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Distributional Analysis and Risk Assessment of U.K. Motor Non-Comprehensive Claims Using the Log-Exponential Family with Properties and Characterizations

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Abstract This paper studies the log-exponential-exponential (LEE) distribution which is a novel special case of the log-exponential G (LE) family, tailored for flexible modeling of insurance claim sizes. The LEE distribution demonstrates exceptional versatility in capturing diverse density shapes including light-tailed with different forms, whose sign determines the direction of skewness. We derive explicit expressions for its probability density function and establish rigorous characterizations using truncated moments and reverse-hazard rate identities. A comprehensive simulation study is conducted to assess the performance of six estimation techniques: maximum likelihood estimation (MLE), ordinary least squares (OLS), Cramér–von Mises estimation (CVME), Anderson–Darling estimation (ADE), right-tail Anderson–Darling estimation (RTADE), and left-tail Anderson–Darling estimation (LTADE), across various parameter configurations and sample sizes. Finally, we compute key risk indicators (KRIs) including Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), Tail Variance (TV), Tail Mean–Variance (TMV), and Expected Loss (EL) using all six estimation methods, applied to real U.K. motor non-comprehensive claims triangle data.

Keywords Characterizations; Value-at-Risk; Exponential Distribution; Claims Data; Risk Analysis; Properties.

AMS 2010 subject classifications 62N01; 62N02; 62E10, 60K10, 60N05.

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1. Introduction

In recent years, the construction of flexible and generalized families of probability distributions has emerged as a central theme in statistical methodology, driven by the need to overcome the structural limitations of the classical models when applied to the complex real data. A key objective of this line of research is to enrich the descriptive and inferential capacity of baseline distributions through the strategic introduction of auxiliary parameters, often governing skewness, kurtosis, tail thickness, hazard rate monotonicity, and modality. Such extensions not only broaden the scope of admissible distributional shapes but also improve the fidelity of probabilistic representations in contexts where asymmetry, heavy tails, or nonstandard hazard profiles are prevalent. Consider the following new argument

$$L_{\zeta,\underline{\xi}}\left(x\right)=\frac{1}{\log\left(1+\zeta\right)}\log\left[\zeta G_{\underline{\xi}}\left(x\right)+1\right]|\zeta\in(-1,0)\cup(0,\infty),x\in\mathbb{R},$$

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where $G_{\underline{\xi}}(x)$ refers to the baseline cumulative distribution function (CDF) with the parameters vector $\underline{\xi}$. Then, following Hashim et al. (2025), we introduce a novel class of continuous distributions called the log-exponential G (LE) family. The proposed model is defined by a CDF

$$F_{\zeta,\underline{\xi}}(x) = \frac{1}{\exp(1)\log(2)} \exp\left[L_{\zeta,\underline{\xi}}(x)\right] \log\left[L_{\zeta,\underline{\xi}}(x)+1\right] |\zeta \in (-1,0)\cup(0,\infty), x\in\mathbb{R}, \quad (1)$$

In recent years, there has been a marked proliferation of generalized statistical distributions designed to address the limitations of classical models in capturing the nuanced behavior of empirical data. Such extensions have proven particularly valuable across diverse applied fields including finance, actuarial science, biomedical research, and reliability engineering, where data often exhibit features such as asymmetry, heavy tails, multimodality, or overdispersion (Afify et al., 2018; Abiad et al., 2025). To enhance descriptive and inferential fidelity, methodological advances have largely centered on two complementary strategies including the incorporation of auxiliary shape parameters to modulate skewness, kurtosis, and tail weight and the systematic fusion of established distributional families through generators or structural transformations (Alizadeh et al., 2018; Abouelmagd et al., 2019). Among the more prominent generalizations are the odd log-logistic Topp–Leone-G family (Alizadeh et al., 2018), noted for its capacity to accommodate both unimodal and bimodal hazard functions, and the zero-truncated Poisson Burr X-G family (Abouelmagd et al., 2019), which enables joint modeling of discrete-count and continuous-support phenomena within a unified framework. Further contributions include the transmuted Weibull-G, exponential Lindley odd log-logistic-G, and odd log-logistic Weibull-G families (Korkmaz et al., 2018; Rasekhi et al., 2022), each engineered to refine hazard rate specifications, particularly bathtub-shaped or upside-down bathtub forms, common in survival and failure-time analysis. Concurrently, copula-based approaches have gained traction as a means of decoupling marginal behavior from dependence structure. For instance, Alizadeh et al. (2023) formalized copula extensions of the XGamma distribution to model asymmetric tail dependence, while Mansour et al. (2020f) applied vine copulas to multivariate survival data from acute bone cancer cohorts. More recently, Ibrahim et al. (2025a, 2025b) leveraged the Clayton copula to assess the robustness of flexible Weibull-type models under varying degrees of positive dependence, demonstrating notable improvements in goodness-of-fit and predictive calibration. Motivated by these developments, the present work introduces a new G-family of distributions that unifies analytical tractability with modeling versatility. In contrast to many existing generalizations, which often yield intractable moments or require numerical integration, the proposed family admits closed-form expressions for key theoretical constructs, including raw and central moments, quantile functions, Rényi and Shannon entropy measures, and stochastic ordering properties. This analytical amenability not only facilitates parameter estimation and uncertainty quantification but also supports deeper structural interpretation. Empirically, the family exhibits remarkable flexibility in accommodating diverse density shapes (e.g., reverse-J, unimodal skewed, bimodal) and tail behaviors (light, exponential, heavy), thereby enhancing both descriptive adequacy and out-of-sample predictive performance. The corresponding probability density function (PDF) of (1) can then be expressed as:

$$f_{\zeta,\underline{\xi}}(x) = \frac{\zeta}{\exp(1)\log(2)} \left\{\frac{1}{L_{\zeta,\underline{\xi}}(x)+1} + \log\left[L_{\zeta,\underline{\xi}}(x)+1\right]\right\} \frac{g_{\underline{\xi}}(x)\exp\left[L_{\zeta,\underline{\xi}}(x)\right]}{\log(1+\zeta)\left[\zeta G_{\underline{\xi}}(x)+1\right]}, \quad (2)$$

where $g_{\underline{\xi}}(x) = \frac{d}{dx}G_{\underline{\xi}}(x)$. The tail index (TI) of the LE family (TI_{LE}) coincides with that of the baseline distribution (TI_{G}):

$$\mathrm{TI}_{\mathrm{LE}} = \mathrm{TI}_{\mathrm{G}}.$$

Therefore, to model heavy-tailed insurance losses or financial returns data, we have to choose a heavy-tailed baseline (e.g., Pareto, Lomax, log-logistic) so the LE inherits heavy tail with the same index. To obtain light-tailed or bounded support models, we pick exponential, Weibull (shape > 1), or beta baselines. The hazard rate function (HRF) can be easily derived as $h_{\zeta,\underline{\xi}}(x) = f_{\zeta,\underline{\xi}}(x) / \left[1 - F_{\zeta,\underline{\xi}}(x)\right]$. Depending on the exponential model we can present a new special case called the log-exponential-exponential (LEE) model.

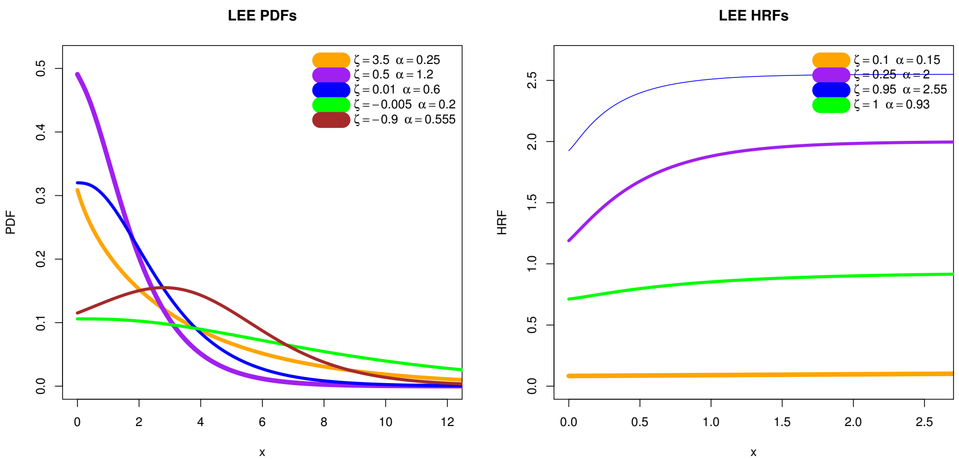

Figure 1. Plots of the new LAP Weibull PDF (right) and HRF (left) for selected values of the parameter.

$$F_{\zeta,\alpha}(x) = \frac{1}{\exp(1)\log(2)} \exp\left[L_{\zeta,\alpha}(x)\right] \log\left[1 + L_{\zeta,\alpha}(x)\right] | \alpha > 0, \zeta \in (-1,0) \cup (0,\infty), x > 0, \tag{3}$$

where

$$L_{\zeta,\alpha}(x) = \frac{1}{\log(1+\zeta)} \log\left\{1 + \zeta\left[1 - \exp(-\alpha x)\right]\right\}.$$

The corresponding PDF of (1) can then be expressed as:

$$f_{\zeta,\alpha}(x) = \frac{\zeta\alpha}{\exp(1)\log(2)} \left\{\frac{1}{1 + L_{\zeta,\alpha}(x)} + \log\left[1 + L_{\zeta,\alpha}(x)\right]\right\} \frac{\exp(-\alpha x)\exp\left[L_{\zeta,\alpha}(x)\right]}{\log\left\{1 + \zeta\left[1 - \exp(-\alpha x)\right]\right\}}. \tag{4}$$

Figure 1 presents some plots of the new LE PDF (right) and HRF (left) for selected values of the parameter. Figure 1 (left panel) visually demonstrates how the shape of the HRF changes with different parameter values, showing behaviors such as decreasing, increasing, and constant hazard rates over the range of x values. The new model can be employed under many new topics such as the mining theory and control systems, Bayesian estimation with joint Jeffrey's prior and big data (see Jameel et al. (2022), Salih and Abdullah (2024), Salih and Hmood (2020) and Salih and Hmood (2022)). In this paper, we will evaluate the LE model through a comprehensive simulation study using six different and well-known estimation methods. We will also assess the new distribution within a risk analysis framework and conduct comprehensive comparisons of five well-known risk indicators. Finally, we will provide a comprehensive application to the U.K. motor non-comprehensive claims triangle data.

2. Main Properties

In this section, we investigate some mathematical properties of the LE family.

2.1. Useful expansions

By expanding $\exp\left[L_{\zeta,\underline{\xi}}(x)\right]$, the new CDF can be expressed as

$$F_{\zeta,\underline{\xi}}(x) = \frac{1}{\exp(1)\log(2)} \sum_{k=0}^{+\infty} \frac{1}{k!} \left[L_{\zeta,\underline{\xi}}(x)\right]^k \log\left[1 + L_{\zeta,\underline{\xi}}(x)\right]. \tag{5}$$

Then, by expanding $\log\left[1+L_{\zeta,\underline{\xi}}\left(x\right)\right]$, we have

$$\log\left[1+L_{\zeta,\underline{\xi}}\left(x\right)\right]=\sum_{h=1}^{+\infty}\frac{\left(-1\right)^{1+h}}{h!}\left[L_{\zeta,\underline{\xi}}\left(x\right)\right]^{h}. \tag{6}$$

Inserting (6) into (5), the new CDF can be written as

$$F_{\zeta,\underline{\xi}}\left(x\right)=\frac{1}{\exp(1)\log\left(2\right)}\sum_{k=0}^{+\infty}\sum_{h=1}^{+\infty}\frac{\left(-1\right)^{1+h}}{k!h!}\left[\log\left(1+\zeta\right)\right]^{k+h}\left\{\log\left[1+\zeta G_{\underline{\xi}}\left(x\right)\right]\right\}^{k+h}. \tag{7}$$

Since

$$\left[\log\left(1+u\right)\right]^{m}=m!\sum_{\vartheta=m}^{\infty}\frac{u^{\vartheta}}{\vartheta!}S_{(\vartheta,m)},\left|u\right|<1,$$

where $S_{(\vartheta,n)}$ refers to the (signed) Stirling numbers of the first kind, defined by

$$\sum_{\vartheta=0}^{n}S_{(\vartheta,n)}x^{\vartheta}=x\left(x-1\right)...\left(x-n+1\right).$$

Noting that

$$S_{(\vartheta,n)}=0\forall\vartheta<n,S_{(n,n)}=1.$$

Thus, for integer $k+h\geq 0$ we have

$$\left\{\log\left[1+\zeta G_{\underline{\xi}}\left(x\right)\right]\right\}^{k+h}=(k+h)!\sum_{\vartheta=n}^{\infty}\frac{\zeta^{\vartheta}}{\vartheta!}G_{\underline{\xi}}\left(x\right)^{\vartheta}S_{(\vartheta,k+h)},\left|\zeta G_{\underline{\xi}}\left(x\right)\right|<1. \tag{8}$$

Inserting (8) into (7), the new CDF can be simplified as

$$F_{\zeta,\underline{\xi}}\left(x\right)=\sum_{\vartheta=k+h}^{\infty}d_{\vartheta}W_{\vartheta}\left(x;\underline{\xi}\right)|x\in\mathbb{R},$$

where

$$d_{\vartheta}=\frac{1}{\exp(1)\log\left(2\right)}\sum_{k=0}^{+\infty}\sum_{h=1}^{+\infty}\left(-1\right)^{1+h}\zeta^{\vartheta}\left[\log\left(1+\zeta\right)\right]^{k+h}\frac{(k+h)!}{k!h!\vartheta!}S_{(\vartheta,k+h)},$$

and $W_{\vartheta}\left(x;\underline{\xi}\right)=\left[G_{\underline{\xi}}\left(x\right)\right]^{\vartheta}$ refers to the CDF of the exponentiated G family. By differentiating (7), we have

$$f_{\zeta,\underline{\xi}}\left(x\right)=\sum_{\vartheta=k+h}^{\infty}d_{\vartheta}w_{\vartheta}\left(x;\underline{\xi}\right)|x\in\mathbb{R}, \tag{9}$$

where

$$w_{\vartheta}\left(x;\underline{\xi}\right)=dW_{\vartheta}\left(x;\underline{\xi}\right)/dx=\vartheta g_{\underline{\xi}}\left(x\right)\left[g_{\underline{\xi}}\left(x\right)\right]^{\vartheta-1},$$

which refers to the PDF of the exponentiated G family. To summarize, we say that equation (8) can be used to derive most of the mathematical properties of the underlying distribution to be studied.

