

New Types of Integral Contractions in Supra Metric Space

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Abstract In the present article, we shall define the new notions of generalized $(\mathcal{S} - \psi)$ contractions of integral type A and B and prove the related fixed point theorems in the setting of supra metric space. Then, we shall deduce some new results from the proved results in the form of corollaries. Some examples will also be given to show the real existence of proved results.

Keywords fixed point, integral type A, integral type B, supra metric space

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1. Introduction

By seeing the utility of Banach contraction principle in metric space, many authors tried to generalize the notion of Banach contraction principle [5] and metric space. In 2002, Branciari gave the new notion of integral type contraction which was an analogue result for Banach contraction principle. Many fixed point results by various authors are given in ([1, 3, 4, 8, 13, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 36]).

In 2012, Samet *et al.* [35] defined the $\mathcal{S} - \psi$ -contraction by defining the \mathcal{S} admissible mappings as follows:

Definition 1. [35] Suppose \mathfrak{N} be a non empty set and a map $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$. Then, a self map \mathcal{T} on \mathfrak{N} is said to be \mathcal{S} -admissible if for $a, b \in \mathfrak{N}$

$$\mathcal{S}(a, b) \geq 1 \Rightarrow \mathcal{S}(\mathcal{T}a, \mathcal{T}b).$$

Berzig and Rus [9] gave the following definition

Definition 2. [9] Suppose $N \in \mathbb{N}$. Then, we say that \mathcal{S} is N -transitive on \mathfrak{N} if

$$a_0, a_1, \dots, a_{N+1} \in \mathfrak{N} : \mathcal{S}(a_i, a_{i+1}) \geq 1$$

for all $i = 0, 1, 2, \dots, N$

implies that

$$\mathcal{S}(a_0, a_{N+1}) \geq 1.$$

From Definition 2, we can conclude that

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Remark 1. [9]

1. Any function $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is 0-transitive.
2. If \mathcal{S} is N -transitive then it is pN -transitive for all $p \in \mathbb{N}$.
3. $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is transitive then it is N -transitive for all $N \in \mathbb{N}$.
4. If \mathcal{S} is transitive then it is not necessary that it is transitive for all $N \in \mathbb{N}$.

Let $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

1. ψ is monotonically increasing;
2. $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$.

The collection of all such type of function will be denoted by Ψ .

Define $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ a non-negative Lebesgue integrable function such that

$$\int_0^\epsilon \phi(t)dt > 0 \text{ for all } \epsilon > 0.$$

Collection of such function will be denoted by Φ .

In 2014, Shahi *et al.* [15] gave the new notions of $\mathcal{S} - \psi$ contractions of integral type I and II as stated below:

Definition 3. [15] Suppose \mathcal{T} be a self map on a metric space (\mathfrak{N}, σ) . We say that \mathcal{T} is a generalized $\mathcal{S} - \psi$ -contractive mapping of integral type I if there exists two functions $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that for all $a, b \in \mathfrak{N}$,

$$\mathcal{S}(a, b) \int_0^{\sigma(\mathcal{T}a, \mathcal{T}b)} \phi(t)dt \leq \psi \left(\int_0^{M(a,b)} \phi(t)dt \right)$$

where $\phi \in \Phi$ and $M(a, b) = \max\{\sigma(a, b), \sigma(a, \mathcal{T}a), \sigma(b, \mathcal{T}b), \frac{\sigma(a, \mathcal{T}b) + \sigma(b, \mathcal{T}a)}{2}\}$.

Definition 4. [15] Suppose \mathcal{T} be a self map on a metric space (\mathfrak{N}, σ) . We say that \mathcal{T} is a generalized $\mathcal{S} - \psi$ -contractive mapping of integral type II if there exists two functions $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that for all $a, b \in \mathfrak{N}$,

$$\mathcal{S}(a, b) \int_0^{\sigma(\mathcal{T}a, \mathcal{T}b)} \phi(t)dt \leq \psi \left(\int_0^{M(a,b)} \phi(t)dt \right)$$

where $\phi \in \Phi$ and $M(a, b) = \max\{\sigma(a, b), \frac{\sigma(a, \mathcal{T}b) + \sigma(a, \mathcal{T}a)}{2}, \frac{d(b, \mathcal{T}b) + d(b, \mathcal{T}a)}{2}\}$.

Then, they proved the following theorems.

Theorem 1. [15] Suppose (\mathfrak{N}, σ) be a complete metric space and $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is a transitive mapping. Suppose that $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ is a generalized $\mathcal{S} - \psi$ -contractive mapping of integral type I and satisfies the following conditions:

1. \mathcal{T} is \mathcal{S} -admissible;
2. there exists $a_0 \in \mathfrak{N}$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$;
3. \mathcal{T} is continuous.

Then, \mathcal{T} has a fixed point, that is, there exists $a \in \mathfrak{N}$ such that $\mathcal{T}a = a$.

Theorem 2. [15] Suppose (\mathfrak{N}, σ) be a complete metric space and $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is a transitive mapping. Suppose that $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ be a a generalized $\mathcal{S} - \psi$ -contractive mapping of integral type II and satisfies the following conditions:

1. \mathcal{T} is \mathcal{S} -admissible;
2. there exists $a_0 \in \mathfrak{N}$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$;
3. \mathcal{T} is continuous.

Then, \mathcal{T} has a fixed point, that is, there exists $a \in \mathfrak{N}$ such that $\mathcal{T}a = a$.

Then, many authors generalize the notion of integral contraction in metric space and other generalize spaces see ([6, 11, 15, 12, 14, 16, 18, 19, 20, 21, 22, 34, 37]).

