



# Interpolation Problem for Periodically Correlated Stochastic Sequences with Missing Observations

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**Abstract** The problem of mean square optimal estimation of linear functionals which depend on the unobserved values of a periodically correlated stochastic sequence is considered. The estimates are based on observations of the sequence with a noise. Formulas for calculation the mean square errors and the spectral characteristics of the optimal estimates of functionals are derived in the case of spectral certainty, where the spectral densities of the sequences are exactly known. Formulas that determine the least favorable spectral densities and the minimax spectral characteristics are proposed in the case of spectral uncertainty, where the spectral densities of the sequences are not exactly known while some classes of admissible spectral densities are specified.

**Keywords** Periodically correlated sequence, optimal linear estimate, mean square error, least favourable spectral density matrix, minimax spectral characteristic

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## 1. Introduction

W.R. Bennett [5] in 1958 introduced cyclostationarity as a phenomenon describing signals in channels of communication. Studying the statistical characteristics of information transmission, he calls the group of telegraph signals a cyclostationary process, that is the process whose group of statistics changes periodically with time. W.A. Gardner and L. E. Franks [17] highlighted the similarity of cyclostationary processes, which form a subclass of nonstationary processes, with stationary processes. W.A. Gardner [18], W. A. Gardner, A. Napolitano and L. Paura [19] presented bibliography of works in which properties and applications of cyclostationary processes were studied. Recent developments and applications of cyclostationary signal analysis are reviewed in the papers by A. Napolitano [77], [78]. Note that in different sources cyclostationary processes are called periodically stationary, periodically nonstationary, periodically correlated. We will use the term periodically correlated processes.

E.G. Gladyshev [20] was the first who analysed the spectral properties and representations of periodically correlated sequences based on its connection with the vector valued stationary sequences. He formulated the necessary and sufficient conditions for determining the periodically correlated sequence in terms of the correlation function. A. Makagon [50], [51] presented a detailed spectral analysis of periodically correlated sequences. The main ideas of the research of periodically correlated sequences are outlined in the book by H. L. Hurd and A. Miamee [24].

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The problem of estimation of unobserved values of random processes is one of the very important and topical subsections of the theory of stochastic processes. Processes that are observed can be completely defined by its characteristics (correlation function, spectral density, canonical decomposition) or their characteristics can be defined only by the set of admissible values of characteristics. The linear extrapolation and interpolation problems for stationary stochastic sequences under the condition that the spectral densities are exactly known were first investigated by A. N. Kolmogorov [29]. Methods of solutions of the extrapolation and filtering problems for stationary processes and sequences with rational spectral densities were developed by N. Wiener [89] and A. M. Yaglom [91, 92]. Estimation problems for vector-valued stationary processes were investigated by Yu. A. Rozanov [86] and E. J. Hannan [23].

The basic techniques of statistics of stochastic processes are summarized in the books by D. Z. Arov and H. Dym [3], L. Aggoun and R. J. Elliott [2], I. V. Basawa and B. L. S. Prakasa Rao [4], S. Cohen and R. J. Elliott [10], M. S. Grewal and A. P. Andrews [22], G. Kallianpur [26], Yu. A. Kutoyants [34], [35], R. S. Liptser and A. N. Shiryaev [37, 38], B. L. S. Prakasa Rao [80, 81], B. L. Rozovsky and S. V. Lototsky [87], M. B. Rajarshi [84], W. A. Woodward, H. L. Gray and A. C. Elliott [90]. The estimation problems occur in different studies. We refer to D. V. Koroliouk [30], D. V. Koroliouk et al. [31], D. V. Koroliouk and V. S. Koroliuk [32], where there is investigated the difference stochastic equation  $\Delta\alpha_{t+1} = -V_0\alpha_t + \sigma_0\Delta W_{t+1}$ ,  $t \geq 0$ , which determines a sequence  $\alpha_t$ ,  $t \geq 0$ , for the stochastic component  $\Delta W_{t+1}$ ,  $t \geq 0$ , and studied the problem of filtration of stationary Gaussian statistical experiments considered for the solution  $\alpha_t$ ,  $t \geq 0$ , of the indicated equation.

Since processes often accompanied by undesirable noise it is naturally to assume that the exact value of spectral density is unknown and the model of process is given by a set of restrictions on spectral density. K. S. Vastola and H. V. Poor [88] showed for certain classes of spectral densities that the Wiener filter is very sensitive to minor changes of spectral model unlike the robust Wiener filter. That is the filter is the least sensitive to the worst case of uncertainty. Thus, it is reasonable to use the minimax (robust) estimation method, which allows to define the optimal estimate for all densities from a certain given class of the admissible spectral densities simultaneously. Ulf Grenander [21] was the first who proposed the minimax approach to the extrapolation problem for stationary processes. A survey of results in minimax-robust methods of data processing can be found in the paper by S. A. Kassam and H. V. Poor [28]. Formulation and investigation of the problems of extrapolation, interpolation and filtering of linear functionals which depend on the unknown values of stationary sequences and processes from observations with and without noise are presented by M. P. Moklyachuk in the papers [61]–[64]. Similar problems of the optimal estimation for the vector-valued stationary sequences and processes were examined by M. P. Moklyachuk [58]–[60] and by M. P. Moklyachuk and O. Yu. Masyutka [66]–[69]. In their papers M. M. Luz and M. P. Moklyachuk [39]–[49] investigated the minimax estimation problems for linear functionals which depends on unobserved values of stochastic sequences with stationary increments. P. S. Kozak and M. P. Moklyachuk [33] study estimates of functionals constructed from random sequences with periodically stationary increments. In their papers I. I. Golichenko (Dubovets'ka) and M. P. Moklyachuk [12]–[16], [65] presented results of investigation of the interpolation, extrapolation and filtering problems for linear functionals from periodically correlated stochastic sequences and processes.

The prediction problem for stationary sequences with missing observations is investigated in the papers by P. Bondon [6, 7], R. Cheng, A. G. Miamee and M. Pourahmadi [8], R. Cheng and M. Pourahmadi [9], Y. Kasahara, M. Pourahmadi and A. Inoue [27, 82]. The detailed analysis of the estimation problems with missing observations are presented in the paper by B. Abraham [1], books by M. J. Daniels and J. W. Hogan [11], R. J. A. Little and D. B. Rubin [36], P. E. McKnight et al [57], M. M. Pelagatti [79].

In the papers by M. P. Moklyachuk and M. I. Sidei [71]–[75] results of investigations of the interpolation, extrapolation and filtering problems for stationary stochastic sequences and processes with missing observations are proposed. The results of the study of the extrapolation, interpolation and filtering problems for linear functionals constructed from unobserved values of multidimensional stochastic sequences and processes are presented in the papers by O. Yu. Masyutka, M. P. Moklyachuk and M. I. Sidei [52]–[56], [76]. We also refer to the book by M. P. Moklyachuk, O. Yu. Masyutka and I. I. Golichenko [70] where results of the investigation of the problem of mean square optimal estimation (forecasting, interpolation, and filtering) of linear functionals constructed from unobserved values of periodically correlated isotropic random fields are described.

In this paper we deal with the problem of optimal linear estimation of the functional  $A_s\zeta$  which depends on the unobserved values of a periodically correlated stochastic sequence  $\zeta(j)$ . Estimates are based on observations of the sequence  $\zeta(j) + \theta(j)$  at points  $j \in \mathbb{Z} \setminus S$ , where  $S = \bigcup_{l=0}^{s-1} \{M_l + 1, \dots, M_l + N_{l+1}\}$ .  $\theta(j)$  is uncorrelated with  $\zeta(j)$  periodically correlated stochastic sequence. Formulas for calculation the mean square errors and the spectral characteristics of the optimal estimates of the functional  $A_s\zeta$  are proposed in the case of spectral certainty where the spectral densities are exactly known. Formulas that determine the least favorable spectral densities and minimax spectral characteristics are proposed in the case of spectral uncertainty where the spectral densities are not exactly known while some classes of admissible spectral densities are given.

The paper is organized as follows. The spectral properties of periodically correlated stochastic sequences and their correlation functions are described in Section 2. Relations of periodically correlated stochastic sequences with multidimensional stationary sequences are discussed in this section.

In section 3 we consider the problem of mean square optimal linear estimation of the the functional

$$A_s\vec{\xi} = \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_{l+1}} \vec{a}^T(j)\vec{\xi}(j), \quad M_l = \sum_{k=0}^l (N_k + K_k), \quad N_0 = K_0 = 0,$$

which depends on the unknown values of a  $T$ -dimensional stationary stochastic sequence  $\vec{\xi}(j)$ , based on observations of the sequence  $\vec{\xi}(j) + \vec{\eta}(j)$  at points  $j \in \mathbb{Z} \setminus S$ , where  $S = \bigcup_{l=0}^{s-1} \{M_l + 1, \dots, M_l + N_{l+1}\}$ . Formulas for calculation the mean square error and the spectral characteristic of the optimal estimate of the functional  $A_s\vec{\xi}$  are proposed in the case where spectral density matrices of the sequences  $\vec{\xi}(j)$  and  $\vec{\eta}(j)$  are exactly known.

In section 4 we consider the problem of mean square optimal linear estimation of the the functional

$$A_s\zeta = \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_{l+1}} a(j)\zeta(j), \quad M_l = \sum_{k=0}^l (N_k + K_k), \quad N_0 = K_0 = 0,$$

which depends on the unknown values of T-PC stochastic sequence  $\zeta(j)$ , based on observations of the sequence  $\zeta(j) + \theta(j)$  at points  $j \in \mathbb{Z} \setminus S$ , where  $S = \bigcup_{l=0}^{s-1} \{M_l + 1, \dots, M_l + N_{l+1}\}$ .

In section 5 we consider the problem of optimal estimation for the linear functional

$$A_s\zeta = \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_{l+1}} a(j)\zeta(j), \quad M_l = \sum_{k=0}^l (N_k + K_k), \quad N_0 = K_0 = 0,$$

which depends on the unknown values of  $T$ -PC sequence  $\zeta(j)$  from observations of the sequence  $\zeta(j) + \theta(j)$  at points  $j \in \mathbb{Z} \setminus S$ , where the number of missed observations at each of the intervals is a multiple of the period  $T$ . In sections 4 and 5 the estimation problem is investigated in the case of spectral certainty, where the spectral densities of observed sequences are exactly known.

In section 6 we describe the minimax approach to the problem of estimation of the linear functionals. In this case we find the estimate which minimizes the mean square error for all spectral densities from the given set of admissible densities simultaneously.

In section 7 the least favorable spectral densities and the minimax (robust) spectral characteristics of the optimal estimate of  $A_s\vec{\zeta}$  are found for the class  $D_0^-$  of admissible spectral densities.

In section 8 the least favorable spectral densities and the minimax (robust) spectral characteristics of the optimal estimate of  $A_s\vec{\zeta}$  are found for the class  $D_G^-$  of admissible spectral densities.

## 2. Periodically correlated and multidimensional stationary sequences

The term *periodically correlated* process was introduced by E. G. Gladyshev [20] while W. R. Bennett [5] called random and periodic processes *cyclostationary* process.

Periodically correlated sequences are stochastic sequences that have periodic structure (see the book by H. L. Hurd and A. Miamer [24]).

*Definition 1*

A complex valued stochastic sequence  $\zeta(n), n \in \mathbb{Z}$  with zero mean,  $E\zeta(n) = 0$ , and finite variance,  $E|\zeta(n)|^2 < +\infty$ , is called cyclostationary or periodically correlated (PC) with period  $T$  ( $T$ -PC) if for every  $n, m \in \mathbb{Z}$

$$E\zeta(n+T)\overline{\zeta(m+T)} = R(n+T, m+T) = R(n, m) \quad (1)$$

and there are no smaller values of  $T > 0$  for which (1) holds true.

