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# Modifying spectral conjugate gradient method for solving Iteration Problems

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**Abstract** One well-known and simple technique for minimizing the models is to use spectral conjugate gradients. In this work, we construct a unique Spectrum using the gradient approach to expansion a novel search Path, we have proven that the novel spectral approach Possesses the descent property and that the spectrum methodology is globally convergent. The experimental results indicate that for the test problems, the suggested methodology may be computed in conjunction with alternative conjugate gradient techniques.

**Keywords** Conjugate Gradient Methods, Global convergence, parameter, Iteration problems.

**AMS 2010 subject classifications** 90C06, 65K05, 90C26

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## 1. Introduction

We will examine the problem of optimization minimization as delineated below:

$$min_{x\in R^n} f(x) \tag{1}$$

where $f : R^n \longrightarrow R$ is a smooth function, whose gradient is assumed for convenience. For further information, consult [22]. The most effective categories of iterative methodologies are the conjugate gradient (CG) procedures utilized to address iterative challenges, as they do not necessitate the utilization of matrices and are typically characterized by high efficiency. For further insight, refer to [21]. In the context of CG-algorithms, a sequence $\{x_k\}$ is effectively determined by the following iterative process:

$$x_0 \in R^n,\ x_{\tau+1} = x_\tau + \alpha_\tau d_\tau \tag{2}$$

where $\alpha_\tau > 0$ represents the step size and $d_\tau$ denotes the direction. The search direction is generated by:

$$d_{\tau+1} = -g_{\tau+1} + \beta_\tau d_\tau \tag{3}$$

where $\beta_\tau$ is designated as the CG-parameter. The step-size $\alpha_k$ adheres to specific line search criteria, such as the conventional Wolfe line search:

$$f(x_\tau + \alpha_\tau d_\tau) \le f(x_\tau) + \delta\alpha_\tau g_\tau^T d_\tau \tag{4}$$

$$g(x_\tau + \alpha_\tau d_\tau)^T d_\tau \ge \sigma g_\tau^T d_\tau \tag{5}$$

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where $0 < \delta < \sigma < 1$ , see in [17]. The most recognized expressions for the parameter $\beta_\tau$ in CG-methods is the Hestenes-Stiefel method [13], which is defined as follows:

$$\beta_\tau^{HS} = \frac{g_{\tau+1}^T y_\tau}{d_\tau^T y_\tau} \tag{6}$$

In contrast to conventional CG-methods, in the spectral CG- approach, the search direction $d_{\tau+1}$ is delineated as:

$$d_{\tau+1} = -\theta_\tau g_{\tau+1} + \beta_\tau d_\tau \tag{7}$$

where $\theta_\tau$ is referred to as a spectral constant. It is readily apparent that (7) simplifies to (3) when $\theta_\tau = 1$. Additional details can be located in [14].

In this influence, enhancements are introduced to the CG-methods, thereby augmenting the effectiveness of the foundational techniques. The ensuing alterations incorporate spectral constants:

$$\theta_\tau^{DY} = \theta_\tau^{HS} = \frac{y_\tau^T d_\tau}{y_\tau^T d_\tau} = 1 \ , \ \theta_\tau^{PR} = \theta_\tau^{FR} = \frac{y_\tau^T d_\tau}{g_\tau^T g_\tau} \ , \ \theta_\tau^{LS} = \beta_\tau^{CD} = \frac{y_\tau^T d_\tau}{|g_\tau^T d_\tau|} \tag{8}$$

further information can be located in [8, 5, 3, 7, 18].

CG-methods have been associated with various global convergence results, which are validated in [25, 19, 23]. The objective of this manuscript is to derive the spectral constant, as well as to investigate its convergence properties, arithmetical results, and subsequent dialogue.

## 2. A novel CG-method

Basim A. Hassan et al. [8] Suggest a CG parameter based on the Taylor Series (quadratic model), which we represent as follows:

$$\beta_\tau = \frac{\|g_{\tau+1}\|^2}{(f_\tau - f_{\tau+1})/\alpha_\tau + 1/2 d_\tau^T y_\tau} \tag{9}$$

On the basis of a parameter conjugate gradient, we drive a unique spectral conjugate gradient.

