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Expectation Properties of Generalized Order Statistics from Poisson Lomax Distribution

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Abstract The Poisson Lomax distribution was proposed by [3], as a useful model for analyzing lifetime data. In this paper, we have derived recurrence relations for single and product moments of generalized order statistics for this distribution. Further, characterization of the distribution is carried out. Some deductions and particular cases are also discussed.

Keywords Poisson Lomax Distribution; Generalized Order Statistics; Single Moments; Product Moments; Characterization.

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1. Introduction

The Poisson Lomax distribution was proposed by [3], as a three-parameter lifetime distribution with upside-down bathtub shaped failure rate and heavy tailed, and can be used in modelling many practical situation. It is a compound distribution of the zero-truncated Poisson and the Lomax distributions.

A random variable X is said to follow the Poisson Lomax distribution (PLD) if its probability density function (pdf) is of the form

$$f(x) = \frac{\alpha\beta\lambda(1+\beta x)^{-(1+\alpha)}e^{-\lambda(1+\beta x)^{-\alpha}}}{1-e^{-\lambda}}, \quad x > 0, \alpha > 0, \beta > 0, \lambda > 0, \tag{1}$$

and the corresponding survival function is

$$\bar{F}(x) = \frac{1-e^{-\lambda(1+\beta x)^{-\alpha}}}{1-e^{-\lambda}}, \quad x > 0, \alpha > 0, \beta > 0, \lambda > 0, \tag{2}$$

where $\bar{F}(x) = 1 - F(x)$.

In view of (1) and (2), it can be seen that

$$f(x) = c_1(1+\beta x)^{-(\alpha+1)} - c_2(1+\beta x)^{-(\alpha+1)}\bar{F}(x). \tag{3}$$

where $c_1 = \frac{\alpha\beta\lambda}{1-e^{-\lambda}}$ and $c_2 = \alpha\beta\lambda$.

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Relation (3) can also be expressed as

$$\frac{\bar{F}(x)}{f(x)} = \frac{1}{\alpha\beta\lambda}\sum_{a=0}^{\infty}\sum_{b=0}^{[\alpha(1-a)+1)]}\frac{\lambda^a}{a!}\binom{[\alpha(1-a)+1)]}{b}\beta^b x^b \tag{4}$$

where $[\alpha(1-a)+1)]$ is an integer.

1.1. Generalized order statistics

The concept of generalized order statistics was introduced and extensively studied by [17], which includes different ordered random schemes, such as order statistics, record values, sequential order statistics, progressively type II censored order statistics and Pfeifer's records as its special cases.

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $k \geq 1$ be the parameters, such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j \geq 0 \text{ for } 1 \leq i \leq n-1.$$

The random variables $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k), \ldots, X(n,n,\tilde{m},k)$ are said to be generalized order statistics from an absolutely continuous distribution function $F()$ with the probability density function (*pdf*) $f()$, if their joint *pdf* is of the form

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}\left[1-F(x_i)\right]^{m_i}f(x_i)\Big)\left[1-F(x_n)\right]^{k-1}f(x_n) \tag{5}$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$.

If $m_i = 0;\ i = 1\ldots n-1$, $k = 1$, we obtain the joint *pdf* of the order statistics and for $m_i \to -1,\ k \in N$, we get joint *pdf* of k^{th} upper record values.

Here we may consider two cases:

Case I. $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \ldots, n-1, i \neq j$.

In view of (5), the *pdf* of r^{th} *gos* $X(r,n,\tilde{m},k)$ is given as ([18])

$$f_{X(r,n,\tilde{m},k)}(x) \ = \ C_{r-1}f(x)\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i-1}, \tag{6}$$

where

$$C_{r-1} \ = \ \prod_{i=1}^{r}\gamma_i, \qquad \gamma_i = k+n-i+\sum_{j=1}^{n-1}m_j > 0,$$

and

$$a_i(r) = \prod_{\substack{j=1 \\ j\neq i}}^{r}\frac{1}{(\gamma_j-\gamma_i)}, \quad 1 \leq i \leq r \leq n.$$

The joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \leq r < s \leq n$, is given as ([18])

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) \ = \ C_{s-1}\sum_{j=r+1}^{s}{a_j}^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_j}\left[\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}, \ \ x<y, \tag{7}$$

where

$$a_i{}^{(r)}(s) = \prod_{\substack{I=r+1 \\ I\neq j}}^{s} \frac{1}{(\gamma_I - \gamma_j)}, \quad r+1 \leq j \leq s \leq n.$$

Case II : $m_i = m$, $i = 1, 2, \ldots, n-1$.

The *pdf* of r^{th} *gos* $X(r,n,m,k)$ is given as ([17])

$$f_{X(r,n,m,k)}(x) \;=\; \frac{C_{r-1}}{(r-1)!}\,[\bar{F}(x)]^{\gamma_r - 1}\, f(x)\, g_m^{r-1}(F(x)), \tag{8}$$

where

$$C_{r-1} \;=\; \prod_{i=1}^{r} \gamma_i \quad , \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) \;=\; \begin{cases} -\dfrac{1}{m+1}\,(1-x)^{m+1} & , \;\; m \neq -1 \\[2ex] \log\left(\dfrac{1}{1-x}\right) & , \;\; m = -1 \end{cases}$$

and

$$g_m(x) \;=\; h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \;\; x \in [0,1).$$

The joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \leq r < s \leq n$, is given as ([17])

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) \;=\; \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}\,[\bar{F}(x)]^m\, g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1}$$
$$\times\, [\bar{F}(y)]^{\gamma_s - 1}\, f(x)\, f(y), \;\; -\infty \leq x < y \leq \infty. \tag{9}$$

.

