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Infeasible primal–dual interior point methods based on the kernel function for convex *QCQPs*

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Abstract In this paper, we study convex quadratically constrained quadratic programming (*QCQP*) problems through the primal–dual interior-point methods based on kernel functions. In contrast to standard feasible interior-point approaches, we develop an infeasible method whose iterates do not necessarily satisfy the primal or dual constraints throughout the iterations. When the problem admits a feasible solution, primal feasibility and optimality are achieved simultaneously at convergence. When the feasible set is empty, infeasibility is automatically detected and an approximate solution is obtained via penalized relaxation minimizing the constraint violation, weighted by the chosen kernel function. Under standard convexity assumptions and the existence of optimal solutions, the resulting convex *QCQP* enjoys strong duality, and its optimal solutions are fully characterized by the Karush–Kuhn–Tucker (*KKT*) conditions. We introduce a kernel-function-based barrier framework that replaces the classical logarithmic barrier, leading to a parametrized perturbed *KKT* system with explicit primal and dual residuals. This system defines an infeasible central path, whose neighborhood is followed using exact Newton directions derived from the chosen kernel function. We demonstrate that this approach provides a flexible and unified framework for designing and analyzing efficient Newton-based algorithms for *QCQP*, with potential extensions to broader classes of conic optimization problems.

Keywords Quadratically constrained quadratic programming, Infeasible methods, Kernel functions, Primal–dual methods, Karush–Kuhn–Tucker, Infeasible central path, Newton directions, Conic optimization.

AMS 2020 subject classifications 90C20, 90C51, 90C30

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1. Introduction

Mathematical optimization plays a fundamental role in many scientific and industrial domains, ranging from financial portfolio management to engineering system design. Among the different classes of optimization problems, Quadratically Constrained Quadratic Programs (*QCQP*) constitute a particularly important family that generalizes both linear programming and quadratic programming, and arises naturally in a wide range of applications including signal processing, combinatorial optimization, robust optimization, and control theory.

The history of numerical resolution of convex optimization problems dates back to the pioneering work of Legendre [15] and Gauss [9] in the early 19th century on the least squares method, considered the first algorithmically solved convex problem. Later, Cauchy [3] proposed in 1847 the first gradient descent method, paving the way for modern iterative approaches. The general theoretical framework was established by Karush [11] in 1939 and formalized by Kuhn and Tucker [12] in 1951 with the famous *KKT* optimality conditions.

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In the field of solution methods, interior-point methods (*IPM*) have occupied a central place since the pioneering work of Frisch [8] in the 1950s, who introduced the logarithmic barrier function. Carroll [2] developed practical applications in 1961, while Fiacco and McCormick [7] formalized the complete theory of barrier and penalty methods in 1968. A significant step was made by Dikin [5] in 1967, who proposed an affine scaling method for linear programming, later recognized as a precursor to modern interior-point methods. However, it was the publication of Karmarkar's [10] revolutionary algorithm in 1984 that renewed interest in these methods by demonstrating their superior efficiency over the simplex method for certain large-scale problems. Since then, primal–dual interior-point methods have been extensively studied and extended to broader classes of convex optimization problems [19].

Modern interior-point methods rely on the use of kernel functions that maintain iterates inside the feasible domain by penalizing approach to the boundaries. The use of kernel functions as a generalization of the classical logarithmic barrier was first systematically studied by Peng, Roos and Terlaky [17], who introduced the concept of self-regular functions and demonstrated that kernel based barriers yield improved theoretical complexity bounds. This line of research was subsequently extended by Bai, El Ghami and Roos [1], who proposed a unified framework for designing and analyzing kernel function based interior-point methods, and showed that different choices of kernel functions lead to algorithms with distinct theoretical and practical properties. Further contributions in this direction include the works of Djeffal and Laouar [6], who extended kernel-based *IPM* to the linear complementarity problem, Chalekh and Djeffal [4], who developed a kernel-based approach for convex quadratic programming, and more recently Laouar et al., who combined kernel functions with *BFGS* quasi-Newton methods for nonlinear semidefinite programming [13] and proposed a generalized self-regular kernel function for large-scale nonlinear optimization problems [14]. In the same spirit, Lesaja et al. [16] introduced a new kernel function for monotone linear complementarity problems and demonstrated its applicability to the control tabular adjustment problem, further illustrating the versatility of kernel-based *IPM* across diverse application domains. These methods present several advantages: polynomial convergence for convex problems, efficiency on large-scale problems, and ability to exploit sparse matrix structure.

However, a fundamental limitation of classical interior-point approaches, including most kernel-function-based methods, is that they require a strictly feasible initial point satisfying all primal and dual constraints. In practice, finding such a point can be as difficult as solving the original problem itself. This limitation has motivated the development of infeasible-start methods that allow the algorithm to begin with an arbitrary point that may violate the constraints, or even to handle problems that are not necessarily feasible. A pioneering contribution in this direction was made by Roos [18], who proposed a full-Newton step $\mathcal{O}(n)$ infeasible interior-point algorithm for linear optimization, establishing polynomial convergence without requiring a feasible starting point. These methods introduce explicit primal and dual residuals that are driven to zero alongside the complementarity gap, defining a so-called infeasible central path whose neighborhood is tracked throughout the iterations [19]. A key theoretical question is whether the infeasibility of the starting point degrades the iteration complexity of the algorithm. We show that this is not the case: our infeasible method recovers the same complexity bounds as feasible kernel-based methods, and even improves upon them under an aggressive barrier update strategy.

Despite significant progress in both kernel-function-based methods and infeasible-start interior-point methods, the combination of these two approaches for Quadratically Constrained Quadratic Programs has not been thoroughly investigated in the literature. Existing kernel-based works on *QCQP*, such as [4], rely on feasible-start assumptions and employ specific kernel functions without addressing the infeasibility of the starting point or of the problem itself. In particular, no existing work has established, for an infeasible kernel-based interior-point method applied to *QCQP*, complexity bounds that match or improve upon those of their feasible counterparts. This gap motivates the present work.

In this work, we consider a Quadratically Constrained Quadratic Program ($QCQP$) under convexity assumptions. Problem (P1) is formulated as follows:

$$\begin{cases} \min\limits_{x\in\mathbb{R}^n} & f(x) \\ \text{subject to} & g_i(x) \le 0, \quad i = 1,\dots,m, \end{cases} \tag{P1}$$

where $f(x) := \frac{1}{2}x^\top Q_0 x + c_0^\top x + d_0$ is the convex quadratic objective function (with $Q_0 \succeq 0$) and the constraints $g_i(x) := \frac{1}{2}x^\top Q_i x + c_i^\top x + d_i$ are also quadratic and convex (with $Q_i \succeq 0$ for $i = 1,\dots,m$). The matrices $Q_i \in \mathbb{R}^{n\times n}$ are symmetric.