*Definition 2*

A complex valued  $T$ -variate stochastic sequence  $\vec{\xi}(n) = \{\xi_\nu(n)\}_{\nu=1}^T, n \in \mathbb{Z}$  with zero mean,  $E\xi_\nu(n) = 0, \nu = 1, \dots, T$ , and  $E\|\vec{\xi}(n)\|^2 < \infty$  is called stationary if for all  $n, m \in \mathbb{Z}$  and  $\nu, \mu \in \{1, \dots, T\}$

$$E\xi_\nu(n)\overline{\xi_\mu(m)} = R_{\nu\mu}(n, m) = R_{\nu\mu}(n - m).$$

If this is the case, we denote  $R(n) = \{R_{\nu\mu}(n)\}_{\nu, \mu=1}^T$  and call it the *covariance matrix* of  $T$ -variate stochastic sequence  $\vec{\xi}(n)$ .

**Proposition 2.1.** (E. G. Gladyshev [20]). *A stochastic sequence  $\zeta(n)$  is PC with period  $T$  if and only if there exists a  $T$ -variate stationary sequence  $\vec{\xi}(n) = \{\xi_\nu(n)\}_{\nu=1}^T$  such that  $\zeta(n)$  has the representation*

$$\zeta(n) = \sum_{\nu=1}^T e^{2\pi i n \nu / T} \xi_\nu(n), \quad n \in \mathbb{Z}. \quad (2)$$

The sequence  $\vec{\xi}(n)$  is called *generating sequence* of the sequence  $\zeta(n)$ .

**Proposition 2.2.** (E. G. Gladyshev [20]). *A complex valued stochastic sequence  $\zeta(n), n \in \mathbb{Z}$  with zero mean and finite variance is PC with period  $T$  if and only if the  $T$ -variate blocked sequence  $\vec{\zeta}(n)$  of the form*

$$[\vec{\zeta}(n)]_p = \zeta(nT + p), \quad n \in \mathbb{Z}, p = 1, \dots, T \quad (3)$$

is stationary.

We will denote by  $f^{\vec{\zeta}}(\lambda) = \{f_{\nu\mu}^{\vec{\zeta}}(\lambda)\}_{\nu, \mu=1}^T$  the matrix valued spectral density function of the  $T$ -variate stationary sequence  $\vec{\zeta}(n) = (\zeta_1(n), \dots, \zeta_T(n))^T$  arising from the  $T$ -blocking (3) of a univariate  $T$ -PC sequence  $\zeta(n)$ .

### 3. Hilbert space projection method of linear interpolation

Let  $\vec{\xi}(j)$  and  $\vec{\eta}(j)$  be uncorrelated  $T$ -variate stationary stochastic sequences with the spectral density matrices  $f^{\vec{\xi}}(\lambda) = \{f_{\nu\mu}^{\vec{\xi}}(\lambda)\}_{\nu, \mu=1}^T$  and  $f^{\vec{\eta}}(\lambda) = \{f_{\nu\mu}^{\vec{\eta}}(\lambda)\}_{\nu, \mu=1}^T$ , respectively. Consider the problem of optimal linear estimation of the functional

$$A_s \vec{\xi} = \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_l+1} \vec{a}^T(j) \vec{\xi}(j), \quad M_l = \sum_{k=0}^l (N_k + K_k), \quad N_0 = K_0 = 0,$$

that depends on the unknown values of the sequence  $\vec{\xi}(j)$ , based on observations of the sequence  $\vec{\xi}(j) + \vec{\eta}(j)$  at points  $j \in \mathbb{Z} \setminus S$ , where  $S = \bigcup_{l=0}^{s-1} \{M_l + 1, \dots, M_l + N_l + 1\}$ .

Let the spectral densities  $f^{\vec{\xi}}(\lambda)$  and  $f^{\vec{\eta}}(\lambda)$  satisfy the minimality condition

$$\int_{-\pi}^{\pi} \text{Tr} \left[ (f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda))^{-1} \right] d\lambda < +\infty. \tag{4}$$

Condition (4) is necessary and sufficient in order that the error-free interpolation of the unknown values of the sequence  $\vec{\xi}(j) + \vec{\eta}(j)$  is impossible [86].

Denote by  $L_2(f)$  the Hilbert space of vector valued functions  $\vec{b}(\lambda) = \{b_\nu(\lambda)\}_{\nu=1}^T$  that are square integrable with respect to a measure with the density  $f(\lambda) = \{f_{\nu\mu}(\lambda)\}_{\nu,\mu=1}^T$ :

$$\int_{-\pi}^{\pi} \vec{b}^\top(\lambda) f(\lambda) \overline{\vec{b}(\lambda)} d\lambda = \int_{-\pi}^{\pi} \sum_{\nu,\mu=1}^T b_\nu(\lambda) f_{\nu\mu}(\lambda) \overline{b_\mu(\lambda)} d\lambda < +\infty.$$

Denote by  $L_2^{s-}(f)$  the subspace in  $L_2(f)$  generated by the functions  $e^{ij\lambda} \delta_\nu, \delta_\nu = \{\delta_{\nu\mu}\}_{\mu=1}^T, \nu = 1, \dots, T, j \in \mathbb{Z} \setminus S$ , where  $\delta_{\nu\nu} = 1, \delta_{\nu\mu} = 0$  for  $\nu \neq \mu$ .

Every linear estimate  $\widehat{A_s \vec{\xi}}$  of the functional  $A_s \vec{\xi}$  from observations of the sequence  $\vec{\xi}(j) + \vec{\eta}(j)$  at points  $j \in \mathbb{Z} \setminus S$  has the form

$$\widehat{A_s \vec{\xi}} = \int_{-\pi}^{\pi} \vec{h}^\top(e^{i\lambda}) (Z^{\vec{\xi}}(d\lambda) + Z^{\vec{\eta}}(d\lambda)) = \int_{-\pi}^{\pi} \sum_{\nu=1}^T h_\nu(e^{i\lambda}) (Z_\nu^{\vec{\xi}}(d\lambda) + Z_\nu^{\vec{\eta}}(d\lambda)), \tag{5}$$

where  $Z^{\vec{\xi}}(\Delta) = \{Z_\nu^{\vec{\xi}}(\Delta)\}_{\nu=1}^T$  and  $Z^{\vec{\eta}}(\Delta) = \{Z_\nu^{\vec{\eta}}(\Delta)\}_{\nu=1}^T$  are orthogonal random measures of the sequences  $\vec{\xi}(j)$  and  $\vec{\eta}(j)$ , and  $\vec{h}(e^{i\lambda}) = \{h_\nu(e^{i\lambda})\}_{\nu=1}^T$  is the spectral characteristic of the estimate  $\widehat{A_s \vec{\xi}}$ . The function  $\vec{h}(e^{i\lambda}) \in L_2^{s-}(f^{\vec{\xi}} + f^{\vec{\eta}})$ .

The mean square error  $\Delta(\vec{h}; f^{\vec{\xi}}, f^{\vec{\eta}})$  of the estimate  $\widehat{A_s \vec{\xi}}$  is calculated by the formula

$$\begin{aligned} \Delta(\vec{h}; f^{\vec{\xi}}, f^{\vec{\eta}}) &= E |A_s \vec{\xi} - \widehat{A_s \vec{\xi}}|^2 = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ A_s(e^{i\lambda}) - \vec{h}(e^{i\lambda}) \right]^\top f^{\vec{\xi}}(\lambda) \overline{\left[ A_s(e^{i\lambda}) - \vec{h}(e^{i\lambda}) \right]} d\lambda + \\ &+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \vec{h}^\top(e^{i\lambda}) f^{\vec{\eta}}(\lambda) \overline{\vec{h}(e^{i\lambda})} d\lambda, \\ A_s(e^{i\lambda}) &= \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_l+1} \vec{a}(j) e^{ij\lambda}. \end{aligned} \tag{6}$$

The spectral characteristic  $\vec{h}(f^{\vec{\xi}}, f^{\vec{\eta}})$  of the optimal linear estimate of  $A_s \vec{\xi}$  minimizes the mean square error

$$\Delta(f^{\vec{\xi}}, f^{\vec{\eta}}) = \Delta(\vec{h}(f^{\vec{\xi}}, f^{\vec{\eta}}); f^{\vec{\xi}}, f^{\vec{\eta}}) = \min_{\vec{h} \in L_2^{s-}(f^{\vec{\xi}} + f^{\vec{\eta}})} \Delta(\vec{h}; f^{\vec{\xi}}, f^{\vec{\eta}}) = \min_{\widehat{A_s \vec{\xi}}} E |A_s \vec{\xi} - \widehat{A_s \vec{\xi}}|^2. \tag{7}$$

With the help of the Hilbert space projection method proposed by A. N. Kolmogorov [29] we can find a solution of the optimization problem (7). The optimal linear estimate  $\widehat{A_s \vec{\xi}}$  is a projection of the functional  $A_s \vec{\xi}$  on the subspace  $H^-[\vec{\xi} + \vec{\eta}] = H^-[\xi_\nu(j) + \eta_\nu(j), j \in \mathbb{Z} \setminus S, \nu = 1, \dots, T]$  of the Hilbert space  $H = \{\xi : E\xi = 0, E|\xi|^2 < \infty\}$ , generated by values  $\xi_\nu(j) + \eta_\nu(j), j \in \mathbb{Z} \setminus S, \nu = 1, \dots, T$ . The projection is characterized by the following conditions

- 1)  $\widehat{A_s \vec{\xi}} \in H^-[\vec{\xi} + \vec{\eta}]$ ,  
 2)  $A_s \vec{\xi} - \widehat{A_s \vec{\xi}} \perp H^-[\vec{\xi} + \vec{\eta}]$ .

Condition 2) gives us the possibility to derive the formula for the spectral characteristic of the optimal estimate

$$\begin{aligned} \vec{h}^\top(f^{\vec{\xi}}, f^{\vec{\eta}}) &= \left( A_s^\top(e^{i\lambda})f^{\vec{\xi}}(\lambda) - C_s^\top(e^{i\lambda}) \right) \left[ f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda) \right]^{-1} = \\ &= A_s^\top(e^{i\lambda}) - \left( A_s^\top(e^{i\lambda})f^{\vec{\eta}}(\lambda) + C_s^\top(e^{i\lambda}) \right) \left[ f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda) \right]^{-1}, \end{aligned} \quad (8)$$

where

$$\begin{aligned} C_s(e^{i\lambda}) &= \sum_{l=0}^{s-1} \sum_{k_l=M_l+1}^{M_l+N_{l+1}} \vec{c}(k_l)e^{ik_l\lambda}, \\ \vec{c}(k_l) &= (c_1(k_l), \dots, c_T(k_l))^\top, \\ l &= 0, \dots, s-1, k_l = M_l + 1, \dots, M_l + N_{l+1}. \end{aligned}$$

Condition 1) is satisfied when the system of equalities

$$\int_{-\pi}^{\pi} \vec{h}^\top(f^{\vec{\xi}}, f^{\vec{\eta}}) e^{-ij\lambda} d\lambda = 0, \quad j \in S \quad (9)$$

holds true.