Also from [23], for $m_i = m \neq -1$, $i = 1, 2, \ldots, n-1$.

$$a_i(r) = \frac{(-1)^{r-i}}{(m+1)^{r-1}(r-1)!}\binom{r-1}{i} \tag{10}$$

$$a_i^{(r)}(s) = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}(r-1)!}\binom{s-r-1}{i} \tag{11}$$

A large volume of work has been done on the study of moments and recurrence relations between moments of generalized order statistics. The moments of ordered random schemes assume considerable importance in the statistical literature. Many authors have investigated and derived several recurrence relations and identities satisfied by the single as well as product moments. [24],[25] studied the recurrence relations and identities for moments of order statistics for some specific distributions. Recurrence relations for the expected values of certain functions of order statistics are considered by [1], [2]. [7] investigated the relations between expected values of functions of *gos*. For more detailed survey, one may refer to [4], [5], [8], [12], [15], [18], [19], [26], [27], [29], [30], [32] and references therein.

The characterization of a probability distribution has always been the important topic in statistics and mathematical sciences. Several approaches are available to characterize a probability distribution. In this paper, first we established recurrence relations between single and product moments of *gos* from PLD. Then, these relation are used to characterize the said distribution. Also a characterization theorem based on conditional expectation is presented. For related results on characterization, one can see [6], [9], [10], [11], [13], [20], [22] and [28] among others.

1.2. Gauss hypergeometric function

Gauss hypergeometric function is defined as

$$
{}_2F_1(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!} \tag{12}
$$

where $c\neq 0,-1,-2,\ldots$. It converges if one of the following conditions holds:
$(i)\ \ |x|<1;$
$(ii)\ \ |x|=1, Re(c-a-b)>0.$

2. Single Moments

Theorem1
Let case I be satisfied. For the PLD given in (1) and for $n\,\epsilon\, N, \tilde{m}\,\epsilon\,\mathbb{R}, k\ >\ 0, 1\leq r\leq n,\ p=1,2,\ldots$

$$
E[X^p(r,n,\tilde{m},k)]=\alpha C_{r-1}\sum_{i=1}^{r}\sum_{j=0}^{\gamma_i-1}\sum_{l=0}^{\infty}a_i(r)\binom{\gamma_i-1}{j}\frac{(-1)^{j+l}\lambda^l(j+1)^l}{\beta^p(1-e^{-\lambda})^{\gamma_i}l!}\mathbf{B}(p+1,\alpha+\alpha l-p), \tag{13}
$$

where $B(x,y)$ is complete beta function.
Proof
We have

$$
\begin{aligned}
E[X^p(r,n,\tilde{m},k)] &= C_{r-1}\int_0^{\infty}x^p\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i-1}f(x)\mathrm{d}x \\
&= C_{r-1}\sum_{i=1}^{r}a_i(r)\int_0^{\infty}x^p[\bar{F}(x)]^{\gamma_i-1} \\
&\qquad\times\left\{\frac{\alpha\beta\lambda(1+\beta x)^{-(\alpha+1)}}{1-e^{-\lambda}}-\alpha\beta\lambda(1+\beta x)^{-(\alpha+1)}\bar{F}(x)\right\}\mathrm{d}x \\
&= \alpha\beta\lambda C_{r-1}\sum_{i=1}^{r}\frac{a_i(r)}{(1-e^{-\lambda})^{\gamma_i}}\int_0^{\infty}x^p(1+\beta x)^{-(\alpha+1)} \\
&\qquad\times\{1-e^{-\lambda(1+\beta x)^{-\alpha}}\}^{\gamma_i-1}e^{-\lambda(1+\beta x)^{-\alpha}}\mathrm{d}x \\
&= \alpha\beta\lambda C_{r-1}\sum_{i=1}^{r}\frac{a_i(r)}{(1-e^{-\lambda})^{\gamma_i}}\sum_{j=0}^{\gamma_i-1}\binom{\gamma_i-1}{j}(-1)^j\sum_{l=0}^{\infty}\frac{(-1)^l\gamma^l(j+1)^l}{l!} \\
&\qquad\times\int_0^{\infty}x^p(1+\beta x)^{-(\alpha+\alpha l+1)}\mathrm{d}x.
\end{aligned}
$$

Now, by using the result from [14] p. 315 given as,

$$
\int_0^{\infty}x^{\mu-1}(1+\beta x)^{-\nu}\mathrm{d}x=\beta^{-\mu}\mathbf{B}(\mu,\nu-\mu) \tag{14}
$$

we get

$$
E[X^p(r,n,\tilde{m},k)]=\alpha C_{r-1}\sum_{i=1}^{r}\sum_{j=0}^{\gamma_i-1}\sum_{l=0}^{\infty}\binom{\gamma_i-1}{j}\frac{a_i(r)\lambda^l(j+1)^l(-1)^{j+l}}{\beta^p(1-e^{-\lambda})^{\gamma_i}l!}\mathbf{B}(p+1,\alpha+\alpha l-p). \tag{15}
$$

Hence the theorem.