The main contributions of this work are summarized as follows:

- We propose an *infeasible-start primal–dual interior-point method* for convex $QCQP$ that does not require a feasible initial point, neither for the primal problem nor for its dual, thus considerably extending the practical applicability of kernel-based methods beyond the works of [4, 6, 13, 14].
- We develop an *automatic infeasibility detection and penalized relaxation procedure*: when the feasible set is empty, the algorithm automatically detects infeasibility through a numerical certificate, and an approximate solution is recovered via a kernel-weighted penalized relaxation that minimizes the total constraint violation $\sum_{i=1}^m \max(g_i(x), 0)$, thereby extending the practical scope of the method to intrinsically infeasible $QCQP$ instances.
- We introduce a *kernel-function-based barrier framework* that replaces the classical logarithmic barrier, leading to a parametrized perturbed KKT system with explicit primal and dual residuals, and we define the associated *infeasible central path* for convex $QCQP$.
- We derive *exact Newton directions* from the chosen kernel function and establish a neighborhood framework for tracking the infeasible central path, providing a flexible and unified algorithmic framework in the spirit of [17, 1].
- We establish *polynomial complexity bounds* for the proposed algorithm under four barrier update strategies. Strategies *(S1)–(S3)* achieve the best-known bound

  $$\mathcal{O}\Big(\sqrt{m}\,\log\frac{m}{\varepsilon}\Big),$$

  matching that of feasible kernel-based interior-point methods [17, 1], while strategy *(S4)* yields the improved bound

  $$\mathcal{O}\left(\sqrt{m}\sqrt{\log\frac{m}{\varepsilon}}\right).$$

  Remarkably, the infeasibility of the starting point and of the problem itself does *not* degrade these complexity bounds.
- We *validate the theoretical results* through numerical experiments on representative $QCQP$ instances, illustrating the practical efficiency and robustness of the proposed approach.

The remainder of this paper is organized as follows: Section 2 presents the solution of convex *QCQPs*. Section 3 describes the theoretical properties of the new kernel. Section 4 analyzes the theoretical complexity. Section 5 presents numerical results and performance analysis. Finally, Section 6 concludes this work.

2. Solving convex *QCQP*

This section presents the convexity and regularity conditions that guarantee strong duality and the validity of the KKT conditions for $QCQP$. We then introduce the Lagrangian formulation and the associated dual problem. Based on this, an infeasible primal–dual interior-point scheme grounded in kernel functions is developed. Finally, we describe the general algorithmic framework for following the central path and ensuring convergence to the optimal solution.

Regularity assumptions

In order to ensure that problem (P1) is convex and that strong duality holds, we impose the following assumptions:

(A1) The objective function $f(x)$ is convex, which holds if the matrix Q_0 is positive semidefinite:

$$Q_0 \succeq 0.$$

(A2) Each inequality constraint function $g_i(x)$ is convex, which requires that

$$Q_i \succeq 0, \quad i = 1, \ldots, m.$$

(A3) (Slater condition) There exists a strictly feasible point $\bar{x} \in \mathbb{R}^n$ such that

$$g_i(\bar{x}) < 0, \quad i = 1, \ldots, m.$$

Under (A1)–(A3), strong duality holds ($p^\star = d^\star$), and the Karush–Kuhn–Tucker (KKT) conditions are necessary and sufficient for optimality.

Under these assumptions, the $QCQP$ becomes a convex optimization problem for which every local minimizer is a global minimizer.

Lagrangian and primal–dual formulation

The Lagrangian function associated with problem (P1) is defined as

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) = \tfrac{1}{2} x^\top Q_0 x + c_0^\top x + d_0 + \sum_{i=1}^{m} \lambda_i \left(\tfrac{1}{2} x^\top Q_i x + c_i^\top x + d_i \right),$$

where $\lambda \in \mathbb{R}^m$ (with $\lambda_i \geq 0$) are the Lagrange multipliers associated with the inequality constraints.

The dual function is defined as

$$\theta(\lambda) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda),$$

and the corresponding dual problem is

$$\begin{aligned} \max_{\lambda} \quad & \theta(\lambda) \\ \text{subject to} \quad & \lambda \geq 0. \end{aligned}$$

If assumptions (A1)–(A3) hold, then strong duality applies, i.e.,

$$p^\star = d^\star,$$

and any optimal primal–dual solution $(x^\star, \lambda^\star)$ of problem (P1) satisfies the Karush–Kuhn–Tucker (KKT) optimality conditions:

$$\begin{cases} g_i(x^\star) \leq 0, \quad i = 1, \ldots, m, & \text{(Primal feasibility)} \\ \lambda_i^\star \geq 0, \quad i = 1, \ldots, m, & \text{(Dual feasibility)} \\ -\lambda_i^\star g_i(x^\star) = 0, \quad i = 1, \ldots, m, & \text{(Complementary slackness)} \\ (Q_0 + \sum_{i=1}^m \lambda_i^\star Q_i)\, x^\star + (c_0 + \sum_{i=1}^m \lambda_i^\star c_i) = 0. & \text{(Stationarity)} \end{cases} \tag{P2}$$

These conditions are necessary and sufficient for optimality in the convex $QCQP$ case.

2.1. An infeasible primal–dual interior-point approach

We now develop an infeasible primal–dual interior-point method (*IPM*) for solving the convex *QCQP* problem (P1). This approach does not require the starting point to satisfy the primal or dual constraints.

To apply the interior-point framework, we introduce slack variables $s \in \mathbb{R}^m$ to convert the inequality constraints into equality constraints coupled with nonnegativity conditions:

$$g_i(x) + s_i = 0, \quad s_i \geq 0, \quad i = 1, \ldots, m.$$

The feasible region of the perturbed problem is thus characterized by $(\lambda, s) > 0$, and the method operates strictly in its interior.

Rather than solving the *KKT* system exactly (which would correspond to $\mu = 0$), the interior-point approach tracks a *central path* parameterized by $\mu > 0$. For each μ, we seek a triple (x, λ, s) satisfying the perturbed *KKT* system $\mathbf{F}_\mu(x, \lambda, s) = 0$, where

$$\mathbf{F}_\mu(x, \lambda, s) = \begin{pmatrix} \nabla_x \mathcal{L}(x, \lambda) \\ g(x) + s \\ \Lambda s - \mu e \end{pmatrix} = \begin{pmatrix} \mathbf{r}_d \\ \mathbf{r}_p \\ \mathbf{r}_{cs} \end{pmatrix} = 0,$$

with $\Lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$, $e := (1, \ldots, 1)^\top \in \mathbb{R}^m$, and $\mu > 0$ a barrier parameter that is driven to zero as the algorithm progresses.

The three components of $\mathbf{F}_\mu$ are interpreted as follows:

- $\mathbf{r}_d = \nabla_x \mathcal{L}(x, \lambda) = \big(Q_0 + \sum_{i=1}^m \lambda_i Q_i\big)x + \big(c_0 + \sum_{i=1}^m \lambda_i c_i\big)$ is the *dual residual* (stationarity condition);
- $\mathbf{r}_p = g(x) + s$ is the *primal residual* (equality form of the inequality constraints, with $s_i = -g_i(x)$ at feasibility);
- $\mathbf{r}_{cs} = \Lambda s - \mu e$ is the *complementarity residual*, which replaces the exact complementarity condition $\lambda_i s_i = 0$ by the *centering condition* $\lambda_i s_i = \mu$ for all $i = 1, \ldots, m$.

Since the starting point need not be feasible, the residuals $\mathbf{r}_d$ and $\mathbf{r}_p$ are generally nonzero at initialization. The algorithm simultaneously reduces all three residuals to zero, achieving primal–dual feasibility and complementarity in the limit $\mu \to 0$.

2.2. Kernel function and infeasible central path

Let $\psi : (0, \infty) \to \mathbb{R}$ be a twice continuously differentiable kernel function satisfying the following properties:

(K1) ψ is strictly convex, i.e., $\psi''(t) > 0$ for all $t > 0$;
(K2) $\psi(1) = 0$ and $\psi'(1) = 0$;
(K3) $\lim\limits_{t \to 0^+} \psi(t) = \lim\limits_{t \to \infty} \psi(t) = +\infty$.

Such a function naturally induces a barrier for the nonnegativity constraints $(\lambda, s) > 0$ that arise in the interior-point formulation of problem (P1).

