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An Integer Optimal Control Model of Production-Inventory System

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Abstract The optimal control model of the production-inventory system has investigated in several past studies, but without taking into account the integer condition. This study suggested a new approach to find the integer solution of production-inventory control model under periodic review policy. A new approach is based on the modified some equations of Pontryagin maximum principle that used to find the solution of the non-integer model. Our numerical results showed the efficiency of the new approach by saving the paths of inventory level and production rate up to reach its goals over time. The total penalty costs of the model were the same, despite a difference in the values of initial inventory level. Also, we testified a new approach by formulating the quadratic programming problem of the production-inventory system. The solution was the same for the two problems; quadratic programming and new approach.

Keywords Integer optimal control, Production-inventory system, Periodic review policy, Deteriorating items.

AMS 2010 subject classifications 49J21, 90B05.

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1. introduction

An inventory plays a major role in production planning through its effect on the costs and demand satisfied. Therefore, several researchers have investigated the production-inventory system with many practical situations, such as lead time, shortage, defective, deterioration and so on. Many mathematical techniques have applied to suggest a policy of production and inventory.

Several studies have investigated the production-inventory problem by constructing integer programming model. [13] considered the distribution problem of production-inventory system with many suppliers and multiple periods. The aim is to determine the production rate, inventory level and supplied quantities at each supply chain by branch-and-bound approach. Also with multiple periods, [12] discussed a model with many constraints, such as shortage, batch size and price discount. The amount of purchase at each period and minimized the total cost of materials was the target. The relationship between times of production and ordering in the inventory system with single item where the deteriorating rate is a constant and happened at a retailer was studied [30]. [28] investigated the steel production with slab allocation problem to maximize the total profit. They have taken into account many production factors, such as allocation quality, order-fulfill and the performance of slab and order. The minimize the total cost of setup, holding and transportation in the production system with multi-stage was addressed [11]. They assume a constant demand, as well as the greatest and smallest rates of production at each stage. Also with minimizing the total cost of production, inventory and transportation, [17] developed a vehicle routing model with multiple products, split deliveries, and a limitation for each route. Multi-Echelon model of the

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production-distribution problem discussed [22, 8]. [22] developed a model for perishable products, which have fixed lifetime, with multi-vehicle, production lines and multi-distributors. The aim is to minimize the total cost that includes storage, production and transportation. Meanwhile, [8] developed approach to find integer points from their neighborhood points, which is non-integer. For more references, the works of [24, 31, 16, 9] can be reviewed.
Other models of inventory control were used to investigate the optimality of production rate and inventory level. Researchers have discussed many assumptions to the inventory system to find the optimal policy of production. [1] discussed two policies of inventory review; continuous and periodic, with demand, deterioration, and holding cost varies with time. A production - inventory system formulated as a Markov decision problem with exponential distribution of production time [2, 23]. [2] considered a balance between the costs of inventory and lost sales for a single product and two customer kinds. They discussed decisions of produce an item or not, and production use to satisfy demand or increase inventory at each decision stage.[23] assumed exponential distribution at the lifetime of items, and the customer demand is satisfying from inventory, otherwise the customer must be waiting. [18, 29] have investigated model to minimize the penalty costs of inventory and production with stochastic demand and capacity limit by using the Pontryagins maximum principles and Hamilton-Jacobi-Bellman, respectively. [6, 10] have discussed a periodic-review policy with deterministic and stochastic demand, respectively. [10] minimized the total cost of inspection, backorder and inventory. Poisson process used to describe the stochastic demand of a single item [7]. They discussed an optimal stopping problem with a single machine, and formulated a problem as a free boundary. [27] considered inventory level that depends on the demand with multi-product, the time function of demand and production, continuous review policy, and market segmentation approach. A continuous review policy with deteriorating items and single product was investigated [14, 19]. [14] developed a model with tradable emission permits, and nonlinear costs functions of inventory and production that depend on the inventory level and production rate, respectively. Meanwhile, [19] discussed a stochastic model with emission tax, pollution abatement, and different functions of demand. They showed the effect of the model parameters, such as rates of demand, deterioration, tax, and costs of inventory and production, on the results by sensitivity analysis. [3] developed a mathematical model of inventory system that integrates a second and first order dynamical system together, which include production rate, inventory level, and the associated costs of the system. For more references, the works of [15, 26, 21, 20, 4] can be reviewed.
Above-mentioned studies have found the optimal solution of an optimal control model without taking into account the integer values of production and inventory. Our model contributes knowledge of literature suggested a new approach to find the solution of optimal control model with integer values of production rates and inventory levels.