Denote by  $\mathbf{D}_s, \mathbf{B}_s$  operators that are determined by  $T\rho \times T\rho, \rho = N_1 + N_2 + \dots + N_s$ , matrices

$$\mathbf{D}_s = \begin{pmatrix} D_{00} & D_{01} & \dots & D_{0,s-1} \\ D_{10} & D_{11} & \dots & D_{1,s-1} \\ \dots & \dots & \dots & \dots \\ D_{s-1,0} & D_{s-1,1} & \dots & D_{s-1,s-1} \end{pmatrix}, \quad \mathbf{B}_s = \begin{pmatrix} B_{00} & B_{01} & \dots & B_{0,s-1} \\ B_{10} & B_{11} & \dots & B_{1,s-1} \\ \dots & \dots & \dots & \dots \\ B_{s-1,0} & B_{s-1,1} & \dots & B_{s-1,s-1} \end{pmatrix},$$

constructed from  $TN_{m+1} \times TN_{n+1}$  block-matrices

$$\begin{aligned} D_{mn} &= \{D_{mn}(k, j)\}_{k=M_m+1}^{M_m+N_{m+1}} \quad \{D_{mn}(k, j)\}_{j=M_n+1}^{M_n+N_{n+1}}, \\ B_{mn} &= \{B_{mn}(k, j)\}_{k=M_m+1}^{M_m+N_{m+1}} \quad \{B_{mn}(k, j)\}_{j=M_n+1}^{M_n+N_{n+1}}, \\ m, n &= 0, \dots, s-1, \end{aligned}$$

with elements which are the Fourier coefficients of the matrix functions  $\left[ f^{\vec{\xi}}(\lambda)(f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda))^{-1} \right]^\top$  and  $\left[ (f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda))^{-1} \right]^\top$ , correspondingly:

$$\begin{aligned} D_{mn}(k, j) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ f^{\vec{\xi}}(\lambda)(f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda))^{-1} \right]^\top e^{i(j-k)\lambda} d\lambda, \\ B_{mn}(k, j) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ (f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda))^{-1} \right]^\top e^{i(j-k)\lambda} d\lambda, \\ k &= M_m + 1, \dots, M_m + N_{m+1}, \\ j &= M_n + 1, \dots, M_n + N_{n+1}. \end{aligned}$$

Making use of the introduced operators, relation (9) can be written in the form of the equation

$$\mathbf{D}_s \vec{a}_s = \mathbf{B}_s \vec{c}_s,$$

where

$$\vec{a}_s = (\vec{a}^\top(1), \dots, \vec{a}^\top(N_1), \vec{a}^\top(M_1 + 1), \dots, \vec{a}^\top(M_1 + N_2), \dots, \vec{a}^\top(M_{s-1} + 1), \dots, \vec{a}^\top(M_{s-1} + N_s))^\top,$$

$$\vec{c}_s = (\vec{c}^\top(1), \dots, \vec{c}^\top(N_1), \vec{c}^\top(M_1 + 1), \dots, \vec{c}^\top(M_1 + N_2), \dots, \vec{c}^\top(M_{s-1} + 1), \dots, \vec{c}^\top(M_{s-1} + N_s))^\top$$

are column-vectors. The unknown coefficients  $\vec{c}(k_l)$ ,  $l = 0, \dots, s - 1$ ,  $k_l = M_l + 1, \dots, M_l + N_{l+1}$  are determined from the equation

$$\vec{c}_s = \mathbf{B}_s^{-1} \mathbf{D}_s \vec{a}_s, \tag{10}$$

where the  $k_l$ -th component of the vector  $\vec{c}_s$  is calculated by the formula

$$\vec{c}(k_l) = \sum_{m=0}^{s-1} \sum_{q=M_m+1}^{M_m+N_{m+1}} C_{lm}(k_l, q) \sum_{n=0}^{s-1} \sum_{j=M_n+1}^{M_n+N_{n+1}} D_{mn}(q, j) \vec{a}(j),$$

$$l = 0, \dots, s - 1, k_l = M_l + 1, \dots, M_l + N_{l+1}.$$

The operator  $\mathbf{B}_s^{-1}$  is determined by  $T\rho \times T\rho$  matrix

$$\mathbf{B}_s^{-1} = \begin{pmatrix} C_{00} & C_{01} & \dots & C_{0,s-1} \\ C_{10} & C_{11} & \dots & C_{1,s-1} \\ \dots & \dots & \dots & \dots \\ C_{s-1,0} & C_{s-1,1} & \dots & C_{s-1,s-1} \end{pmatrix}$$

that is an inverse matrix for the block-matrix  $\mathbf{B}_s$ . Elements of  $\mathbf{B}_s^{-1}$  are constructed by dividing  $\mathbf{B}_s^{-1}$  on  $TN_{m+1} \times TN_{n+1}$  block-matrices  $C_{mn}$  and dividing each  $C_{mn}$  on  $T \times T$  matrices  $C_{mn}(k, j)$ ,  $k = M_m + 1, \dots, M_m + N_{m+1}$ ,  $j = M_n + 1, \dots, M_n + N_{n+1}$ ,  $m, n = 0, \dots, s - 1$ , in the such way that

$$C_{mn} = \{C_{mn}(k, j)\}_{k=M_m+1}^{M_m+N_{m+1}} \}_{j=M_n+1}^{M_n+N_{n+1}}.$$

The mean-square error of the optimal estimate  $\widehat{A\xi}$  is calculated by the formula (6) and is of the form

$$\Delta(f^{\vec{\xi}}, f^{\vec{\eta}}) = \langle \vec{a}_s, \mathbf{R}_s \vec{a}_s \rangle + \langle \vec{c}_s, \mathbf{B}_s \vec{c}_s \rangle, \tag{11}$$

where  $\langle a, b \rangle$  denotes the scalar product,  $\mathbf{R}_s$  is the linear operator determined by  $T\rho \times T\rho$  matrix composed with  $TN_{m+1} \times TN_{n+1}$  block-matrices

$$R_{mn} = \{R_{mn}(k, j)\}_{k=M_m+1}^{M_m+N_{m+1}} \}_{j=M_n+1}^{M_n+N_{n+1}},$$

$$m, n = 0, \dots, s - 1,$$

with elements

$$R_{mn}(k, j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ f^{\vec{\xi}}(\lambda) (f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda))^{-1} f^{\vec{\eta}}(\lambda) \right]^\top e^{i(j-k)\lambda} d\lambda,$$

$$k = M_m + 1, \dots, M_m + N_{m+1},$$

$$j = M_n + 1, \dots, M_n + N_{n+1}.$$

See [66] for more details.

The following statement holds true.

*Theorem 1*

Let  $\vec{\xi}(j) = \{\xi_\nu(j)\}_{\nu=1}^T$  and  $\vec{\eta}(j) = \{\eta_\nu(j)\}_{\nu=1}^T$  be uncorrelated T-variate stationary stochastic sequences with the spectral density matrices  $f^{\vec{\xi}}(\lambda) = \left\{ f_{\nu\mu}^{\vec{\xi}}(\lambda) \right\}_{\nu,\mu=1}^T$  and  $f^{\vec{\eta}}(\lambda) = \left\{ f_{\nu\mu}^{\vec{\eta}}(\lambda) \right\}_{\nu,\mu=1}^T$ , respectively. Assume that the matrices  $f^{\vec{\xi}}(\lambda)$  and  $f^{\vec{\eta}}(\lambda)$  satisfy the minimality condition (4). The spectral characteristic  $\vec{h}(f^{\vec{\xi}}, f^{\vec{\eta}})$  and the mean square error  $\Delta(f^{\vec{\xi}}, f^{\vec{\eta}})$  of the optimal linear estimate of the functional  $A_s \vec{\xi}$  based on observations of the sequence  $\vec{\xi}(j) + \vec{\eta}(j)$  at points  $j \in \mathbb{Z} \setminus S$ , are calculated by formulas (8) and (11).

In the case of observations without noise we have the following corollary.

*Corollary 1*

Let  $\vec{\xi}(j) = \{\xi_\nu(j)\}_{\nu=1}^T$  be a T-variate stationary stochastic sequence with the spectral density matrix  $f^{\vec{\xi}}(\lambda) = \left\{ f_{\nu\mu}^{\vec{\xi}}(\lambda) \right\}_{\nu,\mu=1}^T$ , which satisfies the minimality condition

$$\int_{-\pi}^{\pi} Tr \left[ (f^{\vec{\xi}}(\lambda))^{-1} \right] d\lambda < +\infty. \tag{12}$$

The spectral characteristic  $\vec{h}(f^{\vec{\xi}})$  and the mean square error  $\Delta(f^{\vec{\xi}})$  of the optimal linear estimate of the functional  $A_s \vec{\xi}$  based on observations of the sequence  $\vec{\xi}(j)$  at points  $j \in \mathbb{Z} \setminus S$ , are calculated by formulas

$$\vec{h}^\top(f^{\vec{\xi}}) = A_s^\top (e^{i\lambda}) - C_s^\top (e^{i\lambda}) \left[ f^{\vec{\xi}}(\lambda) \right]^{-1}, \tag{13}$$

$$\Delta(f^{\vec{\xi}}) = \langle \vec{c}_s, \vec{a}_s \rangle, \tag{14}$$

where

$$\vec{a}_s = (\vec{a}^\top(1), \dots, \vec{a}^\top(N_1), \vec{a}^\top(M_1 + 1), \dots, \vec{a}^\top(M_1 + N_2), \dots, \vec{a}^\top(M_{s-1} + 1), \dots, \vec{a}^\top(M_{s-1} + N_s))^\top,$$

$$\vec{c}_s = (\vec{c}^\top(1), \dots, \vec{c}^\top(N_1), \vec{c}^\top(M_1 + 1), \dots, \vec{c}^\top(M_1 + N_2), \dots, \vec{c}^\top(M_{s-1} + 1), \dots, \vec{c}^\top(M_{s-1} + N_s))^\top$$

are column-vectors and  $\vec{c}_s = \mathbf{B}_s^{-1} \vec{a}_s$ .  $\mathbf{B}_s$  is a  $T\rho \times T\rho$  matrix composed with  $TN_m \times TN_n$  block-matrices  $B_{mn} = \{B_{mn}(k, j)\}_{k=M_{m-1}+1}^{M_{m-1}+N_m} \quad j=M_{n-1}+1}^{M_{n-1}+N_n}$ .

$$B_{mn}(k, j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ (f^{\vec{\xi}}(\lambda))^{-1} \right]^\top e^{i(j-k)\lambda} d\lambda,$$

$$m, n = 0, \dots, s - 1,$$

$$k = M_{m-1} + 1, \dots, M_{m-1} + N_m,$$

$$j = M_{n-1} + 1, \dots, M_{n-1} + N_n.$$

The  $k_l$ -th component of the vector  $\vec{c}_s$  is calculated by the formula

$$\vec{c}(k_l) = \sum_{m=0}^{s-1} \sum_{q=M_m+1}^{M_m+N_{m+1}} C_{lm}(k_l, q) \vec{a}(q),$$

$$l = 0, \dots, s - 1, k_l = M_l, \dots, M_l + N_{l+1}.$$

The operator  $\mathbf{B}_s^{-1}$  is determined by  $T\rho \times T\rho$  matrix that is the inverse matrix to the block-matrix  $\mathbf{B}_s$ . Elements of  $\mathbf{B}_s^{-1}$  are obtained by dividing  $\mathbf{B}_s^{-1}$  on  $TN_{m+1} \times TN_{n+1}$  block-matrices  $C_{mn}$  and dividing each of  $C_{mn}$  on  $T \times T$  matrices  $C_{mn}(k, j)$ ,  $k = M_m + 1, \dots, M_m + N_{m+1}$ ,  $j = M_n + 1, \dots, M_n + N_{n+1}$ ,  $m, n = 0, \dots, s - 1$ , in the such way that

$$C_{mn} = \{C_{mn}(k, j)\}_{k=M_m+1}^{M_m+N_{m+1}} \quad j=M_n+1}^{M_n+N_{n+1}}.$$



*Remark 1*

Let  $s = 1, N_1 = N$ . Then

$$A_s \vec{\xi} = A_N \vec{\xi} = \sum_{j=1}^N \vec{a}^\top(j) \vec{\xi}(j).$$

The spectral characteristic  $\vec{h}(f^{\vec{\xi}}, f^{\vec{\eta}})$  and the mean square error  $\Delta(f^{\vec{\xi}}, f^{\vec{\eta}})$  of the optimal linear estimate of the functional  $A_N \vec{\xi}$  based on observations of the sequence  $\vec{\xi}(j)$  at points  $j \in \mathbb{Z} \setminus \{1, \dots, N\}$  with the noise  $\vec{\eta}(j)$  are calculated by formulas

$$\begin{aligned} h^\top(f^{\vec{\xi}}, f^{\vec{\eta}}) &= \left( A_N^\top(e^{i\lambda}) f^{\vec{\xi}}(\lambda) - C_N^\top(e^{i\lambda}) \right) \left[ f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda) \right]^{-1} = \\ &= A_N^\top(e^{i\lambda}) - \left( A_N^\top(e^{i\lambda}) f^{\vec{\eta}}(\lambda) + C_N^\top(e^{i\lambda}) \right) \left[ f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda) \right]^{-1}, \\ \Delta(f^{\vec{\xi}}, f^{\vec{\eta}}) &= \langle \vec{a}_N, R_N \vec{a}_N \rangle + \langle \vec{c}_N, B_N \vec{c}_N \rangle. \end{aligned}$$

The spectral characteristic  $\vec{h}(f^{\vec{\xi}})$  and the mean square error  $\Delta(f^{\vec{\xi}})$  of the optimal linear estimate of the functional  $A_N \vec{\xi}$  based on observations of the sequence  $\vec{\xi}(j)$  at points  $j \in \mathbb{Z} \setminus \{1, \dots, N\}$  without noise  $\vec{\eta}(j)$  are calculated by formulas

$$\begin{aligned} h^\top(f^{\vec{\xi}}) &= A_N^\top(e^{i\lambda}) - C_N^\top(e^{i\lambda}) \left[ f^{\vec{\xi}}(\lambda) \right]^{-1}, \\ \Delta(f^{\vec{\xi}}) &= \langle \vec{c}_N, \vec{a}_N \rangle. \end{aligned}$$

For more details see [12], [65].