For a given barrier parameter $\mu > 0$ and contraction parameter $\theta \in (0, 1)$, the kernel-based infeasible primal–dual framework incorporates residuals from the initial iterate. Let $\mathbf{r}_d^0$ and $\mathbf{r}_p^0$ denote the initial dual and primal residuals, respectively, evaluated at the starting point (x^0, λ^0, s^0). At iteration k, the infeasible central path is defined by the perturbed *KKT* system with slack variables $s \in \mathbb{R}^m$:

$$\begin{cases} \nabla_x \mathcal{L}(x, \lambda) = (1 - \theta)^k \, \mathbf{r}_d^0, & \text{(Dual residual)} \\ g(x) + s = (1 - \theta)^k \, \mathbf{r}_p^0, & \text{(Primal residual)} \\ \lambda_i s_i = \mu, \quad i = 1, \ldots, m, & \text{(Perturbed complementarity condition)} \end{cases} \tag{1}$$

where the third equation represents the standard perturbed complementarity condition. To establish its connection with the kernel framework, we introduce the following proximity function based on the kernel function.

Specifically, for strictly positive $\lambda_i > 0$ and $s_i > 0$, the proximity function

$$\Psi_\mu(x, \lambda, s) := \sum_{i=1}^{m} \psi\left(\frac{\lambda_i s_i}{\mu}\right)$$

measures the deviation of the products $\lambda_i s_i$ from the target value μ. The first-order optimality condition with respect to λ_i reads

$$\frac{\partial \Psi_\mu}{\partial \lambda_i} = \frac{s_i}{\mu}\,\psi'\left(\frac{\lambda_i s_i}{\mu}\right) = 0, \qquad i = 1, \ldots, m.$$

Since $s_i > 0$, this reduces to

$$\psi'\left(\frac{\lambda_i s_i}{\mu}\right) = 0, \qquad i = 1, \ldots, m.$$

By properties (K1)–(K2), the equation $\psi'(t) = 0$ has the unique solution $t = 1$, yielding the centrality condition

$$\lambda_i s_i = \mu, \qquad i = 1, \ldots, m,$$

which coincides precisely with the third equation of system (1). Thus, the perturbed complementarity condition is equivalent to $\Psi_\mu(x, \lambda, s) = 0$, and the kernel function ψ governs how the iterates approach the central path.

Infeasible central path. Under assumptions (K1)–(K3) and the regularity conditions (A1)–(A3), for each $\mu > 0$ and each $k \geq 0$, system (1) admits a unique solution $(x^k(\mu), \lambda^k(\mu), s^k(\mu))$ with $(\lambda^k, s^k) > 0$, without requiring the starting point to be feasible. As $k \to \infty$ with $\mu = \mu_k \to 0$, the residuals satisfy

$$(1-\theta)^k\, \mathbf{r}_d^0 \;\to\; 0 \qquad \text{and} \qquad (1-\theta)^k\, \mathbf{r}_p^0 \;\to\; 0,$$

and the sequence of iterates converges to the optimal solution:

$$\lim_{k\to\infty} \left(x^k(\mu_k),\, \lambda^k(\mu_k)\right) = (x^\star, \lambda^\star),$$

where $(x^\star, \lambda^\star)$ satisfies the *KKT* system (P2) for problem (P1).

2.3. Proximity measure and Newton direction

For the *QCQP* problem (P1), we introduce slack variables to convert the inequality constraints into equality form:

$$g_i(x) + s_i = 0, \quad s_i > 0, \quad i = 1, \ldots, m.$$

Along the feasible manifold, this implies

$$s_i = -g_i(x), \quad i = 1, \ldots, m.$$

At points satisfying strict feasibility ($g_i(x) < 0$), we have $s_i > 0$.

Let $s \in \mathbb{R}^m$ be the vector of slack variables, and $\lambda \in \mathbb{R}^m$ be the vector of Lagrange multipliers, both assumed to be strictly positive ($s > 0$, $\lambda > 0$). We define the scaled variables

$$t_i := \frac{\lambda_i s_i}{\mu}, \quad i = 1, \ldots, m,$$

which measure the proximity of the i-th complementarity product to the central value μ. The *kernel-based proximity measure* is

$$\Psi(t) := \sum_{i=1}^{m} \psi(t_i),$$

which is consistent with the proximity function Ψ_μ introduced in Section 2.1, since $\Psi(t) = \Psi_\mu(x, \lambda, s)$ with $t_i = \lambda_i s_i/\mu$. The *norm-based proximity measure* is defined as

$$\delta(t) := \frac{1}{2}\,\|\nabla\Psi(t)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{m}\big(\psi'(t_i)\big)^2}.$$

The ℓ_2-norm of the vector $\mathbf{h}(t) = \big(\psi'(t_1), \ldots, \psi'(t_m)\big)^\top$ is a key metric for analyzing the complexity and convergence of kernel-based interior-point methods. Note that by property (K2), we have $\psi'(1) = 0$, so $\delta(t) = 0$ if and only if $t_i = 1$ for all i, i.e., $\lambda_i s_i = \mu$ for all i, which corresponds to a point on the central path.

Newton direction. At each iteration k, we compute the Newton direction $(\Delta x, \Delta\lambda, \Delta s)$ by linearizing the perturbed infeasible *KKT* system (1).

The Hessian of the Lagrangian is

$$H(x, \lambda) := Q_0 + \sum_{i=1}^{m} \lambda_i Q_i,$$

which is symmetric and positive semidefinite under assumptions (A1)–(A2).

The gradient of the Lagrangian with respect to x is

$$\nabla_x \mathcal{L}(x, \lambda) = H(x, \lambda)\, x + \left(c_0 + \sum_{i=1}^{m} \lambda_i c_i\right).$$

The Jacobian matrix of the constraint functions is

$$J(x) = \begin{bmatrix} \nabla g_1(x)^\top \\ \vdots \\ \nabla g_m(x)^\top \end{bmatrix} = \begin{bmatrix} (Q_1 x + c_1)^\top \\ \vdots \\ (Q_m x + c_m)^\top \end{bmatrix} \in \mathbb{R}^{m\times n},$$

where $\nabla g_i(x) = Q_i x + c_i$ for each $i = 1, \ldots, m$.

The Newton system for the infeasible primal–dual method is:

$$\begin{bmatrix} H(x,\lambda) & J(x)^\top & 0 \\ J(x) & 0 & I \\ 0 & S & \Lambda \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{bmatrix} = - \begin{bmatrix} \mathbf{r}_d^k \\ \mathbf{r}_p^k \\ \mathbf{r}_{cs} \end{bmatrix}, \tag{2}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$, $S = \operatorname{diag}(s_1, \ldots, s_m)$, $I \in \mathbb{R}^{m\times m}$ is the identity matrix, and the residuals at iteration k are:

- $\mathbf{r}_d^k = \nabla_x \mathcal{L}(x, \lambda) - (1-\theta)^k\, \mathbf{r}_d^0$ (dual residual),
- $\mathbf{r}_p^k = -(1-\theta)^k\, \mathbf{r}_p^0$ (primal residual),
- $\mathbf{r}_{cs} = \Lambda s - \mu e$ (complementarity residual),

where $e := (1, \ldots, 1)^\top \in \mathbb{R}^m$, and $\mathbf{r}_d^0$, $\mathbf{r}_p^0$ are the initial residuals evaluated at the starting point (x^0, λ^0, s^0).