Corollary1
When $m_i = m, i = 1, 2, \ldots, n-1$, relation (13) reduces to the single moment of gos for Case-II.

$$E[X^p(r,n,m,k)] = \alpha\, C_{r-1} \sum_{i=1}^{r} \sum_{j=0}^{\gamma_i - 1} \sum_{l=0}^{\infty} \binom{\gamma_i - 1}{j} \binom{r-1}{i} \times \frac{(-1)^{r-i+j+l} \lambda^{l+1} (j+1)^l}{(r-1)!\, (m+1)^{r-1} \beta^p (1-e^{-\lambda})^{\gamma_i} l!} \mathbf{B}(p+1, \alpha + \alpha l - p). \quad (16)$$

Theorem2
For the conditions as stated in Theorem 1. The recurrence relation for single moments of gos for PLD is given as

$$E[X^p(r,n,\tilde{m},k)] - E[X^p(r-1,n,\tilde{m},k)] = p\, C_{r-2} \sum_{i=1}^{r} \sum_{j=0}^{\gamma_i} \sum_{l=0}^{\infty} \binom{\gamma_i}{j} \times \frac{a_i(r)\, (-1)^{j+l} \lambda^l j^l}{l!\, \beta^p (1-e^{-\lambda})^{\gamma_i}} \mathbf{B}(p, \alpha l - p). \quad (17)$$

Also,

$$E[X^p(r,n,\tilde{m},k)] - E[X^p(r-1,n,\tilde{m},k)] = \frac{p}{\gamma_r \alpha \beta \lambda} \sum_{a=0}^{\infty} \sum_{b=1}^{[\alpha(1-a)+1]} \binom{[\alpha(1-a)+1]}{b} \frac{\lambda^a \beta^b}{a!} E[X^{p+b-1}(r,n,\tilde{m},k)]. \quad (18)$$

Proof
We have by [7].

$$E[\xi\{X(r,n,\tilde{m},k)\}] - E[\xi\{X(r-1,n,\tilde{m},k)\}] = C_{r-2} \int_{-\infty}^{\infty} \xi'(x) \sum_{i=1}^{r} a_i(r) [\bar{F}(x)]^{\gamma_i} \mathrm{d}x. \quad (19)$$

For $\xi(x) = x^p$ in (19), recurrence relation for single moments of gos is

$$\begin{aligned} E[X^p(r,n,\tilde{m},k)] &- E[X^p(r-1,n,\tilde{m},k)] \\ &= p\, C_{r-2} \int_0^{\infty} x^{p-1} \sum_{i=1}^{r} a_i(r) [\bar{F}(x)]^{\gamma_i} \mathrm{d}x. \\ &= p\, C_{r-2} \sum_{i=1}^{r} a_i(r) \int_0^{\infty} x^{p-1} \left\{ \frac{1 - e^{-\lambda(1+\beta x)^{-\alpha}}}{1 - e^{-\lambda}} \right\}^{\gamma_i} \mathrm{d}x. \\ &= p\, C_{r-2} \sum_{i=1}^{r} \sum_{j=0}^{\gamma_i} \sum_{l=0}^{\infty} \binom{\gamma_i}{j} \frac{a_i(r) (-1)^{j+l} \lambda^l j^l}{l!\, (1-e^{-\lambda})^{\gamma_i}} \int_0^{\infty} x^{p-1} (1+\beta x)^{-\alpha l} \mathrm{d}x. \quad (20) \end{aligned}$$

Now on simplification of (20) and using (14) we get the required result (17).

The expression (18) can be proved in view of [7] using (4).

Corollary2
When $m_i = m, i = 1, 2, \ldots, n-1$, relation (17) and (18) reduces to the single moment of gos for Case-II.

$$E[X^p(r,n,m,k)] - E[X^p(r-1,n,m,k)] = p\, C_{r-2} \sum_{i=1}^{r} \sum_{j=0}^{\gamma_i} \sum_{l=0}^{\infty} \binom{\gamma_i}{j} \binom{r-1}{i} \times \frac{(-1)^{j+l+r-i} \lambda^l j^l}{(m+1)^{r-1}(r-1)!\, l!\, \beta^p (1-e^{-\lambda})^{\gamma_i}} \mathbf{B}(p, \alpha l - p). \quad (21)$$

Also

$$E[X^p(r,n,m,k)] - E[X^p(r-1,n,m,k)] = \frac{p}{\gamma_r \alpha\beta\lambda} \sum_{a=0}^{\infty} \sum_{b=1}^{[\alpha(1-a)+1]} \binom{[\alpha(1-a)+1]}{b} \frac{\lambda^a \beta^b}{a!} E[X^{p+b-1}(r,n,m,k)]. \quad (22)$$