In the spade of generalization of notion of metric space, in 2022, Berzig [7] gave the notion of supra metric space as follows:

Definition 5. [7] Let \mathfrak{N} be a non-empty set and $\sigma_{sm} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ be a map such that

1. $\sigma_{sm}(a, b) = 0$ iff $a = b$;
2. $\sigma_{sm}(a, b) = \sigma_{sm}(b, a)$;
3. $\sigma_{sm}(a, b) \leq \sigma_{sm}(a, c) + \sigma_{sm}(c, b) + \rho \cdot \sigma_{sm}(a, c)\sigma_{sm}(c, b)$;

for all $a, b, c \in \mathfrak{N}$, for some constant $\rho \in [0, \infty)$.

Then, σ_{sm} is called **supra metric** and the ordered pair $(\mathfrak{N}, \sigma_{sm})$ is called **supra metric space**.

Example 1. [7] Suppose (\mathfrak{N}, σ) be a metric space and α, β are positive real numbers. Then

1. $\sigma_{sm_1}^\alpha(a, b) = \sigma(a, b)(\sigma(a, b) + \alpha)$ are supra metrics with constant $\rho = \frac{2}{\alpha}$.
2. $\sigma_{sm_2}^\alpha(a, b) = \beta(e^{\sigma(a, b)} - 1)$ are supra metrics with constant $\rho = \frac{1}{\beta}$.

But, $\sigma_{sm_1}^\alpha$ and $\sigma_{sm_2}^\beta$ are not necessary usual metrics. For example for $\alpha = 2$ and $\sigma(a, b) = |a - b|$
 $\sigma_{sm_1}^2(4, 0) > \sigma_{sm_1}^2(4, 2) + \sigma_{sm_1}^2(2, 0)$.

Definition 6. [7] Let $(\mathfrak{N}, \sigma_{sm})$ be a supra metric space. The set

$$B(a_0, r) := \{a \in \mathfrak{N} : \sigma_{sm}(a_0, a) < r\},$$

where $r > 0$ and $a_0 \in \mathfrak{N}$, is called open of radius r and center a_0 . A subset \mathfrak{M} of \mathfrak{N} is called open if for any point $a \in \mathfrak{M}$ there exists $r > 0$ such that $B(a, r) \subset \mathfrak{M}$. The family of all open subsets of \mathfrak{N} will be denoted by τ .

Proposition 1. [7] Let $(\mathfrak{N}, \sigma_{sm})$ be a supra metric space. Then, each open ball is an open set.

Definition 7. [7] Let $(\mathfrak{N}, \sigma_{sm})$ be a supra metric space. A sequence $\{a_n\} \in \mathfrak{N}$ converges to a if for all $\epsilon > 0$ the ball $B(a, \epsilon)$ contains all but a finite numbers of terms of the sequence. So, a is the limit of the sequence and we say that

$$\lim_{n \rightarrow \infty} \sigma_{sm}(a_n, a) = 0.$$

Proposition 2. [7] Let $(\mathfrak{N}, \sigma_{sm})$ be a supra metric space. If the sequence $\{a_n\}$ has a limit, then it is unique.

Definition 8. [7] Let $(\mathfrak{N}, \sigma_{sm})$ be a supra metric space. A mapping $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ is said to be continuous at a if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\sigma_{sm}(\mathcal{T}b, \mathcal{T}a) < \epsilon$ whenever $\sigma_{sm}(b, a) < \delta$. If \mathcal{T} is continuous at all the points of \mathfrak{N} then, \mathcal{T} is continuous on \mathfrak{N} .

Definition 9. [7] Let $(\mathfrak{N}, \sigma_{sm})$ be a supra metric space. A sequence $\{a_n\} \in \mathfrak{N}$ is a Cauchy sequence if for all $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $\sigma_{sm}(a_n, a_m) < \epsilon$ for all $m, n \geq k$.

A supra metric is complete if every Cauchy sequence is convergent.

Proposition 3. [7] Let $(\mathfrak{N}, \sigma_{sm})$ be a supra metric space. Then the set \mathfrak{N} with respect to supra metrics in Example 1 is complete supra metric space.

Proof

To prove the proposition, we prove the following claim first.

Claim: If $(\mathfrak{N}, \sigma_{sm})$ be a supra metric space and (\mathfrak{N}, σ) be a metric space. Suppose $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a homeomorphism such that $\sigma_{sm} = h \circ \sigma$, then, a convergent sequence with respect to σ converges with respect to σ_{sm} to the same point.

Since, h is homeomorphism

1. h is continuous;
2. h^{-1} is continuous;
3. $h(0) = 0$;
4. h is strictly increasing.

Now, assume $a_n \rightarrow a$ with respect to σ , that is

$$\sigma(a_n, a) \rightarrow 0.$$

Now, apply h :

$$\sigma_{sm}(a_n, a) = h(\sigma(a_n, a)).$$

From the properties of h , we get

$$h(\sigma(a_n, a)) \rightarrow h(0) = 0.$$

Thus,

$$\sigma_{sm}(a_n, a) \rightarrow 0.$$

Therefore,

$a_n \rightarrow a$ with respect to σ_{sm} .

Hence, the claim is proved.

The 2nd claim is that the functions $h_1(t) = t(t + \alpha)$, $h_2(t) = \beta(e^t - 1)$, ($\alpha, \beta > 0$) are homeomorphism of \mathbb{R}^+ .

It is easy to verify that h_1, h_2 are continuous, bijective and their inverse is also continuous.

Thus, one can easily show that they are homeomorphism of \mathbb{R}^+ .

Hence, the proof is complete. □

Lemma 1. [7] Every supra metric is continuous.

Remark 2. [7] If a sequence $\{a_n\} \in \mathfrak{N}$ is a Cauchy sequence in \mathfrak{N} then there exists $a \in \mathfrak{N}$ such that $\lim_{n \rightarrow \infty} \sigma_{sm}(a_n, a) = 0$ and by the property (σ_3) every subsequence converges to a .