Reduced system. To reduce the computational cost, system (2) can be simplified by eliminating Δs. From the third block-row:

$$S\,\Delta\lambda + \Lambda\,\Delta s = -\mathbf{r}_{cs} = -(\Lambda s - \mu e) = \mu e - \Lambda s.$$

Solving for Δs:

$$\Delta s = \Lambda^{-1}(\mu e - \Lambda s) - \Lambda^{-1} S\,\Delta\lambda = \Lambda^{-1}(\mu e - \Lambda s) - \Lambda^{-1} S\,\Delta\lambda.$$

Substituting into the second block-row ($J(x)\,\Delta x + \Delta s = -\mathbf{r}_p^k$):

$$J(x)\,\Delta x + \Lambda^{-1}(\mu e - \Lambda s) - \Lambda^{-1} S\,\Delta\lambda = -\mathbf{r}_p^k.$$

Rearranging yields the following $(n+m)$-dimensional reduced system in $(\Delta x, \Delta\lambda)$:

$$\begin{bmatrix} H(x,\lambda) & J(x)^\top \\ J(x) & -\Lambda^{-1} S \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} -\mathbf{r}_d^k \\ -\mathbf{r}_p^k - \Lambda^{-1}(\mu e - \Lambda s) \end{bmatrix}. \tag{3}$$

Once Δx and $\Delta\lambda$ are computed from (3), the slack step Δs is recovered via:

$$\Delta s = \Lambda^{-1}(\mu e - \Lambda s) - \Lambda^{-1} S\,\Delta\lambda.$$

The matrix $\Lambda^{-1}S = \mathrm{diag}(s_1/\lambda_1, \ldots, s_m/\lambda_m)$ is a positive diagonal matrix that acts as a scaling factor in the $(2,2)$-block of the reduced system. This scaling adapts to the current iterate and plays a crucial role in the convergence properties of the method.

2.4. Algorithmic framework

We now present the complete kernel-based infeasible primal–dual interior-point method. At each *outer* iteration, the barrier parameter μ is reduced by a factor $(1-\theta)^j$ with $j \in \{1,\ 2,\ \lceil \log(k+1) \rceil,\ k\}$, where k denotes the current outer iteration index (see [13]). The *inner* loop applies full Newton steps to decrease the kernel-based proximity measure below a prescribed threshold τ.

Algorithm 1 Kernel-based infeasible primal–dual interior-point method for convex *QCQP* (Part I)

Input: tolerance $\varepsilon > 0$; parameters $\theta \in (0,1)$, $\tau > 0$, $\eta \in (0,1)$
Choose an initial (possibly infeasible) point (x^0, λ^0, s^0) with $\lambda^0 > 0$ and $s^0 > 0$
Compute initial residuals:

$$\mathbf{r}_d^0 = \nabla_x \mathcal{L}(x^0, \lambda^0), \qquad \mathbf{r}_p^0 = g(x^0) + s^0$$

Set $\mu_0 > 0$, $k \leftarrow 0$, and $(x, \lambda, s) \leftarrow (x^0, \lambda^0, s^0)$
while $\mu_k > \varepsilon$ **do**
    Set current target residuals and update barrier parameter:

$$\widehat{\mathbf{r}}_d^k = (1-\theta)^k\,\mathbf{r}_d^0, \qquad \widehat{\mathbf{r}}_p^k = (1-\theta)^k\,\mathbf{r}_p^0, \qquad \mu_{k+1} = (1-\theta)^{j_k}\,\mu_0$$

    où $j_k \in \{1,\ 2,\ \lceil \log(k+1) \rceil,\ k\}$

Algorithm 1 Kernel-based infeasible primal–dual interior-point method for convex *QCQP* (Part II)

Compute scaled variables $t_i = \dfrac{\lambda_i s_i}{\mu_k}$ and the proximity measure $\delta(t)$
while $\delta(t) > \tau$ **do** ▷ Inner centering loop
  Form the Newton residuals:

$$\mathbf{r}_d^k = \nabla_x \mathcal{L}(x,\lambda) - \widehat{\mathbf{r}}_d^k, \qquad \mathbf{r}_p^k = -\widehat{\mathbf{r}}_p^k, \qquad \mathbf{r}_{cs} = \Lambda s - \mu_k\, e$$

  Solve for the Newton direction $(\Delta x, \Delta\lambda, \Delta s)$ via (2) or (3)
  Compute maximal step lengths $\alpha_{\mathrm{p}}^{\max}$, $\alpha_{\mathrm{d}}^{\max} \in (0,1]$ such that:

$$s + \alpha_{\mathrm{p}}^{\max}\, \Delta s > 0, \qquad \lambda + \alpha_{\mathrm{d}}^{\max}\, \Delta\lambda > 0$$

  Set $\alpha \leftarrow \eta \min\{\alpha_{\mathrm{p}}^{\max}, \alpha_{\mathrm{d}}^{\max}\}$
  Update the iterates:

$$x \leftarrow x + \alpha\, \Delta x, \qquad \lambda \leftarrow \lambda + \alpha\, \Delta\lambda, \qquad s \leftarrow s + \alpha\, \Delta s$$

  Update $t_i = \dfrac{\lambda_i s_i}{\mu_k}$ and recompute $\delta(t)$
end while
$k \leftarrow k+1$ ▷ Proceed to next outer iteration
end while
Output: approximate optimal solution $(x^\star, \lambda^\star, s^\star) = (x, \lambda, s)$

Remarks on the algorithm.

1. This algorithm is infeasible in the sense that the initial point does not need to satisfy the primal or dual constraints; feasibility and complementarity are recovered asymptotically along the kernel-based infeasible central path.
2. At each outer iteration k, the barrier parameter is updated according to the chosen strategy as
$$\mu_{k+1} = \mu_0\, (1-\theta)^{j_k}, \tag{4}$$
where $j_k \in \{1,\, 2,\, \lceil \log(k+1) \rceil,\, k\}$ depends on the selected update strategy. The target residuals decrease geometrically with rate $(1-\theta)^{j_k}$.
3. The inner iterations ensures that the iterates remain in a neighborhood of the infeasible central path by maintaining $\delta(v) \le \tau$.
4. The step length parameter $\eta \in (0,1)$ ensures that the iterates remain strictly positive ($\lambda > 0$, $s > 0$). In practice, η is typically chosen close to 1 (e.g., $\eta = 0.99$).

3. Theoretical properties of the new kernel function

In this section, we introduce a new kernel function designed to satisfy all the classical conditions required by primal-dual interior-point methods. We then examine its fundamental analytical properties, which are essential for establishing the convergence and stability of the algorithm.

The kernel function is defined, for $t > 0$, by

$$\begin{aligned} \psi : \ \mathbb{R}_+^* &\longrightarrow \mathbb{R} \\ t &\longmapsto \frac{\mathbf{a}}{te^{t/p}} + \frac{\beta}{t} + \left(\frac{\mathbf{a}(\frac{1}{p}+1)}{2e^{1/p}} + \frac{\beta}{2} \right) t^2 - \frac{\mathbf{a}(\frac{1}{p}+3)}{2e^{1/p}} - \frac{3\beta}{2}, \end{aligned} \tag{5}$$

where $p > 1$, $\mathbf{a} > 0$, and $\beta \geq 1$. We also define the constant

$$A := \frac{\mathbf{a}\left(\frac{1}{p}+1\right)}{e^{1/p}} + \beta.$$

For $t > 0$, the derivatives of ψ are given by

$$\begin{cases} \psi'(t) = -\dfrac{\mathbf{a}(\frac{t}{p}+1)}{t^2 e^{t/p}} - \dfrac{\beta}{t^2} + At, \\ \psi''(t) = \dfrac{\mathbf{a}\left((\frac{t}{p})^2 + 2\frac{t}{p} + 2\right)}{t^3 e^{t/p}} + \dfrac{2\beta}{t^3} + A, \\ \psi'''(t) = -\dfrac{\mathbf{a}\left((\frac{t}{p})^3 + 3(\frac{t}{p})^2 + 6\frac{t}{p} + 6\right)}{t^4 e^{t/p}} - \dfrac{6\beta}{t^4}. \end{cases} \tag{6}$$

Lemma 1. *The function ψ defined in (5) and with these derivatives defined in (6) satisfy:*

1. *$\psi''(t) > 0$ for all $t > 0$;*
2. *$\psi(1) = 0$ and $\psi'(1) = 0$;*
3. *$\psi'''(t) < 0$ for all $t > 0$;*
4. *$t\psi''(t) + \psi'(t) > 0$ for all $t > 0$;*
5. *$t\psi''(t) - \psi'(t) > 0$ for all $t > 0$;*
6. *For all $t > 1$ and $\beta_0 > 1$,*
$$\psi''(t)\psi'(\beta_0 t) - \beta_0\, \psi'(t)\psi''(\beta_0 t) > 0.$$

Proof
We verify the required properties.