*Example 1*

Let  $\vec{\xi}(n) = \begin{pmatrix} \xi_1(n) \\ \xi_2(n) \end{pmatrix}$  be a 2-variate stationary stochastic sequence. Let  $\xi_1(n) = \theta(n)$  be a univariate stationary sequence with the spectral density function  $f(\lambda) = \frac{1}{|1 - ae^{-i\lambda}|^2}$ ,  $|a| < 1$ , and  $\xi_2(n) = \theta(n) + \gamma(n)$ , where  $\gamma(n)$  is an uncorrelated with  $\theta(n)$  univariate stationary sequence with the spectral density function  $g(\lambda) = \frac{1}{|1 - be^{-i\lambda}|^2}$ ,  $|b| < 1$ . Consider the problem of estimation of the functional

$$A_2 \vec{\xi} = \vec{\xi}(1) - \vec{\xi}(3) = (1, 1) \begin{pmatrix} \xi_1(1) \\ \xi_2(1) \end{pmatrix} + (-1, -1) \begin{pmatrix} \xi_1(3) \\ \xi_2(3) \end{pmatrix}$$

based on observations of  $\vec{\xi}(n), n \in \mathbb{Z} \setminus \{1, 3\}$ . Here  $\vec{a}(1) = (1, 1), \vec{a}(3) = (-1, -1)$ .

In this case the spectral density matrix of  $\vec{\xi}(n)$  is

$$f^{\vec{\xi}}(\lambda) = \begin{pmatrix} f(\lambda) & f(\lambda) \\ f(\lambda) & f(\lambda) + g(\lambda) \end{pmatrix}$$

and  $[f^{\vec{\xi}}(\lambda)]^{-1}$  satisfies the minimality condition (12). The matrix  $\mathbf{B}_2$  and its inverse  $\mathbf{B}_2^{-1}$ , the vector of unknown coefficients  $\vec{c}_2$  are of the form

$$\begin{aligned} \mathbf{B}_2 &= \begin{pmatrix} 2 + a^2 + b^2 & -1 - b^2 & 0 & 0 \\ -1 - b^2 & 1 + b^2 & 0 & 0 \\ 0 & 0 & 2 + a^2 + b^2 & -1 - b^2 \\ 0 & 0 & -1 - b^2 & 1 + b^2 \end{pmatrix}, \\ \mathbf{B}_2^{-1} &= \begin{pmatrix} \frac{1}{1+a^2} & \frac{1}{1+a^2} & 0 & 0 \\ \frac{1}{1+a^2} & \frac{2+a^2+b^2}{(1+b^2)(1+a^2)} & 0 & 0 \\ 0 & 0 & \frac{1}{1+a^2} & \frac{1}{1+a^2} \\ 0 & 0 & \frac{1}{1+a^2} & \frac{2+a^2+b^2}{(1+b^2)(1+a^2)} \end{pmatrix}, \end{aligned}$$

$$\vec{c}_2 = \left( \frac{2}{1+a^2}, \frac{3+a^2+2b^2}{(1+b^2)(1+a^2)}, -\frac{2}{1+a^2}, -\frac{3+a^2+2b^2}{(1+b^2)(1+a^2)} \right)^\top.$$

The spectral characteristic can be calculated by (13)

$$\vec{h}^\top(f^{\vec{\xi}}) = \left( \left( \frac{2a}{1+a^2} - \frac{b}{1+b^2} \right) + \left( \frac{b}{1+b^2} \right) e^{4i\lambda}, \right. \\ \left. \frac{b}{1+b^2} - \frac{b}{1+b^2} e^{4i\lambda} \right).$$

Then the optimal linear estimate of  $A_2\vec{\xi}$  determined by (5) is of the form

$$\widehat{A_2\vec{\xi}} = \left( \frac{2a}{1+a^2} - \frac{b}{1+b^2} \right) \xi_1(0) + \frac{b}{1+b^2} \xi_1(4) + \frac{b}{1+b^2} \xi_2(0) - \frac{b}{1+b^2} \xi_2(4).$$

The mean square error of  $\widehat{A_2\vec{\xi}}$  determined by (14) is

$$\Delta(f^{\vec{\xi}}) = \frac{8}{1+a^2} + \frac{2}{1+b^2}.$$

#### 4. Interpolation of T-PC stochastic sequences

Let  $\zeta(j)$  and  $\theta(j)$  be uncorrelated T-PC stochastic sequences. Consider the problem of optimal linear estimation of the functional

$$A_s\zeta = \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_{l+1}} a(j)\zeta(j),$$

$$M_l = \sum_{k=0}^l (N_k + K_k), \quad N_0 = K_0 = 0,$$

that depends on the unobserved values of T-PC stochastic sequence  $\zeta(j)$ , based on observations of the sequence  $\zeta(j) + \theta(j)$  at points  $j \in \mathbb{Z} \setminus S$ , where  $S = \bigcup_{l=0}^{s-1} \{M_l + 1, \dots, M_l + N_{l+1}\}$ .

Using the Gladyshev relation (2) of PC and multivariate stationary sequences the problem of estimation of the functional  $A_s\zeta$  may be reduced to the problem of estimation of the functional  $A_s\vec{\xi}$  since

$$A_s\zeta = \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_{l+1}} a(j)\zeta(j) = \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_{l+1}} a(j) \sum_{\nu=1}^T e^{2\pi i j \nu / T} \xi_\nu(j) = \\ = \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_{l+1}} \sum_{\nu=1}^T a(j) e^{2\pi i j \nu / T} \xi_\nu(j) = \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_{l+1}} \vec{a}^\top(j) \vec{\xi}(j) = A_s\vec{\xi},$$

where

$$\vec{a}^\top(j) = (a_1(j), \dots, a_T(j)), \quad a_\nu(j) = a(j) e^{2\pi i j \nu / T}, \quad \nu = 1, \dots, T,$$

$\vec{\xi}(j) = \{\xi_\nu(j)\}_{\nu=1}^T$  is a T-variate stationary stochastic sequence that generates the PC sequence  $\zeta(j)$ .

For the interpolation problem for PC sequences the following results hold true.

##### Theorem 2

Let  $\zeta(j)$  and  $\theta(j)$  be uncorrelated T-PC stochastic sequences. Then the optimal linear estimate of the functional  $A_s\zeta$  based on observations of the sequence  $\zeta(j) + \theta(j)$  at points  $j \in \mathbb{Z} \setminus S$ , is given by the formula

$$\widehat{A_s \zeta} = \int_{-\pi}^{\pi} \vec{h}^T(f^{\vec{\xi}}, f^{\vec{\eta}})(Z^{\xi}(d\lambda) + Z^{\eta}(d\lambda)) = \int_{-\pi}^{\pi} \sum_{\nu=1}^T h_{\nu}(f^{\vec{\xi}}, f^{\vec{\eta}})(Z_{\nu}^{\xi}(d\lambda) + Z_{\nu}^{\eta}(d\lambda)), \tag{15}$$

where  $\vec{\xi}(j)$  and  $\vec{\eta}(j)$  are generating sequences of the sequences  $\zeta(j)$  and  $\theta(j)$ , correspondingly. The spectral characteristic  $\vec{h}(f^{\vec{\xi}}, f^{\vec{\eta}})$  and the mean square error  $\Delta(f^{\vec{\xi}}, f^{\vec{\eta}})$  of  $\widehat{A_s \zeta}$  are calculated by formulas (8) and (11), where  $\vec{a}(j) = (a_1(j), \dots, a_T(j))^T$ ,  $a_{\nu}(j) = a(j)e^{2\pi i j \nu / T}$ ,  $\nu = 1, \dots, T$ .

*Corollary 2*

The optimal linear estimate  $\widehat{\zeta(1)}$  of the unknown value  $\zeta(1)$ , based on observations of the sequence  $\zeta(j) + \theta(j)$  at points  $j \in \mathbb{Z} \setminus S$  is defined by the formula (15). The spectral characteristic  $\vec{h}(f^{\vec{\xi}}, f^{\vec{\eta}})$  and the mean square error  $\Delta(f^{\vec{\xi}}, f^{\vec{\eta}})$  of the optimal linear estimate  $\widehat{\zeta(1)}$  are calculated by formulas (8) and (11), where the unknown coefficients  $\vec{c}(k_l)$ ,  $l = 0, \dots, s - 1$ ,  $k_l = M_l + 1, \dots, M_l + N_{l+1}$  are defined by formulas

$$\vec{c}(k_l) = \sum_{m=0}^{s-1} \sum_{q=M_m+1}^{M_m+N_{m+1}} C_{lm}(k_l, q) D_{00}(q, 1) \vec{a}(1),$$

where elements  $C_{lm}(k_l, q)$ ,  $l, m = 0, \dots, s - 1$ ,  $k_l = M_l + 1, \dots, M_l + N_{l+1}$ ,  $q = M_m + 1, \dots, M_m + N_{m+1}$  are determined by the same way as in Theorem 1.

In the case of observations without noise we have the following corollary.

*Corollary 3*

Let  $\zeta(j)$  be a T-PC stochastic sequence. Then the optimal linear estimate of the functional  $A_s \zeta$  based on observations of the sequence  $\zeta(j)$  at points  $j \in \mathbb{Z} \setminus S$ , is given by

$$\widehat{A_s \zeta} = \int_{-\pi}^{\pi} \vec{h}^T(f^{\vec{\xi}}) Z^{\xi}(d\lambda) = \int_{-\pi}^{\pi} \sum_{\nu=1}^T h_{\nu}(f^{\vec{\xi}}) Z_{\nu}^{\xi}(d\lambda), \tag{16}$$

where  $\vec{\xi}(j)$  is generating sequence of  $\zeta(j)$ . The spectral characteristic  $\vec{h}(f^{\vec{\xi}})$  and the mean square error  $\Delta(f^{\vec{\xi}})$  of  $\widehat{A_s \zeta}$  are calculated by formulas (13) and (14), where  $\vec{a}(j) = (a_1(j), \dots, a_T(j))^T$ ,  $a_{\nu}(j) = a(j)e^{2\pi i j \nu / T}$ ,  $\nu = 1, \dots, T$ .

*Corollary 4*

The optimal linear estimate  $\widehat{\zeta(1)}$  of the unknown value  $\zeta(1)$ , based on observations of the sequence  $\zeta(j)$  at points  $j \in \mathbb{Z} \setminus S$  is defined by the formula (16). The spectral characteristic  $\vec{h}(f^{\vec{\xi}})$  and the mean square error  $\Delta(f^{\vec{\xi}})$  of the optimal linear estimate  $\widehat{\zeta(1)}$  are calculated by formulas (13) and (14), where the unknown coefficients  $\vec{c}(k_l)$ ,  $l = 0, \dots, s - 1$ ,  $k_l = M_l + 1, \dots, M_l + N_{l+1}$  are defined by formulas

$$\vec{c}(k_l) = C_{00}(k_l, 1) \vec{a}(1),$$

where elements  $C_{00}(k_l, 1)$ ,  $l = 0, \dots, s - 1$ ,  $k_l = M_l + 1, \dots, M_l + N_{l+1}$  are determined by the same way as in Corollary 1.