2. Assumptions of the Model

We took into account the following:

1. A firm produces a product $N(t)$ to satisfy the demand $D(t)$ and store part of it in warehouse $y(t)$.
2. An optimal control problem includes two variables; control variable (production) that's controlled on the value of the state variable (inventory).
3. The administration determines a specific level of inventory, goal level $\hat{y}$, therefore, the production goal rate $\hat{n}(t)$. The safety stock that faced the demand fluctuation.
4. The demand has an effect on the production rate.
5. A penalty cost is incurred when the inventory level deviates from its goal level (symbolized by h), and for production (symbolized by k).
6. Values of penalty cost determine the priority of achieving the goals; inventory or production.
7. No shortage and items are subjected to deterioration during storage $\delta(t)$.

8. The target is to determine the production rate that satisfies the demand without lost sales and put inventory at a specific level, which determines by the administration.

3. optimal control model

The production-inventory control problem is as follows [25]:

$$Min \quad 2j = \sum_{t=0}^{T-1} h[y(t) - \hat{y}]^2 + k[N(t) - \hat{n}(t)]^2 \tag{1}$$

subject to,

$$\triangle\, y(t) = N(t) - D(t); \qquad t = 0, 1, ..., t_1 \tag{2}$$

$$\triangle\, y(t) = N(t) - D(t) - \delta(t) * y(t); \qquad t = t_1 + 1, ..., T - 1 \tag{3}$$

$$N(t) \geq 0; \qquad t = 0, 1, ..., T - 1 \tag{4}$$

Eq.(1) represents minimize of the total of mean square deviation of inventory levels and production rate from its goals, respectively. Meanwhile, Eq.(2) represents the inventory level that increase by production, and decreased by the demand. Items increased after a specific time by production, and decreased by the demand and deterioration (Eq.3).
To determine the production rate, we must first determine the production goal rate. From Eqs. (2 & 3), the production goal rate is given by:

$$\hat{n}(t) = D(t); \qquad t = 0, 1, ..., t_1 \tag{5}$$

$$\hat{n}(t) = D(t) + \delta(t) * \hat{y}; \qquad t = t_1 + 1, ..., T - 1 \tag{6}$$

In this paper, we suggest a new approach to find the solution of integer optimal control model, therefore we do not derive the optimality conditions of the non-integer model, which it clarified in many previous studies.
The production rate after many mathematical steps on Eqs.(1 to 6) is as follows [5]:

$$N(t) = \hat{n}(t) + \frac{1}{k}\lambda(t + 1); \qquad t = 0, ..., T - 1 \tag{7}$$

Also, the optimality conditions of the non-integer model are as follows:

$$\Delta y(t) = -\delta(t)[y(t) - \hat{y}] + \frac{1}{k}\lambda(t + 1); \qquad t = 0, ..., T - 1 \tag{8}$$

$$\Delta\lambda(t) = h[y(t) - \hat{y}] + \lambda(t + 1)\delta(t); \qquad t = 0, ..., T - 1 \tag{9}$$

where,
T: The length of planning period.
y(0): The initial inventory level.
$\lambda(t)$: The adjoint variable.
$\Delta y(t) = y(t + 1) - y(t)$.

The above system (Eqs. 8& 9) can be solved numerically with conditions $y(0) = y_0$ and $\lambda(T) = 0$ to find the inventory level and adjoint variable. Then, substitute the adjoint variable value into Eq.(7) to determine the production rate.
[25] solved the system (Eqs. 8& 9) by using the goal seek function in Microsoft Excel. They supposed initial value of $\lambda(0)$, and then used goal seek function with a condition $\lambda(T) = 0$ to find the adjusted

value of $\lambda(0)$, which means the optimal solution. Meanwhile, [5]) solved the system by finding the relation between $\lambda(0)$ and $\lambda(T)$,then used to condition $\lambda(T) = 0$ to determine $\lambda(0)$ is as follow:

$$\lambda(T) = E(T-1)[y_0 - \hat{y}] + C(T-1)\lambda(0) \tag{10}$$

E(T-1) represents the total value of the many paths that showed in Fig.(1)

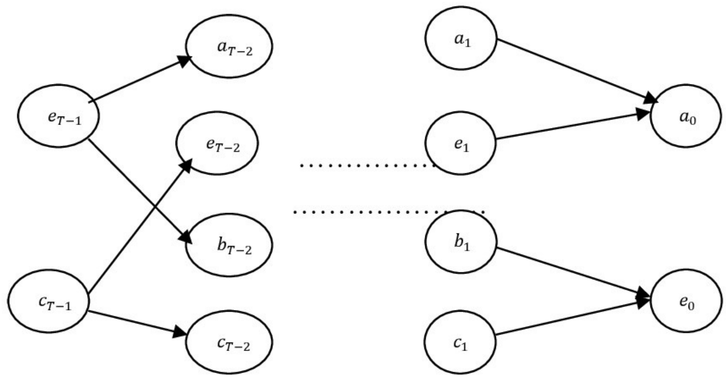

Figure 1. The network of E

where:
a and e: Connect with a and b.
b and c: Connect with c and e.