1. We have
$$\psi''(t) = \frac{\mathbf{a}\left(\left(\frac{t}{p}\right)^2 + 2\frac{t}{p} + 2\right)}{t^3 e^{t/p}} + \frac{2\beta}{t^3} + A.$$
All terms are strictly positive for $t > 0$, since $p > 1$, $\mathbf{a} > 0$, and $\beta \geq 1$. Hence,
$$\psi''(t) > 0, \quad \forall t > 0.$$
2. A direct substitution shows that $\psi(1) = 0$ and $\psi'(1) = 0$. The verification is straightforward.
3. We compute
$$\psi'''(t) = -\frac{\mathbf{a}\left(\left(\frac{t}{p}\right)^3 + 3\left(\frac{t}{p}\right)^2 + 6\frac{t}{p} + 6\right)}{t^4 e^{t/p}} - \frac{6\beta}{t^4}.$$
Each term is negative for $t > 0$, thus
$$\psi'''(t) < 0, \quad \forall t > 0.$$
4. A direct simplification yields
$$t\psi''(t) + \psi'(t) = \frac{\mathbf{a}\left(\left(\frac{t}{p}\right)^2 + \frac{t}{p} + 1\right)}{t^2 e^{t/p}} + \frac{\beta}{t^2} + 2At > 0,$$
which holds for all $t > 0$.

5. Similarly,
$$t\psi''(t) - \psi'(t) = \frac{\mathbf{a}\left(\left(\frac{t}{p}\right)^2 + 3\frac{t}{p} + 3\right)}{t^2 e^{t/p}} + \frac{3\beta}{t^2} > 0, \quad \forall t > 0.$$
6. Define
$$f(\beta_0) = \psi''(t)\psi'(\beta_0 t) - \beta_0 \psi'(t)\psi''(\beta_0 t),$$
so that $f(1) = 0$. Differentiating, we obtain
$$f'(\beta_0) = \psi''(\beta_0 t)\big(t\psi''(t) - \psi'(t)\big) - \beta_0 t\psi'(t)\psi'''(\beta_0 t).$$
From the previous parts, we have
$$t\psi''(t) - \psi'(t) > 0, \quad \psi''(\beta_0 t) > 0, \quad \psi'''(\beta_0 t) < 0,$$
and for $t > 1$, $\psi'(t) > 0$. Hence $f'(\beta_0) > 0$, i.e., f is strictly increasing. Since $f(1) = 0$, it follows that
$$f(\beta_0) > 0, \quad \forall \beta_0 > 1.$$

□

Remark 1. The function ψ satisfies the standard kernel-function conditions (K1)–(K3): strict convexity, normalization at $t = 1$, and divergence as $t \to 0^+$ or $t \to +\infty$.

Definition 1. For a vector $v \in \mathbb{R}^m_{++}$, the associated barrier function is
$$\Psi(v) = \sum_{i=1}^{m} \psi(v_i).$$
The normalized proximity measure is defined as
$$\delta(v) = \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{m}\big(\psi'(v_i)\big)^2}.$$

Lemma 2 ([17])**.** *For a twice differentiable function $\psi : (0, \infty) \to \mathbb{R}$, the following statements are equivalent:*

1. *$\psi(e^{\zeta})$ is convex in ζ;*
2. *$\psi(\sqrt{\zeta})$ is convex in ζ;*
3. *$t\psi''(t) + \psi'(t) > 0$ for all $t > 0$;*
4. *$t\psi''(t) - \psi'(t) > 0$ for all $t > 0$.*

Lemma 3. *Using the fact that $\psi'''(t) < 0$ for all $t > 0$ together with the convexity of ψ , the following inequalities hold Since , the function ψ satisfies:*

If $t < 1$, then
$$\frac{\psi''(1)}{2}(t-1)^2 < \psi(t) < \psi'(t)(t-1)$$
and
$$\frac{\psi''(1)}{2}(t-1)^2 < \psi(t) < \frac{\psi''(t)}{2}(t-1)^2.$$

If $t > 1$, then

$$\frac{\psi''(t)}{2}(t-1)^2 < \psi(t) < \frac{\psi''(1)}{2}(t-1)^2.$$

and

$$\frac{\psi''(t)}{2}(t-1)^2 < \psi(t) < \psi'(t)(t-1).$$

Proof
We distinguish the cases $t < 1$ and $t > 1$.

Case $t < 1$**.** We prove

$$\frac{\psi''(1)}{2}(t-1)^2 < \psi(t) < \psi'(t)(t-1), \quad \frac{\psi''(1)}{2}(t-1)^2 < \psi(t) < \frac{\psi''(t)}{2}(t-1)^2.$$

1. Let
$$f(t) = \frac{\psi''(1)}{2}(t-1)^2 - \psi(t).$$
Then $f(1) = f'(1) = f''(1) = 0$ and $f'''(t) = -\psi'''(t) > 0$. Hence f'' is increasing, so $f''(t) < 0$ for $t < 1$, which implies $f'(t) > 0$ and thus $f(t) < 0$.
2. Let
$$g(t) = \psi(t) - \psi'(t)(t-1).$$
Then $g(1) = 0$ and
$$g'(t) = -\psi''(t)(t-1) > 0 \quad (t < 1),$$
since $\psi''(t) > 0$. Hence $g(t) < 0$ for $t < 1$.
3. Let
$$h(t) = \psi(t) - \frac{\psi''(t)}{2}(t-1)^2.$$
Then $h(1) = 0$ and
$$h'(t) = -\frac{\psi'''(t)}{2}(t-1)^2 + \big(\psi'(t) - \psi''(t)(t-1)\big).$$
Since $\psi'''(t) < 0$, the first term is positive. Moreover,
$$T(t) := \psi'(t) - \psi''(t)(t-1)$$
satisfies $T(1) = 0$ and $T'(t) = -\psi'''(t)(t-1) < 0$ for $t < 1$, hence $T(t) > 0$. Therefore $h'(t) > 0$, which implies $h(t) < 0$ for $t < 1$.

Case $t > 1$**.** We prove

$$\frac{\psi''(t)}{2}(t-1)^2 < \psi(t) < \frac{\psi''(1)}{2}(t-1)^2, \quad \frac{\psi''(t)}{2}(t-1)^2 < \psi(t) < \psi'(t)(t-1).$$

1. Let
$$f(t) = \psi(t) - \frac{\psi''(1)}{2}(t-1)^2.$$
Then $f(1) = f'(1) = f''(1) = 0$ and $f'''(t) = \psi'''(t) < 0$. Hence $f''(t) < 0$ for $t > 1$, which implies $f'(t) < 0$ and thus $f(t) < 0$.