Then, many authors proved fixed point results in supra metric space and its other generalized spaces.

In 2025, Kumar *et al.* [17] α -admissible E-type contractions with respect to ξ and related fixed point theorems in supra metric space as stated below:

Definition 10. [17] Suppose \mathcal{T} be a self map on a supra metric space $(\mathfrak{N}, \sigma_{sm})$. If there exist $\xi \in \mathcal{Z}$ and $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ such that

$$\xi(\mathcal{S}(a, b)\sigma_{sm}(\mathcal{T}a, \mathcal{T}b), E(a, b)) \geq 0, \tag{1.1}$$

where

$$E(a, b) = \sigma_{sm}(a, b) + |\sigma_{sm}(a, \mathcal{T}a) - \sigma_{sm}(b, \mathcal{T}b)|, \text{ for all } a, b \in \mathfrak{N}.$$

Then, we say that \mathcal{T} is an \mathcal{S} -admissible E-type \mathcal{Z} contraction with respect to ξ .

If $\mathcal{S}(a, b) = 1$, then \mathcal{T} is said to be E-type \mathcal{Z} contraction with respect to ξ .

Theorem 3. [17] Suppose $(\mathfrak{N}, \sigma_{sm})$ be a complete supra metric space and \mathcal{T} is an \mathcal{S} -admissible E-type \mathcal{Z} contraction with respect to ξ . Assume

1. \mathcal{T} is \mathcal{S} -admissible;
2. there exists $a_0 \in \mathfrak{N}$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$;
3. \mathcal{T} is continuous.

Then \mathcal{T} has a unique fixed point.

Again in 2025, Ahmad and Albargi [2] proved the following fixed point theorem in extended b -metric space.

Theorem 4. [2] Let (\mathfrak{N}, σ) be a complete extended b -supra metric space with $b \geq 1$ and consider a mapping $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$. Suppose that there exists the functions $\alpha, \beta : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ and a constant $\varpi \in [0, 1)$ such that $\alpha(a, \mathcal{T}a)\alpha(b, \mathcal{T}b) \geq \beta(a, \mathcal{T}a)\beta(b, \mathcal{T}b)$ implies

$$\sigma(\mathcal{T}a, \mathcal{T}b) \leq \frac{\varpi}{b} M(a, b)$$

where

$$M(a, b) = \max\{\sigma(a, b), \min\{\frac{\sigma(a, \mathcal{T}a)\sigma(b, \mathcal{T}b)}{1 + \sigma(a, b)}, \frac{\sigma(a, \mathcal{T}b)\sigma(b, \mathcal{T}a)}{1 + \sigma(a, b)}\}\}$$

holds for all $a, b \in \mathfrak{N}$. Assume that the following conditions hold:

1. the mapping \mathcal{T} is an α -admissible mapping with respect to β ;
2. there exists $a_0 \in \mathfrak{N}$ such that $\alpha(a_0, \mathcal{T}a_0) \geq \beta(a_0, \mathcal{T}a_0)$;
3. either \mathcal{T} is continuous or if $\{a_n\}$ is a sequence in \mathfrak{N} such that $a_n \rightarrow a$, $\alpha(a_n, a_{n+1}) \geq \beta(a_n, a_{n+1})$ then $\alpha(a, \mathcal{T}a) \geq \beta(a, \mathcal{T}a)$.

Then, \mathcal{T} possesses a fixed point.

Moreover $\alpha(a, \mathcal{T}a)\alpha(b, \mathcal{T}b) \geq \beta(a, \mathcal{T}a)\beta(b, \mathcal{T}b)$ for all fixed points a, b . Then, the fixed point is unique.

Then, using this results they shown the existence of solution of non-linear voltaerra integral equation. Again in 2025, Shammaky and Hakami [38] gave new notion of complex-valued suprametric space and gave the following common fixed point theorem.

Theorem 5. [38] Let $(\mathfrak{N}, \mathfrak{z})$ be a complete complex valued supra metric space and $\mathcal{T}, \mathcal{S} : \mathfrak{N} \rightarrow \mathfrak{N}$. Assuming the existence of functions $k_1, k_2, k_3 : \mathfrak{N} \rightarrow [0, 1)$ such that

1. $k_i(\mathcal{T}a) \leq k_i(a)$ and $k_i(\mathcal{S}a) \leq k_i(a)$, for all $i = 1, 2, 3$.
2. $k_1(a) + 2k_2(a) + k_3(a) < 1$.
- 3.

$$\mathfrak{z}(\mathcal{T}a, \mathcal{S}b) \leq k_1(a)\mathfrak{z}(a, b) + k_2(a)(\mathfrak{z}(a, \mathcal{T}a) + \mathfrak{z}(b, \mathcal{S}b)) + k_3(a) \frac{\mathfrak{z}(a, \mathcal{T}a)\mathfrak{z}(b, \mathcal{S}b)}{1 + \mathfrak{z}(a, b)}$$

for all $a, b \in \mathfrak{N}$ with $\mathfrak{z}(a, b) \neq -1$, then \mathcal{T}, \mathcal{S} possesses a unique common fixed point.

Using it, they solved a Volterra-Hammerstein non-linear integral equation.

Now onward, the article is divided into two sections 2,3 respectively. In section 2, we shall define the generalized integral type contractions and proved the related fixed point theorems. Some consequences are discussed in the form of corollaries in section 3.

2. Results

Here, we shall give the notion of generalized rational $\mathcal{S} - \psi$ -contractive mapping of integral type A and type B and proved related fixed point theorems.