Remark1
Putting $m_i = 0, i = 1, 2, \ldots, n-1$ and $k = 1$, we get the recurrence relation for single moments of order statistics

$$E[X^p_{r:n}] - E[X^p_{r-1:n}] = \frac{p}{(n-r-1)\alpha\beta\lambda} \sum_{a=0}^{\infty} \sum_{b=1}^{[\alpha(1-a)+1]} \binom{[\alpha(1-a)+1]}{b} \frac{\lambda^a \beta^b}{a!} E[X^{p+b-1}_{r:n}]$$

Remark2
When $m_i \to -1, i = 1, 2, \ldots, n-1$, the recurrence relation for single moments of k^{th} upper record values will be

$$E(X^{(k)}_{U(r)})^p - E(X^{(k)}_{U(r-1)})^p = \frac{p}{k\alpha\beta\lambda} \sum_{a=0}^{\infty} \sum_{b=1}^{[\alpha(1-a)+1]} \binom{[\alpha(1-a)+1]}{b} \frac{\lambda^a \beta^b}{a!} E(X^{(k)}_{U(r)})^{p+b-1}.$$

3. Product Moments

Theorem3
Let case-I be satisfied For the Poisson Lomax distribution given as in (1) and for $n \in N, \tilde{m} \in \mathbb{R}, k > 0, 1 \le r < s \le n, p, q = 1, 2, \ldots$, then the $(p,q)th$ product moment is given by

$$\mu^{p,q}_{r,s,n,\tilde{m},k} = \alpha^2\, C_{s-1} \sum_{i=1}^{r} \sum_{j=r+1}^{s} \sum_{l=0}^{\gamma_j - 1} \sum_{u=0}^{\infty} \sum_{v=0}^{\gamma_i-\gamma_j-1} \sum_{w=0}^{\infty} a_i(r)\, a_j^{(r)}(s) \times \binom{\gamma_j - 1}{l} \binom{\gamma_i - \gamma_j - 1}{v} \frac{(\alpha+\alpha t+1)_u\, (\alpha+\alpha t - q)_w}{u!\, (\alpha+\alpha t - q+1)_u} \times \frac{(-1)^{l+t+v+w+u}\, \lambda^{t+w+2} (l+1)^t (v+1)^w}{t!\, (1-e^{-\lambda})^{\gamma_i} \beta^{p+q-1} (\alpha+\alpha t - q)} \quad (23)$$

Proof
We have

$$\mu^{p,q}_{r,s,n,\tilde{m},k} = E[X^p(r,n,\tilde{m},k), X^q(s,n,\tilde{m},k)]$$
$$= C_{s-1} \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s) \int_0^{\infty} \int_x^{\infty} x^p y^q \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_j} [\bar{F}(x)]^{\gamma_i} \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)} \mathrm{d}x\, \mathrm{d}y$$

$$= C_{s-1}\sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s) \int_0^{\infty}\int_x^{\infty} x^p y^q [\bar{F}(y)]^{\gamma_j-1}[\bar{F}(x)]^{\gamma_i-\gamma_j-1} f(x)\, f(y) \mathrm{d}x\, \mathrm{d}y$$

$$= C_{s-1}\sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s) \int_0^{\infty} x^p [\bar{F}(x)]^{\gamma_i-\gamma_j-1} f(x) \mathrm{d}x \int_x^{\infty} y^q [\bar{F}(y)]^{\gamma_j-1}\, f(y)\mathrm{d}y\,. \tag{24}$$

Consider

$$I(y) = \int_x^{\infty} y^q [\bar{F}(y)]^{\gamma_j-1}\, f(y)\mathrm{d}y$$

$$= \int_x^{\infty} y^q [\bar{F}(y)]^{\gamma_j-1} \left\{ \frac{\alpha\beta\lambda(1+\beta y)^{-(\alpha+1)}}{1-e^{-\lambda}} - \alpha\beta\lambda(1+\beta y)^{-(\alpha+1)}\, \bar{F}(y) \right\} \mathrm{d}y$$

$$= \frac{\alpha\beta\lambda}{(1-e^{-\lambda})^{\gamma_i}} \int_x^{\infty} y^q (1+\beta y)^{-(\alpha+1)} [1-e^{-\lambda(1+\beta y)^{-\alpha}}]^{\gamma_j-1}\, e^{-\lambda(1+\beta y)^{-\alpha}} \mathrm{d}y$$

$$= \frac{\alpha\beta\lambda}{(1-e^{-\lambda})^{\gamma_i}} \sum_{l=0}^{\gamma_j-1} \binom{\gamma_j-1}{l} \sum_{t=0}^{\infty} \frac{(-1)^{l+t}\lambda^t (l+1)^t}{t!} \times \int_x^{\infty} y^q (1+\beta y)^{-(\alpha+\alpha t+1)} \mathrm{d}y. \tag{25}$$

Since we have from [14], p. 315.

$$\int_u^{\infty} x^{\mu-1}(1+\beta x)^{-\nu} \mathrm{d}x = \frac{u^{\mu-\nu}}{\beta^{\nu}(\mu-\nu)}\, {}_2F_1\left(\nu, \nu-\mu; \nu-\mu+1; -\frac{1}{\beta u}\right). \tag{26}$$

