-1, 1), 0, \xi)$ is a feasible solution of associated second-order Mond-Weir type dual (MD), for every $\xi \in \mathbb{R}_+$.

Conclusion

In this article, we have studied nonsmooth vector optimization problem (VOP) wherein the functions are first and second-order directionally differentiable. New classes of second-order cone-semi(pseudoconvex)quasiconvex functions have been introduced in terms of second-order directional derivative. These functions generalize the ones studied by Suneja et al. [26]. Further, these functions are used to establish second-order KKT type sufficient optimality conditions for (VOP). Second-order Mond-Weir type and Wolfe type duals are formulated and duality results are proved. It may be explored that whether some conditions in the Strong Duality Theorem (Non-trivial Strong Duality Theorem) can be relaxed.

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