Definition 11. Let $(\mathfrak{N}, \sigma_{sm})$ be a supra metric space and \mathcal{T} be a self map on it. We say that \mathcal{T} is a generalized $\mathcal{S} - \psi$ -contractive mapping of integral type A if there exist two functions $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that for all $a, b \in \mathfrak{N}$,

$$\mathcal{S}(a, b) \int_0^{\sigma(\mathcal{T}a, \mathcal{T}b)} \phi(t) dt \leq \psi \left(\int_0^{M(a, b)} \phi(t) dt \right) \quad (2.1)$$

where $\phi \in \Phi$ and

$$M(a, b) = \max\{\sigma_{sm}(a, b), \sigma_{sm}(a, \mathcal{T}a), \sigma_{sm}(b, \mathcal{T}b), \frac{\sigma_{sm}(a, \mathcal{T}a) + \sigma_{sm}(b, \mathcal{T}b)}{2}, \frac{\sigma_{sm}(a, \mathcal{T}a) \cdot \sigma_{sm}(b, \mathcal{T}b)}{\sigma_{sm}(a, b)}\}.$$

Theorem 6. Suppose $(\mathfrak{N}, \sigma_{sm})$ be a complete supra metric space and $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is a transitive mapping. Suppose that $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ is a generalized $\mathcal{S} - \psi$ -contractive mapping of integral type A and satisfies the following conditions:

1. \mathcal{T} is \mathcal{S} -admissible;
2. there exists $a_0 \in \mathfrak{N}$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$;
3. \mathcal{T} is continuous.

Then, \mathcal{T} has a fixed point, that is, there exists $a \in \mathfrak{N}$ such that $\mathcal{T}a = a$.

Proof

Suppose $a_0 \in \mathfrak{N}$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$. Construct a sequence $\{a_n\}$ as $a_{n+1} = \mathcal{T}a_n$.

Now, if possible assume that there exists $N \in \mathbb{N}$ such that $a_{N+1} = a_N$ then $\mathcal{T}a_N = a_N$, which implies that a_N is the fixed point of \mathcal{T} and the proof is done.

So, we conclude that $a_{n+1} \neq a_n$ for all $n \in \mathbb{N}$.

Since, \mathcal{T} is \mathcal{S} -admissible

$$\begin{aligned} \mathcal{S}(a_0, a_1) = \mathcal{S}(a_0, \mathcal{T}a_0) \geq 1 &\Rightarrow \mathcal{S}(a_1, a_2) \geq 1; \\ \mathcal{S}(a_1, a_2) = \mathcal{S}(a_1, \mathcal{T}a_1) \geq 1 &\Rightarrow \mathcal{S}(a_2, a_3) \geq 1 \end{aligned}$$

Continuing in this way, we find that

$$\mathcal{S}(a_n, a_{n+1}) \geq 1, \tag{2.2}$$

for all $n \in \mathbb{N}$.

Using $a = a_{n-1}, b = a_n$, equation (3), equation (2) implies that

$$\begin{aligned} \int_0^{\sigma_{sm}(a_n, a_{n+1})} \phi(t) dt &= \int_0^{\sigma_{sm}(\mathcal{T}a_{n-1}, \mathcal{T}a_n)} \phi(t) dt; \\ &\leq \mathcal{S}(a_{n-1}, a_n) \int_0^{\sigma_{sm}(\mathcal{T}a_{n-1}, \mathcal{T}a_n)} \phi(t) dt; \\ &\leq \psi\left(\int_0^{M(a_{n-1}, a_n)} \phi(t) dt\right), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} M(a_{n-1}, a_n) &= \max\left\{\sigma_{sm}(a_{n-1}, a_n), \sigma_{sm}(a_{n-1}, \mathcal{T}a_{n-1}), \sigma_{sm}(a_n, \mathcal{T}a_n), \right. \\ &\quad \left. \frac{\sigma_{sm}(a_{n-1}, \mathcal{T}a_{n-1}) + \sigma_{sm}(a_n, \mathcal{T}a_n)}{2}, \frac{\sigma_{sm}(a_{n-1}, \mathcal{T}a_{n-1}) \cdot \sigma_{sm}(a_n, \mathcal{T}a_n)}{\sigma_{sm}(a_{n-1}, a_n)}\right\} \\ &= \max\left\{\sigma_{sm}(a_{n-1}, a_n), \sigma_{sm}(a_{n-1}, a_n), \sigma_{sm}(a_n, a_{n+1}) \right. \\ &\quad \left. , \frac{\sigma_{sm}(a_{n-1}, a_n) + \sigma_{sm}(a_n, a_{n+1})}{2}, \frac{\sigma_{sm}(a_{n-1}, a_n) \cdot \sigma_{sm}(a_n, a_{n+1})}{\sigma_{sm}(a_{n-1}, a_n)}\right\}, \\ &= \max\left\{\sigma_{sm}(a_{n-1}, a_n), \sigma_{sm}(a_n, a_{n+1}) \right. \\ &\quad \left. , \frac{\sigma_{sm}(a_{n-1}, a_n) + \sigma_{sm}(a_n, a_{n+1})}{2}, \sigma_{sm}(a_n, a_{n+1})\right\} \\ &= \max\left\{\sigma_{sm}(a_{n-1}, a_n), \sigma_{sm}(a_n, a_{n+1})\right\} \end{aligned}$$

Now, if possible assume that $\sigma_{sm}(a_{n-1}, a_n) < \sigma_{sm}(a_n, a_{n+1})$ then

$$M(a_{n-1}, a_n) = \sigma_{sm}(a_n, a_{n+1}).$$

Using this value in equation (4), we get

$$\int_0^{\sigma_{sm}(a_n, a_{n+1})} \phi(t) dt \leq \psi\left(\int_0^{M(a_{n-1}, a_n)} \phi(t) dt\right) < \int_0^{\sigma_{sm}(a_n, a_{n+1})} \phi(t) dt,$$

By using the fact that $\psi(t) < t$ for all $t > 0$.

which is a contradiction.