$$e_{T-1} \longrightarrow a_{T-2} \quad means \quad e_{T-1} * a_{T-2}$$

$$a_t = \{1 - \delta(t)\} + \frac{h}{k[1-\delta(t)]} \tag{11}$$

$$b_t = \frac{1}{k[1-\delta(t)]} \tag{12}$$

$$c_t = \frac{1}{1-\delta(t)} \tag{13}$$

$$e_t = \frac{h}{1-\delta(t)} \tag{14}$$

The network of $C(T-1)$ is similar to the network of $E(T-1)$, only replace the ending nodes; a_0 by b_0 and e_0 by c_0.
By applying the condition $\lambda(T) = 0$, we can find $\lambda(0)$ from Eq.(10).
Substitute $t = 0$ into Eq. (9), we get:

$$\lambda(1) = \frac{h}{1-\delta(0)}\{y(0) - \hat{y}\} + \frac{1}{1-\delta(0)}\lambda(0) \tag{15}$$

By substituting Eqs.(13 & 14) into Eq.(15), yields:

$$\lambda(1) = e_0\{y(0) - \hat{y}\} + c_0\lambda(0) \tag{16}$$

From Eq.(8), we get:

$$y(1) = \{1 - \delta(0)\}y(0) + \frac{1}{k}\lambda(1) + \delta(0)\hat{y} \tag{17}$$

By substituting Eq.(15) into Eq.(17), then substitute Eqs. (11 & 12) into result, yields:

$$y(1) = a_0 y(0) + b_0 \lambda(0) + \{1 - a_0\}\hat{y} \tag{18}$$

We can find $\lambda(1)$ from Eq.(16), then find $y(1)$ from Eq.(18). The same manner can be used to find other values of inventory level and adjust variable.
Remember the solution of the above-mentioned equations represents the non-integer solution of the problem, therefore, we showed it with brief details.

4. An Integer Optimal Control Model

$\lambda(t)$ represents the per unit change in the objective function for a small change in the optimal value of the state variable at time t [25]. Therefore, $\lambda(T) = 0$ means the optimal solution of the problem. Our suggested approach based on this idea, $\lambda(T) = 0$ means the non-integer optimal solution, so the integer optimal solution happen when $\lambda(T)$ negative and other values $\lambda(0, 1, 2, ..)$ are positive, and vice verse.
A new approach to find the solution of optimal control model with integer values of inventory level and production rate is as follows:

1. Modify Eq.(9) as follow:

$$\lambda(t+1) = Max[(\frac{h\{y(t) - \hat{y}\} + \lambda(t)}{1 - \lambda(t)}), \quad 0]; \qquad y(0) < \hat{y}$$
$$\lambda(t+1) = Min[(\frac{h\{y(t) - \hat{y}\} + \lambda(t)}{1 - \lambda(t)}), \quad 0]; \qquad y(0) > \hat{y}$$

   The terminal condition $\lambda(T) = 0$ will achieve automatically by Eq.(??). Also, Eq.(??) will save the path of the inventory level, which means decrease or increase, to reach its goal.
2. Modify Eq.(6) as follow:

$$\hat{n}(t) = D(t) + integ.\{\delta(t)\hat{y}\}; \qquad t = t_1 + 1, ..., T - 1. \tag{19}$$

3. From Eqs. (3 & 19), the production rate is as follows:

$$N(t) = \hat{n}(t) + y(t+1) - y(t); \qquad t = 0, 1, ..., T - 1. \tag{20}$$

4. From Eq.(18), we can find $\lambda(1)$, then chose two integer values; the lowest integer that higher than $\lambda(1)$, and the highest integer that lower than $y(1)$.
5. We can find two values of $\lambda(2)$ by substituting two integer values of $y(1)$ into Eq. (19).
6. We choose the nearest value from two values of $\lambda(2)$, extracted in (5), to $\lambda(2)$ extracted from the non-integer solution.
7. $y(1)$ corresponding to the $\lambda(2)$ that selected in (6) represents the integer value of the inventory level.
8. Repeat steps 4 to 7 to find other integer values of inventory levels. Sometimes, we test three integer values of $y(t)$ based on its value of the non-integer solution and its adjusted non-integer values during integer solution steps.
9. We can find the integer production rate from Eq.(20).