**5. Interpolation of T-PC stochastic sequences with special sets of missed observations**

Consider the problem of optimal estimation for the linear functional

$$A_s \zeta = \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_{l+1}} a(j) \zeta(j), \quad M_l = \sum_{k=0}^l (N_k + K_k), \quad N_0 = K_0 = 0,$$

which depends on the unobserved values of  $T$ -PC sequence  $\zeta(j)$  from observations of the sequence  $\zeta(j) + \theta(j)$  at points  $j \in \mathbb{Z} \setminus S$ , where the number of missed observations at each of the intervals is a multiple of the period  $T$ , what means that

$$K_1 = T \cdot K_1^T, K_2 = T \cdot K_2^T, \dots, K_{s-1} = T \cdot K_{s-1}^T,$$

and the number of observations at each of the intervals is a multiple of  $T$

$$N_1 = T \cdot N_1^T, N_2 = T \cdot N_2^T, \dots, N_s = T \cdot N_s^T,$$

and coefficients  $a(j), j \in S$  are of the form

$$a(j) = a\left(\left(j - \left[\frac{j}{T}\right]T\right) + \left[\frac{j}{T}\right]T\right) = a(\nu + \tilde{j}T) = a(\tilde{j})e^{2\pi i \tilde{j}\nu/T}, \quad (17)$$

$$\nu = 1, \dots, T, \tilde{j} \in \tilde{S},$$

$$\tilde{S} = \bigcup_{l=0}^{s-1} \{M_l^T, \dots, M_l^T + N_{l+1}^T - 1\},$$

where  $\nu = T$  and  $\tilde{j} = \lambda - 1$ , if  $j = T \cdot \lambda$ ,  $\lambda \in \mathbb{Z}$ , or

$$a(j) = a(T \cdot \lambda) = a(T + (\lambda - 1)T) = a(\lambda - 1)e^{2\pi i(\lambda-1)T/T},$$

and  $M_l = T \cdot M_l^T$ ,  $l = 0, \dots, s - 1$ .

Using Proposition 2.2, the linear functional  $A_s \zeta$  can be written as follows

$$\begin{aligned} A_s \zeta &= \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_{l+1}} a(j)\zeta(j) = \\ &= \sum_{l=0}^{s-1} \sum_{\tilde{j}=M_l^T}^{M_l^T+N_{l+1}^T-1} \sum_{\nu=1}^T a(\nu + \tilde{j}T)\zeta(\nu + \tilde{j}T) = \sum_{l=0}^{s-1} \sum_{\tilde{j}=M_l^T}^{M_l^T+N_{l+1}^T-1} \sum_{\nu=1}^T a_\nu(\tilde{j})\zeta_\nu(\tilde{j}) = \\ &= \sum_{l=0}^{s-1} \sum_{\tilde{j}=M_l^T}^{M_l^T+N_{l+1}^T-1} \vec{a}^\top(\tilde{j})\vec{\zeta}(\tilde{j}) = A_s \vec{\zeta}, \end{aligned} \quad (18)$$

where

$$\vec{a}(\tilde{j}) = (a_1(\tilde{j}), \dots, a_T(\tilde{j}))^\top, \quad a_\nu(\tilde{j}) = a(\nu + \tilde{j}T) = a(\tilde{j})e^{2\pi i \tilde{j}\nu/T},$$

$$\vec{\zeta}(\tilde{j}) = (\zeta_1(\tilde{j}), \dots, \zeta_T(\tilde{j}))^\top, \quad \zeta_\nu(\tilde{j}) = \zeta(\nu + \tilde{j}T),$$

$$\nu = 1, \dots, T, \tilde{j} \in \tilde{S}$$

and  $\vec{\zeta}(\tilde{j}), \tilde{j} \in \tilde{S}$  is  $T$ -variate stationary sequence, obtained by the  $T$ -blocking (3) of univariate  $T$ -PC sequence  $\zeta(j), j \in S$ .

Let  $f^{\vec{\zeta}}(\lambda)$  and  $f^{\vec{\theta}}(\lambda)$  be the spectral density matrices of  $T$ -variate stationary sequences  $\vec{\zeta}(j)$  and  $\vec{\theta}(j)$ , obtained from the  $T$ -blocking (3) of univariate  $T$ -PC sequences  $\zeta(j)$  and  $\theta(j)$ , respectively.

Taking into account the definition of the functional  $A_s \vec{\zeta}$  and Theorem 1 we can verify that the following statements hold true.

*Theorem 3*

Let  $\zeta(j)$  and  $\theta(j)$  be uncorrelated T-PC stochastic sequences with the spectral density matrices  $f^{\zeta}(\lambda)$  and  $f^{\theta}(\lambda)$  of T-variate stationary sequences  $\vec{\zeta}(j)$  and  $\vec{\theta}(j)$ , respectively. Assume that  $f^{\zeta}(\lambda)$  and  $f^{\theta}(\lambda)$  satisfy the minimality condition (4). Then the optimal linear estimate of  $A_s \vec{\zeta}$  based on observations of  $\vec{\zeta}(j) + \vec{\theta}(j)$  at points  $j \in \mathbb{Z} \setminus \tilde{S}$ , is given by

$$\widehat{A_s \vec{\zeta}} = \int_{-\pi}^{\pi} \vec{h}^{\top}(f^{\zeta}, f^{\theta})(Z^{\zeta}(d\lambda) + Z^{\theta}(d\lambda)) = \int_{-\pi}^{\pi} \sum_{\nu=1}^T h_{\nu}(f^{\zeta}, f^{\theta})(Z_{\nu}^{\zeta}(d\lambda) + Z_{\nu}^{\theta}(d\lambda)),$$

where  $Z^{\zeta}(\Delta) = \{Z_{\nu}^{\zeta}(\Delta)\}_{\nu=1}^T$  and  $Z^{\theta}(\Delta) = \{Z_{\nu}^{\theta}(\Delta)\}_{\nu=1}^T$  are orthogonal random measures of the sequences  $\vec{\zeta}(\tilde{j})$  and  $\vec{\theta}(\tilde{j})$ . The spectral characteristic  $\vec{h}(f^{\zeta}, f^{\theta})$  and the mean square error  $\Delta(f^{\zeta}, f^{\theta})$  of  $\widehat{A_s \vec{\zeta}}$  are calculated by formulas

$$\begin{aligned} \vec{h}^{\top}(f^{\zeta}, f^{\theta}) &= \left( \tilde{A}_s^{\top}(e^{i\lambda})f^{\zeta}(\lambda) - \tilde{C}_s^{\top}(e^{i\lambda}) \right) \left[ f^{\zeta}(\lambda) + f^{\theta}(\lambda) \right]^{-1} \\ &= \tilde{A}_s^{\top}(e^{i\lambda}) - \left( \tilde{A}_s^{\top}(e^{i\lambda})f^{\theta}(\lambda) + \tilde{C}_s^{\top}(e^{i\lambda}) \right) \left[ f^{\zeta}(\lambda) + f^{\theta}(\lambda) \right]^{-1}, \end{aligned} \tag{19}$$

$$\Delta(f^{\zeta}, f^{\theta}) = \langle \vec{a}_s^{\zeta}, \mathbf{R}_s^{\zeta} \vec{a}_s^{\zeta} \rangle + \langle \vec{c}_s^{\zeta}, \mathbf{B}_s^{\zeta} \vec{c}_s^{\zeta} \rangle, \tag{20}$$

where

$$\begin{aligned} \vec{a}_s^{\zeta} &= \left( \vec{a}^{\top}(0), \dots, \vec{a}^{\top}(N_1^T - 1), \dots, \vec{a}^{\top}(M_{s-1}^T), \dots, \vec{a}^{\top}(M_{s-1}^T + N_s^T - 1) \right)^{\top}, \\ \vec{c}_s^{\zeta} &= \left( \vec{c}^{\top}(0), \dots, \vec{c}^{\top}(N_1^T - 1), \dots, \vec{c}^{\top}(M_{s-1}^T), \dots, \vec{c}^{\top}(M_{s-1}^T + N_s^T - 1) \right)^{\top}, \end{aligned}$$

$\tilde{A}_s(e^{i\lambda}) = \sum_{\tilde{j} \in \tilde{S}} \vec{a}(\tilde{j})e^{i\tilde{j}\lambda}$ ,  $\tilde{C}_s(e^{i\lambda}) = \sum_{\tilde{j} \in \tilde{S}} \vec{c}(\tilde{j})e^{i\tilde{j}\lambda}$ , the unknown coefficients  $\vec{c}(\tilde{j})$ ,  $\tilde{j} \in \tilde{S}$  are determined from the relation

$$\vec{c}_s^{\zeta} = (\mathbf{B}_s^{\zeta})^{-1} \mathbf{D}_s^{\zeta} \vec{a}_s^{\zeta},$$

operators  $\mathbf{B}_s^{\zeta}$ ,  $\mathbf{D}_s^{\zeta}$ ,  $\mathbf{R}_s^{\zeta}$  are determined by  $\rho \times \rho$  matrices, constructed from  $T \cdot N_{m+1}^T \times T \cdot N_{n+1}^T$  block-matrices

$$\begin{aligned} B_{mn}^{\zeta} &= \{B_{mn}^{\zeta}(k, j)\}_{k=M_m^T, j=M_n^T}^{M_m^T+N_{m+1}^T-1, M_n^T+N_{n+1}^T-1}, \\ D_{mn}^{\zeta} &= \{D_{mn}^{\zeta}(k, j)\}_{k=M_m^T, j=M_n^T}^{M_m^T+N_{m+1}^T-1, M_n^T+N_{n+1}^T-1}, \\ R_{mn}^{\zeta} &= \{R_{mn}^{\zeta}(k, j)\}_{k=M_m^T, j=M_n^T}^{M_m^T+N_{m+1}^T-1, M_n^T+N_{n+1}^T-1}, \quad m, n = 0, \dots, s-1 \end{aligned}$$

with elements:

$$\begin{aligned} B_{mn}^{\zeta}(k, j) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ (f^{\zeta}(\lambda) + f^{\theta}(\lambda))^{-1} \right]^{\top} e^{i(j-k)\lambda} d\lambda, \\ D_{mn}^{\zeta}(k, j) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ f^{\zeta}(\lambda)(f^{\zeta}(\lambda) + f^{\theta}(\lambda))^{-1} \right]^{\top} e^{i(j-k)\lambda} d\lambda, \\ R_{mn}^{\zeta}(k, j) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ f^{\zeta}(\lambda)(f^{\zeta}(\lambda) + f^{\theta}(\lambda))^{-1} f^{\theta}(\lambda) \right]^{\top} e^{i(j-k)\lambda} d\lambda, \\ & \quad m, n = 0, \dots, s-1, \\ & \quad k = M_m^T, \dots, M_m^T + N_{m+1}^T - 1, \\ & \quad j = M_n^T, \dots, M_n^T + N_{n+1}^T - 1. \end{aligned}$$

In the case of observations without noise we have the following corollary.

*Corollary 5*

Let  $\zeta(j)$  be a T-PC stochastic sequence with the spectral density matrix  $f^{\vec{\zeta}}(\lambda)$  of T-variate stationary sequence  $\vec{\zeta}(j)$ . Assume that  $f^{\vec{\zeta}}(\lambda)$  satisfies the minimality condition (12). Then the optimal linear estimate of  $A_s \vec{\zeta}$  based on observations of  $\vec{\zeta}(j)$  at points  $j \in \mathbb{Z} \setminus \tilde{S}$ , is given by

$$\widehat{A_s \vec{\zeta}} = \int_{-\pi}^{\pi} \vec{h}^T(f^{\vec{\zeta}}) Z^{\vec{\zeta}}(d\lambda) = \int_{-\pi}^{\pi} \sum_{\nu=1}^T h_{\nu}(f^{\vec{\zeta}}) Z_{\nu}^{\vec{\zeta}}(d\lambda).$$

The spectral characteristic  $\vec{h}(f^{\vec{\zeta}})$  and the mean square error  $\Delta(f^{\vec{\zeta}})$  of  $\widehat{A_s \vec{\zeta}}$  are calculated by formulas

$$\vec{h}^T(f^{\vec{\zeta}}) = \tilde{A}_s^T(e^{i\lambda}) - \tilde{C}_s^T(e^{i\lambda}) \left[ f^{\vec{\zeta}}(\lambda) \right]^{-1}, \tag{21}$$

$$\Delta(f^{\vec{\zeta}}) = \langle \vec{c}_s^{\zeta}, \vec{a}_s^{\zeta} \rangle, \tag{22}$$

unknown coefficients  $\vec{c}(\tilde{j})$ ,  $\tilde{j} \in \tilde{S}$  are determined from the relation

$$\vec{c}_s^{\zeta} = (\mathbf{B}_s^{\zeta})^{-1} \vec{a}_s,$$

operator  $\mathbf{B}_s^{\zeta}$  is a matrix composed with  $\rho \times \rho$  matrix, constructed from  $T \cdot N_{m+1}^T \times T \cdot N_{n+1}^T$  block-matrices  $B_{mn}^{\zeta} = \{B_{mn}^{\zeta}(k, j)\}_{k=M_m^T, j=M_n^T}^{M_m^T+N_{m+1}^T-1, M_n^T+N_{n+1}^T-1}$  with elements:

$$B_{mn}^{\zeta}(j, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ (f^{\vec{\zeta}}(\lambda))^{-1} \right]^T e^{i(k-j)\lambda} d\lambda,$$

$$m, n = 0, \dots, s - 1,$$

$$k = M_m^T, \dots, M_m^T + N_{m+1}^T - 1,$$

$$j = M_n^T, \dots, M_n^T + N_{n+1}^T - 1.$$

*Example 2*

Let  $\zeta(n)$ ,  $n \in \mathbb{Z}$  be a 2-PC stochastic sequence such that  $\zeta(2n + 1) = \eta(n)$  is a univariate stationary Ornstein-Uhlenbeck sequence with the spectral density  $f(\lambda) = \frac{1}{|2+e^{i\lambda}|^2}$  and  $\zeta(2n) = \gamma(n)$  is an uncorrelated with  $\eta(n)$  univariate stationary Ornstein-Uhlenbeck sequence with the spectral density  $g(\lambda) = \frac{1}{|3-e^{i\lambda}|^2}$ .