Hence,

$$M(a_{n-1}, a_n) = \sigma_{sm}(a_{n-1}, a_n).$$

Thus, equation (4) becomes

$$\int_0^{\sigma_{sm}(a_n, a_{n+1})} \phi(t) dt \leq \psi \left(\int_0^{\sigma_{sm}(a_{n-1}, a_n)} \phi(t) dt \right).$$

Continuing in this way we conclude that

$$\int_0^{\sigma_{sm}(a_n, a_{n+1})} \phi(t) dt \leq \psi^n \left(\int_0^{\sigma_{sm}(a_0, a_1)} \phi(t) dt \right).$$

Applying limit as $n \rightarrow \infty$ in the above setting and using property of ψ function, we get

$\int_0^{\sigma_{sm}(a_n, a_{n+1})} \phi(t) dt \rightarrow 0$ and this from property of ϕ function implies that

$\sigma_{sm}(a_n, a_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$.

Now, we shall prove that sequence $\{a_n\}$ is a Cauchy sequence. So, if possible assume that it is not Cauchy then there exists a $\epsilon > 0$ such that for each natural number k , there are natural numbers $m(k), n(k)$ such that $m(k) > n(k) > k$, ($m(k)$ is the smallest) with

$$\sigma_{sm}(a_{m(k)}, a_{n(k)}) \geq \epsilon, \sigma_{sm}(a_{m(k)-1}, a_{n(k)}) < \epsilon. \quad (2.4)$$

Therefore, using Definition 5 and inequality (5) for all $r \in \mathbb{N}$, we have

$$\begin{aligned} \sigma_{sm}(a_{m(k)-1}, a_{n(k)-1}) &\leq \sigma_{sm}(a_{m(k)-1}, a_{n(k)}) + \sigma_{sm}(a_{n(k)-1}, a_{n(k)}) + \rho \sigma_{sm}(a_{m(k)-1}, a_{n(k)}) \sigma_{sm}(a_{n(k)-1}, a_{n(k)}) \\ &\leq \epsilon + \sigma_{sm}(a_{n(k)-1}, a_{n(k)}) + \rho \epsilon \sigma_{sm}(a_{n(k)-1}, a_{n(k)}), \end{aligned}$$

Letting $k \rightarrow \infty$ in the above inequality and using equation (5), we obtain

$$\lim_{k \rightarrow \infty} \sigma_{sm}(a_{m(k)-1}, a_{n(k)-1}) \leq \epsilon. \quad (2.5)$$

Letting $k \rightarrow \infty$ from the above inequality, we conclude that

$$\lim_{k \rightarrow \infty} \int_0^{\sigma_{sm}(a_{m(k)-1}, a_{n(k)-1})} \phi(t) dt \leq \int_0^\epsilon \phi(t) dt. \quad (2.6)$$

Transitive property of \mathcal{S} implies that

$$\mathcal{S}(a_{m(k)-1}, a_{n(k)-1}) \geq 1. \quad (2.7)$$

Thus

$$\begin{aligned} \int_0^\epsilon \phi(t) dt &\leq \int_0^{\sigma_{sm}(a_{m(k)}, a_{n(k)})} \phi(t) dt = \int_0^{\sigma_{sm}(\mathcal{T} a_{m(k)-1}, \mathcal{T} a_{n(k)-1})} \phi(t) dt; \\ &\leq \mathcal{S}(a_{m(k)-1}, a_{n(k)-1}) \int_0^{\sigma_{sm}(\mathcal{T} a_{m(k)-1}, \mathcal{T} a_{n(k)-1})} \phi(t) dt; \\ &\leq \psi \left(\int_0^{M(a_{m(k)-1}, a_{n(k)-1})} \phi(t) dt \right). \end{aligned} \quad (2.8)$$

where

$$M(a_{m(k)}, a_{n(k)}) = \max\{\sigma_{sm}(a_{m(k)}, a_{n(k)}), \sigma_{sm}(a_{m(k)}, \mathcal{T} a_{m(k)}), \sigma_{sm}(a_{n(k)}, \mathcal{T} a_{n(k)})\}$$

$$\begin{aligned}
 & , \frac{\sigma_{sm}(a_{m(k)}, \mathcal{T}a_{m(k)}) + \sigma_{sm}(a_{n(k)}, \mathcal{T}a_{n(k)})}{2}, \frac{\sigma_{sm}(a_{m(k)}, \mathcal{T}a_{m(k)}) \cdot \sigma_{sm}(a_{n(k)}, \mathcal{T}a_{n(k)})}{\sigma_{sm}(a_{m(k)}, a_{n(k)})} \} \\
 = & \max\{\sigma_{sm}(a_{m(k)}, a_{n(k)}), \sigma_{sm}(a_{m(k)}, a_{m(k)+1}), \sigma_{sm}(a_{n(k)}, a_{n(k)+1}) \\
 & , \frac{\sigma_{sm}(a_{m(k)}, a_{m(k)+1}) + \sigma_{sm}(a_{n(k)}, a_{n(k)+1})}{2}, \frac{\sigma_{sm}(a_{m(k)}, a_{m(k)+1}) \cdot \sigma_{sm}(a_{n(k)}, a_{n(k)+1})}{\sigma_{sm}(a_{m(k)}, a_{n(k)})} \}
 \end{aligned}$$

Taking limit as $k \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} M(a_{m(k)}, a_{n(k)}) = \epsilon.$$

Thus, after applying limit $k \rightarrow \infty$, equation (9) becomes

$$\int_0^\epsilon \phi(t)dt \leq \int_0^{\sigma_{sm}(a_{m(k)}, a_{n(k)})} \phi(t)dt < \int_0^\epsilon \phi(t)dt.$$

a contradiction.