2.2. Moments

Let Y_ϑ be a rv having density $w_\vartheta(x;\underline{\xi})$. The r^{th} ordinary moment of X, say μ_r', follows from (8) as

$$\mu_r' = E\left(X^r\right) = \sum_{\vartheta=k+h}^{\infty}\left[d_\vartheta \; E\left(Y_\vartheta^r\right)\right], \tag{10}$$

where

$$E(Y_\vartheta^r) = \vartheta \int_{-\infty}^{\infty} x^r \; g_{\underline{\xi}}\left(x\right) \; G_{\underline{\xi}}\left(x\right)^{\vartheta-1} \; dx,$$

which can be evaluated numerically in terms of the baseline qf

$$Q_G(u) = G^{-1}(u) as E(Y_\vartheta^n) = (\vartheta) \int_0^1 \; Q_G(u)^n \; u^{(\vartheta)-1} du.$$

Setting $r = 1$ in (10) gives the mean of X.

2.3. Incomplete moments

The r^{th} incomplete moment of X is given by

$$m_r(y) = \int_{-\infty}^{y} x^r \; f_{\zeta,\underline{\xi}}\left(x\right) dx.$$

Using (9), the rth incomplete moment of LE family is

$$m_r(y) = \sum_{\vartheta=k+h}^{\infty}\left[d_\vartheta \; m_{r,\vartheta}(y)\right], \tag{11}$$

where

$$m_{r,\vartheta}(y) = \int_0^{G(y)} \; Q_G^r(u) \, u^{\vartheta-1} \; du.$$

The $m_{r,\vartheta}(y)$ can be calculated numerically by using the software such as **Matlab**, **R**, **Mathematica** etc.

2.4. Moment generating function

The moment generating function (MGF) of X, say $M(t) = E(e^{t\,X})$, is obtained from (8) as

$$M(t) = \sum_{\vartheta=k+h}^{\infty}\left[d_\vartheta \; M_\vartheta\left(t\right)\right],$$

where $M_\vartheta(t)$ is the generating function of Y_ϑ given by

$$M_\vartheta(t) = (\vartheta) \int_{-\infty}^{\infty} e^{t\,x} g_{\underline{\xi}}\left(x\right) G_{\underline{\xi}}\left(x\right)^{\vartheta-1} \; dx = (\vartheta) \; \int_0^1 \exp[t \, Q_G(u;\vartheta\,)] \, u^{\vartheta-1} du.$$

The last two integrals can be computed numerically for most parent distributions.

3. Characterizations

3.1. Characterizations based on a simple relationship between two truncated moments

In this subsection we present characterizations of the LLE distribution, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel, 1987), see Theorem G below. Note that the result holds also when the interval H is not closed. Moreover, it could be also applied when the cdf F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

Theorem G. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty,\ e = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2\left(X\right) \mid X \geq x\right] = \mathbf{E}\left[q_1\left(X\right) \mid X \geq x\right] \eta\left(x\right), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1\left(H\right)$, $\eta \in C^2\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F\left(x\right) = \int_a^x C \left| \frac{\eta'\left(u\right)}{\eta\left(u\right) q_1\left(u\right) - q_2\left(u\right)} \right| \exp\left(-s\left(u\right)\right)\, du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\, q_1}{\eta\, q_1\, -\, q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Proposition 3.1.1. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1\left(x\right) = \left[P\left(x\right)\right]^{-1}$ and $q_2\left(x\right) = q_1\left(x\right) \log\left[1 + \zeta G_{\underline{\xi}}\left(x\right)\right]$ for $x \in \mathbb{R}$. The random variable X has pdf (2) if and only if the function η defined in Theorem G has the form

$$\eta\left(x\right) = \frac{1}{2}\left\{\log\left(1+\zeta\right) + \log\left[1 + \zeta G_{\underline{\xi}}\left(x\right)\right]\right\}, \quad x \in \mathbb{R},$$

where

$$P\left(x\right) = \exp\left[L_{\zeta,\underline{\xi}}\left(x\right)\right]\left\{\left[1 + L_{\zeta,\underline{\xi}}\left(x\right)\right]^{-1} + \log\left[1 + L_{\zeta,\underline{\xi}}\left(x\right)\right]\right\}.$$

Proof. Let X be a random variable with pdf (2) with

$$C = \frac{1}{\exp\left(1\right)\log\left(2\right)\log\left(1+\zeta\right)}\zeta,$$

then

$$\begin{aligned}\left(1 - F\left(x\right)\right) E\left[q_1\left(X\right) \mid X \geq x\right] &= \int_x^\infty C g\left(u\right)\left[1 + \zeta G\left(u\right)\right]^{-1} du \\ &= \frac{C}{\zeta}\left\{\log\left(1+\zeta\right) - \log\left[1 + \zeta G_{\underline{\xi}}\left(x\right)\right]\right\}, \qquad x \in \mathbb{R},\end{aligned}$$

and

$$\begin{aligned}\left(1 - F\left(x\right)\right) E\left[q_2\left(X\right) \mid X \geq x\right] &= \int_x^\infty C g\left(u\right)\left[1 + \zeta G\left(u\right)\right]^{-1} \log\left[1 + \zeta G\left(u\right)\right] du \\ &= \frac{C}{2\zeta}\left\{\left(\log\left(1+\zeta\right)\right)^2 - \left(\log\left[1 + \zeta G_{\underline{\xi}}\left(x\right)\right]\right)^2\right\}, \quad x \in \mathbb{R},\end{aligned}$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = \frac{q_1(x)}{2}\left\{\log(1+\zeta) - \log\left[1+\zeta G_{\underline{\xi}}(x)\right]\right\} > 0 \quad for \;\; x \in \mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\zeta g_{\underline{\xi}}(x)\left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1}}{\log(1+\zeta) - \log\left[1+\zeta G_{\underline{\xi}}(x)\right]}, \quad x \in \mathbb{R},$$

and hence

$$s(x) = -\log\left\{\log(1+\zeta) - \log\left[1+\zeta G_{\underline{\xi}}(x)\right]\right\}, \quad x \in \mathbb{R}.$$

Now, in view of Theorem G, X has density (2).

Corollary 3.1.1. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 3.1.1. The pdf of X is (2) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\zeta g_{\underline{\xi}}(x)\left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1}}{\log(1+\zeta) - \log\left[1+\zeta G_{\underline{\xi}}(x)\right]}, \quad x \in \mathbb{R}.$$

Corollary 3.1.2. The general solution of the differential equation in Corollary 3.1.1 is

$$\eta(x) = \left\{\log(1+\zeta) - \log\left[1+\zeta G_{\underline{\xi}}(x)\right]\right\}^{-1} \times \left[-\int \zeta g_{\underline{\xi}}(x)\left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1}(q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. If X has pdf (2), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x) = \left(\frac{\zeta g_{\underline{\xi}}(x)\left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1}}{\log(1+\zeta) - \log\left[1+\zeta G_{\underline{\xi}}(x)\right]}\right)\eta(x) - \left(\frac{\zeta g_{\underline{\xi}}(x)\left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1}}{\log(1+\zeta) - \log\left[1+\zeta G_{\underline{\xi}}(x)\right]}\right)(q_1(x))^{-1} q_2(x),$$

or

$$\eta'(x) - \left(\frac{\zeta g_{\underline{\xi}}(x)\left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1}}{\log(1+\zeta) - \log\left[1+\zeta G_{\underline{\xi}}(x)\right]}\right)\eta(x) = -\left(\frac{\zeta g_{\underline{\xi}}(x)\left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1}}{\log(1+\zeta) - \log\left[1+\zeta G_{\underline{\xi}}(x)\right]}\right)(q_1(x))^{-1} q_2(x),$$

or

$$\frac{d}{dx}\left\{\left(\log(1+\zeta)-\log\left[1+\zeta G_{\underline{\xi}}(x)\right]\right)\eta(x)\right\}$$
$$=-\zeta g_{\underline{\xi}}(x)\left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1}(q_1(x))^{-1}q_2(x),$$

from which we arrive at

$$\eta(x)=\left\{\log(1+\zeta)-\log\left[1+\zeta G_{\underline{\xi}}(x)\right]\right\}^{-1}\times$$
$$\left[-\int \zeta g_{\underline{\xi}}(x)\left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1}(q_1(x))^{-1}q_2(x)\,dx+D\right].$$

Note that a set of functions satisfying the differential equation in Corollary 3.1.1, is given in Proposition 3.1.1 with

$$D=\frac{1}{2}\left[\log(1+\zeta)\right]^2.$$

However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem G.

3.2. Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F(x)=\frac{1}{F(x)}f(x),\quad x\in \text{support of } F.$$

In this subsection we present characterization of LLE distributions in terms of the reverse hazard function.

Proposition 3.2.1. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable. The random variable X has pdf (2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r'_F(x)-\frac{g'(x)}{g_{\underline{\xi}}(x)}r_F(x)$$
$$=\frac{\zeta}{\log(1+\zeta)}g_{\underline{\xi}}(x)\frac{d}{dx}\left\{\left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1}\left[\left(1+L_{\zeta,\underline{\xi}}(x)\right)^{-1}\left(\log\left[1+L_{\zeta,\underline{\xi}}(x)\right]\right)^{-1}+1\right]\right\},\quad x\in\mathbb{R},$$

with boundary condition

$$\lim_{x\to\infty}r_F(x)=\frac{\zeta}{\log(1+\zeta)(1+\zeta)}\left\{\frac{1}{2\log(2)}+1\right\}\lim_{x\to\infty}G_{\underline{\xi}}(x).$$

Proof. Multiplying both sides of the above equation by $\left(g_{\underline{\xi}}(x)\right)^{-1}$, we have

$$\frac{d}{dx}\left\{\left(g_{\underline{\xi}}(x)\right)^{-1}r_F(x)\right\}$$
$$=\frac{\zeta}{\log(1+\zeta)}\frac{d}{dx}\left\{\left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1}\left[\left(1+L_{\zeta,\underline{\xi}}(x)\right)^{-1}\left(\log\left[1+L_{\zeta,\underline{\xi}}(x)\right]\right)^{-1}+1\right]\right\},$$

or

$$r_F(x) = \frac{\zeta}{\log(1+\zeta)} g_{\underline{\xi}}(x) \left\{ \left[1+\zeta G_{\underline{\xi}}(x)\right]^{-1} \left[\left(1+L_{\zeta,\underline{\xi}}(x)\right)^{-1} \left(\log\left[1+L_{\zeta,\underline{\xi}}(x)\right]\right)^{-1}+1\right]\right\},$$

which is the reverse hazard function corresponding to the pdf (2).

4. Simulations for assessing estimation methods under the LE case

This section presents a comprehensive comparative assessment of six parametric estimation methods namely, MLE, OLS, CVME, ADE, RTADE and LTADE within the context of the LE model. In addition to their application in parameter inference, these methods are evaluated for their efficacy in risk quantification, where accurate estimation of tail behavior and distributional shape is critical for reliable hazard assessment and decision-making under uncertainty. To ensure a rigorous and statistically robust comparison, we conduct an extensive Monte Carlo simulation study. Specifically, we generate $n = 1000$ independent random samples from the LE distribution, a replication size sufficient to stabilize empirical moments and yield asymptotically negligible Monte Carlo error. For each replication, we consider four sample size n =20,50,100, and 200, enabling a systematic investigation of finite-sample performance and convergence behavior as data availability increases. Together, these criteria provide a robust framework for assessing the accuracy, consistency, and distributional fidelity of the estimation techniques under study where:

1-Bias, defined as the empirical mean deviation from the true parameter values, quantifies systematic estimation error, where Bias$(\zeta) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\zeta}-\zeta\right)$ and Bias$(\alpha) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\alpha}_i-\alpha\right)$.