2. For
$$g(t) = \psi(t) - \psi'(t)(t-1),$$
we have $g(1) = 0$ and
$$g'(t) = -\psi''(t)(t-1) < 0 \quad (t > 1),$$
hence $g(t) < 0$.
3. Let
$$h(t) = \frac{\psi''(t)}{2}(t-1)^2 - \psi(t).$$
Then $h(1) = 0$ and
$$h'(t) = \frac{\psi'''(t)}{2}(t-1)^2 + \big(-\psi'(t) + \psi''(t)(t-1)\big).$$
Since $\psi'''(t) < 0$, the first term is negative. Moreover,
$$T(t) := -\psi'(t) + \psi''(t)(t-1)$$
satisfies $T(1) = 0$ and $T'(t) = \psi'''(t)(t-1) < 0$ for $t > 1$, hence $T(t) < 0$. Therefore $h'(t) < 0$, which implies $h(t) < 0$ for $t > 1$.

□

Lemma 4. *For ψ we have for all $t > 0$:*
$$\frac{A}{2}(t-1)^2 \le \psi(t) \le \frac{1}{2A}\psi'^2(t).$$
With $A = \dfrac{\mathbf{a}(\frac{1}{p}+1)}{e^{\frac{1}{p}}} + \beta$.

Proof

1. We prove
$$\frac{A}{2}(t-1)^2 \le \psi(t), \quad \forall t > 0. \tag{7}$$
Let
$$f(t) := \psi(t) - \frac{A}{2}(t-1)^2.$$
Then
$$f''(t) = \psi''(t) - A.$$
From the expression of $\psi''(t)$, it follows directly that $f''(t) > 0$ for all $t > 0$. Hence f is strictly convex.
Moreover, $f(1) = 0$ and $f'(1) = 0$. Therefore, $t = 1$ is the unique minimizer of f, which yields
$$f(t) \ge 0, \quad \forall t > 0,$$
and (7) follows.
2. We prove
$$\psi(t) \le \frac{1}{2A}\psi'^2(t), \quad \forall t > 0. \tag{8}$$
Let
$$g(t) := \frac{1}{2A}\psi'^2(t) - \psi(t).$$

Then

$$g'(t) = \psi'(t)\left(\frac{\psi''(t)}{A} - 1\right).$$

Since $\psi''(t)$ is strictly decreasing and satisfies $\lim_{t\to\infty} \psi''(t) = A$, we have

$$\frac{\psi''(t)}{A} - 1 \begin{cases} > 0, & t < 1, \\ < 0, & t > 1. \end{cases}$$

Moreover, $\psi'(1) = 0$, hence $g(1) = 0$. It follows that g' changes sign at $t = 1$, so $t = 1$ is the global minimizer of g. Therefore,

$$g(t) \geq 0, \quad \forall t > 0,$$

which proves (8).

3. Combining (7) and (8), we obtain

$$\frac{A}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2A}\psi'^2(t), \quad \forall t > 0,$$

where

$$A = \frac{\mathbf{a}\left(\frac{1}{p}+1\right)}{e^{1/p}} + \beta.$$

□

Lemma 5. *Let $\varrho : [0, +\infty[\to [1, +\infty[$ be the inverse of ψ restricted to $t \geq 1$. Then, for all $s \geq 0$,*

$$1 + \sqrt{\frac{2s}{A}} \geq \varrho(s) \geq 1 + \sqrt{\frac{2s}{B}}.$$

With $A = \dfrac{\mathbf{a}\left(\frac{1}{p}+1\right)}{e^{1/p}} + \beta$, $B = \psi''(1)$.

Proof

Let $s \geq 0$ and let $t = \varrho(s) \geq 1$ such that $\psi(t) = s$.

From the left-hand inequality in Lemma 4, which holds for all $t > 0$, we have

$$\frac{A}{2}(t-1)^2 \leq \psi(t).$$

Which implies

$$t \leq 1 + \sqrt{\frac{2\psi(t)}{A}}.$$

Hence,

$$\varrho(s) \leq 1 + \sqrt{\frac{2s}{A}}.$$

On the other hand, by Lemma 3,

$$\psi(t) \leq \frac{\psi''(1)}{2}(t-1)^2 = \frac{B}{2}(t-1)^2,$$

which yields

$$1 + \sqrt{\frac{2\psi(t)}{B}} \leq t.$$

Thus,

$$\varrho(s) \geq 1 + \sqrt{\frac{2s}{B}}.$$

Combining the two bounds, we obtain

$$1 + \sqrt{\frac{2s}{B}} \leq \varrho(s) \leq 1 + \sqrt{\frac{2s}{A}}, \quad \forall s \geq 0.$$

□

Lemma 6. *Let $\eta : [0, +\infty[\to]0, 1]$, be The inverse function of $-\frac{1}{2}\psi'$ restricted to $1 \geq t > 0$, and q is an upper bound for $1 + \frac{2}{t}$ over $t \in (0, 1]$, then for all $s \geq 0$:*

$$\eta(s) \geq \sqrt[3]{\frac{\beta}{q(s + A)}}.$$

Proof
Let $t \in (0, 1]$ and define

$$s := -\frac{1}{2}\psi'(t).$$

By definition of η, we have $t = \eta(s)$ for all $s \geq 0$.

Using the explicit expression of $\psi'(t)$, we obtain

$$s = \frac{\mathbf{a}\left(\frac{t}{p} + 1\right)}{2t^2 e^{t/p}} + \frac{\beta}{2t^2} - \frac{A}{2}t.$$

Rearranging terms gives

$$s + \frac{A}{2}t = \frac{\mathbf{a}\left(\frac{t}{p} + 1\right)}{2t^2 e^{t/p}} + \frac{\beta}{2t^2}.$$

Since $\mathbf{a} > 0$ and all terms are nonnegative for $t \in (0, 1]$, it follows that

$$s + \frac{A}{2}t \geq \frac{\beta}{2t^2}.$$

Moreover, since $t \leq 1$, we have

$$s + \frac{A}{2} \geq s + \frac{A}{2}t.$$

Combining the two inequalities yields

$$s + \frac{A}{2} \geq \frac{\beta}{2t^2} \geq \frac{\beta}{2t^2 + t^3} = \frac{\beta}{t^3(\frac{2}{t} + 1)}.$$

Assuming $t \in (0, 1]$, we can define $t_0 > 0$ (however small) such that $t \in [t_0, 1]$. We denote:

$$q = 1 + \frac{2}{t_0} \quad \text{then} \quad 1 + \frac{2}{t} \leq q,$$

which implies:

$$s + \frac{A}{2} \geq \frac{\beta}{t^3 q}.$$

Multiplying both sides by $t^3 > 0$, we obtain

$$t^3(s + \frac{A}{2}) \geq \frac{\beta}{q}.$$

Hence,

$$t \geq \sqrt[3]{\frac{\beta}{q(s+A)}}.$$

Since $t = \eta(s)$, we conclude that

$$\eta(s) \geq \sqrt[3]{\frac{\beta}{q(s+A)}}.$$

□

Lemma 7. *From Definition 1, we have:*

1. $\delta(v) \geq \sqrt{\frac{A}{2}\Psi(v)}$;
2. $\|v\| \leq \sqrt{m} + \sqrt{\frac{2}{A}\Psi(v)} \leq \sqrt{m} + \frac{2}{A}\delta(v)$.

Proof

1. We use the Lemma 4,

$$\Psi(v) = \sum_{i=1}^{m} \psi(v_i) \leq \frac{1}{A}\sum_{i=1}^{m} \frac{\psi'(v_i)^2}{2} = \frac{1}{2A}\|\nabla\Psi(v)\|^2 = \frac{2}{A}\delta(v)^2.$$

And we get the outcome of lemma.