Thus, the sequence $\{a_n\}$ is a Cauchy sequence. Since, the space is complete. So, $\{a_n\}$ is a convergent sequence. This shows that there is $a \in \mathfrak{N}$ such that $a_n \rightarrow a$. It is also given that \mathcal{T} is continuous. So, $\mathcal{T}a_n = a_{n+1} \rightarrow \mathcal{T}a$, and the uniqueness of the limit implies that $\mathcal{T}a = a$. \square

In the next theorem, we shall remove the condition of continuity.

Theorem 7. Suppose $(\mathfrak{N}, \sigma_{sm})$ be a complete supra metric space and $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is a transitive mapping. Suppose that $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ is a generalized $\mathcal{S} - \psi$ -contractive mapping of integral type A and satisfies the following conditions:

1. \mathcal{T} is \mathcal{S} -admissible;
2. there exists $a_0 \in \mathfrak{N}$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$;
3. If $\{a_n\}$ is a sequence in \mathfrak{N} such that $\mathcal{S}(a_n, a_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $a_n \rightarrow a \in \mathfrak{N}$ as $n \rightarrow \infty$, then there exists a subsequence $\{a_{n(k)}\}$ of $\{a_n\}$ such that $\mathcal{S}(a_{n(k)}, a) \geq 1$ for all k .

Then \mathcal{T} has a fixed point, that is, there exists $a \in \mathfrak{N}$ such that $\mathcal{T}a = a$.

Proof

From the proof of above theorem, we can find a sequence $\{a_n\}$ such that $a_{n+1} = \mathcal{T}^n a_n$ which is converging to a . Now, by the condition 3, there exists a subsequence $\{a_{n(k)}\}$ of $\{a_n\}$ such that $\mathcal{S}(a_{n(k)}, a) \geq 1$ for all k . Now, by the equation (2) and condition 3, we get

$$\begin{aligned}
 \int_0^{\sigma_{sm}(\mathcal{T}a, a_{n(k)+1})} \phi(t)dt & = \int_0^{\sigma_{sm}(\mathcal{T}a, \mathcal{T}a_{n(k)})} \phi(t)dt; \\
 & \leq \mathcal{S}(a, a_{n(k)}) \int_0^{\sigma_{sm}(\mathcal{T}a, \mathcal{T}a_{n(k)})} \phi(t)dt; \\
 & \leq \psi\left(\int_0^{M(a, a_{n(k)})} \phi(t)dt\right),
 \end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
 M(a, a_{n(k)}) & = \max\{\sigma_{sm}(a, a_{n(k)}), \sigma_{sm}(a, \mathcal{T}a), \sigma_{sm}(a_{n(k)}, \mathcal{T}a_{n(k)}) \\
 & , \frac{\sigma_{sm}(a, \mathcal{T}a) + \sigma_{sm}(a_{n(k)}, \mathcal{T}a_{n(k)})}{2}, \frac{\sigma_{sm}(a, \mathcal{T}a) \cdot \sigma_{sm}(a_{n(k)}, \mathcal{T}a_{n(k)})}{\sigma_{sm}(a, a_{n(k)})} \}
 \end{aligned}$$

Since, $\{a_n\}$ is a convergent sequence, so by applying limit as $k \rightarrow \infty$, we get $\lim_{k \rightarrow \infty} M(a, a_{n(k)}) = \sigma_{sm}(a, \mathcal{T}a)$.

Assume that if possible $\sigma_{sm}(a, \mathcal{T}a) > 0$ then, from the above we can say that there exist large k for which $M(a, a_{n(k)}) > 0$, which implies that

$$\int_0^{\sigma_{sm}(\mathcal{T}a, a_{n(k)+1})} \phi(t) dt \leq \psi \left(\int_0^{M(a, a_{n(k)})} \phi(t) dt \right). \quad (2.10)$$

Letting $k \rightarrow \infty$ in the above, we find that

$$\int_0^{\sigma_{sm}(\mathcal{T}a, a)} \phi(t) dt \leq \psi \left(\int_0^{\sigma_{sm}(\mathcal{T}a, a)} \phi(t) dt \right) < \int_0^{\sigma_{sm}(\mathcal{T}a, a)} \phi(t) dt.$$

a contradiction.

Thus, $\sigma_{sm}(\mathcal{T}a, a) = 0$ implies that $\mathcal{T}a = a$. □

Definition 12. Let (\mathfrak{N}, σ) be a supra metric space and \mathcal{T} be a self map on it. We say that \mathcal{T} is a generalized $\mathcal{S} - \psi$ -contractive mapping of integral type B if there exist two functions $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that for all $a, b \in \mathfrak{N}$,

$$\mathcal{S}(a, b) \int_0^{\sigma_{sm}(\mathcal{T}a, \mathcal{T}b)} \phi(t) dt \leq \psi \left(\int_0^{M(a, b)} \phi(t) dt \right) \quad (2.11)$$

where $\phi \in \Phi$ and

$$M(a, b) = \max \left\{ \sigma_{sm}(a, b), \sigma_{sm}(a, \mathcal{T}a), \sigma_{sm}(b, \mathcal{T}b), \frac{\sigma_{sm}(a, \mathcal{T}a) + \sigma_{sm}(b, \mathcal{T}b)}{2}, \frac{\sigma_{sm}(a, \mathcal{T}a) \cdot \sigma_{sm}(b, \mathcal{T}b)}{\sigma_{sm}(a, b)}, \frac{(1 + \sigma_{sm}(a, \mathcal{T}a)) \sigma_{sm}(b, \mathcal{T}b)}{1 + \sigma_{sm}(a, b)} \right\}.$$

Theorem 8. Suppose $(\mathfrak{N}, \sigma_{sm})$ be a complete supra metric space and $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is a transitive mapping. Suppose that $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ is a generalized $\mathcal{S} - \psi$ -contractive mapping of integral type B and satisfies the following conditions:

1. \mathcal{T} is \mathcal{S} -admissible;
2. there exists $a_0 \in \mathfrak{N}$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$;
3. \mathcal{T} is continuous.