We will articulate a conceptual framework for a novel methodology pertaining to the spectral conjugate gradients, which facilitates the generation of a new algorithm. Consequently, the parameters of the conjugate gradient method satisfy the relationship:

$$\beta_\tau \leq \frac{-g_{\tau+1}^T d_{\tau+1}}{-g_\tau^T d_\tau} \tag{10}$$

then it can be regarded as:

$$\beta_\tau g_\tau^T d_\tau = g_{\tau+1}^T d_{\tau+1} = \frac{\|g_{\tau+1}\|^2}{(f_\tau - f_{\tau+1})/\alpha_\tau + 1/2 d_\tau^T y_\tau} g_\tau^T d_\tau \tag{11}$$

From the definition of $y_\tau$, we obtain:

$$y_\tau^T d_\tau = g_{\tau+1}^T d_\tau - g_\tau^T d_\tau \tag{12}$$

It is obvious that:

$$\begin{aligned} g_{\tau+1}^T d_{\tau+1} &= \frac{\|g_{\tau+1}\|^2}{\frac{f_\tau - f_{\tau+1}}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau} \left[ \frac{y_\tau^T d_\tau}{\frac{f_\tau - f_{\tau+1}}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau} \left( \frac{f_\tau - f_{\tau+1}}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau \right) - g_{\tau+1}^T d_\tau \right] \\ &= - \frac{y_\tau^T d_\tau}{\frac{f_\tau - f_{\tau+1}}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau} \|g_{\tau+1}\|^2 + \frac{\|g_{\tau+1}\|^2}{\frac{f_\tau - f_{\tau+1}}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau} g_{\tau+1}^T d_\tau \end{aligned} \tag{13}$$

$$g_{\tau+1}^T d_{\tau+1} = -\theta_\tau \|g_{\tau+1}\|^2 + \frac{\|g_{\tau+1}\|^2}{\frac{(f_\tau - f_{\tau+1})}{\alpha_\tau} + \frac{1}{2d_\tau^T y_\tau}} g_{\tau+1}^T d_\tau \tag{14}$$

where:

$$\theta_\tau^{NBN} = \frac{y_\tau^T d_\tau}{[(f_\tau - f_{\tau+1})/\alpha_\tau + 1/2d_\tau^T y_\tau]} \tag{15}$$

which we term the NBN method. Based on equation (15), this can provide an innovative search trajectory:

$$d_{\tau+1} = -\theta_\tau^{NBN} g_{\tau+1} + \frac{\|g_{\tau+1}\|^2}{(f(x_\tau) - f(x_{\tau+1}))/\alpha_\tau + 1/2d_\tau^T y_\tau} d_\tau \tag{16}$$

Employing the aforementioned process, an existing method is articulated as follows:

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NBN Algorithm:

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1: **Step 0:** Choose $x_0 \in \mathbb{R}^n$, $0 < \delta_1 < \delta_2 < 1$, set $d_0 = -g_0$.
2: **Step 1:** If $\|g_{\tau+1}\|^2 \leq 10^{-6}$, stop.
3: **Step 2:** Compute $\beta_\tau$ by (9) with $\theta^{NBN}$ by (15).
4: **Step 3:** Set $x_{\tau+1} = x_\tau + \alpha_\tau d_\tau$, where $\alpha_\tau$ satisfies (4) and (5).
5: **Step 4:** Compute

$$d_{\tau+1} = -\theta_\tau g_{\tau+1} + \beta_\tau d_\tau.$$

6: **Step 5:** Set $\tau = \tau + 1$ and go to Step 1.

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The descent property of the proposed process is demonstrated by the following theorem.

*Theorem 1*
Let Algorithm **??** yield sequence xo, then $g_{\tau+1}^T d_{\tau+1} \leq -C\|g_{\tau+1}\|^2$ for every $\tau$.