Thus using (26) and (12) in (25), we get

$$I(y) = \frac{\alpha}{(1-e^{-\lambda})^{\gamma_i}} \sum_{l=0}^{\gamma_j-1} \binom{\gamma_j-1}{l} \sum_{t=0}^{\infty} \frac{(-1)^{l+t}\lambda^{t+1}(l+1)^t}{t!} \times \frac{x^{q-\alpha-\alpha t}}{\beta^{\alpha+\alpha t}(q-\alpha-\alpha t)}\, {}_2F_1\left(\alpha+\alpha t+1, \alpha+\alpha t-q; \alpha+\alpha t+1; -\frac{1}{\beta x}\right).$$

$$I(y) = \frac{\alpha}{(1-e^{-\lambda})^{\gamma_i}} \sum_{l=0}^{\gamma_j-1} \binom{\gamma_j-1}{l} \sum_{t=0}^{\infty}\sum_{u=0}^{\infty} \frac{(-1)^{l+t+u}\lambda^{t+1}(l+1)^t x^{q-\alpha-\alpha t-1}}{t!u!\, \beta^{\alpha+\alpha t}(q-\alpha-\alpha t)} \times \frac{(\alpha+\alpha t+1)_u(\alpha+\alpha t-q)_u}{(q-\alpha-\alpha t-q+1)_u}. \tag{27}$$

Now using (27) in (24), we get

$$\mu_{r,s,n,\tilde{m},k}^{p,q} = \alpha\, C_{s-1} \sum_{i=1}^{r} \frac{a_i(r)}{(1-e^{-\lambda})^{\gamma_i}} \sum_{j=r+1}^{s} a_j^{(r)}(s) \sum_{l=0}^{\gamma_j-1} \binom{\gamma_j-1}{l} \times \sum_{t=0}^{\infty}\sum_{u=0}^{\infty} \frac{(-1)^{l+t+u}\lambda^{t+1}(l+1)^t}{t!u!\, \beta^{\alpha+\alpha t}(q-\alpha-\alpha t)} \frac{(\alpha+\alpha t+1)_u(\alpha+\alpha t-q)_u}{(q-\alpha-\alpha t-q+1)_u} \times \int_0^{\infty} x^{p+q-\alpha-\alpha t-u}[\bar{F}(x)]^{\gamma_i-\gamma_j-1} f(x)\mathrm{d}x. \tag{28}$$

Again consider the integral of (28) as

$$I(x) = \int_0^{\infty} x^{p+q-\alpha-\alpha t-u}[\bar{F}(x)]^{\gamma_i-\gamma_j-1} f(x)\mathrm{d}x.$$

$$= \int_0^\infty x^{p+q-\alpha-\alpha t-u}[\bar{F}(x)]^{\gamma_i-\gamma_j-1}\left\{\frac{\alpha\beta\lambda(1+\beta x)^{-(\alpha+1)}}{1-e^{-\lambda}} - \alpha\beta\lambda(1+\beta x)^{-(\alpha+1)}\bar{F}(x)\right\}\mathrm{d}x.$$

$$= \frac{\alpha\beta\lambda}{(1-e^{-\lambda})^{\gamma_i-\gamma_j}}\int_0^\infty x^{p+q-\alpha-\alpha t-u}(1+\beta x)^{-(\alpha+1)}e^{-\lambda(1+\beta x)^{-\alpha}}(1-e^{-\lambda(1+\beta x)^{-\alpha}})^{\gamma_i-\gamma_j-1}\mathrm{d}x.$$

$$= \frac{\alpha\beta\lambda}{(1-e^{-\lambda})^{\gamma_i-\gamma_j}}\sum_{v=0}^{\gamma_i-\gamma_j-1}\binom{\gamma_i-\gamma_j-1}{v}\sum_{w=0}^{\infty}\frac{(-1)^{v+w}\lambda^w(v+1)^w}{w!}$$
$$\times\int_0^\infty x^{p+q-\alpha-\alpha t-u}(1+\beta x)^{-(\alpha+\alpha t+1)}\mathrm{d}x.$$

$$= \frac{\alpha\beta\lambda}{(1-e^{-\lambda})^{\gamma_i-\gamma_j}}\sum_{v=0}^{\gamma_i-\gamma_j-1}\binom{\gamma_i-\gamma_j-1}{v}\sum_{w=0}^{\infty}\frac{(-1)^{v+w}\lambda^w(v+1)^w}{w!\;\beta^{p+q-u-\alpha-\alpha t+1}}$$
$$\times\,\mathbf{B}(p+q-u-\alpha-\alpha t+1,\;\; \alpha(t+w+2)-p-q+u). \quad (29)$$

Now using (29) in (28), we get the required result.