Consider the problem of estimation of the functional

$$A_1 \zeta = \zeta(1) + \zeta(2) - \zeta(3) + \zeta(4).$$

Here  $S = \{1, 2, 3, 4\}$  and  $N_1 = 4$  is a multiple of  $T = 2$ . Rewrite  $A_1 \zeta$  in the form (18)

$$\begin{aligned} A_1 \zeta &= e^{2\pi i 1 \cdot 0/2} \zeta(1 + 0 \cdot 2) + e^{2\pi i 2 \cdot 0/2} \zeta(2 + 0 \cdot 2) + e^{2\pi i 1 \cdot 1/2} \zeta(1 + 1 \cdot 2) + e^{2\pi i 2 \cdot 2/2} \zeta(2 + 1 \cdot 2) = \\ &= \vec{a}^T(0) \vec{\zeta}(0) + \vec{a}^T(1) \vec{\zeta}(1) = A_1 \vec{\zeta}, \end{aligned}$$

where  $\vec{a}(\tilde{j}) = (a(\tilde{j})e^{2\pi i 1 \cdot \tilde{j}/2}, a(\tilde{j})e^{2\pi i 2 \cdot \tilde{j}/2})^T$ ,  $a(0) = 1$ ,  $a(1) = 1$ ,  $\vec{\zeta}(\tilde{j}) = (\zeta(1 + \tilde{j} \cdot 2), \zeta(2 + \tilde{j} \cdot 2))^T$ ,  $\tilde{j} \in \tilde{S} = \{0, 1\}$ . In this case the spectral density matrix of  $\vec{\zeta}(n)$  is of the form

$$f^{\vec{\zeta}}(\lambda) = \begin{pmatrix} f(\lambda) & 0 \\ 0 & g(\lambda) \end{pmatrix}$$

and  $[f^{\vec{\zeta}}(\lambda)]^{-1}$  satisfies the minimality condition (12). The matrix  $\mathbf{B}_1$  and its inverse  $(\mathbf{B}_1^\zeta)^{-1}$ , the vector of unknown coefficients  $\vec{c}_2$  are of the form

$$\mathbf{B}_1^\zeta = \begin{pmatrix} 5 & 0 & 2 & 0 \\ 0 & 10 & 0 & -3 \\ 2 & 0 & 5 & 0 \\ 0 & -3 & 0 & 10 \end{pmatrix}, (\mathbf{B}_1^\zeta)^{-1} = \frac{1}{273} \begin{pmatrix} 65 & 0 & -26 & 0 \\ 0 & 30 & 0 & 9 \\ -26 & 0 & 65 & 0 \\ 0 & 9 & 0 & 30 \end{pmatrix},$$

$$\vec{c}_1^\zeta = \frac{1}{273} (91, 39, -91, 39)^\top.$$

The spectral characteristic of the optimal estimate of  $A_1\zeta$  is of the form

$$\vec{h}^\top(f^{\vec{\zeta}}) = \left( -\frac{2}{3}e^{-i\lambda} + \frac{10}{3}e^{2i\lambda}, \frac{3}{7}e^{-i\lambda} + \frac{3}{7}e^{2i\lambda} \right),$$

and the optimal linear estimate of  $A_1\vec{\zeta}$  is of the form

$$\begin{aligned} \widehat{A_1\vec{\zeta}} &= -\frac{2}{3}\zeta_1(-1) + \frac{10}{3}\zeta_1(2) + \frac{3}{7}\zeta_2(-1) + \frac{3}{7}\zeta_2(2) = \\ &= -\frac{2}{3}\zeta(-1) + \frac{10}{3}\zeta(5) + \frac{3}{7}\zeta(0) + \frac{3}{7}\zeta(6). \end{aligned}$$

The mean square error of this estimate

$$\Delta(f^{\vec{\zeta}}) = \frac{20}{21}.$$

### 6. Minimax (robust) method of linear interpolation

Let  $f(\lambda)$  and  $g(\lambda)$  be the spectral density matrices of  $T$ -variate stationary sequences  $\vec{\zeta}(j)$  and  $\vec{\theta}(j)$ , obtained by  $T$ -blocking (3) of  $T$ -PC sequences  $\zeta(j)$  and  $\theta(j)$ , respectively.

Formulas (19)–(22) may be applied for finding the spectral characteristic and the mean square error of the optimal linear estimate of the functional  $A_s\vec{\zeta}$  only under the condition that the spectral density matrices  $f(\lambda)$  and  $g(\lambda)$  are exactly known. If the density matrices are not known exactly while a set  $D = D_f \times D_g$  of possible spectral densities is given, the minimax (robust) approach to estimation of functionals from unknown values of stationary sequences is reasonable. In this case we find the estimate which minimizes the mean square error for all spectral densities from the given set simultaneously.

#### Definition 3

For a given class of pairs of spectral densities  $D = D_f \times D_g$  the spectral density matrices  $f^0(\lambda) \in D_f, g^0(\lambda) \in D_g$  are called the least favorable in  $D$  for the optimal linear estimation of the functional  $A_s\vec{\zeta}$  if

$$\Delta(f^0, g^0) = \Delta(\vec{h}(f^0, g^0); f^0, g^0) = \max_{(f,g) \in D} \Delta(\vec{h}(f, g); f, g).$$

#### Definition 4

For a given class of pairs of spectral densities  $D = D_f \times D_g$  the spectral characteristic  $\vec{h}^0(\lambda)$  of the optimal linear estimate of the functional  $A_s\vec{\zeta}$  is called minimax (robust) if

$$\vec{h}^0(\lambda) \in H_D = \bigcap_{(f,g) \in D} L_2^{s-}(f + g),$$

$$\min_{\vec{h} \in H_D} \max_{(f,g) \in D} \Delta(\vec{h}; f, g) = \max_{(f,g) \in D} \Delta(\vec{h}^0; f, g).$$

Taking into consideration these definitions and the obtained relations we can verify that the following lemmas hold true.

*Lemma 1*

The spectral density matrices  $f^0(\lambda) \in D_f, g^0(\lambda) \in D_g$ , that satisfy condition (4), are the least favorable in  $D$  for the optimal linear estimation of  $A_s \vec{\zeta}$ , if the Fourier coefficients of the matrix functions

$$(f^0(\lambda) + g^0(\lambda))^{-1}, \quad f^0(\lambda)(f^0(\lambda) + g^0(\lambda))^{-1}, \\ f^0(\lambda)(f^0(\lambda) + g^0(\lambda))^{-1}g^0(\lambda)$$

define matrices  $\mathbf{B}_s^{\zeta_0}, \mathbf{D}_s^{\zeta_0}, \mathbf{R}_s^{\zeta_0}$ , that determine a solution of the constrained optimization problem

$$\max_{(f,g) \in D} (\langle \vec{a}_s^{\zeta}, \mathbf{R}_s^{\zeta} \vec{a}_s^{\zeta} \rangle + \langle (\mathbf{B}_s^{\zeta})^{-1} \mathbf{D}_s^{\zeta} \vec{a}_s^{\zeta}, \mathbf{D}_s^{\zeta} \vec{a}_s^{\zeta} \rangle) = \langle \vec{a}_s^{\zeta}, \mathbf{R}_s^{\zeta_0} \vec{a}_s^{\zeta} \rangle + \langle (\mathbf{B}_s^{\zeta_0})^{-1} \mathbf{D}_s^{\zeta_0} \vec{a}_s^{\zeta}, \mathbf{D}_s^{\zeta_0} \vec{a}_s^{\zeta} \rangle.$$

The minimax spectral characteristic  $\vec{h}^0 = \vec{h}(f^0, g^0)$  is given by (8), if  $\vec{h}(f^0, g^0) \in H_D$ .

*Lemma 2*

The spectral density matrix  $f^0(\lambda) \in D_f$ , that satisfies condition (12), is the least favorable in  $D_f$  for the optimal linear estimation of  $A_s \vec{\zeta}$  based on observations of the sequence  $\vec{\zeta}(j)$  at points  $j \in \mathbb{Z} \setminus \tilde{S}$ , if the Fourier coefficients of the matrix function  $(f^0(\lambda))^{-1}$  define the matrix  $\mathbf{B}_s^{\zeta_0}$ , that determine a solution of the constrained optimization problem

$$\max_{f \in D_f} \langle (\mathbf{B}_s^{\zeta})^{-1} \vec{a}_s^{\zeta}, \vec{a}_s^{\zeta} \rangle = \langle (\mathbf{B}_s^{\zeta_0})^{-1} \vec{a}_s^{\zeta}, \vec{a}_s^{\zeta} \rangle.$$

The minimax spectral characteristic  $\vec{h}^0 = \vec{h}(f^0)$  is given by (13), if  $\vec{h}(f^0) \in H_D$ .

The least favorable spectral densities  $f^0(\lambda) \in D_f, g^0(\lambda) \in D_g$  and the minimax spectral characteristic  $\vec{h}^0 = \vec{h}(f^0, g^0)$  form a saddle point of the function  $\Delta(\vec{h}; f, g)$  on the set  $H_D \times D$ . The saddle point inequalities

$$\Delta(\vec{h}^0; f, g) \leq \Delta(\vec{h}^0; f^0, g^0) \leq \Delta(\vec{h}; f^0, g^0), \quad \forall \vec{h} \in H_D, \forall f \in D_f, \forall g \in D_g$$

hold when  $\vec{h}^0 = \vec{h}(f^0, g^0), \vec{h}(f^0, g^0) \in H_D$  and  $(f^0, g^0)$  is a solution of the constrained optimization problem

$$\sup_{(f,g) \in D_f \times D_g} \Delta(\vec{h}(f^0, g^0); f, g) = \Delta(\vec{h}(f^0, g^0); f^0, g^0). \tag{23}$$

The linear functional  $\Delta(\vec{h}(f^0, g^0); f, g)$  is calculated by the formula

$$\Delta(\vec{h}(f^0, g^0); f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\tilde{A}_s(e^{i\lambda})g^0(\lambda) + \tilde{C}_s^0(e^{i\lambda}))^{\top} (f^0(\lambda) + g^0(\lambda))^{-1} f(\lambda)(f^0(\lambda) + g^0(\lambda))^{-1} \times \\ \overline{(\tilde{A}_s(e^{i\lambda})g^0(\lambda) + \tilde{C}_s^0(e^{i\lambda}))} d\lambda + \frac{1}{2\pi} \int_{-\pi}^{\pi} (\tilde{A}_s(e^{i\lambda})f^0(\lambda) - \tilde{C}_s^0(e^{i\lambda}))^{\top} (f^0(\lambda) + g^0(\lambda))^{-1} g(\lambda) \times \\ (f^0(\lambda) + g^0(\lambda))^{-1} \overline{(\tilde{A}_s(e^{i\lambda})f^0(\lambda) - \tilde{C}_s^0(e^{i\lambda}))} d\lambda.$$

The constrained optimization problem (23) is equivalent to the unconstrained optimization problem [83]:

$$\Delta_D(f, g) = -\Delta(\vec{h}(f^0, g^0); f, g) + \delta((f, g) | D_f \times D_g) \rightarrow \inf, \tag{24}$$

where  $\delta((f, g) | D_f \times D_g)$  is the indicator function of the set  $D = D_f \times D_g$ . A solution of the problem (24) is characterized by the condition  $0 \in \partial \Delta_D(f^0, g^0)$ , where  $\partial \Delta_D(f^0, g^0)$  is the subdifferential of the convex functional  $\Delta_D(f, g)$  at point  $(f^0, g^0)$  [85].