*Proof*
So $d_o = -g_o$ we take $g_o^T d_o = \|g_o\|^2 < 0$. Let $g_\tau^T d_\tau < -C_1\|g_\tau\|^2$ . Increasing (16) by $g_{\tau+1}$ and using (12), we have:

$$g_{\tau+1}^T d_{\tau+1} = -\theta^{NBN} g_{\tau+1}^T g_{\tau+1} + \frac{\|g_{\tau+1}\|^2}{(f(x_\tau) - f(x_{\tau+1}))/\alpha_\tau + 1/2d_\tau^T y_\tau} g_{\tau+1}^T d_\tau \tag{17}$$

$$\begin{aligned} g_{\tau+1}^T d_{\tau+1} &= \frac{\|g_{\tau+1}\|^2}{\frac{f(x_\tau) - f(x_{\tau+1})}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau} \left[-y_\tau^T d_\tau + g_{\tau+1}^T d_\tau\right] \\ &= \frac{\|g_{\tau+1}\|^2}{\frac{f(x_\tau) - f(x_{\tau+1})}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau} g_\tau^T d_\tau \\ &= \frac{g_\tau^T d_\tau}{\frac{f(x_\tau) - f(x_{\tau+1})}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau} \|g_{\tau+1}\|^2 \end{aligned} \tag{18}$$

Since $g_\tau^T d_\tau < -C_1\|g_\tau\|^2$ then we have:

$$g_{\tau+1}^T d_{\tau+1} < -C_1 \frac{\|g_\tau\|^2}{(f(x_\tau) - f(x_{\tau+1}))/\alpha_\tau + 1/2d_\tau^T y_\tau} \|g_{\tau+1}\|^2 \tag{19}$$

where $C = C_1 \frac{\|g_\tau\|^2}{(f(x_\tau)-f(x_{\tau+1}))/\alpha_\tau + 1/2 d_\tau^T y_\tau}$. Since $C_1, f(x_\tau) - f(x_{\tau+1})/\alpha_\tau + 1/2 d_\tau^T y_\tau$ and $\|g_\tau\|^2$ are positive, therefore the value of C is similarly positive.

$$\therefore g_{\tau+1}^T d_{\tau+1} \leq -C\|g_{\tau+1}\|^2 \tag{20}$$

□

## 3. Global convergence

We will examine the global convergence of NBN. In order to support the primary conclusions of this study, we first provide the following modest hypothesis.

*Assumption 1*
i- Let level set $L = \{x \in R^n \setminus f(x) \leq f(x_0)\}$ is bounded.
ii- In some locality $U$ and $L$, $f(x)$ is smooth and its gradient is Lipschitz continuous, viz, there exists a constant $\mu_1 > 0$ obtain:

$$\|g_{\tau+1} - g_\tau\| \leq \mu_1 \|x_{\tau+1} - x_\tau\|, \ \forall x_{\tau+1}, x_\tau \in U \tag{21}$$

See [25, 20].

The global convergence of the suggested algorithms is demonstrated using the outcome of the following lemma, commonly known as the Zoutendijk condition. This was first provided by Zoutendijk [26].

*Lemma 1*
Let assumptions holds. $If\ g_{\tau+1}^T d_{\tau+1} \leq 0$ with $\alpha_\tau$ satisfies the (4),(5). Then

$$\sum_{\tau=1}^{\infty} \frac{(g_\tau^T d_\tau)^2}{\|d_\tau\|^2} < +\infty \tag{22}$$

The following theorem highlights the suggested approaches' global convergence.