Corollary3
When $m_i = m, i = 1, 2, \ldots, n-1$, relation (23) reduces to the product moment of gos for Case-II.

$$\mu^{p,q}_{r,s,n,m,k} = \alpha^2\, C_{s-1}\sum_{i=1}^{r}\sum_{j=r+1}^{s}\sum_{l=0}^{\gamma_j-1}\sum_{u=0}^{\infty}\sum_{v=0}^{\gamma_i-\gamma_j-1}\sum_{w=0}^{\infty}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times\binom{\gamma_j-1}{l}\binom{\gamma_i-\gamma_j-1}{v}\frac{(\alpha+\alpha t+1)_u\;(\alpha+\alpha t-h)_w}{u!\;(\alpha+\alpha t-q+1)_u}$$
$$\times\frac{(-1)^{r+s-i-jl+t+v+w+u}\;\lambda^{t+w+2}(l+1)^t(v+1)^w}{(m+1)^{s-2}\;(r-1)!^2t!\;(1-e^{-\lambda})^{\gamma_i}\beta^{p+q-1}(\alpha+\alpha t-q)}$$

Theorem4
Under the condition as stated in Theorem 3 The recurrence relation for product moments is given as

$$E[X^p(r,n,\tilde{m},k)X^q(s,n,\tilde{m},k)] - E[X^p(r,n,\tilde{m},k)X^q(s-1,n,\tilde{m},k)]$$
$$= q\;\alpha\;C_{s-1}\sum_{i=1}^{r}a_i(r)\sum_{j=r+1}^{s}a_j^{(r)}(s)\sum_{l=0}^{\gamma_j}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{t=0}^{\gamma_i-\gamma_j-1}\sum_{u=0}^{\infty}$$
$$\times\frac{\binom{\gamma_j}{l}\binom{\gamma_i-\gamma_j-1}{t}}{(1-e^{-\lambda})^{\gamma_i}}\frac{(\alpha c)_d(\alpha c-q)_d}{(\alpha c-q+1)_d}\times\frac{\lambda^{c+u+1}l^c(t+1)^u(-1)^{l+c+d+t+q}}{u!\;d!\;\beta^{p+q}(\alpha c-q)}$$
$$\times\,\mathbf{B}(p+\alpha c-d+1;\alpha-\alpha u-p-q+\alpha c+d). \quad (30)$$

Also,

$$E[X^p(r,n,\tilde{m},k)X^q(s,n,\tilde{m},k)] - E[X^p(r,n,\tilde{m},k)X^q(s-1,n,\tilde{m},k)]$$
$$= \frac{q}{\gamma_s\;\alpha\beta\lambda}\sum_{a=0}^{\infty}\sum_{b=1}^{[\alpha(1-a)+1]}\binom{[\alpha(1-a)+1]}{b}\frac{\lambda^a\beta^b}{a!}E[X^p(r,n,\tilde{m},k)X^{q+b-1}(s,n,\tilde{m},k)]. \quad (31)$$

Proof
We have by [7],

$$E[\xi\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}] - E[\xi\{X(r,n,\tilde{m},k),X(s-1,n,\tilde{m},k)\}]$$
$$= q\,C_{s-1}\int_{-\infty}^{\infty}\int_{x}^{\infty}\frac{\partial}{\partial y}\xi(x,y)\sum_{j=r+1}^{s}a_j^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_j}\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i}\frac{f(x)}{\bar{F}(x)}dy\,dx.$$

Let $\xi(x,y)=\xi_1(x)\xi_2(y)=x^py^q$, we get

$$E[X^p(r,n,\tilde{m},k)X^q(s,n,\tilde{m},k)] - E[X^p(r,n,\tilde{m},k)X^q(s-1,n,\tilde{m},k)]$$

$$= q\,C_{s-1}\sum_{i=1}^{r}a_i(r)\sum_{j=r+1}^{s}a_j^{(r)}(s)\int_0^{\infty}x^p[\bar{F}(x)]^{\gamma_i-\gamma_j-1}f(x)\mathrm{d}x\int_x^{\infty}y^{q-1}[\bar{F}(y)]^{\gamma_j-1}\,f(y)\mathrm{d}y\,.$$

Proceeding on the lines of Theorem 3.1, the relation (30) yields on, using(26) and (12) and simplifications.

The relation (31) can be proved in view of [7] and using (4).

Corollary 4
When $m_i=m, i=1,2,\ldots,n-1$, relation (30) and (31) reduces to the product moment of gos for Case-II.

$$E[X^p(r,n,m,k)X^q(s,n,m,k)] - E[X^p(r,n,m,k)X^q(s-1,n,m,k)]$$
$$= q\,\alpha\,C_{s-1}\sum_{i=1}^{r}\sum_{j=r+1}^{s}\sum_{l=0}^{\gamma_j}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{t=0}^{\gamma_i-\gamma_j-1}\sum_{u=0}^{\infty}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times\frac{\binom{\gamma_j}{l}\binom{\gamma_i-\gamma_j-1}{t}}{(m+1)^{s-2}(r-1)!^2}\frac{(\alpha c)_d(\alpha c-q)_d}{(\alpha c-q+1)_d}\times\frac{\lambda^{c+u+1}l^c(t+1)^u(-1)^{l+c+d+t+q+r+s-i-j}}{u!\,d!\,\beta^{p+q}(1-e^{-\lambda})^{\gamma_i}(\alpha c-q)}$$
$$\times\,\mathbf{B}(p+\alpha c-d+1;\alpha-\alpha u-p-q+\alpha c+d).$$