Then \mathcal{T} has a fixed point, that is, there exists $a \in \mathfrak{N}$ such that $\mathcal{T}a = a$.

Proof

The proof will be the same as the proof of Theorem 7. □

Theorem 9. Suppose $(\mathfrak{N}, \sigma_{sm})$ be a complete supra metric space and $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is a transitive mapping. Suppose that $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ is a generalized $\mathcal{S} - \psi$ -contractive mapping of integral type B and satisfies the following conditions:

1. \mathcal{T} is \mathcal{S} -admissible;
2. there exists $a_0 \in \mathfrak{N}$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$;
3. If $\{a_n\}$ is a sequence in \mathfrak{N} such that $\mathcal{S}(a_n, a_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $a_n \rightarrow a \in \mathfrak{N}$ as $n \rightarrow \infty$, then there exists a subsequence $\{a_{n(k)}\}$ of $\{a_n\}$ such that $\mathcal{S}(a_{n(k)}, a) \geq 1$ for all k .

Then \mathcal{T} has a fixed point, that is, there exists $a \in \mathfrak{N}$ such that $\mathcal{T}a = a$.

Proof

Proof will be same like Theorem 8. □

3. Consequences

In this section, we shall discuss some direct consequences of our proved results.

Definition 13. Let (\mathfrak{N}, σ) be a supra metric space and \mathcal{T} be a self map on it. We say that \mathcal{T} is a $\mathcal{S} - \psi$ -contractive mapping of integral type if there exist two functions $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that for all $a, b \in \mathfrak{N}$,

$$\mathcal{S}(a, b) \int_0^{\sigma_{sm}(\mathcal{T}a, \mathcal{T}b)} \phi(t) dt \leq \psi \left(\int_0^{\sigma_{sm}(a, b)} \phi(t) dt \right) \tag{3.1}$$

where $\phi \in \Phi$.

Corollary 1. Suppose $(\mathfrak{N}, \sigma_{sm})$ be a complete supra metric space and $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is a transitive mapping. Suppose that $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ is a $\mathcal{S} - \psi$ -contractive mapping of integral type and satisfies the following conditions:

1. \mathcal{T} is \mathcal{S} -admissible;
2. there exists $a_0 \in X$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$;
3. \mathcal{T} is continuous.

Then \mathcal{T} has a fixed point, that is, there exists $a \in \mathfrak{N}$ such that $\mathcal{T}a = a$.

Proof

Taking $M(a, b) = \sigma_{sm}(a, b)$ in Theorem 9. □

Corollary 2. Suppose $(\mathfrak{N}, \sigma_{sm})$ be a complete supra metric space and $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is a transitive mapping. Suppose that $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ is a $\mathcal{S} - \psi$ -contractive mapping of integral type and satisfies the following conditions:

1. \mathcal{T} is \mathcal{S} -admissible;
2. there exists $a_0 \in \mathfrak{N}$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$;
3. If $\{a_n\}$ is a sequence in \mathfrak{N} such that $\mathcal{S}(a_n, a_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $a_n \rightarrow a \in \mathfrak{N}$ as $n \rightarrow \infty$, then there exists a subsequence $\{a_{n(k)}\}$ of $\{a_n\}$ such that $\mathcal{S}(a_{n(k)}, a) \geq 1$ for all k .

Then \mathcal{T} has a fixed point, that is, there exists $a \in \mathfrak{N}$ such that $\mathcal{T}a = a$.

Proof

Taking $M(a, b) = \sigma_{sm}(a, b)$ in Theorem 10. □

Definition 14. Let (\mathfrak{N}, σ) be a supra metric space and \mathcal{T} be a self map on it. We say that \mathcal{T} is a $\mathcal{S} - \psi$ -contractive mapping if there exist two functions $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that for all $a, b \in \mathfrak{N}$,

$$\mathcal{S}(a, b) \sigma_{sm}(\mathcal{T}a, \mathcal{T}b) \leq \psi(\sigma_{sm}(a, b)). \tag{3.2}$$

Corollary 3. Suppose $(\mathfrak{N}, \sigma_{sm})$ be a complete supra metric space and $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is a mapping. Suppose that $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ is a $\mathcal{S} - \psi$ -contractive mapping and satisfies the following conditions:

1. \mathcal{T} is \mathcal{S} -admissible;
2. there exists $a_0 \in \mathfrak{N}$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$;
3. \mathcal{T} is continuous.

Then \mathcal{T} has a fixed point, that is, there exists $a \in \mathfrak{N}$ such that $\mathcal{T}a = a$.

Proof

Taking $\phi(t) = 1$ in Corollary 1. □

Corollary 4. Suppose $(\mathfrak{N}, \sigma_{sm})$ be a complete supra metric space and $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ is a mapping. Suppose that $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ is a $\mathcal{S} - \psi$ -contractive mapping that satisfies the following conditions:

1. \mathcal{T} is \mathcal{S} -admissible;
2. there exists $a_0 \in \mathfrak{N}$ such that $\mathcal{S}(a_0, \mathcal{T}a_0) \geq 1$;
3. If $\{a_n\}$ is a sequence in \mathfrak{N} such that $\mathcal{S}(a_n, a_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $a_n \rightarrow a \in \mathfrak{N}$ as $n \rightarrow \infty$, then there exists a subsequence $\{a_{n(k)}\}$ of $\{a_n\}$ such that $\mathcal{S}(a_{n(k)}, a) \geq 1$ for all k .

Then \mathcal{T} has a fixed point, that is, there exists $a \in \mathfrak{N}$ such that $\mathcal{T}a = a$.

Proof

Taking $\phi(t) = 1$ in Corollary 2. □

Many more results from the literature can be extracted from the main results in the form of corollaries.