*Theorem 2*
Let assumptions holds and $\{g_{\tau+1}\}$ and $\{d_{k+1}\}$ be generated by Algorithm ??. Then

$$\lim_{\tau\to\infty} inf\ \|g_{\tau+1}\| = 0 \tag{23}$$

*Proof*
Under the given conditions, Lemma 1 is true. In the following, the inference will be made via implication. Assume, by inconsistency, that near is a positive constant $\in_1 > 0$.

$$\|g_{\tau+1}\| > \in_1 \tag{24}$$

Rewriting (18) as:

$$d_{\tau+1} + \theta_\tau^{NBN} g_{\tau+1} = B_\tau^\beta d_\tau \tag{25}$$

After squaring its two sides, we obtain:

$$\|d_{\tau+1}\|^2 + (\theta_\tau^{NBN})^2 \|g_{\tau+1}\|^2 + 2\theta_\tau^{NBN} d_{\tau+1} g_{\tau+1} = (B_\tau^\beta)^2 \|d_\tau\|^2 \tag{26}$$

From (26), we get

$$\|d_{\tau+1}\|^2 = (B_\tau^\beta)^2 \|d_\tau\|^2 - 2\theta_\tau^{NBN} d_{\tau+1}^T g_{\tau+1} - (\theta_\tau^{NBN})^2 \|g_{\tau+1}\|^2 \tag{27}$$

From the above equation and (10), we have:

$$\|d_{\tau+1}\|^2 \leq \left(\frac{g_{\tau+1}^T d_{\tau+1}}{g_\tau^T d_\tau}\right)^2 \|d_\tau\|^2 - 2\theta_\tau^{NBN} d_{\tau+1}^T g_{\tau+1} - (\theta_\tau^{NBN})^2 \|g_{\tau+1}\|^2 \tag{28}$$

Diving the both side of the inequality by $(g_{\tau+1}^T d_{\tau+1})^2$ we have:

$$\begin{aligned}
\frac{\|d_{\tau+1}\|^2}{(d_{\tau+1}^T g_{\tau+1})^2} &\le \frac{\|d_\tau\|^2}{(d_\tau^T g_\tau)^2} - (\theta_\tau^{NBN})^2 \frac{\|d_{\tau+1}\|^2}{(d_{\tau+1}^T g_{\tau+1})^2} - 2\theta_\tau^{NBN} \frac{1}{d_{\tau+1}^T g_{\tau+1}} \\
&\le \frac{\|d_\tau\|^2}{(d_\tau^T g_\tau)^2} - (\theta_\tau^{NBN})^2 \frac{\|g_{\tau+1}\|^2}{C^2\|g_{\tau+1}\|^4} - 2\theta_\tau^{NBN} \frac{1}{C\|g_{\tau+1}\|^2} \\
&\quad - \frac{1}{\|g_{\tau+1}\|^2} + \frac{1}{\|g_{\tau+1}\|^2} \\
&\le \frac{\|d_\tau\|^2}{(d_\tau^T g_\tau)^2} - \left(\theta_\tau^{NBN} \frac{\|g_{\tau+1}\|}{C\|g_{\tau+1}\|^2} + \frac{1}{\|g_{\tau+1}\|}\right) + \frac{1}{\|g_{\tau+1}\|^2} \\
\frac{\|d_{\tau+1}\|^2}{(d_{\tau+1}^T g_{\tau+1})^2} &\le \frac{\|d_\tau\|^2}{(d_\tau^T g_\tau)^2} + \frac{1}{\|g_{\tau+1}\|^2} \qquad (29)
\end{aligned}$$

Applying (29) to recurrence, see that $\|d_1\|^2 = -g_1^T d_1 = \|g_1\|^2$ , we get:

$$\frac{\|d_{\tau+1}\|^2}{\left(d_{\tau+1}^T g_{\tau+1}\right)^2} \le \sum_{i=1}^{\tau} \frac{1}{\|g_i\|^2} \qquad (30)$$

Then we get from (29) and (24) that

$$\frac{\left(g_\tau^T d_\tau\right)^2}{\|d_\tau\|^2} \ge \frac{\varepsilon_1^2}{\tau} \qquad (31)$$

which indicates

$$\sum_{\tau=1}^{\infty} \frac{\left(g_\tau^T d_\tau\right)^2}{\|d_\tau\|^2} \ge \sum_{\tau=1}^{\infty} \frac{\varepsilon_1^2}{\tau} = \infty \qquad (32)$$

□

## 4. Arithmetical Results

By calculating the value of the performance measure, each of the following strategies arrived at the best outcome. Depending on the number on the number of function (FUNC) and the number of iterations (ITER) each function has attest.