5. Numerical Solution

Consider an inventory system, with the following parameter values, $\hat{y} = 50 item; \quad T = 6 month; \quad k = 30\$; \quad h = 20\$; \quad D(t) = 150 + 5t; \quad \{\delta(t) = 0; \quad t = 0, 1, 2\}; \quad \{\delta(t) = 0.05t; \quad t = 3, 4, 5, 6\}$

- At first, we must find the value of initial adjust variable (see appendix A): $\lambda(0) = 1819.985$
- We can find the non-integer solution of Eqs.(5 to 9) by using Eqs.(16 & 17), which is shown in Figures 2 to 6 with two cases; $y(0) = 0$ and $y(0) = 100$.
- We suppose $y(0) = 0$, this means the initial inventory level is lower than the inventory goal level (50 items).
- The solution of integer problem as follows:

$$y(1) = 27 \quad or \quad 28$$

From Eq.(19):

$$\lambda(2) = \frac{h}{1-\delta(1)}\{y(1) - \hat{y}\} + \frac{1}{1-\delta(1)}\lambda(1)$$

$$\lambda(2) = 359.98 \quad or \quad 379.98$$

359.98 corresponding to $y(1) = 27$ is the nearest value to $\lambda(2)$ extracted from non-integer solution, thus:

$$y(1) = 27; \qquad \lambda(2) = 359.98$$

From Eq.(20):

$$N(0) = \hat{n} + y(1) - y(0)$$

$$N(0) = 150 + 27 - 0 = 177$$

Using Eq.(18) to find $y(2)$:

$$y(2) = a_1 y(1) + b_1 \lambda(1) + (1 - a_1)\hat{y} = 38.99$$

$y(2) = 39.55$ from non-integer solution, thus:

$$y(2) = 38 \quad or \quad 39 \quad or \quad 40$$

From Eq.(19):

$$\lambda(3) = \frac{h}{1-\delta(2)}\{y(2) - \hat{y}\} + \frac{1}{1-\delta(2)}\lambda(2)$$

$$\lambda(3) = 119.98 \quad or \quad 139.98 \quad or \quad 159.98$$

159.98 corresponding to $y(2) = 40$ is the nearest value to $\lambda(3)$ extracted from non-integer solution, thus:

$$y(2) = 40; \lambda(3) = 159.98$$

From Eq. (21):

$$N(1) = \hat{n}(1) + y(2) - y(1)$$

$$N(1) = 155 + 40 - 27 = 168$$

- By applying the same manner, we can find the solution:

$$\lambda(2) = 359.98; \quad y(2) = 40; \quad N(2) = 165,$$

$$\lambda(3) = 159.98; \quad y(3) = 45; \quad N(3) = 176;$$

$$\lambda(4) = 59.98; \quad y(4) = 48; \quad N(4) = 173;$$

$$\lambda(5) = 19.98; \quad y(5) = 49; \quad N(5) = 180;$$

$$\lambda(6) = 0; \quad y(6) = 49.$$

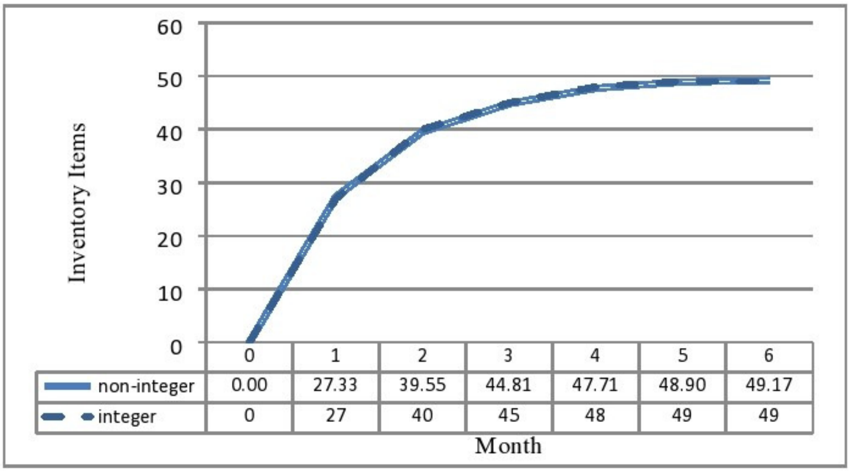

Figure 2. The inventory level (initial inventory less than goal)