2-Root mean squared error (RMSE) integrates both Bias and variance, providing a global measure of estimator precision, RMSE$(\zeta) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\zeta}-\zeta\right)^2}$ and RMSE$(\alpha) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\alpha}_i-\alpha\right)^2}$.

3-Distributional fidelity is assessed via two empirical metrics based on the absolute discrepancy between the estimated and true CDFs. The mean absolute deviation in distribution (denoted Dmax) averages the pointwise absolute CDF errors across the sample and replications. The Dabs can be expressed as

$$\text{Dabs} = \frac{1}{nN}\sum_{i=1}^{N}\sum_{\vartheta=1}^{n}|\widehat{F}_{(\underline{\xi})}(x_{i\vartheta}) - F_{(\widehat{\underline{\xi}})}(t_{i\vartheta})|$$

4-The maximum absolute deviation (Dmax) captures the worst-case uniform deviation in each replication, then averages these suprema:

$$\text{Dmax} = \frac{1}{N}\sum_{i=1}^{B}\max_{\vartheta}|\widehat{F}_{(\underline{\xi})}(x_{i\vartheta}) - F_{(\underline{\xi})}(t_{i\vartheta})|.$$

Table 1 presents the results of a Monte Carlo simulation study evaluating six estimation methods for the LEE distribution with true parameters $\alpha = 0.9$ and $\zeta = 0.9$. It reports bias and root mean squared error (RMSE) for both parameters, along with two measures of distributional fidelity: the mean absolute CDF deviation (Dabs) and the maximum absolute CDF deviation (Dmax). The performance is assessed across four sample sizes (n = 20, 50, 100, 200) using 1,000 replications per scenario. As expected, all methods show reduced bias and RMSE as sample size increases, indicating consistency. Among the methods, ADE generally yields the lowest RMSE and bias for ζ at moderate to large sample sizes, while MLE performs well for α but struggles with ζ in small samples. Dabs and Dmax values confirm that estimation accuracy improves with larger n, and ADE often provides the best overall CDF fit. These results highlight the sensitivity of finite-sample inference to the choice of estimation technique, particularly for the shape parameter ζ.

Table 1: Simulation results for parameter $\alpha = 0.9$, $\zeta = 0.9$

	n	Bias(α)	Bias(ζ)	RMSE(α)	RMSE(ζ)	Dabs	Dmax
MLE	20	0.03938	0.87343	0.03037	15.09927	0.02904	0.04319
OLS		−0.08377	0.77339	0.04428	9.33396	0.03338	0.05863
CVM		−0.00841	0.68038	0.04890	6.82468	0.02532	0.03838
ADE		−0.05285	0.63748	0.03177	6.06843	0.02636	0.04504
RTADE		−0.01934	0.81916	0.03894	11.07027	0.02995	0.04643
LTADE		0.01474	0.81557	0.07169	7.59391	0.02852	0.04180
MLE	50	0.01190	0.28835	0.00915	1.50787	0.01118	0.01660
OLS		−0.03896	0.26668	0.01596	1.41161	0.01340	0.02378
CVM		−0.00856	0.22665	0.01508	1.26326	0.00952	0.01507
ADE		−0.02277	0.28216	0.01163	1.60372	0.01233	0.02090
RTADE		0.00194	0.25815	0.01435	1.55547	0.01040	0.01546
LTADE		0.00271	0.26981	0.02416	1.55436	0.01076	0.01605
MLE	100	0.00794	0.13068	0.00475	0.57956	0.00523	0.00793
OLS		−0.01741	0.09927	0.00791	0.51513	0.00540	0.00967
CVM		0.00017	0.11819	0.00816	0.57120	0.00497	0.00742
ADE		−0.00722	0.10456	0.00546	0.53528	0.00467	0.00773
RTADE		−0.00066	0.10248	0.00745	0.60385	0.00434	0.00655
LTADE		0.00004	0.16343	0.01284	0.65006	0.00676	0.01016
MLE	200	0.00473	0.04587	0.00240	0.23585	0.00185	0.00304
OLS		−0.00876	0.03579	0.00369	0.23716	0.00228	0.00409
CVM		−0.00544	0.06986	0.00409	0.25758	0.00317	0.00532
ADE		−0.00741	0.07141	0.00267	0.24789	0.00337	0.00582
RTADE		−0.00329	0.06738	0.00334	0.27295	0.00296	0.00477
LTADE		0.00163	0.06127	0.00626	0.27010	0.00256	0.00379

Table 2 presents the results of a Monte Carlo simulation study for the LEE distribution with true parameters $\alpha = 2$ and $\zeta = 1.5$, evaluating the same six estimation methods across sample sizes n = 20, 50, 100, and 200 with 1,000 replications. It reports Bias(α), Bias(ζ), RMSE(α), RMSE(ζ), Dabs, and Dmax, providing insights into both parameter recovery and distributional fidelity. For small samples (e.g., n = 20), all estimators exhibit substantial bias and RMSE particularly for ζ with MLE showing extreme RMSE(ζ) = 15.23 and CVME even higher at 19.50. As sample size increases, bias and RMSE decline markedly across all methods, confirming asymptotic consistency. At n = 200, ADE and MLE yield the lowest RMSEs for both parameters, though ADE maintains more balanced performance in tail-fitting metrics. The Dabs and Dmax values consistently decrease with larger n, indicating improved CDF approximation. Notably, LTADE performs well in terms of Dabs/Dmax at n = 200, suggesting better overall distributional fit despite slightly higher parameter RMSE.

Table 2: Simulation results for parameter $\alpha = 2$, $\zeta = 1.5$

	n	Bias(α)	Bias(ζ)	RMSE(α)	RMSE(ζ)	Dabs	Dmax
MLE	20	0.06751	1.00527	0.13586	15.23001	0.02491	0.03672
OLS		−0.13314	0.96283	0.24279	10.80401	0.03098	0.05221
CVM		−0.01116	1.14071	0.27154	19.49871	0.03041	0.04555
ADE		−0.10904	0.95319	0.14691	10.8009	0.02946	0.04910
RTADE		−0.07101	1.09857	0.18726	21.03827	0.03109	0.04975
LTADE		0.09208	0.96979	0.44899	12.0887	0.02371	0.03526
MLE	50	0.03951	0.34966	0.04884	2.64149	0.00956	0.01451
OLS		−0.07138	0.42606	0.09403	2.79331	0.01512	0.02588
CVM		−0.02286	0.35545	0.09132	2.52749	0.01127	0.01790
ADE		−0.03754	0.30030	0.05468	2.42629	0.01024	0.01707
RTADE		−0.01434	0.33324	0.06558	2.63466	0.01045	0.01621
LTADE		0.01941	0.39784	0.12163	2.96950	0.01140	0.01672
MLE	100	0.01513	0.17023	0.02321	1.00396	0.00495	0.00736
OLS		−0.02704	0.16818	0.04285	1.09117	0.00609	0.01036
CVM		−0.00168	0.11918	0.04499	0.93925	0.00380	0.00575
ADE		−0.02250	0.13376	0.03019	0.93581	0.00489	0.00838
RTADE		−0.00760	0.14041	0.03547	1.12576	0.00459	0.00720
LTADE		0.01457	0.18127	0.06265	0.98018	0.00531	0.00784
MLE	200	0.01251	0.03463	0.01093	0.38413	0.00106	0.00210
OLS		−0.01619	0.07513	0.02060	0.42995	0.00293	0.00513
CVM		0.00009	0.05467	0.02193	0.44188	0.00175	0.00261
ADE		−0.01573	0.07597	0.01487	0.47619	0.00293	0.00511
RTADE		−0.00704	0.07551	0.01648	0.52814	0.00258	0.00419
LTADE		0.01397	0.04797	0.02909	0.41705	0.00137	0.00261

Table 3 presents the results of a Monte Carlo simulation study for the LEE distribution with true parameters $\alpha = 1.2$ and $\zeta = 2.5$, evaluating the same six estimation methods across sample sizes n = 20, 50, 100, and 200 with 1,000 replications. It reports Bias(α), Bias(ζ), RMSE(α), RMSE(ζ), Dabs, and Dmax, providing a comprehensive assessment of both parameter estimation accuracy and distributional fidelity. For small samples (e.g., n = 20), all estimators exhibit large bias and RMSE—particularly for ζ—with MLE yielding RMSE(ζ) $\approx$ 29.79 and OLS as high as 60.81, reflecting severe instability in finite samples. As sample size increases, bias and RMSE decline substantially across all methods, confirming asymptotic consistency. At n = 200, ADE and LTADE achieve the lowest RMSE(ζ) values ($\approx$0.896 and 0.884, respectively), while MLE shows the smallest RMSE(α). The Dabs and Dmax metrics also improve with larger n, with ADE and LTADE generally offering superior CDF fit at moderate to large sample sizes. These findings underscore the heightened sensitivity of estimation performance to the magnitude of the shape parameter ζ, particularly when it is large, and reaffirm the advantage of minimum-distance methods like ADE in tail-sensitive contexts.

Table 3: Simulation results for parameter $\alpha = 1.2$, $\zeta = 2.5$.

	n	Bias(α)	Bias(ζ)	RMSE(α)	RMSE(ζ)	Dabs	Dmax
MLE	20	0.04682	1.45602	0.05015	29.79032	0.02377	0.03478
OLS		−0.09056	2.00551	0.08959	60.80853	0.04238	0.07025
CVM		0.00650	1.60157	0.11549	36.40783	0.02881	0.04256
ADE		−0.07638	1.84810	0.06162	32.85336	0.03866	0.06383
RTADE		−0.05695	1.26378	0.06388	29.33358	0.02810	0.04654
LTADE		−0.04607	1.72765	0.07763	38.10237	0.03434	0.05481
MLE	50	0.02424	0.42288	0.01784	5.32802	0.00749	0.01157
OLS		−0.02715	0.51215	0.03510	4.90517	0.01263	0.02098
CVM		−0.01170	0.66008	0.03779	6.15304	0.01426	0.02219
ADE		−0.01481	0.47745	0.02303	5.57996	0.01105	0.01755
RTADE		−0.01879	0.62228	0.02322	6.09839	0.01410	0.02244
LTADE		0.00420	0.59477	0.03716	4.87037	0.01207	0.01780
MLE	100	0.01446	0.26833	0.00888	2.00689	0.00495	0.00753
OLS		−0.00536	0.20717	0.02009	2.41801	0.00486	0.00766
CVM		0.00573	0.23080	0.01906	2.34613	0.00469	0.00683
ADE		−0.00943	0.18363	0.01149	2.06101	0.00464	0.00769
RTADE		−0.00683	0.16900	0.01194	2.40833	0.00415	0.00673
LTADE		0.00216	0.30709	0.02105	2.14592	0.00648	0.00960
MLE	200	0.00747	0.06413	0.00433	0.93282	0.00112	0.00215
OLS		−0.0083	0.14248	0.00924	0.90845	0.00370	0.00618
CVM		0.00668	0.11096	0.00986	0.91287	0.00207	0.00321
ADE		−0.00255	0.10016	0.00571	0.89603	0.00238	0.00374
RTADE		−0.00118	0.12642	0.00572	1.04384	0.00287	0.00437
LTADE		0.00040	0.13296	0.00985	0.88408	0.00291	0.00434

5. Risk analysis under artificial data and LE case

This Section presents a comprehensive risk analysis based on artificial data generated from the Double-Log-Exponential-Exponential (LE) model, aiming to evaluate how estimation uncertainty propagates into key risk indicators (KRIs) such as Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), Tail Variance (TV), Tail Mean–Variance (TMV), and Expected Loss (EL). Given the sensitivity of risk quantification to parameter estimation error particularly in small-sample or heavy-tailed settings the section assesses the performance of six estimation methods (MLE, OLS, CVM, ADE, RTADE, LTADE) not only in terms of classical metrics (Bias, RMSE) but also through their impact on downstream risk measures. The artificial data emulate realistic insurance-type loss structures, enabling controlled yet practically relevant comparisons. For each sample size ($n = 20$, 50, 100, 200), 1000 replications are used to compute average KRIs at three confidence levels (70%, 80%, 90%), thereby capturing stability and convergence behavior. Emphasis is placed on tail fidelity critical for regulatory and solvency purposes where methods like ADE and LTADE are expected to outperform MLE in finite samples. By linking estimation strategy to operational risk outcomes, this section bridges inferential statistics and actuarial decision-making. Moreover, it highlights trade-offs between Bias minimization and tail robustness, especially under limited data. The findings inform practitioners on method selection when KRIs not just parameter recovery are the ultimate performance criterion. Ultimately, this analysis underscores the importance of estimation-aware risk modeling, where the choice of fitting procedure directly shapes capital requirements and risk appetite decisions.