2. From Lemma 4, we have:

$$\|v\| \leq \|v - I\| + \|I\| = \sqrt{\sum_{i=1}^{m}(v_i - 1)^2} + \sqrt{m} \leq \sqrt{\frac{2}{A}\Psi(v)} + \sqrt{m}$$

so $\|v\| \leq \sqrt{m} + \sqrt{\frac{2}{A}\Psi(v)} \leq \sqrt{m} + \frac{2}{A}\delta(v)$.

□

Remark 2. Throughout the entire paper, it is assumed that $\tau \geq 1$. By applying the aforementioned lemma and considering the assumption $\Psi(v) \geq \tau$, it follows that $\delta(v) \geq \sqrt{\frac{A}{2}}$.

Lemma 8. *For all $\beta_0 \geq 1$ and $t > 0$,*

$$\psi(\beta_0 t) \leq \psi(t) + (\beta_0^2 - 1)\frac{A}{2}t^2.$$

Lemma 9. *Let $0 < \theta < 1$ and define $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Psi(v) \leq \tau$, then*

$$\Psi(v_+) \leq \Psi(v) + \frac{A}{2}(\frac{\theta}{1-\theta})(m + \frac{2}{A}\Psi(v) + 2\sqrt{\frac{2}{A}m\Psi(v)}).$$

Proof
From Lemma 8, with $\beta_0 = \frac{1}{\sqrt{(1-\theta)}}$, we obtain

$$\Psi(v_+) = \Psi(\beta_0 v) = \sum_{i=1}^{m} \psi(\beta_0 v_i) \leq \sum_{i=1}^{m} \psi(v_i) + (\beta_0^2 - 1)\frac{A}{2} v_i^2 = \Psi(v) + (\beta_0^2 - 1)\,\|v\|^2,$$

and from Lemma 7, we get the result.

$$\Psi(v_+) \leq \Psi(v) + \frac{A}{2}(\frac{\theta}{1-\theta})(\sqrt{m} + \sqrt{\frac{2}{A}\Psi(v)})^2 = \Psi(v) + \frac{A}{2}(\frac{\theta}{1-\theta})(m + \frac{2}{A}\Psi(v) + 2\sqrt{\frac{2}{A}m\Psi(v)}).$$

This completes the proof.

□

Theorem 1 ([17]). *Let ϱ be the inverse of ψ, for all $t \geq 1$, Then :*

$$\Psi(\beta_0 v) \leq m\,\psi\left(\beta_0\,\varrho\left(\frac{\Psi(v)}{m}\right)\right),\ v \in \mathbb{R}^m_{++},\ \beta_0 \geq 1.$$

Lemma 10. *Let $0 \leq \theta < 1$ and $v_+ = \dfrac{v}{\sqrt{1-\theta}}$. If $\Psi(v) \leq \tau$, then*

$$\Psi(v_+) \leq \frac{1}{1-\theta}\left(\theta\sqrt{\frac{Bm}{2}} + \sqrt{\tau}\right)^2 =: \Psi_0.$$

Moreover, Ψ_0 provides a global upper bound for Ψ throughout the algorithm.

Proof
We set $\beta_0 = \frac{1}{\sqrt{1-\theta}}$, so $\beta_0 \geq 1$. From Theorem 1,

$$\Psi(v_+) \leq m\psi\left(\beta_0\varrho\left(\frac{\Psi(v)}{m}\right)\right).$$

Since the argument of ψ is $t = \beta_0\varrho\left(\dfrac{\Psi(v)}{m}\right) \geq 1$, we use the upper bound from Lemma 3 (Case $t > 1$), $\psi(t) \leq \dfrac{B}{2}(t-1)^2$:

$$\Psi(v_+) \leq \frac{Bm}{2}\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(v)}{m}\right) - 1\right)^2.$$

From Lemma 5, we use the lower bound $\varrho\left(\dfrac{\Psi(v)}{m}\right) \geq 1 + \sqrt{\dfrac{2\Psi(v)}{mB}}$. Therefore:

$$\begin{aligned}\Psi(v_+) &\leq \frac{Bm}{2}\left(\frac{1}{\sqrt{1-\theta}}\left(1 + \sqrt{\frac{2\Psi(v)}{mB}}\right) - 1\right)^2 \\ &= \frac{Bm}{2(1-\theta)}\left(1 + \sqrt{\frac{2\Psi(v)}{mB}} - \sqrt{1-\theta}\right)^2.\end{aligned}$$

We use the inequality $1 - \sqrt{1-\theta} \leq \theta$ for $\theta \in [0,1]$, which leads to:

$$1 + \sqrt{\frac{2\Psi(v)}{mB}} - \sqrt{1-\theta} \leq \theta + \sqrt{\frac{2\Psi(v)}{mB}}.$$

Substituting this into the previous expression for $\Psi(v_+)$:

$$\Psi(v_+) \leq \frac{Bm}{2(1-\theta)} \left(\theta + \sqrt{\frac{2\Psi(v)}{mB}} \right)^2.$$

The final step uses the identity $\frac{Bm}{2} \left(\theta + \sqrt{\frac{2\Psi(v)}{mB}} \right)^2 = \left(\theta \sqrt{\frac{Bm}{2}} + \sqrt{\Psi(v)} \right)^2$:

$$\Psi(v_+) \leq \frac{1}{1-\theta} \left(\theta \sqrt{\frac{Bm}{2}} + \sqrt{\Psi(v)} \right)^2.$$

Since $\Psi(v) \leq \tau$, we finally obtain the desired upper bound:

$$\Psi(v_+) \leq \frac{1}{1-\theta} \left(\theta \sqrt{\frac{Bm}{2}} + \sqrt{\tau} \right)^2 =: \Psi_0.$$

□

4. Analysis of complexity

For each external iteration, we will compute in this section the number of inner iterations for the convex *QCQP* problem (P1) in the infeasible primal–dual setting, using the proximity function defined by the new kernel function proposed in this paper, following the proximity function approach in [19].

4.1. The step size

We calculate an approximation of the value of a step α and the resulting decrease of the barrier function. We use the full primal–dual step, but for simplicity, we focus on the update of the scaled variables

$$v_i = \sqrt{\frac{\lambda_i s_i}{\mu}}, \quad i = 1, \ldots, m,$$

where $s_i > 0$ are the slack variables satisfying

$$g_i(x) + s_i = \mathbf{r}_{p,i}^0 (1-\theta)^k,$$

along the infeasible central path at outer iteration k. Note that this relation converges to $s_i = -g_i(x)$ as $k \to \infty$, recovering the feasible case asymptotically.

After the update from (x, λ, s) to $(x + \alpha\Delta x, \lambda + \alpha\Delta\lambda, s + \alpha\Delta s)$, the new scaled variable $v_{+,i}$ is:

$$v_{+,i} = \sqrt{\frac{(\lambda_i + \alpha\Delta\lambda_i)(s_i + \alpha\Delta s_i)}{\mu_+}}, \quad i = 1, \ldots, m,$$

where

$$\mu_+ = (1-\theta)^j \mu, \quad j \in \{1,\ 2,\ \lceil \log(k+1) \rceil,\ k\},$$

and $(\Delta\lambda, \Delta s)$ is the Newton direction computed from the infeasible *KKT* system (2), incorporating the dual residual $\mathbf{r}_d^k$ and primal residual $\mathbf{r}_p^k$ at outer iteration k.