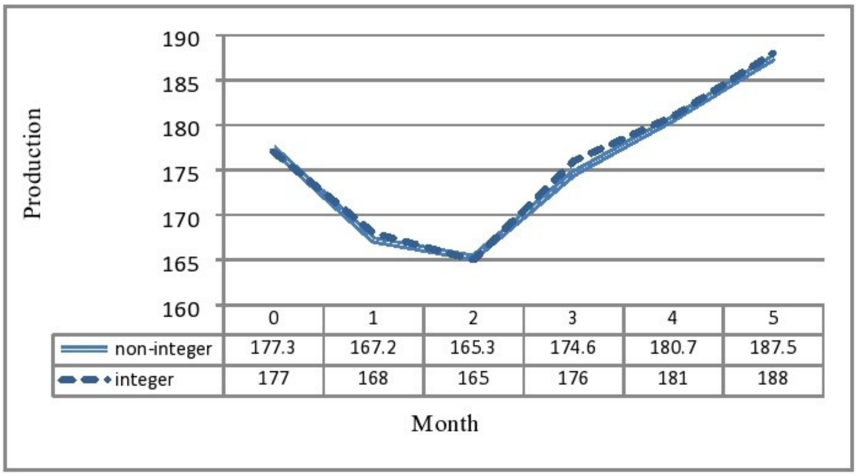

Figure 3. The production rate (initial inventory less than goal)

We can see the condition of the integer optimal solution $\lambda(T) = 0$, means the negative value according to Eq.(19), and other values $\lambda(0, 1, 2, ..)$ are positive, was achieved. Figures (2, 3 & 4) show the comparison between two solutions; integer and non-integer:

The simulation results are given in Figure (2) show that the large convergence between inventory levels for the two problems; integer and non-integer. Also, Figure (3) shows the same conclusion for the production rates.

Basically, the total penalty cost of the integer problem is higher than the non-integer problem, which is shown in Figure (4). The penalty costs for the two cases; $y(0) = 0$ and $y(0) = 100$ are the same, which means the efficiency of the approach in the find the solution of integer production- inventory problem. Figures (5& 6) show the solution when the initial inventory level is higher than the inventory goal level. Appendix B shows the integer solution and two additional examples.

From Figure (5), the inventory level is decreased up to reach inventory goal level against time when the initial inventory level is higher than inventory goal level. The opposite case is shown in Figure (2).

Theoretically, the production rate is a control variable that satisfies the demand, inventory goal level and compensates the deterioration. Therefore, there is a difference between production rates for the first months that are shown in Figures(3& 6) , then a convergence appears at the end of the planning period. For more efficiency measure, we formulated the quadratic programming problem of the production-inventory system with quadratic objective function and linear constraints. The objective function is the same without change. Linear constraints are as follows:

- Inventory levels less than or equal to inventory goal level.
- Eq.(20) for production rates.
- The total penalty cost is greater than or equal to the total penalty cost extracted from non-integer solution.
- Integer condition for all variables (inventory and production).

Solution of the quadratic programming problem is the same solution of our new approach.

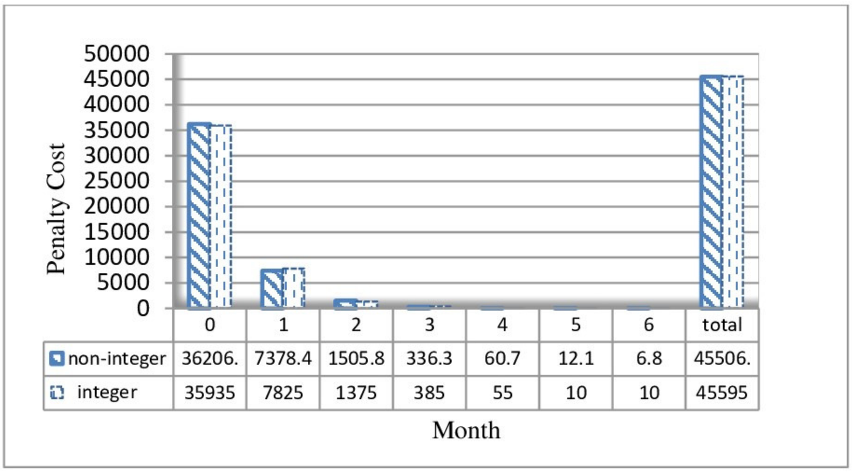

Figure 4. Penalty cost (initial inventory less than goal)