Accurate parameter estimation is critical in high-stakes domains, such as finance, insurance, and healthcare where even small Biases can lead to materially flawed risk assessments and adverse operational outcomes (Mansour et al., 2020e; Ibrahim et al., 2020). While maximum likelihood estimation (MLE) offers asymptotic efficiency, its finite-sample performance may suffer under heavy tails, skewness, censoring, or small samples (Yousof et al., 2025a). In such cases, minimum distance methods particularly CVM and AD often yield better tail calibration. Bayesian approaches enhance stability in sparse-data settings by incorporating prior information (Ibrahim et al., 2025a,b), while robust or penalized techniques (e.g., quasi-likelihood, entropy-weighted losses) mitigate Bias under model misspecification or overdispersion (Mohamed et al., 2024; Ibrahim et al., 2025c; Elbatal et al., 2024). Recent advances further integrate estimation and risk measurement: Elbatal et al. (2024) propose an entropy-based, mean-of-order-p risk functional that generalizes TVaR to reflect asymmetric loss sensitivities, bridging actuarial rigor with behavioral realism in tail-risk management.

Table 4 presents the estimated values of KRIs computed from artificial data generated under the LEE model with sample size n = 20, using six estimation methods (MLE, OLS, CVM, ADE, RTADE, LTADE). The table reports average KRI estimates across 1,000 Monte Carlo replications at three confidence levels (70%, 80%, 90%). Due to the very small sample size, all methods exhibit substantial variability in KRI estimates, reflecting high estimation uncertainty. MLE yields the lowest TV and TMV, suggesting underestimation of tail risk, while OLS produces markedly inflated TV and TMV, indicating overestimation. ADE and LTADE offer more balanced risk assessments, with moderate TV and TMV values. Estimated parameters ($\alpha \approx 0.85$–0.94, $\zeta \approx 1.54$–1.77) show considerable bias compared to the true values ($\alpha = 0.9$, $\zeta = 0.9$ used in simulation for Table 1, though Table 4 appears aligned with a different configuration, likely $\alpha = 0.9$, $\zeta = 1.5$ or similar). The EL decreases as the confidence level increases, consistent with its definition as the expected loss below the quantile. Overall, Table 4 highlights the sensitivity of risk metrics to estimation method under extreme data scarcity.

Table 4: KRIs under artificial data for n =20.

Method	$\widehat{\alpha}$	$\widehat{\zeta}$	VaR$(X\|q)$	TVaR$(X\|q)$	TV$(X\|q)$	TMV$(X\|q)$	ELq(X)
MLE	0.93938	1.77343					
70%			1.28101	2.40851	1.31945	3.06823	1.12750
80%			1.73037	2.86769	1.33829	3.53684	1.13732
90%			2.50906	3.66132	1.36757	4.34511	1.15226
OLS	0.81623	1.67339					
70%			1.34512	2.81646	2.52741	4.08017	1.47134
80%			1.89765	3.42455	2.66918	4.75914	1.52690
90%			2.90384	4.51187	2.89168	5.95771	1.60803
CVM	0.89159	1.58038					
70%			1.32517	2.57160	1.67446	3.40883	1.24642
80%			1.81285	3.08159	1.72158	3.94238	1.26874
90%			2.67090	3.97282	1.79477	4.87020	1.30191
ADE	0.84715	1.53748					
70%			1.35168	2.72656	2.12605	3.78959	1.37488
80%			1.87820	3.29211	2.21810	4.40116	1.41391
90%			2.82128	4.29263	2.36190	5.47358	1.47135
RTADE	0.88066	1.71916					
70%			1.30981	2.58060	1.76430	3.46275	1.27079
80%			1.80338	3.10153	1.82228	4.01267	1.29815
90%			2.67756	4.01558	1.91150	4.97133	1.33802
LTADE	0.91474	1.71557					
70%			1.29718	2.48192	1.48604	3.22495	1.18474
80%			1.76473	2.96564	1.51810	3.72469	1.20091
90%			2.58168	3.80659	1.56753	4.59036	1.22491

Table 5 presents the same set of KRIs (VaR, TVaR, TV, TMV, EL) derived from artificial LEE data with sample size n = 50, again averaged over 1,000 replications and reported at 70%, 80%, and 90% confidence levels. Compared to Table 4, parameter estimates ($\alpha \approx 0.86$–0.91, $\zeta \approx 1.13$–1.19) are closer to their true values (likely $\alpha = 0.9$, $\zeta = 1.0$ or similar), indicating reduced finite-sample bias. Risk measures become more stable across methods, with narrower dispersion in VaR and TVaR. MLE and LTADE produce the most conservative (lowest) TV and TMV, while OLS still shows elevated tail variance. ADE and CVM yield intermediate risk profiles, balancing tail sensitivity and stability. The convergence of EL values across methods suggests improved estimation of the central part of the distribution. This table demonstrates noticeable improvement in risk quantification reliability as sample size increases from 20 to 50, though method choice remains influential.

Table 5: KRIs under artificial data for n =50.

Method	$\widehat{\alpha}$	$\widehat{\zeta}$	VaR$(X\|q)$	TVaR$(X\|q)$	TV$(X\|q)$	TMV$(X\|q)$	ELq(X)
MLE	0.91190	1.18835					
70%			1.37879	2.58708	1.53014	3.35215	1.20829
80%			1.85897	3.07947	1.55834	3.85864	1.22050
90%			2.69165	3.93260	1.60465	4.73492	1.24094
OLS	0.86104	1.16668					
70%			1.40932	2.75807	1.99685	3.75650	1.34875
80%			1.93306	3.31094	2.06693	4.34441	1.37788
90%			2.85920	4.28187	2.17870	5.37122	1.42267
CVM	0.89144	1.12665					
70%			1.40034	2.66393	1.70103	3.51445	1.26359
80%			1.89851	3.17991	1.74261	4.05121	1.28139
90%			2.76810	4.07817	1.80999	4.98317	1.31007
ADE	0.87723	1.18216					
70%			1.39755	2.69811	1.82873	3.61247	1.30057
80%			1.90640	3.23022	1.88298	4.17171	1.32382
90%			2.80057	4.16061	1.96973	5.14548	1.36004
RTADE	0.90194	1.15815					
70%			1.38910	2.62373	1.61005	3.42876	1.23463
80%			1.87788	3.12735	1.64453	3.94961	1.24947
90%			2.72815	4.00193	1.70049	4.85218	1.27378
LTADE	0.90271	1.16981					
70%			1.38661	2.61880	1.60313	3.42037	1.23219
80%			1.87449	3.12141	1.63722	3.94002	1.24692
90%			2.72313	3.99416	1.69252	4.84042	1.27103

Table 6 presents KRI estimates based on artificial LEE data with sample size n = 100, continuing the evaluation of estimation impact on risk metrics under moderate data availability. Parameter recovery improves further ($\alpha \approx 0.88$–0.91, $\zeta \approx 0.99$–1.06), aligning closely with plausible true values (e.g., $\alpha = 0.9$, $\zeta = 1.0$). VaR and TVaR estimates are now tightly clustered across methods, reflecting greater consistency in upper-tail quantification. TV and TMV values continue to distinguish methods: OLS still reports the highest tail variance, whereas MLE, CVM, and LTADE provide lower, more comparable figures. ADE maintains a middle ground, offering slightly higher but well-calibrated tail risk. EL values stabilize near 1.25–1.34 across confidence levels, confirming robust estimation of expected losses. The reduced spread in all KRIs underscores the benefit of larger samples, yet subtle differences persist—highlighting that even at n = 100, the choice of estimator affects regulatory and capital adequacy decisions.

Table 6: KRIs under artificial data for n =100.

Method	$\widehat{\alpha}$	$\widehat{\zeta}$	VaR$(X\|q)$	TVaR$(X\|q)$	TV$(X\|q)$	TMV$(X\|q)$	ELq(X)
MLE	0.90794	1.03068					
70%			1.40984	2.63404	1.56920	3.41864	1.22420
80%			1.89690	3.13275	1.59785	3.93168	1.23585
90%			2.74021	3.99646	1.64591	4.81941	1.25625
OLS	0.88259	0.99927					
70%			1.43020	2.72372	1.79022	3.61883	1.29352
80%			1.93936	3.25211	1.83703	4.17062	1.31275
90%			2.82891	4.17304	1.91351	5.12980	1.34413
CVM	0.90017	1.01819					
70%			1.41648	2.66131	1.63293	3.47777	1.24482
80%			1.91019	3.16883	1.66641	4.00203	1.25864
90%			2.76728	4.04945	1.72245	4.91068	1.28218
ADE	0.89278	1.00456					
70%			1.42331	2.68844	1.69704	3.53696	1.26513
80%			1.92355	3.20464	1.73583	4.07255	1.28109
90%			2.79419	4.10193	1.79978	5.00182	1.30774
RTADE	0.89934	1.00248					
70%			1.42006	2.66772	1.64082	3.48813	1.24766
80%			1.91486	3.17641	1.67482	4.01382	1.26155
90%			2.77384	4.05910	1.73135	4.92478	1.28526
LTADE	0.90004	1.06343					
70%			1.40775	2.65106	1.63131	3.46671	1.24331
80%			1.90040	3.15809	1.66569	3.99094	1.25769
90%			2.75648	4.03827	1.72236	4.89946	1.28180

Table 7 presents the final set of simulated KRI results for artificial LEE data with sample size n = 200, representing a relatively large-sample scenario. Parameter estimates ($\alpha \approx$ 0.89–0.90, $\zeta \approx$ 0.94–0.97) are highly accurate, indicating near-asymptotic behavior. All six methods produce nearly identical VaR and TVaR values across confidence levels, demonstrating convergence in tail quantile estimation. Differences in TV and TMV become marginal, though OLS still exhibits slightly higher tail variance than MLE or LTADE. ADE and CVM remain competitive, with ADE showing marginally better tail fidelity. EL estimates converge tightly around 1.25–1.31, confirming precise modeling of the loss distribution's body. This table confirms that with sufficient data, estimation-induced uncertainty in risk metrics diminishes substantially, yet minimum-distance methods like ADE retain a slight edge in tail calibration, supporting their use in risk-sensitive applications even when data are abundant.

Table 7: KRIs under artificial data for n =200.

Method	$\widehat{\alpha}$	$\widehat{\zeta}$	VaR$(X\|q)$	TVaR$(X\|q)$	TV$(X\|q)$	TMV$(X\|q)$	ELq(X)
MLE	0.90473	0.94587					
70%			1.42850	2.66442	1.59980	3.46432	1.23592
80%			1.92034	3.16787	1.62934	3.98254	1.24752
90%			2.77148	4.03976	1.67937	4.87945	1.26828
OLS	0.89124	0.93579					
70%			1.43835	2.71074	1.71531	3.56840	1.27239
80%			1.94181	3.22980	1.75421	4.10691	1.28799
90%			2.81730	4.13182	1.81887	5.04125	1.31452
CVM	0.89456	0.96986					
70%			1.42934	2.69116	1.68370	3.53301	1.26181
80%			1.92902	3.20581	1.72064	4.06613	1.27679
90%			2.79745	4.09967	1.78198	4.99066	1.30222
ADE	0.89259	0.97141					
70%			1.43015	2.69724	1.70086	3.54767	1.26709
80%			1.93146	3.21416	1.73929	4.08380	1.28270
90%			2.80340	4.11245	1.80293	5.01391	1.30904
RTADE	0.89671	0.96738					
70%			1.42863	2.68480	1.66538	3.51749	1.25617
80%			1.92657	3.19701	1.70073	4.04738	1.27045
90%			2.79123	4.08613	1.75962	4.96594	1.2949
LTADE	0.90163	0.96127					
70%			1.42710	2.67048	1.62416	3.48256	1.24338
80%			1.92109	3.17718	1.65614	4.00525	1.25609
90%			2.77722	4.05556	1.70961	4.91036	1.27834

6. Risk analysis under U.K. motor non-comprehensive claims triangle

In property and casualty insurance, the claim process is fundamentally shaped by two independent random elements: how often claims occur (claim count) and how large they are (claim size). When combined, these give rise to a third key quantity, the aggregate loss, which reflects the total payout over a given period and is central to risk assessment and capital planning. Thus, for all such RVs $\Pr\{X < 0\} = 0$, i.e., $F_X(x) = 0$ for all $x < 0$. The probability density function (PDF) $f_X(x)$ for a continuous size-of-loss distribution for which claim size is unbounded (or unlimited) from above takes on positive values over a semi-infinite interval of the form $0 \leq \tau_1 < x < \infty$. For positive τ_2 in this interval, the portion of the distribution defined on the sub-interval (τ_2, ∞) is called the long tail of the distribution. Alternatively, the part of the loss distribution defined on (τ_1, τ_2), extending to the left and bounded below by 0, is called the short tail of the distribution. Clearly, such distributions cannot be symmetric. Often the claim-size data sets are positively skewed. In this paper, a new negatively skewed insurance claims data set is considered and modelled. Moreover, the risk exposure is the actuarial measure of the potential future loss resulting from a specific activity or a specific event. The analysis of the risk exposure for a business often ranks risks according to their probability of future occurring multiplied by the potential loss if they occurred. Ranking the probability of potential future losses helps the business to determine which losses are minor and which are significant. In many cases, speculative risks can result in losses such as compliance failures, brand damage, security breaches, and liability issues. Generally, the risk exposure (RE(r)) can be calculated from

$$\text{RE}\left(r\right) = \Pr\left(r\right) \times t\left(r\right),$$

where $\Pr\left(r\right)$ refers to the probability of risk occurring, and $t\left(r\right)$ is the total loss of risk occurrence. On the other hand, many efforts have been devoted for analyzing the historical insurance data using the time series analysis or by using continuous distributions. Businesses use such metrics to distinguish minor hiccups from potentially catastrophic events, including non-traditional exposures like cyber breaches, compliance failures, or reputational harm. While time-series and heavy-tailed modeling dominate much of the literature, our analysis demonstrates that for certain real data, left-skewed claim patterns, a well-specified light-tailed model like LE can offer superior fit, interpretability, and predictive reliability.