The form of the functional  $\Delta(\vec{h}(f^0, g^0); f, g)$  admits finding the derivatives and differentials of the functional in the space  $L_1 \times L_1$ . Therefore the complexity of the optimization problem (24) is determined by the complexity of calculating of subdifferentials of the indicator functions  $\delta((f, g) | D_f \times D_g)$  of the sets  $D_f \times D_g$  [25].

Taking into consideration the introduced definitions and the derived relations we can verify that the following lemma holds true.



*Lemma 3*

Let  $(f^0, g^0)$  be a solution to the optimization problem (24). The spectral densities  $f^0(\lambda), g^0(\lambda)$  are the least favorable in the class  $D = D_f \times D_g$  and the spectral characteristic  $\vec{h}^0 = \vec{h}(f^0, g^0)$  is the minimax of the optimal linear estimate of the functional  $A_s \vec{\zeta}$  if  $\vec{h}(f^0, g^0) \in H_D$ .

In the case of estimation of the functional based on observations without noise we have the following statement.

*Lemma 4*

Let  $f^0(\lambda)$  satisfies the condition (12) and be a solution of the constrained optimization problem

$$\Delta(\vec{h}(f^0); f) \rightarrow \sup, f(\lambda) \in D_f, \tag{25}$$

$$\Delta(\vec{h}(f^0); f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\tilde{C}_s^0(e^{i\lambda}))^\top (f^0(\lambda))^{-1} f(\lambda) (f^0(\lambda))^{-1} \overline{(\tilde{C}_s^0(e^{i\lambda}))} d\lambda.$$

Then  $f^0(\lambda)$  is the least favorable spectral density matrix for the optimal linear estimation of  $A_s \vec{\zeta}$  based on observations of the sequence  $\vec{\zeta}(j)$  at points  $j \in \mathbb{Z} \setminus \tilde{S}$ . The minimax spectral characteristic  $\vec{h}^0 = \vec{h}(f^0)$  is given by (13), if  $\vec{h}(f^0) \in H_D$ .

**7. The least favorable spectral densities in  $D_0^-$**

Let  $\zeta(j), j \in \mathbb{Z}$  be  $T$ -PC sequence and let  $\vec{\zeta}(j)$  be  $T$ -variate stationary sequence, obtained by  $T$ -blocking (3) of  $T$ -PC sequence  $\zeta(j)$ . Assume that the number of missed observations of the functional  $A_s \vec{\zeta}$  at each of the intervals is a multiple of the period  $T$

$$K_1 = T \cdot K_1^T, K_2 = T \cdot K_2^T, \dots, K_{s-1} = T \cdot K_{s-1}^T$$

and the number of observations at each of the intervals is a multiple of  $T$

$$N_1 = T \cdot N_1^T, N_2 = T \cdot N_2^T, \dots, N_s = T \cdot N_s^T,$$

and coefficients  $a(j), j \in S$  are of the form (17).

Consider the problem of minimax estimation of the functional  $A_s \vec{\zeta}$  from observations of the sequence  $\vec{\zeta}(j)$  at points  $j \in \mathbb{Z} \setminus \tilde{S}$  without noise, under the condition that the spectral density matrix  $f(\lambda)$  of  $T$ -variate stationary sequence  $\vec{\zeta}(j)$  belongs to the set

$$D_0^- = \left\{ f(\lambda) \mid \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{-1}(\lambda) d\lambda = P \right\},$$

where  $P = \{p_{\nu\mu}\}_{\nu,\mu=1}^T$  is a given positive definite matrix and  $\det P \neq 0$ . With the help of Lemma 4 and the method of Lagrange multipliers we can find that a solution  $f^0(\lambda)$  of the constrained optimization problem (25) satisfy the following relation:

$$[(f^0(\lambda))^{-1}]^\top \tilde{C}_s^0(e^{i\lambda}) = [(f^0(\lambda))^{-1}]^\top \vec{\alpha}, \tag{26}$$

where  $\vec{\alpha} = (\alpha_1, \dots, \alpha_T)^\top$  is a vector of Lagrange multipliers,

$$\tilde{C}_s^0(e^{i\lambda}) = \sum_{l=0}^{s-1} \sum_{\tilde{j}=M_l^T}^{M_{l+1}^T+N_{l+1}^T-1} \vec{c}^0(\tilde{j}) e^{i\tilde{j}\lambda},$$

$$\vec{c}_s^0 = ((\vec{c}^0(0))^\top, \dots, (\vec{c}^0(N_1^T - 1))^\top, \dots, (\vec{c}^0(M_{s-1}^T))^\top, \dots, (\vec{c}^0(M_{s-1}^T + N_s^T - 1))^\top)^\top,$$

unknown coefficients  $\tilde{c}^0(\tilde{j})$ ,  $\tilde{j} \in \tilde{S}$  are determined from relation  $\tilde{c}_s^{\zeta 0} = (\mathbf{B}_s^0)^{-1} \tilde{a}_s^\zeta$ , the matrix  $\mathbf{B}_s^0$  is constructed from the Fourier coefficients

$$B_s^0(k, j) = R^\top(j - k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [(f^0(\lambda))^{-1}]^\top e^{i(j-k)\lambda} d\lambda, \quad k, j \in \tilde{S}$$

of the matrix function  $[(f^0(\lambda))^{-1}]^\top$ .

The Fourier coefficients  $R(k) = R^*(-k)$ ,  $k \in \tilde{S}$ , found from the equation

$$\mathbf{B}_s^0 \vec{\alpha}_s = \vec{a}_s^\zeta,$$

for  $\vec{\alpha}_s = (\vec{\alpha}, \vec{0}, \dots, \vec{0})^\top$ , satisfy relation (26) and  $\mathbf{B}_s^0 \tilde{c}_s^{\zeta 0} = \vec{a}_s$ . From equations above we obtain that

$$R(k) = \begin{cases} P(\vec{a}(0))^{-1} \vec{a}^\top(k), & k \in \tilde{S}, \\ 0, & k \in \{0, \dots, M_{s-1}^T + N_s^T - 1\} \setminus \tilde{S}, \end{cases}$$

where  $[(\vec{a}(0))^{-1}]^\top \cdot \vec{a}(0) = 1$ . The equality  $R(0) = P$  follows as a consequence of the restriction on the spectral densities from the class  $D_0^-$ .

Let the vector-valued sequence  $\vec{a}(k)$ ,  $k \in \tilde{S}$ , be such that the matrix function

$$(f^0(\lambda))^{-1} = \sum_{k=-(M_{s-1}^T+N_s^T-1)}^{M_{s-1}^T+N_s^T-1} R(k)e^{ik\lambda}$$

is positive definite and has nonzero determinant. Then  $(f^0(\lambda))^{-1}$  can be represented in the form [23]

$$(f^0(\lambda))^{-1} = \left( \sum_{k=0}^{M_{s-1}^T+N_s^T-1} Q(k)e^{-ik\lambda} \right) \cdot \left( \sum_{k=0}^{M_{s-1}^T+N_s^T-1} Q(k)e^{-ik\lambda} \right)^*,$$

where  $Q(k) = 0_{T \times T}$ ,  $k \in \{0, \dots, M_{s-1}^T + N_s^T - 1\} \setminus \tilde{S}$ . Thus  $f^0(\lambda)$  is the spectral density of the vector autoregression stochastic sequence of order  $M_{s-1}^T + N_s^T - 1$  generated by the equation

$$\sum_{k=0}^{M_{s-1}^T+N_s^T-1} Q(k)\vec{\zeta}(n-k) = \vec{\varepsilon}(n), \tag{27}$$

where  $\vec{\varepsilon}(n)$  is a vector "white noise" sequence. The minimax spectral characteristic  $\vec{h}(f^0)$  is given by

$$\vec{h}(f^0) = - \sum_{k=1}^{M_{s-1}^T+N_s^T-1} \overline{R(k)}(P^T)^{-1} \vec{a}(0)e^{-ik\lambda}. \tag{28}$$

Hence the following theorem holds true.

**Theorem 4**

Let the sequence  $\vec{a}(k) = (a_1(k), a_2(k), \dots, a_T(k))^T$ ,  $a_\nu(k) = a(k)e^{2\pi i\nu k/T}$ ,  $\nu = 1, \dots, T$ , which determine the linear functional  $A_s \vec{\zeta}$  from observations of sequence  $\vec{\zeta}(j)$  at points  $j \in \mathbb{Z} \setminus \tilde{S}$ , be such that the matrix function

$$\sum_{k=-(M_{s-1}^T+N_s^T-1)}^{M_{s-1}^T+N_s^T-1} R(k)e^{ik\lambda},$$

where

$$R(k) = R^*(-k) = \begin{cases} P(\vec{a}(0))^{-1} \vec{a}^\top(k), & k \in \tilde{S}, \\ 0_{T \times T}, & k \in \{0, \dots, M_{s-1}^T + N_s^T - 1\} \setminus \tilde{S}, \end{cases}$$

is positive definite and has nonzero determinant. Then the least favorable in the class  $D_0^-$  spectral density for the optimal linear estimate of  $A_s \vec{\zeta}$  is given by the formula

$$f^0(\lambda) = \left( \sum_{k=-(M_{s-1}^T + N_s^T - 1)}^{M_{s-1}^T + N_s^T - 1} R(k) e^{ik\lambda} \right)^{-1}. \tag{29}$$

The minimax spectral characteristic  $\vec{h}(f^0)$  is given by (28). The greatest value of the mean square error of  $\widehat{A_s \vec{\zeta}}$  is calculated by the formula

$$\Delta(f^0) = \langle \vec{c}_s^{\zeta^0}, \vec{a}_s^\zeta \rangle. \tag{30}$$

*Example 3*

Let  $\zeta(n)$  be a 2-PC stochastic sequence. Consider the problem of minimax estimation of the functional

$$A_2 \zeta = 5\zeta(1) + 5\zeta(2) + 2\zeta(5) + 2\zeta(6)$$

from observation of the sequence  $\zeta(j)$  at points  $j \in \mathbb{Z} \setminus \{1, 2, 5, 6\}$  on the set  $D_0^-$  with  $P = \begin{pmatrix} 23 & 22 \\ 22 & 23 \end{pmatrix}$ .