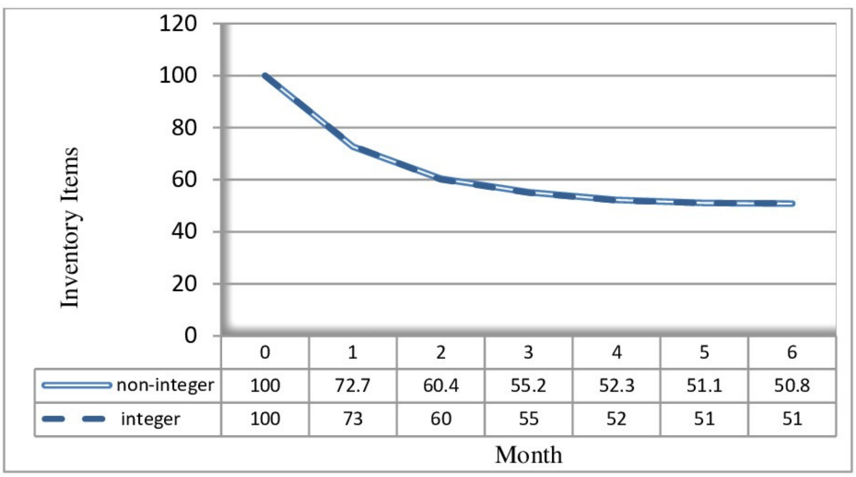

Figure 5. The inventory level (initial inventory greater than goal)

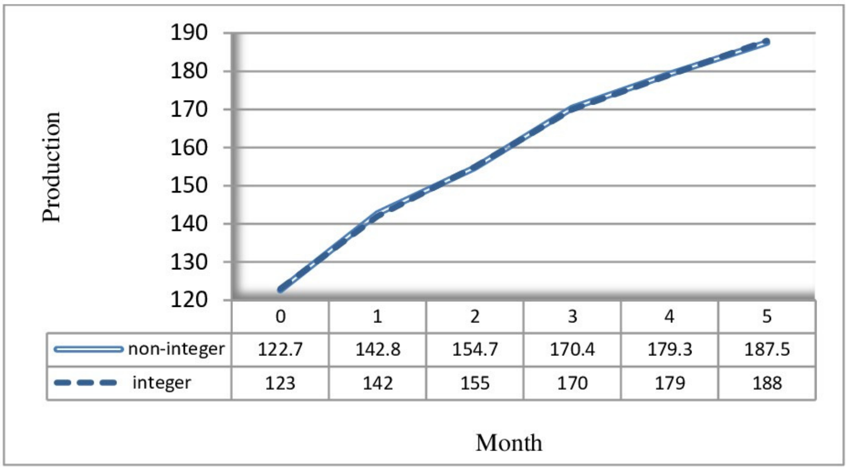

Figure 6. The production rate (initial inventory greater than goal)

6. Conclusion and Recommendations

In this paper, we developed an optimal control model of the production- inventory system under periodic review policy. The model supposed the items subject to deterioration through storage after a specific time period. Past studies that have discussed optimal control model of inventory system do not take into account the integer values of production and inventory. Practically, many fields of production must take integer values through the process of production or inventory.
Our model suggested a new approach to find the solution of optimal control model with integer values of production and inventory. A new approach based on the modified some equations of Pontryagin maximum principle that used to find the explicit solution of the system.
A new approach applied in two cases when the initial inventory level is higher that the inventory goal level, and the opposite case. The results showed achieving the inventory goal level and a production goal rate over time with the same total penalty cost despite a difference in the initial inventory level. We testified a new approach by formulating the quadratic programming problem of the production-inventory system. The solution was the same for the two problems; quadratic programming and new approach. Therefore, the approach was found to be efficient to find the solution of optimal control model with integer values of production and inventory. This study could be extended to include multi-items or multi-production lines.

A. appendix A

From Eq.(10), we have:

$$\lambda(6) = E(5)y(0) + C(5)\lambda(0) - e(5)\hat{y}$$

From Eq.(11):

$$a_0 = (1-0) + \frac{20}{30(1-0)} = 1.667; \qquad a_1 = (1-0)\frac{20}{30(1-0)} = 1.667$$

$$a_2 = (1-0) + \frac{20}{30(1-0)} = 1.667; \qquad a_3 = (1-0.05*3)\frac{20}{30(1-0.05*3)} = 1.634$$