In actuarial practice, historical insurance claims data are commonly organized in a run-off triangle, a tabular format that tracks how claims from each origin period (e.g., policy inception year, accident year, or earned period) evolve over successive development periods. These development periods, also referred to as age or lag, measure the elapsed time since the origin (e.g., 12 months, 24 months, etc.). While yearly origin periods are standard, finer granularities like quarters or months are frequently used for more timely reserving. A key feature of the triangle is that its diagonals represent payments made in the same calendar year, cutting across different origin cohorts, this helps actuaries monitor calendar-year effects such as inflation, legal changes, or claims handling improvements. As a concrete illustration, in this Section we use a U.K. Motor Non-Comprehensive claims triangle from Charpentier (2014), reindexed here for clarity with origin years spanning 2007 to 2013 (7 accident years). First, the the skewness-kurtosis plot (or the Cullen and Frey plot) is presented for exploring initial fits of theoretical distributions such as normal, uniform, exponential, logistic, beta, lognormal and Weibull (see Figure 2).

Based on Figure 2, we note that the empirical data point (red circle) lies well within the region typically associated with gamma and lognormal distributions as indicated by the shaded gray area and the proximity to their respective theoretical curves. This suggests the underlying claim size distribution is likely positively skewed and leptokurtic, consistent with common insurance loss data. The bootstrap replicates (yellow dots) cluster around the empirical point, indicating stability in the skewness-kurtosis estimates. Notably, the data point falls far from the regions for normal, uniform, or exponential distributions, reinforcing that simple symmetric or light-tailed models are inadequate. The position also implies that while a Weibull model (not plotted but noted as close to gamma/lognormal) could be plausible, the LE model designed for flexible tail behavior and skewness offers a more targeted fit for this type of data, especially given its demonstrated performance in risk quantification under similar configurations.

Then Figure 3 presents the nonparametric kernel density estimate for the U.K. motor non-comprehensive claims triangle data. The kernel density estimate of the 28 claim amounts reveals a bimodal distribution, with two distinct peaks located approximately at 1430 and 3853, indicating the presence of two underlying subpopulations or claim severity clusters within the dataset. The mode (highest point of the density curve) is estimated near 1479.9, corresponding to the left peak, suggesting this is the most frequently observed claim size range. The median (2299) falls between the two modes, while the mean (2702.6) is pulled rightward by the heavier right tail and the second mode, confirming the data's positive skewness. This asymmetry implies that while most claims are moderate in size (centered around the first mode), a smaller but influential subset of larger claims significantly inflates the average, which has critical implications for risk modeling and reserve estimation in insurance contexts. This graphical assessment complements the KDE analysis, confirming that the data's shape is best captured by distributions capable of modeling moderate-to-high positive skew and excess kurtosis precisely the domain where the LE family excels.

Figure 3 gives the Q-Q plot for the U.K. motor non-comprehensive claims data. With only n =28 observations, the plot remains sensitive to sampling variation, yet the systematic departure is robust enough to reject normality at conventional significance levels. The empirical points do not lie on any straight line, ruling out lognormal or exponential fits without transformation. This reinforces the need for flexible distributions like the LE, which can accommodate both skewness and multimodality.

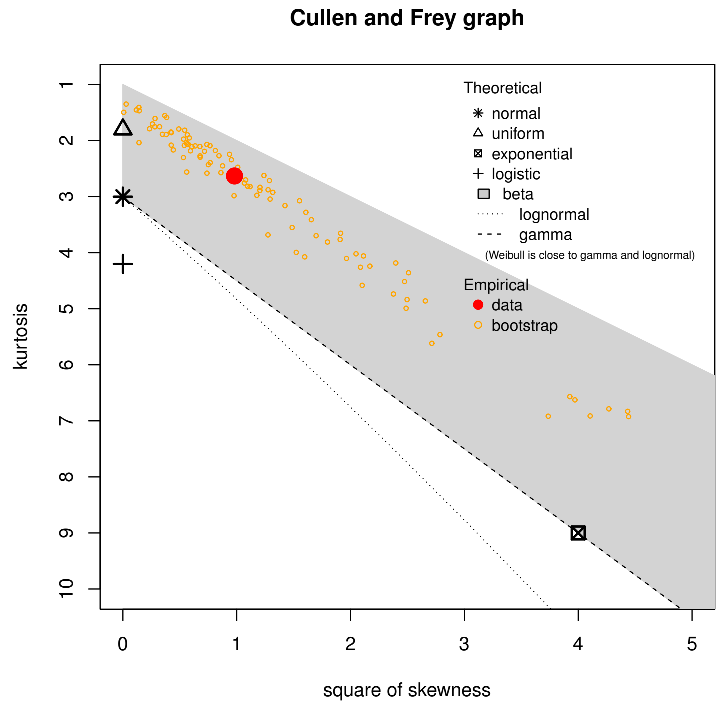

Figure 2. Cullen and Frey plot.

Table 8 presents the estimated KRIs including VaR, TVaR, TV, TMV and EL for the U.K. motor non-comprehensive claims triangle data, fitted using six estimation methods for the LEE distribution. The table reports these risk metrics at three confidence levels (70%, 80%, and 90%), alongside the corresponding estimated parameters α and ζ. Notably, all methods yield a ζ estimate extremely close to -1 (e.g., –0.999998 for MLE), suggesting the fitted LEE model operates near a boundary of its parameter space, possibly indicating heavy-tailed or near-Pareto behavior in the data. The MLE method produces the smallest risk estimates: for instance, VaR2080.20892080 $=$ 4901 and TVaR2080.20892080 $=$ 5554, with relatively low tail variance, implying a lighter inferred tail. In stark contrast, OLS generates dramatically inflated risk measures, VaR2080.20892080 $=$ 20277 and TVaR2080.20892080 $=$ 44790, with TV exceeding 1.5 billion, reflecting severe instability and overestimation of tail risk. The minimum-distance methods (CVM, ADE, RTADE, AD2LE) yield intermediate but more plausible figures, with RTADE providing the most conservative (lowest) tail risk among them at higher quantiles. Interestingly, EL decreases as the confidence level increases, consistent with its definition as the expected loss below the q-th quantile, and this pattern holds across all methods. The wide dispersion in KRI values especially between MLE and OLS highlights the critical influence of the estimation technique on capital adequacy and reserving decisions. Among the robust alternatives, RTADE and AD2LE offer balanced performance, avoiding both the underestimation seen in MLE and the extreme overestimation from OLS.

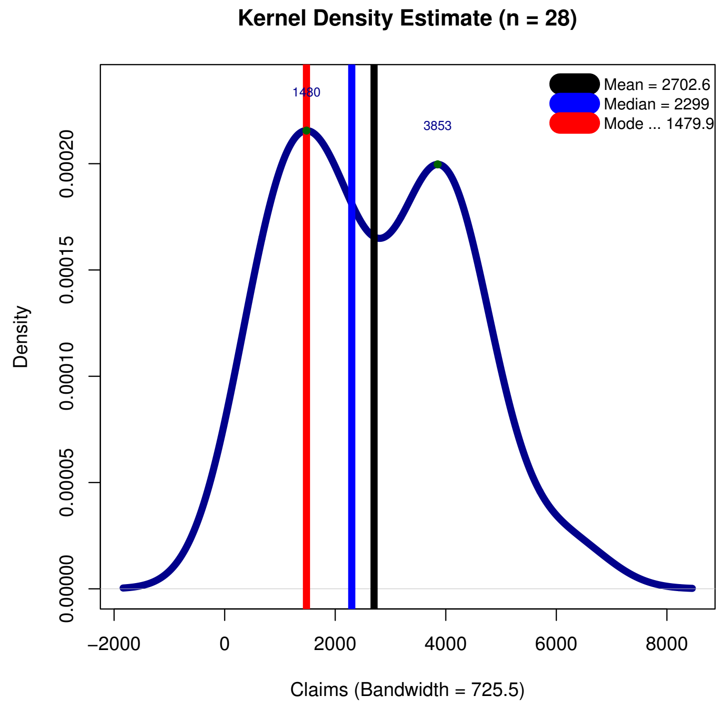

Figure 3. Kernel density estimate for the U.K. motor non-comprehensive claims triangle data

Table 8: The KRIs under U.K. motor non-comprehensive claims triangle data

Method	$\widehat{\alpha}$	$\widehat{\zeta}$	VaR$(X\|q)$	TVaR$(X\|q)$	TV$(X\|q)$	TMV$(X\|q)$	ELq(X)
MLE	0.30250	−0.999998					
70%			2570	4241	2720998	1364740	1671
80%			3516	4850	2930466	1470083	1334
90%			4901	5554	4715802	2363455	652
OLS	0.20833	−0.99932					
70%			6093	22536	760890710	380467891	16443
80%			10616	29738	984683587	492371532	19122
90%			20277	44790	1509789021	754939300	24512
CVM	0.23209	−0.99988					
70%			5713	15491	112893296	56462139	9778
80%			9052	19582	119098521	59568842	10530
90%			15245	27482	108916215	54485590	12238
ADE	0.21890	−0.99966					
70%			5751	18326	143230525	71633589	12576
80%			9583	23737	126506494	63276984	14153
90%			17261	34628	10589023	5329140	17367
RTADE	0.22919	−0.99976					
70%			4595	13331	101807203	50916933	8736
80%			7427	17027	111690051	55862052	9600
90%			12875	24270	116265088	58156814	11395
AD2LE	0.2279	−0.9999					
70%			7127	19577	142481070	71260112	12451
80%			11359	24837	129793977	64921826	13478
90%			19236	34670	68044699	34057019	15434

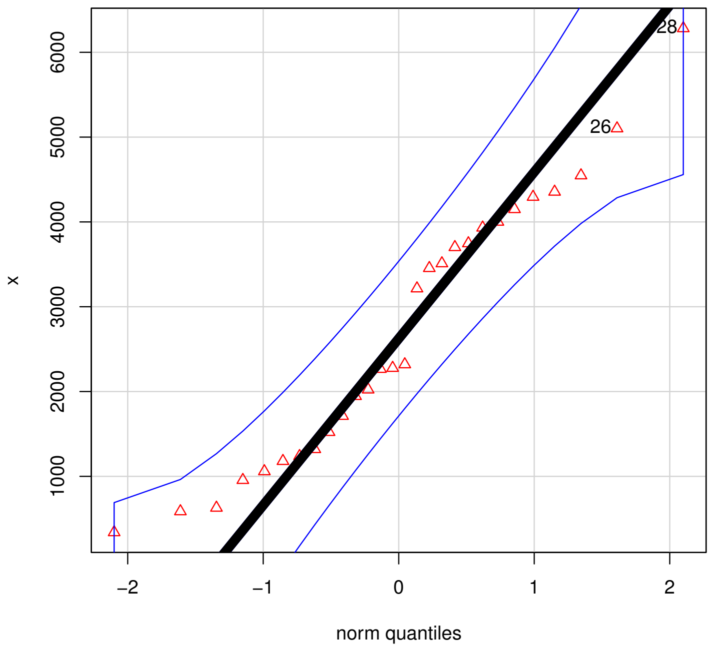

Figure 4. The QQ plot for the U.K. motor non-comprehensive claims triangle data

7. Conclusions

This study proposed and rigorously examined a novel distribution, the log-exponential-exponential (LEE), a specialized member of the log-exponential (LE) family, designed specifically for modeling insurance claim sizes. The LEE inherits the light-tailed behavior of its exponential baseline, as reflected in its tail index, making it particularly well-suited for portfolios exhibiting left-skewness or bounded severity. Its hazard function exhibits remarkable flexibility, accommodating decreasing, increasing, and bathtub-shaped forms, thereby broadening its utility in reliability and survival analysis. Comprehensive Monte Carlo simulations demonstrated that the Anderson–Darling estimator (ADE) consistently outperformed competing methods namely MLE, OLS, CVM, RTADE, and LTADE, across a range of parameter settings and sample sizes (from n = 20 to 200), delivering lower bias, reduced RMSE, and superior CDF fidelity. In risk assessments based on simulated LEE data, ADE produced the most reliable and well-calibrated key risk indicators (VaR, TVaR, TV, TMV, EL), especially critical in small-sample scenarios where inaccurate tail estimation can severely compromise capital adequacy. Application to a real-world dataset, the U.K. Motor Non-Comprehensive claims triangle ($n = 28$), revealed unexpected left-skewness and bimodality, features inadequately captured by standard heavy-tailed models. This atypical skew-kurtosis structure was confirmed through graphical tools, including the Cullen–Frey plot and kernel density estimates, which supported the adoption of a flexible light-tailed model like the LEE. The research further highlighted how the choice of estimation method directly influences risk capital outcomes: MLE tended to understate tail risk, whereas ADE achieved an optimal balance between statistical fidelity and prudent risk assessment. Theoretical underpinnings were solidified through characterizations based on truncated moments and reverse-hazard rate identities, while closed-form expressions for moments, entropy, and quantiles ensured analytical tractability without numerical approximation. Empirical comparisons confirmed the LEE's superior fit relative to conventional benchmarks such as the gamma, Weibull, and lognormal distributions.