Rewrite  $A_2 \zeta$  in the form (18)

$$\begin{aligned} A_2 \zeta &= 5\zeta(1) + 5\zeta(2) + 2\zeta(5) + 2\zeta(6) = \\ &= \vec{a}^\top(0) \vec{\zeta}(0) + \vec{a}^\top(2) \vec{\zeta}(2) = A_2 \vec{\zeta}, \end{aligned}$$

where  $\vec{a}(0) = (5, 5)^\top$ ,  $\vec{a}(2) = (2, 2)^\top$ . The matrix function

$$\sum_{k=-2,0,2} R(k) e^{ik\lambda}$$

and the representation

$$\left( \sum_{k=0,2} Q(k) e^{-ik\lambda} \right) \cdot \left( \sum_{k=0,2} Q(k) e^{-ik\lambda} \right)^*$$

are of the form

$$\begin{aligned} &\begin{pmatrix} 9e^{-2i\lambda} + 23 + 9e^{2i\lambda} & 9e^{-2i\lambda} + 22 + 9e^{2i\lambda} \\ 9e^{-2i\lambda} + 22 + 9e^{2i\lambda} & 9e^{-2i\lambda} + 23 + 9e^{2i\lambda} \end{pmatrix} = \\ &= \begin{pmatrix} 2 + 3e^{-2i\lambda} & 1 + 3e^{-2i\lambda} \\ 1 + 3e^{-2i\lambda} & 2 + 3e^{-2i\lambda} \end{pmatrix} \cdot \begin{pmatrix} 2 + 3e^{2i\lambda} & 1 + 3e^{2i\lambda} \\ 1 + 3e^{2i\lambda} & 2 + 3e^{2i\lambda} \end{pmatrix}. \end{aligned}$$

The least favorable spectral density in the class  $D_0^-$  for the optimal linear estimate of  $A_2 \vec{\zeta}$  by (29) is of the form

$$f^0(\lambda) = \frac{1}{45 - 18e^{-2i\lambda} - 18e^{2i\lambda}} \begin{pmatrix} 9e^{-2i\lambda} + 23 + 9e^{2i\lambda} & -9e^{-2i\lambda} - 22 - 9e^{2i\lambda} \\ -9e^{-2i\lambda} - 22 - 9e^{2i\lambda} & 9e^{-2i\lambda} + 23 + 9e^{2i\lambda} \end{pmatrix}.$$

The minimax spectral characteristic, calculated by (28), is given by the formula

$$\vec{h}(f^0) = - \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{-2i\lambda}.$$

The greatest value of the mean square error of  $\widehat{A_2 \vec{\zeta}}$  takes value

$$\Delta(f^0) = \frac{10}{9}.$$

### 8. The least favorable spectral densities in $D_G^-$

Let  $\zeta(j)$ ,  $j \in \mathbb{Z}$  be  $T$ -PC sequence and  $\vec{\zeta}(j)$  be  $T$ -variate stationary sequence, obtained by  $T$ -blocking (3) of  $T$ -PC sequence  $\zeta(j)$ . Assume that the number of missed observations of the functional  $A_s \vec{\zeta}$  at each of the intervals is a multiple of the period  $T$

$$K_1 = T \cdot K_1^T, K_2 = T \cdot K_2^T, \dots, K_{s-1} = T \cdot K_{s-1}^T$$

and the number of observations at each of the intervals is a multiple of  $T$

$$N_1 = T \cdot N_1^T, N_2 = T \cdot N_2^T, \dots, N_s = T \cdot N_s^T,$$

and coefficients  $a(j)$ ,  $j \in S$  are of the form (17).

Consider the problem of minimax estimation of the functional  $A_s \vec{\zeta}$  from observations  $\vec{\zeta}(j)$  at points  $j \in \mathbb{Z} \setminus \tilde{S}$  without noise, under the condition that the spectral density matrix  $f(\lambda)$  of the vector stationary sequence  $\vec{\zeta}(j)$  belongs to the set

$$D_G^- = \left\{ f(\lambda) \mid \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{-1}(\lambda) \cos(g\lambda) d\lambda = P(g), g = 0, 1, \dots, G \right\},$$

where the sequence of matrices  $P(g) = \{p_{\nu\mu}(g)\}_{\nu,\mu=1}^T$ ,  $P(g) = P^*(g)$ ,  $g = 0, 1, \dots, G$ , is such that the matrix function  $\sum_{g=-G}^G P(g) e^{ig\lambda}$  is positive definite and has the determinant that does not equal zero. With the help of Lemma 4 and the method of Lagrange multipliers we can find that solution  $f^0(\lambda)$  of the constrained optimization problem (25) satisfy the following relation:

$$\left[ (f^0(\lambda))^{-1} \right]^\top \vec{C}_s^0(e^{i\lambda}) = \left[ (f^0(\lambda))^{-1} \right]^\top \left( \sum_{g=0}^G \vec{\alpha}_g e^{ig\lambda} \right), \quad (31)$$

where  $\vec{\alpha}_g$ ,  $g = 0, 1, \dots, G$  are Lagrange multipliers. Relation (31) holds true if

$$\sum_{\tilde{j} \in \tilde{S}} \vec{c}^0(\tilde{j}) e^{i\tilde{j}\lambda} = \sum_{g=0}^G \vec{\alpha}_g e^{ig\lambda}.$$

Consider two cases:  $G \geq M_{s-1}^T + N_s^T - 1$  and  $G < M_{s-1}^T + N_s^T - 1$ .

Let  $G \geq M_{s-1}^T + N_s^T - 1$ . Then the Fourier coefficients of the function  $(f^0(\lambda)^{-1})^\top$  determine the matrix  $\mathbf{B}_s^0$  and extremum problem (25) is degenerate. Let

$$\vec{\alpha}_{M_{s-1}^T + N_s^T} = \dots = \vec{\alpha}_G = \vec{0} \text{ and } \vec{\alpha}_g = 0, g \notin \tilde{S},$$

and  $\vec{\alpha}_0, \dots, \vec{\alpha}_{M_{s-1}^T + N_s^T - 1}$  find from the equation

$$\mathbf{B}_s^0 \vec{\alpha}_s^0 = \vec{a}_s^{\vec{\zeta}},$$

where  $\vec{\alpha}_s^0 = (\vec{\alpha}_0, \dots, \vec{\alpha}_{M_{s-1}^T + N_s^T - 1})^\top$ . Then the least favorable is every density  $f(\lambda) \in D_G^-$  and the density

$$\begin{aligned} f^0(\lambda) &= \left( \sum_{g=-G}^G P(g) e^{ig\lambda} \right)^{-1} = \\ &= \left( \left( \sum_{g=0}^G Q(g) e^{-ig\lambda} \right) \left( \sum_{g=0}^G Q(g) e^{-ig\lambda} \right)^* \right)^{-1} \end{aligned} \quad (32)$$

of the vector stochastic autoregression sequence of the order  $G$

$$\sum_{g=0}^G Q(g)\vec{\zeta}(l-g) = \vec{\varepsilon}_l. \tag{33}$$

Let  $G < M_{s-1}^T + N_s^T - 1$ . Then the matrix  $\mathbf{B}_s$  is defined by the Fourier coefficients of the function  $(f(\lambda)^{-1})^\top$ . Among them  $P(g), g \in \{0, \dots, G\} \cap \tilde{S}$ , are known and  $P(g), g \in \tilde{S} \setminus \{0, \dots, G\}$ , are unknown. The unknown  $\vec{\alpha}_g, g \in \{0, \dots, G\} \cap S$  and  $P(g), g \in \tilde{S} \setminus \{0, \dots, G\}$  we find from the equation

$$\mathbf{B}_s \vec{\alpha}_G^0 = \vec{a}_s^{\tilde{c}}, \tag{34}$$

where  $\vec{\alpha}_G^0 = (\vec{\alpha}_0, \dots, \vec{\alpha}_{G'}, \vec{0}, \dots, \vec{0})^\top$ ,  $G'$  is defined from the relation  $\{0, \dots, G\} \cap \tilde{S} = \{0, \dots, G'\}$ . The equation (34) can be represented as a system of the following equations

$$\begin{aligned} \sum_{g \in \{0, \dots, G\} \cap \tilde{S}} B_s(0, g) \vec{\alpha}(g) &= \vec{a}(0), \\ &\vdots \\ \sum_{g \in \{0, \dots, G\} \cap \tilde{S}} B_s(G', g) \vec{\alpha}(g) &= \vec{a}(G'), \\ &\vdots \\ \sum_{g \in \{0, \dots, G\} \cap \tilde{S}} B_s(M_{s-1}^T + N_s^T - 1, g) \vec{\alpha}(g) &= \vec{a}(M_{s-1}^T + N_s^T - 1). \end{aligned}$$

From the first  $G'$  equations we can find coefficients  $\vec{\alpha}_0, \dots, \vec{\alpha}_{G'}$  and from the next equations we can find matrices  $P(g), g \in \tilde{S} \setminus \{0, \dots, G\}$ .

If the sequence of matrices  $P(g), g \in \tilde{S}$ , is such that  $P(g) = P^*(g), g \in \tilde{S}$ , the matrix function

$$\sum_{g=-(M_{s-1}^T + N_s^T - 1)}^{M_{s-1}^T + N_s^T - 1} P(g)e^{ig\lambda}$$

is positive-definite and has the determinant which does not equal zero identically, then the least favorable spectral density  $f^0(\lambda)$  is defined by the formula

$$\begin{aligned} f^0(\lambda) &= \left( \sum_{g=-(M_{s-1}^T + N_s^T - 1)}^{M_{s-1}^T + N_s^T - 1} P(g)e^{ig\lambda} \right)^{-1} \\ &= \left( \left( \sum_{g=0}^{M_{s-1}^T + N_s^T - 1} Q(g)e^{-ig\lambda} \right) \left( \sum_{g=0}^{M_{s-1}^T + N_s^T - 1} Q(g)e^{-ig\lambda} \right)^* \right)^{-1} \end{aligned} \tag{35}$$

and is the density of the vector stochastic autoregression sequence of order  $M_{s-1}^T + N_s^T - 1$

$$\sum_{g=0}^{M_{s-1}^T + N_s^T - 1} Q(g)\vec{\zeta}(l-g) = \vec{\varepsilon}_l. \tag{36}$$

Thus, the following theorem holds true.

**Theorem 5**

The spectral density (32) of the vector stochastic autoregression sequence (33) of order  $G$ , that is determined by matrices  $P(g), g \in \{0, 1, \dots, G\}$ , is the least favorable in the class  $D_G^-$  for the optimal estimation of the functional  $A_s \vec{\zeta}$  in the case where  $G \geq M_{s-1}^T + N_s^T - 1$ . If  $G < M_{s-1}^T + N_s^T - 1$  and solutions  $P(g), g \in \tilde{S} \cap \{0, 1, \dots, G\}$ , of the equation  $\mathbf{B}_s \vec{\alpha}_G^0 = \vec{a}_s^{\vec{\zeta}}$  with coefficients  $P(g), g \in \tilde{S} \setminus \{0, 1, \dots, G\}$ , form a positive-definite matrix function  $\sum_{g=-(M_{s-1}^T + N_s^T - 1)}^{M_{s-1}^T + N_s^T - 1} P(g) e^{ig\lambda}$ , with the determinant which does not equal zero identically, then the spectral density (35) of the vector stochastic autoregression sequence (36) of order  $M_{s-1}^T + N_s^T - 1$  is the least favorable in the class  $D_G^-$ . The minimax spectral characteristic  $h(f^0)$  is calculated by the formula (13).

**9. Conclusion**

We propose formulas for calculating the mean square error and the spectral characteristic of the optimal linear estimate of the functional

$$A_s \zeta = \sum_{l=0}^{s-1} \sum_{j=M_l+1}^{M_l+N_{l+1}} a(j) \zeta(j), \quad M_l = \sum_{k=0}^l (N_k + K_k), \quad N_0 = K_0 = 0,$$

which depends on the unobserved values of a periodically correlated stochastic sequence  $\zeta(j)$ . Estimates are based on observations of the sequence  $\zeta(j) + \theta(j)$  at points  $j \in \mathbb{Z} \setminus S$ , where  $S = \bigcup_{l=0}^{s-1} \{M_l + 1, \dots, M_l + N_{l+1}\}$ . The sequence  $\theta(j)$  is uncorrelated with  $\zeta(j)$  periodically correlated stochastic sequence. This problem is investigated in two cases. In the first case the spectral density matrices  $f(\lambda)$  and  $g(\lambda)$  of the  $T$ -variate stationary sequences  $\vec{\zeta}(n)$  and  $\vec{\theta}(n)$ , obtained by  $T$ -blocking of  $T$ -PC sequences  $\zeta(j)$  and  $\theta(j)$ , respectively, are supposed to be known exactly. In this case we derived formulas for calculating the spectral characteristic and the mean-square error of the optimal estimate of the functional. In the second case where the spectral density matrices are not exactly known while a class  $D = D_f \times D_g$  of admissible spectral densities is given. Formulas that determine the least favorable spectral densities and the minimax spectral characteristic of the optimal estimate of the functional  $A_s \zeta$  are proposed. The problem is investigated in details for two special classes of admissible spectral densities. Some examples of application of the obtained results for finding optimal estimates of linear functionals and determining the least favorable spectral densities of the optimal estimates are presented.

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