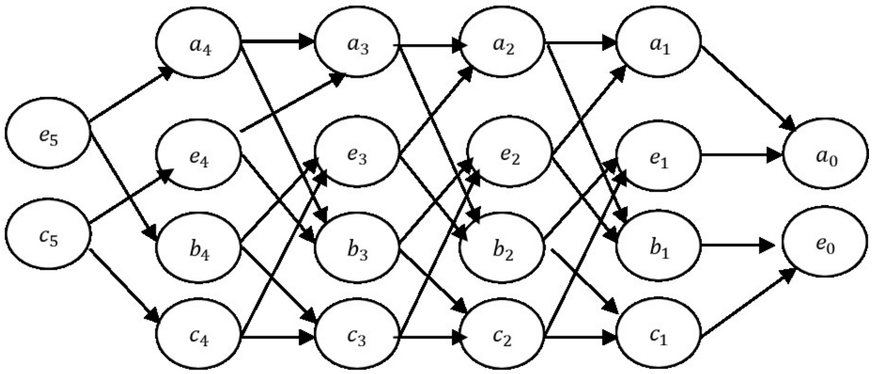

Figure 7. The network of E(5)

$$a_4 = (1 - 0.05 * 4)\frac{20}{30(1 - 0.05 * 4)} = 1.633$$

We can be found b_t from Eq.(12):

$$b_1 = \frac{1}{30(1 - 0)} = 0.033; \qquad b_2 = \frac{1}{30(1 - 0)} = 0.033$$

$$b_3 = \frac{1}{30(1 - 0.05 * 3)} = 0.039; \qquad b_4 = \frac{1}{30(1 - 0.04 * 4)} = 0.042$$

Values of c_t determine from Eq.(13):

$$c_1 = \frac{1}{1 - 0} = 1; \quad c_2 = \frac{1}{1 - 0} = 1; \quad c_3 = \frac{1}{1 - 0.05 * 3} = 1.176$$

$$c_4 = \frac{1}{1 - 0.05 * 4} = 1.25; \quad c_5 = \frac{1}{1 - 0.05 * 5} = 1.333$$

From Eq.(14):

$$e_0 = \frac{20}{1 - 0} = 20; \quad e_1 = \frac{20}{1 - 0} = 20; \quad e_2 = \frac{20}{1 - 0} = 20$$

$$e_3 = \frac{20}{1 - 0.05 * 3} = 23.5; \quad e_1 = \frac{20}{1 - 0.05 * 4} = 25; \quad e_2 = \frac{20}{1 - 0.05 * 5} = 26.7$$

$E(5) = (e_5 * a_4 * a_3 * a_2 * a_1 * a_0) + (c_5 * e_4 * a_3 * a_2 * a_1 * a_0) + (e_5 * a_4 * a_3 * b_2 * e_1 * a_0) + (c_5 * e_4 * a_3 * b_2 * e_1 * a_0) + (e_5 * a_4 * b_3 * e_2 * a_1 * a_0) + (c_5 * e_4 * b_3 * e_2 * a_1 * a_0) + (e_5 * a_4 * b_3 * c_2 * e_1 * a_0) + (c_5 * e_4 * b_3 * c_2 * e_1 * a_0) + (e_5 * b_4 * e_3 * a_2 * a_1 * a_0) + (c_5 * c_4 * e_3 * a_2 * a_1 * a_0) + (e_5 * b_4 * e_3 * b_2 * e_1 * a_0) + (c_5 * c_4 * e_3 * b_2 * e_1 * a_0) + (e_5 * b_4 * c_3 * e_2 * a_1 * a_0) + (c_5 * c_4 * c_3 * e_2 * a_1 * a_0) + (e_5 * b_4 * c_3 * c_2 * e_1 * a_0) + (c_5 * c_4 * c_3 * c_2 * e_1 * a_0) + (e_5 * a_4 * a_3 * a_2 * b_1 * e_0) + (c_5 * e_4 * a_3 * a_2 * b_1 * e_0) + (e_5 * a_4 * a_3 * b_2 * c_1 * e_0) + (c_5 * e_4 * a_3 * b_2 * c_1 * e_0) + (e_5 * a_4 * b_3 * e_2 * b_1 * e_0) + (c_5 * e_4 * b_3 * e_2 * b_1 * e_0) + (e_5 * a_4 * b_3 * c_2 * c_1 * e_0) + (c_5 * e_4 * b_3 * c_2 * c_1 * e_0) + (e_5 * b_4 * e_3 * a_2 * b_1 * e_0) + (c_5 * c_4 * e_3 * a_2 * b_1 * e_0) + (e_5 * b_4 * e_3 * b_2 * c_1 * e_0) + (c_5 * c_4 * e_3 * b_2 * c_1 * e_0) + (e_5 * b_4 * c_3 * e_2 * b_1 * e_0) + (c_5 * c_4 * c_3 * e_2 * b_1 * e_0) + (e_5 * b_4 * c_3 * c_2 * c_1 * e_0) + (c_5 * c_4 * c_3 * c_2 * c_1 * e_0)$

$E(5) = 1655.105 + 549.032 = 2204.137$

$C(5) = allpathsfrome_5toa_0 + allpathsfromc_5toa_0 * (b_0/a_0) + allpathsfrome_5toe_0 + allpathsfromc_5toe_0 * (c_0/e_0)$