The results of our numerical tests comparing the NBN in this work with the algorithm assumed by HS in [13] are presented in this section.

Table 1. Arithmetical Results of NBN-Algorithm and HS-Algorithm

| Test Function | NBN Algorithm | | HS Algorithm | |
|---|---|---|---|---|
| | ITER | FUNC | ITER | FUNC |
| 1 | 33 | 16 | 42 | 26 |
| | 35 | 18 | 44 | 18 |
| 2 | 13 | 7 | 24 | 11 |
| | 12 | 8 | 11 | 7 |
| 3 | 21 | 14 | 23 | 17 |
| | 32 | 13 | 72 | 36 |
| 4 | 33 | 14 | 77 | 34 |

*Continued on next page*

**Table 1 – Continued**

| Test Function | NBN Algorithm FUNC | NBN Algorithm ITER | HS Algorithm FUNC | HS Algorithm ITER |
|---|---|---|---|---|
| | 26 | 11 | 48 | 16 |
| 5 | 59 | 11 | 121 | 42 |
| | 168 | 32 | 435 | 189 |
| 6 | 79 | 27 | 85 | 30 |
| | 67 | 34 | 59 | 28 |
| 7 | 73 | 27 | 95 | 35 |
| | 239 | 81 | 339 | 93 |
| 8 | 71 | 62 | 107 | 42 |
| | 71 | 51 | 103 | 40 |
| 9 | 10 | 7 | 14 | 8 |
| | 16 | 6 | 16 | 6 |
| 10 | 84 | 27 | 122 | 66 |
| | 84 | 27 | 122 | 66 |
| 11 | 7 | 5 | 8 | 5 |
| | 7 | 4 | 13 | 5 |
| 12 | 8 | 4 | 14 | 7 |
| | 6 | 4 | 7 | 5 |
| 13 | 41 | 14 | 38 | 11 |
| | 63 | 22 | 75 | 22 |
| 14 | 30 | 13 | 38 | 11 |
| | 52 | 23 | 62 | 23 |
| 15 | 86 | 36 | 123 | 74 |
| | 241 | 76 | 251 | 105 |
| 16 | 120 | 77 | 187 | 144 |
| | 87 | 18 | 76 | 22 |
| 17 | 143 | 7 | 120 | 32 |
| | 21 | 11 | 26 | 14 |
| 18 | 107 | 33 | 191 | 41 |
| | 101 | 87 | 120 | 117 |
| 19 | 48 | 44 | 85 | 77 |
| | 71 | 66 | 93 | 88 |
| 20 | 106 | 63 | 183 | 89 |
| | 74 | 55 | 98 | 95 |
| 21 | 6 | 5 | 33 | 16 |
| | 144 | 8 | 167 | 17 |
| 22 | 92 | 15 | 121 | 20 |
| | 86 | 8 | 67 | 14 |
| 23 | 40 | 22 | 55 | 24 |
| | 60 | 49 | 125 | 71 |
| 24 | 85 | 17 | 91 | 21 |
| | 123 | 43 | 88 | 18 |
| 25 | 103 | 75 | 145 | 66 |
| | 32 | 7 | 39 | 16 |
| **Total** | **2616** | **2204** | **3800** | **3048** |