Also,

$$E[X^p(r,n,m,k)X^q(s,n,m,k)] - E[X^p(r,n,m,k)X^q(s-1,n,m,k)]$$
$$=\frac{q}{\gamma_s\alpha\beta\lambda}\sum_{a=0}^{\infty}\sum_{b=1}^{[\alpha(1-a)+1]}\binom{[\alpha(1-a)+1]}{b}\frac{\lambda^a\beta^b}{a!}E[X^p(r,n,m,k)X^{q+b-1}(s,n,m,k)].$$

Remark 3
Let $m_i=0, i=1,2,\ldots,n-1$ and $k=1$, then the recurrence relation for product moments of order statistics is

$$E[X^p_{r:n}X^q_{s:n}] - E[X^p_{r:n}X^q_{s-1:n}] = \frac{q}{(n-s-1)\alpha\beta\lambda}\sum_{a=0}^{\infty}\sum_{b=1}^{[\alpha(1-a)+1]}\binom{[\alpha(1-a)+1]}{b}\frac{\lambda^a\beta^b}{a!}E[X^p_{r:n}X^{q+b-1}_{s:n}].$$

Remark 4 When $m_i\to-1, i=1,2,\ldots,n-1$, the recurrence relation for product moments of k^{th} upper record values will be

$$E[(X^{(k)}_{U(r)})^p(X^{(k)}_{U(s)})^q] - E[(X^{(k)}_{U(r)})^p(X^{(k)}_{U(s-1)})^q]$$
$$=\frac{q}{k\,\alpha\beta\lambda}\sum_{a=0}^{\infty}\sum_{b=1}^{[\alpha(1-a)+1]}\binom{[\alpha(1-a)+1]}{b}\frac{\lambda^a\beta^b}{a!}E[(X^{(k)}_{U(r)})^p(X^{(k)}_{U(s)})^{q+b-1}].$$

4. Characterizations

This section contains characterization results for the given distribution through recurrence relations for single and product moments of *gos* as well as through conditional moments.

Theorem 5
Fix a positive integer k and let p be a non-negative integer. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) is that

$$E[X^p(r,n,m,k)] - E[X^p(r-1,n,m,k)] = \frac{p}{\gamma_r \alpha\beta\lambda} \sum_{a=0}^{\infty} \sum_{b=1}^{[\alpha(1-a)+1]} \binom{[\alpha(1-a)+1]}{b} \frac{\lambda^a \beta^b}{a!} E[X^{p+b-1}(r,n,m,k)]. \quad (32)$$

Proof
The necessary part follows from (22). On the other hand, if the relation in (32) is satisfied, then on using [7], for $\xi(x) = x^p$, we have

$$\frac{p}{\gamma_r} \frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^{p-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x)) dx$$
$$= \frac{p}{\gamma_r \alpha\beta\lambda} \frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^{p-1} [\bar{F}(x)]^{\gamma_{r-1}} g_m^{r-1}(F(x)) \times \left\{ \sum_{a=0}^{\infty} \sum_{b=1}^{[\alpha(1-a)+1]} \binom{[\alpha(1-a)+1]}{b} \frac{\lambda^a (\beta x)^b}{a!} f(x) \right\} \mathrm{d}x$$

or

$$\frac{p}{\gamma_r \alpha\beta\lambda} \frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^{p-1} [\bar{F}(x)]^{\gamma_{r-1}} g_m^{r-1}(F(x)) \times \left\{ \alpha\beta\lambda \bar{F}(x) - \sum_{a=0}^{\infty} \sum_{b=1}^{[\alpha(1-a)+1]} \frac{\lambda^a (\beta x)^b}{a!} \binom{[\alpha(1-a)+1]}{b} f(x) \right\} = 0 \quad (33)$$

Applying the extension of M*ü*ntz-Sz*á*sz theorem (see, for example, [16]) to (33), we get

$$\bar{F}(x) = \frac{1}{\alpha\beta\lambda} \sum_{a=0}^{\infty} \sum_{b=0}^{[\alpha(1-a)+1)]} \frac{\lambda^a}{a!} \binom{[\alpha(1-a)+1)]}{b} \beta^b x^b f(x).$$

which proves the theorem.

Theorem 6
Fix a positive integer k and let p and q be non-negative integers. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) is

$$E[X^p(r,n,m,k)X^q(s,n,m,k)] - E[X^p(r,n,m,k)X^q(s-1,n,m,k)] = \frac{q}{\gamma_s \alpha\beta\lambda} \sum_{a=0}^{\infty} \sum_{b=1}^{[\alpha(1-a)+1]} \binom{[\alpha(1-a)+1]}{b} \frac{\lambda^a \beta^b}{a!} E[X^p(r,n,m,k)X^{q+b-1}(s,n,m,k)]. \quad (34)$$