$C(5) = 1655.1 * (0.033333/1.666667) + 549.032 * (1/20) = 60.554$

By applying the condition $\lambda(T) = 0$, we can find $\lambda(0)$ as follows:

$$0 = 2204.137 * 0 + 609.554 * \lambda(0) - 2204.137 * 50$$

$$\lambda(0) = 1819.985$$

B. appendix B

The solution of integer problem as follows:

$$y(1) = 72 \quad or \quad 73$$

From Eq.(19):

$$\lambda(2) = \frac{h}{1-\delta(1)}\{y(1) - \hat{y}\} + \frac{1}{1-\delta(1)}\lambda(1)$$

$$\lambda(2) = -379.98 \quad or \quad -359.98$$

-359.98 corresponding to $y(1) = 73$ is the nearest value to $\lambda(2)$ extracted from non-integer solution, thus:

$$y(1) = 73; \qquad \lambda(2) = -359.98$$

From Eq.(20):

$$N(0) = \hat{n}(0) + y(1) - y(0)$$

$$N(0) = 150 + 73 - 100 = 123$$

Using Eq.(18) to find $y(2)$:

$$y(2) = a_1 y(1) + b_1 \lambda(1) + (1 - a_1)\hat{y} = 61.0005$$

$y(2) = 60.44$ from non-integer solution, thus:

$$y(2) = 60 \quad or \quad 61 \quad or \quad 62$$

From Eq.(19):

$$\lambda(3) = \frac{h}{1-\delta(2)}\{y(2) - \hat{y}\} + \frac{1}{1-\lambda(2)}\lambda(2)$$

$$\lambda(3) = -159.98 \quad or \quad -139.98 \quad or \quad 119.98$$

-159.98 corresponding to $y(2) = 60$ is the nearest value to $\lambda(3)$ extracted from non-integer solution, thus:

$$y(2) = 60; \lambda(3) = -159.98$$

From Eq.(20):

$$N(1) = \hat{n}(1) + y(2) - y(1) = 155 + 60 - 73 = 142$$

By applying the same manner, we can find the solution:

$$\lambda(2) = -159.98; \quad y(2) = 60; \quad N(2) = 155,$$

$$\lambda(3) = -59.98; y(3) = 55; N(3) = 170,$$

$$\lambda(4) = -26.64; y(4) = 52; N(4) = 179,$$

$$\lambda(5) = -11.72; y(5) = 51; N(5) = 188,$$

$$\lambda(6) = 0; y(6) = 51.$$

Table 1. Integer and non-integer solution of the problem 1

Month	Non-integer solution				Integer solution			
	λ	y	N	J	λ	y	N	J
0	-1457.96	70	228.1	23215.3	-1457.96	70	228	23260
1	-657.965	48.06	236.1	4730.7	-657.965	48	236	4740
2	-296.608	38.18	229.6	964.0	-297.965	38	230	880
3	-132.99	33.74	212.1	196.6	-137.965	34	212	220
4	-58.032	31.81	185.3	40.8	-57.9648	32	185	55
5	-21.762	31.08	150	11.8	-17.9648	31	150	10
6	0	31.08		11.8	0	31		10
Total				29171.1				29175

Example 1: Consider an inventory system, with the following parameter values, $\hat{y} = 30item; y(0) = 70item; T = 6month; k = 30\$; h = 20\$; D(t) = 250 - 4t^2$
Sol.
Table (1) shows the solution of integer and non-integer.

Example 2: Consider an inventory system, with the following parameter values, $\hat{y} = 40item; y(0) = 5item; T = 8month; k = 25\$; h = 10\$; D(t) = 0.5t^3 - 1.5t^2 + 10t$
Sol.
Table (2) shows the solution of integer and non-integer.

Table 2. Integer and non-integer solution of the problem 2

Month	Non-integer solution				Integer solution			
	λ	y	N	J	λ	y	N	J
0	755.3069	5	16.21	9410.5	755.3069	5	16	9325
1	405.3069	21.21	17.70	2710.4	405.3069	21	18	2817.5
2	217.4297	29.91	22.66	780.7	215.3069	30	23	812.5
3	116.5243	34.57	32.49	224.8	115.3069	35	32	175
4	62.22869	37.06	49.31	64.8	65.30692	37	49	57.5
5	32.82453	38.37	75.66	18.7	35.30692	38	76	32.5
6	16.55018	39.03	114.28	5.6	15.30692	39	114	5
7	6.89591	39.31	168.00	2.4	5.306915	39	168	5
8	0	39.31		2.4	0	39		5
Total				13220.2				13235

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