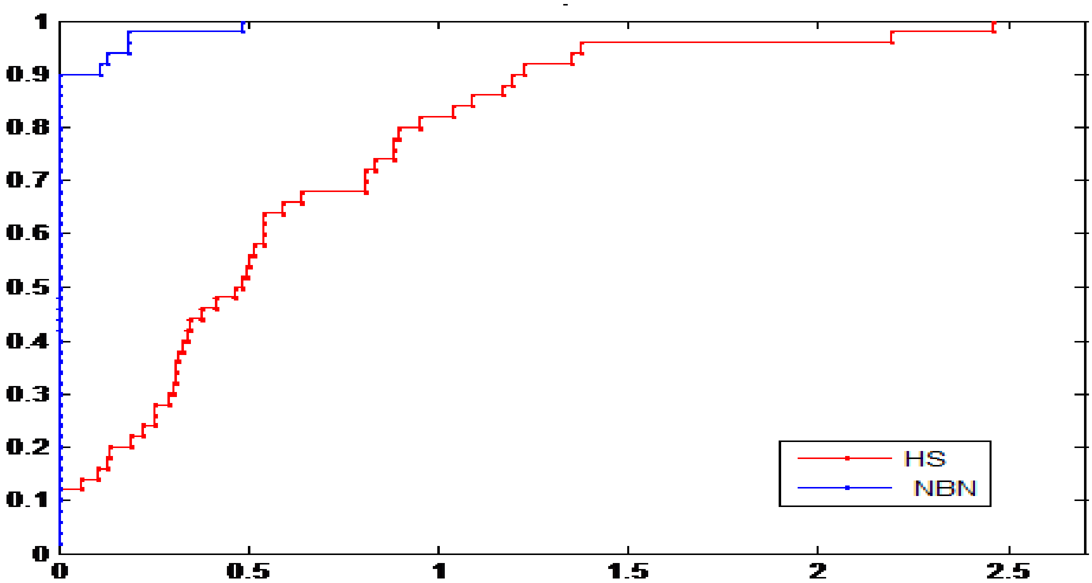
Table 2. Test Functions

| NO. Of Function | Test Function | DIM |
|---|---|---|
| 1 | Extended White & Holst | 100, 1000 |
| 2 | Penalty | 100, 1000 |
| 3 | Extended Beale | 100, 1000 |
| 4 | Extended Wood | 100, 1000 |
| 5 | Quadratic Diagonal P. | 100, 1000 |
| 6 | Extended Cliff | 100, 1000 |
| 7 | Perturbed Quadratic | 100, 1000 |
| 8 | Extended Hiebert | 100, 1000 |
| 9 | NONDIA (CUTE) | 100, 1000 |
| 10 | DIXMAAN E (CUTE) | 100, 1000 |
| 11 | DIXMAAN F (CUTE) | 100, 1000 |
| 12 | Extended PSC1 | 100, 1000 |
| 13 | TRIDIA (CUTE) | 100, 1000 |
| 14 | Generalized Tridiagonal 2 | 100, 1000 |
| 15 | Extended Tridiagonal 1 | 100, 1000 |
| 16 | Raydan 1 | 100, 1000 |
| 17 | Trigonometric | 100, 1000 |
| 18 | Diagonal 1 | 100, 1000 |
| 19 | DIXMAAN K | 100, 1000 |
| 20 | Extended Quadratic QP2 | 100, 1000 |
| 21 | Freudenstein & Roth | 100, 1000 |
| 22 | Rosen brock | 100, 1000 |
| 23 | Broyden Tridiagonal | 100, 1000 |
| 24 | ARWHEAD (CUTE) | 100, 1000 |
| 25 | Diagonal 4 | 100, 1000 |