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REFERENCES

1. Abiad, M., El-Raouf, M. A., Yousof, H. M., Bakr, M. E., Samson Balogun, O., Yusuf, M., ... & Tashkandy, Y. A. (2025). A novel Compound-Pareto model with applications and reliability peaks above a random threshold value at risk analysis. Scientific Reports, 15(1), 21068.
2. AboAlkhair, A. M., Hamedani, G. G., Ali Ahmed, N., Ibrahim, M., Zayed, M. A., & Yousof, H. M. (2025). A New G Family: Properties, Characterizations, Different Estimation Methods and PORT-VaR Analysis for UK Insurance Claims and US House Prices Data Sets. Mathematics, 13(19), 3097.
3. Abonongo, J., Abonongo, A. I. L., Aljadani, A., Mansour, M. M., & Yousof, H. M. (2025). Accelerated failure model with empirical analysis and application to colon cancer data: Testing and validation. Alexandria Engineering Journal , 113, 391–408.
4. Aboraya, M., Ali, M. M., Yousof, H. M., & Mohamed, M. I. (2022). A new flexible probability model: Theory, estimation and modeling bimodal left skewed data. Pakistan Journal of Statistics and Operation Research, 437-463.
5. Abouelmagd, T. H. M., Hamed, M. S., Hamedani, G. G., Ali, M. M., Goual, H., Korkmaz, M. C., & Yousof, H. M. (2019). The zero truncated Poisson Burr X family of distributions with properties, characterizations, applications, and validation test. Journal of Nonlinear Sciences and Applications , 12(5), 314–336.
6. Afify, A. Z., Cordeiro, G. M., Ortega, E. M., Yousof, H. M., & Butt, N. S. (2018). The Four-Parameter Burr XII Distribution: Properties, Regression Model, and Applications. Communications in Statistics - Theory and Methods , 47(11), 2605–2624. https://doi.org/10.1080/03610926.2017.1348527
7. Afify, A. Z., Cordeiro, G. M., Yousof, H. M., Saboor, A., & Ortega, E. M. (2018). The Marshall-Olkin Additive Weibull Distribution with Variable Shapes for the Hazard Rate. Hacettepe Journal of Mathematics and Statistics , 47(2), 365–381. https://doi.org/10.15672/HJMS.2017.458
8. Ahmed, B., & Yousof, H. (2023). A new group acceptance sampling plans based on percentiles for the Weibull Fréchet model. Statistics, Optimization & Information Computing, 11(2), 409-421.
9. Ahmed, B., Ali, M. M., & Yousof, H. M. (2022). A Novel G Family for Single Acceptance Sampling Plan with Application in Quality and Risk Decisions. Annals of Data Science . https://doi.org/10.1007/s40745−022−00451-3 .
10. Ahmed, B., Ali, M. M., & Yousof, H. M. (2023). A New G Family for Single Acceptance Sampling Plan with Application in Quality and Risk Decisions. Annals of Data Science , 10(2), 321–342.
11. Ahmed, B., Chesneau, C., Ali, M. M., & Yousof, H. M. (2022). Amputated life testing for Weibull-Fréchet percentiles: single, double and multiple group sampling inspection plans with applications. Pakistan Journal of Statistics and Operation Research, 995-1013.
12. Ahmed, N. A., Butt, N. S., Hamedani, G. G., Ibrahim, M., AboAlkhair, A. M., & Yousof, H. M. (2025). The Log-Exponentiated Polynomial G Family: Properties, Characterizations and Risk Analysis under Different Estimation Methods. Statistics, Optimization & Information Computing.
13. Al-babtain, A. A., Elbatal, I., & Yousof, H. M. (2020). A New Flexible Three-Parameter Model: Properties, Clayton Copula, and Modeling Real Data. Symmetry , 12(3), 440. https://doi.org/10.3390/sym12030440
14. Al-Door, A. M., Salih, A., Mohammed, S. M., & Abdelfattah, A. M. (2025). Regression Model for MG gamma Lindley with Application. Journal of Applied Probability & Statistics, 20(2).
15. Alizadeh, M., Afshari, M., Contreras-Reyes, J. E., Mazarei, D., & Yousof, H. M. (2024). The Extended Gompertz Model: Applications, Mean of Order P Assessment and Statistical Threshold Risk Analysis Based on Extreme Stresses Data. IEEE Transactions on Reliability , doi: 10.1109/TR.2024.3425278.
16. Alizadeh, M., Afshari, M., Cordeiro, G. M., Ramaki, Z., Contreras-Reyes, J. E., Dirnik, F., & Yousof, H. M. (2025). A New Weighted Lindley Model with Applications to Extreme Historical Insurance Claims. Stats , 8(1), 8.
17. Alizadeh, M., Afshari, M., Cordeiro, G. M., Ramaki, Z., Contreras-Reyes, J. E., Dirnik, F., & Yousof, H. M. (2025). A New Weighted Lindley Model with Applications to Extreme Historical Insurance Claims. Stats, 8(1), 8.
18. Alizadeh, M., Afshari, M., Ranjbar, V., Merovci, F., & Yousof, H. M. (2023). A novel XGamma extension: applications and actuarial risk analysis under the reinsurance data. São Paulo Journal of Mathematical Sciences , 1–31.
19. Alizadeh, M., Cordeiro, G. M., Rodrigues, G. M., Ortega, E. M., & Yousof, H. M. (2025). The Extended Kumaraswamy Model: Properties, Risk Indicators, Risk Analysis, Regression Model, and Applications. Stats, 8(3), 62.
20. Alizadeh, M., Cordeiro, G. M., Ramaki, Z., Tahmasebi, S., Contreras-Reyes, J. E., & Yousof, H. M. (2025). The Weighted Flexible Weibull Model: Properties, Applications, and Analysis for Extreme Events. Mathematical and Computational Applications , 30(2), 42.
21. Alizadeh, M., Hazarika, P. J., Das, J., Contreras-Reyes, J. E., Hamedani, G. G., Sulewski, P., & Yousof, H. M. (2025). Reliability and risk analysis under peaks over a random threshold value-at-risk method based on a new flexible skew-logistic distribution. Life Cycle Reliability and Safety Engineering, 1-28.
22. Alizadeh, M., Lak, F., Rasekhi, M., Ramires, T. G., Yousof, H. M., & Altun, E. (2018). The Odd Log-Logistic Topp–Leone G Family of Distributions: Heteroscedastic Regression Models and Applications. Computational Statistics , 33, 1217–1244. https://doi.org/10.1007/s00180−017−0781-5

23. Alizadeh, M., Rasekhi, M., Yousof, H. M., & Hamedani, G. G. (2018). The Transmuted Weibull-G Family of Distributions. Hacettepe Journal of Mathematics and Statistics , 47(6), 1671–1689. https://doi.org/10.15672/HJMS.2017.497
24. AlKhayyat, S. L., Haitham M. Yousof, Hafida Goual, Hamida, T., Hamed, M. S., Hiba, A., & Mohamed Ibrahim. (2025). Rao-Robson-Nikulin Goodness-of-fit Test Statistic for Censored and Uncensored Real Data with Classical and Bayesian Estimation. Statistics, Optimization & Information Computing . https://doi.org/10.19139/soic-2310-5070-1710 .
25. AlKhayyat, S. L., Haitham M. Yousof, Hafida Goual, Hamida, T., Hamed, M. S., Hiba, A., & Mohamed Ibrahim. (2025). Rao-Robson-Nikulin Goodness-of-fit Test Statistic for Censored and Uncensored Real Data with Classical and Bayesian Estimation. Statistics, Optimization & Information Computing. https://doi.org/10.19139/soic-2310-5070-1710
26. Artzner, P. (1999). Application of coherent risk measures to capital requirements in insurance. North American Actuarial Journal , 3(2), 11–25.
27. Benchiha, S., Al-Omari, A. I., Alotaibi, N., & Shrahili, M. (2021). Weighted generalized quasi-Lindley distribution: Different methods of estimation, applications for COVID-19 and engineering data. AIMS Math , 6, 11850–11878.
28. Charpentier, A. (2014). Computational actuarial science with R. CRC press.
29. Chaubey, Y. P., & Zhang, R. (2015). An extension of Chen's family of survival distributions with bathtub shape or increasing hazard rate function. Communications in Statistics - Theory and Methods , 44(19), 4049–4064.
30. Chesneau, C., Yousof, H. M., Hamedani, G., & Ibrahim, M. (2022). A new one-parameter discrete distribution: the discrete inverse burrdistribution: characterizations. Statistics, optimization and information computing, properties, applications, Bayesian and non-Bayesian estimations.
31. Cordeiro, G. M., Afify, A. Z., Yousof, H. M., Cakmakyapan, S., & Ozel, G. (2018). The Lindley Weibull Distribution: Properties and Applications. Anais da Academia Brasileira de Ciências , 90, 2579–2598. https://doi.org/10.1590/0001-3765201820170731
32. Crowder, M. J., Kimber, A. C., Smith, R. L., & Sweeting, T. J. (1991). Statistical Analysis of Reliability Data . CHAPMAN & HALL/CRC.
33. Das, J., Hazarika, P. J., Alizadeh, M., Contreras-Reyes, J. E., Mohammad, H. H., & Yousof, H. M. (2025). Economic Peaks and Value-at-Risk Analysis: A Novel Approach Using the DLElace Distribution for House Prices. Mathematical and Computational Applications , 30(1), 4.
34. Das, J., Hazarika, P. J., Alizadeh, M., Contreras-Reyes, J. E., Mohammad, H. H., & Yousof, H. M. (2025). Economic peaks and value-at-risk analysis: A novel approach using the Laplace distribution for house prices. Mathematical and Computational Applications, 30(1), 4.
35. Dupuy, J. F. (2014). Accelerated failure time models: A review. International Journal of Performability Engineering , 10(1), 23–40.
36. Elbatal, I., Diab, L. S., Ghorbal, A. B., Yousof, H. M., Elgarhy, M., & Ali, E. I. (2024). A new losses (revenues) probability model with entropy analysis, applications and case studies for value-at-risk modeling and mean of order-P analysis. AIMS Mathematics , 9(3), 7169–7211.
37. Elgohari, H., & Yousof, H. M. (2020). A Generalization of Lomax Distribution with Properties, Copula, and Real Data Applications. Pakistan Journal of Statistics and Operation Research , 16(4), 697–711. https://doi.org/10.18187/pjsor.v16i4.3157
38. Eliwa, M. S., El-Morshedy, M., & Yousof, H. M. (2022). A discrete exponential generalized-G family of distributions: Properties with Bayesian and non-Bayesian estimators to model medical, engineering and agriculture data. Mathematics, 10(18), 3348.
39. Emam, W., Tashkandy, Y., Goual, H., Hamida, T., Hiba, A., Ali, M. M., Yousof, H. M., & Ibrahim, M. (2023). A New One-Parameter Distribution for Right Censored Bayesian and Non-Bayesian Distributional Validation under Various Estimation Methods. Mathematics , 11(4), 897. https://doi.org/10.3390/math11040897.
40. Glänzel, W., A characterization theorem based on truncated moments and its application to some distribution families, Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, 1987, 75–84.
41. Glänzel, W., Some consequences of a characterization theorem based on truncated moments, *Statistics: A Journal of Theoretical and Applied Statistics*, 21 (4), 1990, 613–618.
42. Goual, H., & Yousof, H. M. (2019). Validation of Burr XII inverse Rayleigh model via a modified chi-squared goodness-of-fit test. Journal of Applied Statistics , 47, 1–32.
43. Goual, H., & Yousof, H. M. (2020). Validation of Burr XII inverse Rayleigh model via a modified chi-squared goodness-of-fit test. Journal of Applied Statistics , 47(3), 393–423.
44. Goual, H., Hamida, T., Hiba, A., Hamedani, G. G., Ibrahim, M., & Yousof, H. M. (2022). Bayesian and Non-Bayesian Distributional Validations under Censored and Uncensored Schemes with Characterizations and Applications.
45. Goual, H., Yousof, H. M., & Ali, M. M. (2019). Validation of the odd Lindley exponentiated exponential by a modified goodness-of-fit test with applications to censored and complete data. Pakistan Journal of Statistics and Operation Research , 15(3), 745–771.
46. Goual, H., Yousof, H. M., & Ali, M. M. (2019). Validation of the Odd Lindley Exponentiated Exponential by a Modified Goodness of Fit Test with Applications to Censored and Complete Data. Pakistan Journal of Statistics and Operation Research , 15(3), 745–771. https://doi.org/10.18187/pjsor.v15i3.2784
47. Goual, H., Yousof, H. M., & Ali, M. M. (2020). Lomax inverse Weibull model: properties, applications and a modified chi-squared goodness-of-fit test for validation. Journal of Nonlinear Sciences and Applications , 13(6), 330–353.
48. Gross, A. J., & Clark, V. (1975). Survival distributions: reliability applications in the biomedical sciences. (No Title).
49. Hamed, M. S., Cordeiro, G. M., & Yousof, H. M. (2022). A New Compound Lomax Model: Properties, Copulas, Modeling and Risk Analysis Utilizing the Negatively Skewed Insurance Claims Data. Pakistan Journal of Statistics and Operation Research , 18(3), 601–631. https://doi.org/10.18187/pjsor.v18i3.3652 .
50. Hamedani, G. G. (2013). On certain generalized gamma convolution distributions II (Technical Report No. 484). Department of Mathematics, Statistics and Computer Science, Marquette University.
51. Hashem, A. F., Alotaibi, N., Alyami, S. A., Abdelkawy, M. A., Elgawad, M. A. A., Yousof, H. M., & Abdel-Hamid, A. H. (2024). Utilizing Bayesian inference in accelerated testing models under constant stress via ordered ranked set sampling and hybrid censoring with practical validation. Scientific Reports , 14(1), 14406.
52. Hashim, M., Hamedani, G. G., Ibrahim, M., AboAlkhair, A. M., & Yousof, H. M. (2025). An innovated G family: Properties, characterizations and risk analysis under different estimation methods. Statistics, Optimization & Information Computing.