Example 2. Suppose $\mathfrak{N} = [0, \infty)$. Define $\sigma_{sm} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ as $\sigma_{sm}(a, b) = |a - b|(|a - b| + \frac{1}{3})$. From Example 1 it is clear that $(\mathfrak{N}, \sigma_{sm})$ is a complete supra metric space with $\rho = 6$.

Now, consider the function $\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ as $\mathcal{S}(a, b) = 1$ and self map $\mathcal{T} : \mathfrak{N} \rightarrow \mathfrak{N}$ as

$$\mathcal{T}a = \frac{a}{3}.$$

$$\text{Taking } \psi(t) = \frac{t}{2}.$$

Left hand side of equation (14) becomes

$$\begin{aligned} \mathcal{S}(a, b)\sigma_{sm}(\mathcal{T}a, \mathcal{T}b) &= \left| \frac{a}{3} - \frac{b}{3} \right| \left(\left| \frac{a}{3} - \frac{b}{3} \right| + \frac{1}{3} \right), \\ &= \frac{1}{9} |a - b| (|a - b| + 1). \end{aligned} \tag{3.3}$$

Now, right hand side is

$$\begin{aligned} \psi(\sigma_{sm}(a, b)) &= \frac{1}{2} (|a - b| (|a - b| + \frac{1}{3})) \\ &= \frac{1}{6} |a - b| (3|a - b| + 1). \end{aligned} \tag{3.4}$$

From equations (15) and (16), one can observe that

$$\mathcal{S}(a, b)\sigma_{sm}(\mathcal{T}a, \mathcal{T}b) \leq \psi(\sigma_{sm}(a, b)).$$

Thus, all the conditions of Corollary 3 are satisfied.

Hence, \mathcal{T} has a fixed point.

Clearly, 0 is the fixed point.

Example 3. Consider the set $\mathfrak{N} = \{1, 2, 3\}$ and define the $\sigma_{sm} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ as

$$\sigma_{sm}(1, 2) = 1 = \sigma_{sm}(2, 1), \sigma_{sm}(1, 3) = 2 = \sigma_{sm}(3, 1), \sigma_{sm}(3, 2) = \sigma_{sm}(2, 3) = \frac{1}{2}, \sigma_{sm}(1, 1) = \sigma_{sm}(2, 2) = \sigma_{sm}(3, 3) = 0 \text{ and } \rho = 2.$$

Since,

$$\sigma_{sm}(1, 3) > \sigma_{sm}(2, 3) + \sigma_{sm}(1, 2).$$

Thus, σ_{sm} is not a metric. But, one can easily check that it is a supra metric. Hence,

$(\mathfrak{N}, \sigma_{sm})$ is a complete supra metric space.

Define self map \mathcal{T} on \mathfrak{N} as $\mathcal{T}1 = 3, \mathcal{T}2 = 2, \mathcal{T}3 = 1$.

$\mathcal{S} : \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ as

$$\mathcal{S}(1, 1) = \mathcal{S}(2, 2) = \mathcal{S}(3, 3) = 2, \mathcal{S}(1, 2) = \mathcal{S}(1, 3) = 0 \text{ and } \mathcal{S}(2, 3) = \frac{1}{2}.$$

Now, one can observe easily that

equation (2) holds trivially for all pairs except $a = 2, b = 3$. So, we have to check only for this pair $(2, 3)$. For

$a = 2, b = 3$ left hand side of equation (2) becomes

$$\begin{aligned} \mathcal{S}(a, b) \int_0^{\sigma_{sm}(\mathcal{T}a, \mathcal{T}b)} \phi(t) dt &= \frac{1}{2} \int_0^{\sigma_{sm}(\mathcal{T}2, \mathcal{T}3)} \phi(t) dt, \\ &= \frac{1}{2} \int_0^{\sigma_{sm}(2, 1)} \phi(t) dt \\ &= \frac{1}{2} \int_0^1 \phi(t) dt. \end{aligned} \quad (3.5)$$

and right hand side becomes

$$\psi \left(\int_0^{\sigma_{sm}(2, 3)} \phi(t) dt \right) = \frac{1}{2} \left(\int_0^{\frac{1}{2}} \phi(t) dt \right). \quad (3.6)$$

From equations (17) and (18), we conclude that equation (2) does not hold.

Now, again equation (2) holds trivially for all the pairs except (2, 3). Thus,

$$\begin{aligned} M(2, 3) &= \max\{\sigma_{sm}(2, 3), \sigma_{sm}(2, \mathcal{T}2), \sigma_{sm}(3, \mathcal{T}3), \frac{\sigma_{sm}(2, \mathcal{T}2) + \sigma_{sm}(3, \mathcal{T}3)}{2}, \frac{\sigma_{sm}(2, \mathcal{T}2) \cdot \sigma_{sm}(3, \mathcal{T}3)}{\sigma_{sm}(2, 3)}\} \\ &= \max\{\frac{1}{2}, 0, 2, 1, 0\} \\ &= 2. \end{aligned}$$

So, the right hand side becomes

$$\psi \left(\int_0^{M(2, 3)} \phi(t) dt \right) = \frac{1}{2} \left(\int_0^2 \phi(t) dt \right). \quad (3.7)$$

Hence, from equations (17) and (19) equation (2) holds.

Thus, all the conditions of Theorem 10 holds. Hence, \mathcal{T} has a fixed point. Clearly, 2 is the fixed point of \mathcal{T} .

4. Future Scope

Here, we gave the new notion of generalized integral contractions and proved the related fixed point theorems in the complete supra metric space. Then, in the form of corollaries some consequences of proved results are also given. Then, to validate the proved result examples are given.

In future, the researcher can extend our result for two or maps in supra metric space or any other generalization of supra metric space. One can also apply our proved results for the solution of various integral, differential, and fractional calculus equations. It can be used to study the Ulam-Hyers stability of fixed point problem.

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