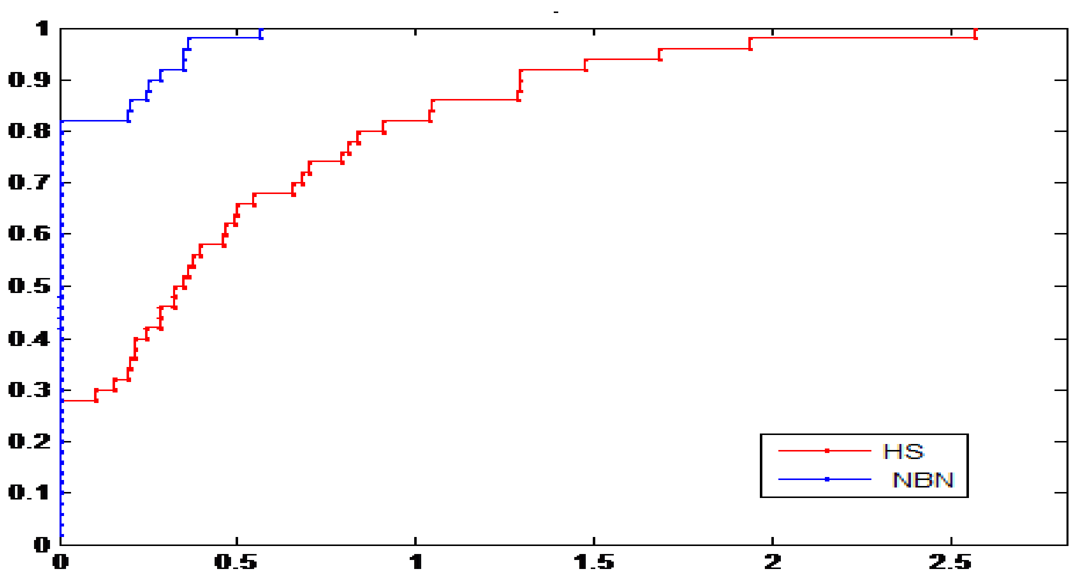
Dolan and More [24] examined how to use a cumulative performance profiling tool (similar to how it was developed) to evaluate our proposed algorithm's performance. We will discuss the application of this tool, which uses a cumulative distribution function (CDF) to describe the ratio of resolution successes for all tested samples to the base reference value. Performance ratios are plotted on the CDF's horizontal axis while each test sample that was resolved is plotted on the CDF's vertical axis. An algorithm's CDF with respect to the reference point has a higher zone than all other algorithms' CDFs and therefore has an advantage because it resolves more samples than any of the other tested algorithms based upon its advantage.

Our first figure presents a comparison of the CDF performance profiles for our modified (NBN) algorithm and the original HS. The cumulative performance profile comparison shows how quickly each algorithm converges to an optimal solution based on the number of iterations needed. The NBN algorithm, despite achieving a higher cumulative performance than the traditional HS algorithm, does so more quickly. As a result, for 25 of the test functions, the proposed NBN algorithm will converge in fewer iterations compared to the traditional HS algorithm.

Conversely, due to converging more slowly than the proposed NBN Algorithm, the traditional HS will require more iterations to converge. Therefore, the traditional HS algorithm is less efficient than the proposed NBN Algorithm.




The data presented in this document indicate that the cost of executing an optimization algorithm is influenced by the number of function evaluations to achieve optimal performance. In addition, the data suggest that the proposed NBN technique has a computational advantage due to a smaller number of function evaluations being needed to reach optimal performance compared to the traditional HS technique. In general, most of the time used to obtain an optimized solution can be attributed to the computational resources consumed to complete function evaluations. As a result, the numerical data supports the recommendation that the NBN technique be employed in actual optimization.



These findings substantiate the endorsement of the proposed NBN method for practical optimization challenges where efficiency is paramount. A wide range of studies have addressed this issue from multiple angles [16, 15, 9, 4, 2, 1], thereby reinforcing the theoretical framework of the current research. Given the recent developments in Quasi-Newton methods reported in [11, 10, 6, 12].

## 5. Conclusions

One famous and simple method for minimizing the functionality is the conjugate Gradient of Spectrum. We constract an unique globally convergent spectral conjugate gradient to meet the acceptable descent Standards.

To do numerical testing, we employ a range of diffuse test functions. By applying to the calculated results there were 31% less, iterations, and 44% less function evaluation total. We drive anew spectral conjugate gradient based on a CG-parameter. In this part, we will describe an idea of a novel approach to the spectral conjugate of gradients and it is generated anew.

Table 3. The new algorithm's relative efficiency

| Tools | **ITER** | **FUNC** |
|---|---|---|
| HS-Algorithm | 100.% | 100.% |
| NBN-Algorithm | 69.23% | 57.73% |

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