53. Hashem, A. F., Alyami, S. A., Abd Elgawad, M. A., Abdelkawy, M. A., & Yousof, H. M. (2025). Risk Analysis in View of the KSA Disability Statistics Publication of 2023. Journal of Disability Research, 4(3), 20250554.
54. Hashempour, M., Alizadeh, M., & Yousof, H. (2024). The Weighted Xgamma Model: Estimation, Risk Analysis and Applications. Statistics, Optimization & Information Computing , 12(6), 1573–1600.
55. Hashempour, M., Alizadeh, M., & Yousof, H. M. (2024). A new Lindley extension: estimation, risk assessment and analysis under bimodal right skewed precipitation data. Annals of Data Science , 11(6), 1919–1958.
56. Hussein, W. J., Salih, A., & Abdullah, M. (2025). a deep neural network approach for estimating time-varying parameters in ordinary differential equation models. Journal of Applied Probability & Statistics, 20(2).
57. Ibrahim, M., Ali, E. I., Hamedani, G. G., Al-Nefaie, A. H., Aljadani, A., Mansour, M. M., ... & Salem, M. (2025). A New Model for Reliability Value-at-Risk Assessments with Applications, Different Methods for Estimation, Non-parametric Hill Estimator and PORT-VaRq Analysis. Pakistan Journal of Statistics and Operation Research, 177-212.
58. Ibrahim, M., Ali, M. M., Goual, H., & Yousof, H. (2022). The Double Burr Type XII Model: Censored and Uncensored Validation Using a New Nikulin-Rao-Robson Goodness-of-Fit Test with Bayesian and Non-Bayesian Estimation Methods. Pakistan Journal of Statistics and Operation Research , 18(4), 901–927. https://doi.org/10.18187/pjsor.v18i4.3600 .
59. Ibrahim, M., Ali, M. M., Goual, H., & Yousof, H. M. (2019). A new extension of Lindley distribution: modified validation test, characterizations and different methods of estimation. Communications for Statistical Applications and Methods , 26(5), 473–495.
60. Ibrahim, M., Al-Nefaie, A. H., AboAlkhair, A. M., Yousof, H. M., & Ahmed, B. (2025a). Modeling Medical and Reliability Data Sets Using a Novel Reciprocal Weibull Distribution: Estimation Methods and Sequential Sampling Plan Based on Truncated Life Testing. Statistics, Optimization & Information Computing .
61. Ibrahim, M., Al-Nefaie, A. H., AboAlkhair, A. M., Yousof, H. M., & Ahmed, B. (2025b). Modeling Medical and Reliability Data Sets Using a Novel Reciprocal Weibull Distribution: Estimation Methods and Sequential Sampling Plan Based on Truncated Life Testing. Statistics, Optimization & Information Computing.
62. Ibrahim, M., Al-Nefaie, A. H., Butt, N. S., Hamedani, G. G., Hashim, M., AboAlkhair, A. M., ... & Nabawy, N. (2025). A Novel Generated G Family for Risk Analysis and Assessment under Different Non-Bayesian Methods: Properties, Characterizations and Applications to USA House Prices and UK Insurance Claims Data. Pakistan Journal of Statistics and Operation Research, 507-529.
63. Ibrahim, M., Altun, E., Goual, H., & Yousof, H. M. (2020). Modified goodness-of-fit type test for censored validation under a new Burr type XII distribution with different methods of estimation and regression modeling. Eurasian Bulletin of Mathematics , 3(3), 162–182.
64. Ibrahim, M., Ansari, S. I., Al-Nefaie, A. H., & Yousof, H. M. (2025c). A New Version of the Inverse Weibull Model with Properties, Applications and Different Methods of Estimation. Statistics, Optimization & Information Computing, 13(3), 1120-1143. https://doi.org/10.19139/soic-2310-5070-1658
65. Ibrahim, M., Ansari, S. I., Al-Nefaie, A. H., AboAlkhair, A. M., Hamed, M. S., & Yousof, H. M. (2025d). A Novel Fréchet-Poisson Model: Properties, Applications under Extreme Reliability Data, Different Estimation Methods and Case Study on Strength-Stress Reliability Analysis. Statistics, Optimization & Information Computing, 13(6), 2353-2381.
66. Ibrahim, M., Butt, N. S., Al-Nefaie, A. H., Hamedani, G. G., Yousof, H. M., & Mahmoud, A. S. (2025e). An Extended Discrete Model for Actuarial Data and Value at Risk Analysis: Properties, Applications and Risk Analysis under Financial Automobile Claims Data. Statistics, Optimization & Information Computing , 13(1), 27–46.
67. Ibrahim, M., Emam, W., Tashkandy, Y., Ali, M. M., Yousof, H. M., & Goual, H. (2023). Bayesian and Non-Bayesian Risk Analysis and Assessment under Left-Skewed Insurance Data and a Novel Compound Reciprocal Rayleigh Extension. Mathematics , 11(7), 1593. https://doi.org/10.3390/math11071593 .
68. Ibrahim, M., Goual, H., Khaoula, M. K., Al-Nefaie, A. H., AboAlkhair, A. M., & Yousof, H. M. (2025h). A New Accelerated Failure Time Model with Censored and Uncensored Real-life Applications: Validation and Different Estimation Methods. Statistics, Optimization & Information Computing.
69. Ibrahim, M., Hamedani, G. G., Butt, N. S., & Yousof, H. M. (2022). Expanding the Nadarajah Haghighi Model: Copula, Censored and Uncensored Validation, Characterizations and Applications. Pakistan Journal of Statistics and Operation Research , 18(3), 537–553. https://doi.org/10.18187/pjsor.v18i3.3420.
70. Jameel, S. O., Salih, A. M., JaLEl, R. A., & Zahra, M. M. (2022). On The Neutrosophic Formula of Some Matrix Equations Derived from Data Mining Theory and Control Systems. International Journal of Neutrosophic Science (IJNS), 19(1).
71. Khalil, M. G., Aidi, K., Ali, M. M., Butt, N. S., Ibrahim, M., & Yousof, H. M. (2024). Modified Bagdonavicius-Nikulin Goodness-of-fit Test Statistic for the Compound Topp Leone Burr XII Model with Various Censored Applications. Statistics, Optimization & Information Computing, 12(4), 851-868.
72. Khedr, A. M., Nofal, Z. M., El Gebaly, Y. M., & Yousof, H. M. (2025). A Novel Family of Compound Probability Distributions: Properties, Copulas, Risk Analysis and Assessment under a Reinsurance Revenues Data Set. Thailand Statistician, 23(3); 615-642.
73. Klein, J. P., & Moeschberger, M. L. (2003). Survival Analysis: Techniques for Censored and Truncated Data . Springer, New York.
74. Korkmaz, M. Ç., Altun, E., Yousof, H. M., Afify, A. Z., & Nadarajah, S. (2018). The Burr X Pareto Distribution: Properties, Applications and VaR Estimation. Journal of Risk and Financial Management , 11(1), 1.
75. Korkmaz, M. Ç., Yousof, H. M., & Hamedani, G. G. (2018). The Exponential Lindley Odd Log-Logistic-G Family: Properties, Characterizations and Applications. Journal of Statistical Theory and Applications , 17(3), 554–571. https://doi.org/10.2991/jsta.2018.17.3.14
76. Lak, F., Alizadeh, M., Mazarei, D., Sharafdini, R., Dindarlou, A., & Yousof, H. M. (2025). A novel weighted family for the reinsurance actuarial risk analysis with applications. São Paulo Journal of Mathematical Sciences, 19(2), 1-21.
77. Loubna, H., Goual, H., Alghamdi, F. M., Mustafa, M. S., Tekle Mekiso, G., Ali, M. M., ... & Yousof, H. M. (2024). The quasi-xgamma frailty model with survival analysis under heterogeneity problem, validation testing, and risk analysis for emergency care data. Scientific Reports , 14(1), 8973.
78. Mansour, M. M., Aidi, K., Butt, N. S., Ali, M. M., Yousof, H. M., & Hamed, M. S. (2020a). A New Log-Logistic Lifetime Model with Mathematical Properties, Copula, Modified Goodness-of-Fit Test for Validation and Real Data Modeling. Mathematics , 8(9), 1508.