Proof
The necessary part follows from (31). Now, suppose that the relation in (34) is satisfied. Then, using [7], for $\xi(x,y) = x^p y^q$, we have

$$\frac{q}{\gamma_s}\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q-1}[\bar{F}(x)]^m f(x) g_m^{r-1}(x)$$
$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s} dydx$$

$$= \frac{q}{\gamma_s\alpha\beta\lambda}\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q-1}[\bar{F}(x)]^m f(x) g_m^{r-1}(x)$$
$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}$$
$$\times\left\{\sum_{a=0}^{\infty}\sum_{b=1}^{[\alpha(1-a)+1]}\frac{\lambda^a(\beta y)^b}{a!}\binom{[\alpha(1-a)+1]}{b}f(y)\right\}dydx$$

which implies,

$$\frac{q}{\gamma_s\alpha\beta\lambda}\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^{q-1}[\bar{F}(x)]^m f(x) g_m^{r-1}(x)$$
$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}$$
$$\times\left\{\alpha\beta\lambda\bar{F}(y) - \sum_{a=0}^{\infty}\sum_{b=1}^{[\alpha(1-a)+1]}\frac{\lambda^a(\beta y)^b}{a!}\binom{[\alpha(1-a)+1]}{b}f(y)\right\}dydx = 0. \tag{35}$$

Now applying the extension of M*ü*ntz-Sz*á*sz theorem (see, for example, [16]) to (35), we get

$$\bar{F}(y) = \sum_{a=0}^{\infty}\sum_{b=1}^{[\alpha(1-a)+1]}\frac{\lambda^a(\beta y)^b}{a!}\binom{[\alpha(1-a)+1]}{b}f(y),$$

which proves the theorem.

Theorem 7
Let $X(r,n,m,k),\ r = 1,2,\ldots,n$ be the the r^{th} gos based on continuous df $F()$ and $E(X)$ exists. Then for two consecutive values r and $r+1$, such that $1 \le r < r+1 \le n$,

$$E\left[e^{-\lambda\left(1+\beta X(r+1,n,m,k)\right)^{-\alpha}}\middle|X(r,n,m,k) = x\right] = \frac{\gamma_{r+1}e^{-\lambda\left(1+\beta x\right)^{-\alpha}}+1}{\gamma_{r+1}+1} \tag{36}$$

if and only if

$$\bar{F}(x) = \frac{1-e^{-\lambda(1+\beta x)^{-\alpha}}}{1-e^{-\lambda}}, \quad x > 0, \alpha > 0, \beta > 0, \lambda > 0. \tag{37}$$

Proof
[21] have shown that

$$E\left[h\big(X(s,n,m,k)\big)|X(r,n,m,k) = x\right] = a^*h(x) + b^* \tag{38}$$

if and only if

$$\bar{F}(x) = \left[ah(x) + b\right]^c \tag{39}$$

with $a^* = \prod_{j=r+1}^{s}\left(\frac{c\gamma_j}{1+c\gamma_j}\right)$ and $b^* = -\frac{b}{a}(1-a^*)$.

Comparing (37) with (39), we get

$$a = -\frac{1}{1-e^{-\lambda}},\, h(x) = e^{-\lambda\left(1+\beta x\right)^{-\alpha}},\, b = \frac{1}{1-e^{-\lambda}},\, c = 1.$$

Thus, the theorem can be proved in view of (38).

Corollary 5

For the r^{th} order statistics $X_{r:n},\ r = 1, 2, ...n$ and under the condition as stated under Theorem 4.3

$$E\big[e^{-\lambda\left(1+\beta X_{r+1:n}\right)^{-\alpha}}\big|X_{r:n} = x\big] = \frac{(n-r)e^{-\lambda\left(1+\beta x\right)^{-\alpha}}+1}{(n-r+1)} \tag{40}$$

and consequently

$$E\big[e^{-\lambda\left(1+\beta X_{n:n}\right)^{-\alpha}}\big|X_{n-1:n} = x\big] = E\big[e^{-\lambda\left(1+\beta X\right)^{-\alpha}}|X \geq x\big]$$

$$= \frac{e^{-\lambda\left(1+\beta x\right)^{-\alpha}}+1}{2} \tag{41}$$

if and only if

$$\bar{F}(x) = \frac{1-e^{-\lambda(1+\beta x)^{-\alpha}}}{1-e^{-\lambda}},\quad x > 0; \alpha, \beta, \lambda > 0. \tag{42}$$

It may be noted that similar characterization result can also be seen for adjacent records as

$$E\big[e^{-\lambda\left(1+\beta X_{U(n)}\right)^{-\alpha}}\big|X_{U(n-1)} = x\big] = E\big[e^{-\lambda\left(1+\beta X\right)^{-\alpha}}|X \geq x\big]$$

$$= \frac{e^{-\lambda\left(1+\beta x\right)^{-\alpha}}+1}{2}. \tag{43}$$

5. Conclusion

The moments of ordered random variables and recurrence relations between them have received great attention in the past few years in statistical literature. We have obtained exact expressions and recurrence relations for single and product moments of generalized order statistics based on Poisson Lomax distribution. Since generalized order statistics is unified approach for several ordered random variables, thus results obtained can be easily deduced for order statistics, record values, sequential order statistics etc. Characterization theorems that use the properties of sample moments, order statistics, record statistics, and reliability properties can be applied to uniquely determine the associated stochastic model.

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