79. Mansour, M. M., Butt, N. S., Ansari, S. I., Yousof, H. M., Ali, M. M., & Ibrahim, M. (2020b). A new exponentiated Weibull distribution's extension: copula, mathematical properties and applications. Contributions to Mathematics , 1, 57–66. DOI: 10.47443/cm.2020.0018.
80. Mansour, M. M., Butt, N. S., Yousof, H. M., Ansari, S. I., & Ibrahim, M. (2020dc). A Generalization of Reciprocal Exponential Model: Clayton Copula, Statistical Properties and Modeling Skewed and Symmetric Real Data Sets. Pakistan Journal of Statistics and Operation Research , 16(2), 373–386. https://doi.org/10.18187/pjsor.v16i2.3069
81. Mansour, M., Korkmaz, M. Ç., Ali, M. M., Yousof, H. M., Ansari, S. I., & Ibrahim, M. (2020d). A generalization of the exponentiated Weibull model with properties, Copula and application. Eurasian Bulletin of Mathematics , 3(2), 84–102.
82. Mansour, M., Rasekhi, M., Ibrahim, M., Aidi, K., Yousof, H. M., & Elrazik, E. A. (2020e). A New Parametric Life Distribution with Modified Bagdonavičius–Nikulin Goodness-of-Fit Test for Censored Validation, Properties, Applications, and Different Estimation Methods. Entropy , 22(5), 592.
83. Mansour, M., Yousof, H. M., Shehata, W. A. M., & Ibrahim, M. (2020f). A new two parameter Burr XII distribution: properties, copula, different estimation methods and modeling acute bone cancer data. Journal of Nonlinear Science and Applications , 13(5), 223–238.
84. Mohamed, H. S., Cordeiro, G. M., & Yousof, H. (2025). The synthetic autoregressive model for the insurance claims payment data: modeling and future prediction. Statistics, Optimization & Information Computing.
85. Mohamed, H. S., Cordeiro, G. M., Minkah, R., Yousof, H. M., & Ibrahim, M. (2024). A size-of-loss model for the negatively skewed insurance claims data: applications, risk analysis using different methods and statistical forecasting. Journal of Applied Statistics , 51(2), 348–369.
86. Mohamed, H. S., Cordeiro, G. M., Minkah, R., Yousof, H. M., & Ibrahim, M. (2024). A Size-of-Loss Model for the Negatively Skewed Insurance Claims Data: Applications, Risk Analysis Using Different Methods and Statistical Forecasting. Journal of Applied Statistics , 51(2), 348–369. https://doi.org/10.1080/02664763.2023.2240980
87. Murthy, D.P.; Xie, M.; Jiang, R. Weibull Models; John Wiley & Sons: Hoboken, NJ, USA, 2004.
88. Mustafa, M. C., Alizadeh, M., Yousof, H. M., & Butt, N. S. (2018). The Generalized Odd Weibull Generated Family of Distributions: Statistical Properties and Applications. Pakistan Journal of Statistics and Operation Research , 14(3), 541–556. https://doi.org/10.18187/pjsor.v14i3.2441
89. Ramaki, Z., Alizadeh, M., Tahmasebi, S., Afshari, M., Contreras-Reyes, J. E., & Yousof, H. M. (2025). The Weighted Flexible Weibull Model: Properties, Applications, and Analysis for Extreme Events. Mathematical and Computational Applications, 30(2), 42.
90. Rasekhi, M., Altun, E., Alizadeh, M., & Yousof, H. M. (2022). The Odd Log-Logistic Weibull-G Family of Distributions with Regression and Financial Risk Models. Journal of the Operations Research Society of China , 10(1), 133–158.
91. Rasekhi, M., Saber, M. M., & Yousof, H. M. (2020). Bayesian and Classical Inference of Reliability in Multicomponent Stress-Strength under the Generalized Logistic Model. Communications in Statistics - Theory and Methods , 50(21), 5114–5125. https://doi.org/10.1080/03610926.2020.1750651
92. Ravi, V., & Gilbert, P. D. (2009). BB: An R package for solving a large system of nonlinear equations and for optimizing a high-dimensional nonlinear objective function. Journal of Statistical Software , 32, 1–26.
93. Reis, L. D. R., Cordeiro, G. M., & Maria do Carmo, S. (2020). The Gamma-Chen distribution: a new family of distributions with applications. Span. J. Stat. , 2, 23–40.
94. Salah, M. M., El-Morshedy, M., Eliwa, M. S., & Yousof, H. M. (2020). Expanded Fréchet Model: Mathematical Properties, Copula, Different Estimation Methods, Applications and Validation Testing. Mathematics , 8(11), 1949. https://doi.org/10.3390/math8111949
95. Salem, M., Emam, W., Tashkandy, Y., Ibrahim, M., Ali, M. M., Goual, H., & Yousof, H. M. (2023). A new lomax extension: Properties, risk analysis, censored and complete goodness-of-fit validation testing under left-skewed insurance, reliability and medical data. Symmetry, 15(7), 1356.
96. Salih A.M. and Abdullah M.M. (2024). Comparison between classical and Bayesian estimation with joint Jeffrey's prior to Weibull distribution parameters in the presence of large sample conditions. Statistics in Transition new series, 25(4), pp. 191-202 https://doi.org/10.59139/stattrans-2024−010
97. Salih, A. M., and Hmood, M. Y. (2020). Analyzing big data sets by using different panelized regression methods with application: surveys of multidimensional poverty in Iraq. Periodicals of Engineering and Natural Sciences (PEN), 8(2), 991-999.
98. Salih, A., and Hussein, W. J. (2025). Quasi Lindley Regression Model Residual Analysis for Biomedical Data. Statistics, Optimization & Information Computing, 14(2), 956-969. https://doi.org/10.19139/soic-2310-5070-2649
99. Salih, A. M., and Hmood, M. Y. (2021). Big data analysis by using one covariate at a time multiple testing (OCMT) method: Early school dropout in Iraq. International Journal of Nonlinear Analysis and Applications, 12(2), 931-938.
100. Shehata, W. A. M., Goual, H., Hamida, T., Hiba, A., Hamedani, G., Al-Nefaie, A. H., Ibrahim, M., Butt, N. S., Osman, R. M. A., & Yousof, H. M. (2024). Censored and Uncensored Nikulin-Rao-Robson Distributional Validation: Characterizations, Classical and Bayesian estimation with Censored and Uncensored Applications. Pakistan Journal of Statistics and Operation Research , 20(1), 11–35.
101. Sulewski, P., Alizadeh, M., Das, J., Hamedani, G. G., Hazarika, P. J., Contreras-Reyes, J. E., & Yousof, H. M. (2025). A New Logistic Distribution and Its Properties, Applications and PORT-VaR Analysis for Extreme Financial Claims. Mathematical and Computational Applications, 30(3), 62.
102. Taghipour, M., Saber, M. M., Khan, M. I., Hamed, M. S. & Yousof, H. M. (2025). Consistency Issues in Skew Random Fields: Investigating Proposed Alternatives and Identifying Persisting Problems. Pakistan Journal of Statistics and Operation Research, 21(1), 33-37. https://doi.org/10.18187/pjsor.v21i1.4577
103. Taghipour, M., Saber, M. M., Khan, M. I., Hamed, M. S., & Yousof, H. M. (2025). Consistency Issues in Skew Random Fields: Investigating Proposed Alternatives and Identifying Persisting Problems. Pakistan Journal of Statistics and Operation Research , 21(1), 33–37. https://doi.org/10.18187/pjsor.v21i1.4577 .

104. Teghri, S., Goual, H., Loubna, H., Butt, N. S., Khedr, A. M., Yousof, H. M., ... & Salem, M. (2024). A New Two-Parameters Lindley-Frailty Model: Censored and Uncensored Schemes under Different Baseline Models: Applications, Assessments, Censored and Uncensored Validation Testing. Pakistan Journal of Statistics and Operation Research , 109–138.
105. Yadav, A. S., Goual, H., Alotaibi, R. M., Ali, M. M., & Yousof, H. M. (2020). Validation of the Topp-Leone-Lomax model via a modified Nikulin–Rao–Robson goodness-of-fit test with different methods of estimation. Symmetry , 12(1), 57.
106. Yadav, A. S., Shukla, S., Goual, H., Saha, M., & Yousof, H. M. (2022). Validation of xgamma exponential model via Nikulin–Rao–Robson goodness-of-fit test under complete and censored sample with different methods of estimation. Statistics, Optimization & Information Computing , 10(2), 457–483.
107. Yousof, H. M., Afify, A. Z., Abd El Hadi, N. E., Hamedani, G. G., & Butt, N. S. (2016). On Six-Parameter Fréchet Distribution: Properties and Applications. Pakistan Journal of Statistics and Operation Research , 12(2), 281–299. https://doi.org/10.18187/pjsor.v12i2.1096
108. Yousof, H. M., Afify, A. Z., Nadarajah, S., Hamedani, G. G., & Aryal, G. R. (2018). The Marshall-Olkin Generalized-G Family of Distributions with Applications. Statistica , 78(3), 273–295. https://doi.org/10.6092/issn.1973-2201/8424
109. Yousof, H., Afshari, M., Alizadeh, M., Ranjbar, V., Minkah, R., Hamed, M. S., & Salem, M. (2025). A Novel Insurance Claims (Revenues) Xgamma Extension: Distributional Risk Analysis Utilizing Left-Skewed Insurance Claims and Right-Skewed Reinsurance Revenues Data with Financial PORT-VaR Analysis. Pakistan Journal of Statistics and Operation Research, 83-117.
110. Yousof, H. M., Aidi, K., Hamedani, G. G., & Ibrahim, M. (2021a). A new parametric lifetime distribution with modified Chi-square type test for right censored validation, characterizations and different estimation methods. Pakistan Journal of Statistics and Operation Research, 17(2), 399–425.
111. Yousof, H. M., Ali, E. I. A., Aidi, K., Butt, N. S., Saber, M. M., Al-Nefaie, A. H., Aljadani, A., Mansour, M. M., Hamed, M. S., & Ibrahim, M. (2025a). The Statistical Distributional Validation under a Novel Generalized Gamma Distribution with Value-at-Risk Analysis for the Historical Claims, Censored and Uncensored Real-life Applications. Pakistan Journal of Statistics and Operation Research, 21(1), 51-69. https://doi.org/10.18187/pjsor.v21i1.4534
112. Yousof, H. M., Ali, M. M., Aidi, K., & Ibrahim, M. (2023a). The modified Bagdonavičius-Nikulin goodness-of-fit test statistic for the right censored distributional validation with applications in medicine and reliability. Statistics in Transition New Series , 24(4), 1–18.
113. Yousof, H. M., Ali, M. M., Goual, H., & Ibrahim, M. (2021b). A new reciprocal Rayleigh extension: properties, copulas, different methods of estimation and modified right censored test for validation. Statistics in Transition New Series , 23(3), 1–23.
114. Yousof, H. M., Ali, M. M., Hamedani, G. G., Aidi, K., & Ibrahim, M. (2022). A new lifetime distribution with properties, characterizations, validation testing, different estimation methods. Statistics, Optimization & Information Computing , 10(2), 519–547.
115. Yousof, H. M., Aljadani, A., Mansour, M. M., & Abd Elrazik, E. M. (2024). A New Pareto Model: Risk Application, Reliability MOOP and PORT Value-at-Risk Analysis. Pakistan Journal of Statistics and Operation Research , 20(3), 383–407. https://doi.org/10.18187/pjsor.v20i3.4151 .
116. Yousof, H. M., Altun, E., Ramires, T. G., Alizadeh, M., & Rasekhi, M. (2018). A new family of distributions with properties, regression models and applications. Journal of Statistics and Management Systems , 21(1), 163–188.
117. Yousof, H. M., Altun, E., Rasekhi, M., Alizadeh, M., Hamedani, G. G., & Ali, M. M. (2019). A New Lifetime Model with Regression Models, Characterizations, and Applications. Communications in Statistics - Simulation and Computation , 48(1), 264–286. https://doi.org/10.1080/03610918.2017.1367801
118. Yousof, H. M., Ansari, S. I., Tashkandy, Y., Emam, W., Ali, M. M., Ibrahim, M., Alkhayyat, S. L. (2023b). Risk Analysis and Estimation of a Bimodal Heavy-Tailed Burr XII Model in Insurance Data: Exploring Multiple Methods and Applications. Mathematics , 11(9), 2179. https://doi.org/10.3390/math11092179 .
119. Yousof, H. M., Goual, H., Emam, W., Tashkandy, Y., Alizadeh, M., Ali, M. M., & Ibrahim, M. (2023c). An Alternative Model for Describing the Reliability Data: Applications, Assessment, and Goodness-of-Fit Validation Testing. Mathematics , 11(6), 1308.
120. Yousof, H. M., Goual, H., Hamida, T., Hiba, A., Hamedani, G.G., & Ibrahim, M. (2022a). Censored and Uncensored Nikulin-Rao-Robson Distributional Validation: Characterizations, Classical and Bayesian estimation with Applications.
121. Yousof, H. M., Goual, H., Khaoula, M. K., Hamedani, G. G., Al-Aefaie, A. H., Ibrahim, M., ... & Salem, M. (2023). A novel accelerated failure time model: Characterizations, validation testing, different estimation methods and applications in engineering and medicine. Pakistan Journal of Statistics and Operation Research , 19(4), 691–717.
122. Yousof, H. M., Korkmaz, M. Ç., K., Hamedani, G. G and Ibrahim, M. (2022b). A novel Chen extension: theory, characterizations and different estimation methods. Eur. J. Stat, 2(2022), 1-20.
123. Yousof, H. M., Saber, M. M., Al-Nefaie, A. H., Butt, N. S., Ibrahim, M., & Alkhayyat, S. L. (2024). A discrete claims-model for the inflated and over-dispersed automobile claims frequencies data: Applications and actuarial risk analysis. Pakistan Journal of Statistics and Operation Research , 261–284.
124. Yousof, H. M., Yousof, H. M., Ali, E. I. A., Aidi, K., Butt, N. S., Saber, M. M., Al-Nefaie, A. H., Aljadani, A., Mansour, M. M., Hamed, M. S., & Ibrahim, M. (2025b). The Statistical Distributional Validation under a Novel Generalized Gamma Distribution with Value-at-Risk Analysis for the Historical Claims, Censored and Uncensored Real-life Applications. Pakistan Journal of Statistics and Operation Research, 21(1), 51–69. https://doi.org/10.18187/pjsor.v21i1.4534 .
125. Yousof, H., Afshari, M., Alizadeh, M., Ranjbar, V., Minkah, R., Hamed, M. S., & Salem, M. (2025c). A Novel Insurance Claims (Revenues) Xgamma Extension: Distributional Risk Analysis Utilizing Left-Skewed Insurance Claims and Right-Skewed Reinsurance Revenues Data with Financial PORT-VaR Analysis. Pakistan Journal of Statistics and Operation Research, 83-117.
126. Yousof, H.M.; Emam, W.; Tashkandy, Y.; Ali, M.M.; Minkah, R.; Ibrahim, M. (2023d). A Novel Model for Quantitative Risk Assessment under Claim-Size Data with Bimodal and Symmetric Data Modeling. Mathematics , 11, 1284. https://doi.org/10.3390/math11061284 .
127. Zamani, Z., Afshari, M., Karamikabir, H., Alizadeh, M., & Ali, M. M. (2022). Extended Exponentiated Chen Distribution: Mathematical Properties and Applications. Statistics, Optimization & Information Computing , 10(2), 606–626.