We choose α as the largest step such that the iterates remain strictly positive: $(\lambda, s) \in \mathring{\mathcal{F}}$, where

$$\alpha_\lambda = \max\{\alpha \in [0,1] : \lambda + \alpha\Delta\lambda > 0\},$$

and

$$\alpha_s = \max\{\alpha \in [0,1] : s + \alpha \Delta s > 0\},$$

then

$$\alpha = \min(0.95 \min(\alpha_\lambda, \alpha_s), 1).$$

For all $\alpha > 0$, we define the reduction function

$$\mathbb{F}(\alpha) = \Psi(v_+) - \Psi(v).$$

$\mathbb{F}(\alpha)$ measures the change in proximity between the new and the old iterates. For the theoretical analysis, we consider an upper bound $\mathbb{F}_1(\alpha)$ based on a linearization argument (see [1]):

$$\mathbb{F}_1(\alpha) = \frac{\Psi(v + \alpha d_\lambda) + \Psi(v + \alpha d_s)}{2} - \Psi(v), \tag{9}$$

where d_λ and d_s are appropriate search directions satisfying

$$v_+ \leq \frac{(v + \alpha d_\lambda) + (v + \alpha d_s)}{2}.$$

By the convexity of Ψ, we have

$$\mathbb{F}(\alpha) \leq \mathbb{F}_1(\alpha), \quad \text{and} \quad \mathbb{F}_1(0) = \mathbb{F}(0) = 0.$$

The first two derivatives of $\mathbb{F}_1(\alpha)$ are:

$$\mathbb{F}_1'(\alpha) = \frac{1}{2} \sum_{i=1}^{m} \left[\psi'(v_{\lambda,i})\, d_{\lambda,i} + \psi'(v_{s,i})\, d_{s,i}\right],$$

where $v_{\lambda,i} = v_i + \alpha d_{\lambda,i}$ and $v_{s,i} = v_i + \alpha d_{s,i}$, and

$$\mathbb{F}_1''(\alpha) = \frac{1}{2} \sum_{i=1}^{m} \left[\psi''(v_{\lambda,i})\, d_{\lambda,i}^2 + \psi''(v_{s,i})\, d_{s,i}^2\right].$$

In the infeasible setting, the Newton direction $(\Delta x, \Delta\lambda, \Delta s)$ is derived from system (2) with nonzero residuals $\mathbf{r}_d^k$ and $\mathbf{r}_p^k$. Nevertheless, the third block equation of (2) gives

$$S\Delta\lambda + \Lambda \Delta s = \mu e - \Lambda s,$$

which, in terms of the scaled variables $v_i = \sqrt{\lambda_i s_i / \mu}$, yields the perturbed complementarity condition

$$\frac{\Delta\lambda_i}{\lambda_i} + \frac{\Delta s_i}{s_i} = \frac{\mu}{\lambda_i s_i} - 1 = \frac{1}{v_i^2} - 1.$$

Under this condition, the search directions satisfy $d_{\lambda,i} + d_{s,i} = -2\psi'(v_i)$ in the scaled variables, so that at $\alpha = 0$:

$$\mathbb{F}_1'(0) = \sum_{i=1}^{m} \psi'(v_i) \cdot \frac{d_{\lambda,i} + d_{s,i}}{2} = -\sum_{i=1}^{m} (\psi'(v_i))^2 = -2\delta^2(v).$$

To simplify notation in what follows, we write $\delta(v) = \delta$ and $v_{\min} = \min\limits_{i \in \{1,\ldots,m\}} v_i$.

Remark 3. Lemmas 11–15 below concern the inner centering loop, in which μ is fixed and the infeasibility residuals $\mathbf{r}_d^k = \mathbf{r}_d^0(1-\theta)^k$ and $\mathbf{r}_p^k = -\mathbf{r}_p^0(1-\theta)^k$ are treated as fixed right-hand sides. These results therefore hold identically in the infeasible setting.

The following lemmas are based on [17]:

Lemma 11. *Let $\mathbb{F}_1(\alpha)$ be defined in (9). Then, for all $\alpha \geq 0$ satisfying $v_{\min} - 2\alpha\delta > 0$, we have*

$$\mathbb{F}_1''(\alpha) \leq 2\delta^2\, \psi''(v_{\min} - 2\alpha\delta).$$

Lemma 12. *For all $\alpha \geq 0$ satisfying $v_{\min} - 2\alpha\delta > 0$, the condition $\mathbb{F}_1'(\alpha) \leq 0$ holds whenever α satisfies the inequality*

$$-\psi'(v_{\min} - 2\alpha\delta) + \psi'(v_{\min}) \leq 2\delta. \tag{10}$$

This follows by integrating the bound of Lemma 11 from 0 to α and using $\mathbb{F}_1'(0) = -2\delta^2$.

Lemma 13. *The largest step size α_1 satisfying inequality (10) is given by:*

$$\alpha_1 := \frac{1}{2\delta}\big(\eta(\delta) - \eta(2\delta)\big),$$

where $\eta : [0, +\infty) \to (0, 1]$ is the inverse function of $-\frac{1}{2}\psi'$ restricted to $(0, 1]$, as defined in Lemma 6, provided that 2δ lies in the range of $-\frac{1}{2}\psi'$.

Lemma 14. *Let α_1 be as defined in Lemma 13. Then*

$$\alpha_1 \geq \frac{1}{\psi''(\eta(2\delta))} =: \alpha_2.$$

Proof
By the mean value theorem applied to η on $[\delta, 2\delta]$:

$$\eta(\delta) - \eta(2\delta) \geq \delta \cdot \frac{2}{\psi''(\eta(2\delta))},$$

where we used the fact that $|\eta'(s)| = \frac{2}{\psi''(\eta(s))}$ and that ψ'' is strictly decreasing (since $\psi'''(t) < 0$ for all $t > 0$ by Lemma 1 (3)), so $\psi''(\eta(s)) \leq \psi''(\eta(2\delta))$ for all $s \in [\delta, 2\delta]$. Therefore:

$$\alpha_1 = \frac{\eta(\delta) - \eta(2\delta)}{2\delta} \geq \frac{1}{\psi''(\eta(2\delta))} = \alpha_2.$$

□

Lemma 15. *If the step size α satisfies $\alpha \leq \alpha_1$, then*

$$\mathbb{F}(\alpha) \leq -\alpha\delta^2. \tag{11}$$

Proof
Since $\alpha \leq \alpha_1$, Lemma 12 ensures that $\mathbb{F}_1'(\beta) \leq 0$ for all $\beta \in [0, \alpha]$. Using $\mathbb{F}_1(0) = 0$ and $\mathbb{F}_1'(0) = -2\delta^2$, we get

$$\mathbb{F}(\alpha) \leq \mathbb{F}_1(\alpha) \leq \alpha\, \mathbb{F}_1'(0) = -2\alpha\delta^2 \leq -\alpha\delta^2.$$

□

4.2. Theoretical number of iterations

4.2.1. Outer iterations

Lemma 16 ([17])**.** *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$, and let $h(t)$ attain its global minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$, then*

$$h(t^*) = \frac{t^* h'(0)}{2}.$$

Theorem 2. *If $\Psi \geq 1$ and $\delta \leq \tau$, then the maximal decrease in the proximity measure after one Newton step satisfies*

$$\mathbb{F}(\alpha_2) \leq -C_{\text{red}}\, \Psi^{1/2},$$

where

$$C_{\text{red}} = \frac{\beta \sqrt{\dfrac{A}{2}}}{\left(\mathbf{a}\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A\right) q(2+\sqrt{2A})} > 0,$$

with $A = \dfrac{\mathbf{a}\left(\frac{1}{p}+1\right)}{e^{1/p}} + \beta$.

Proof
The proof is based on Lemmas 6, 7, 14, and 15.

Step 1: Estimate of the step size.
From (6) and Lemma 14, for all $t \in (0,1]$, we have

$$\psi''(t) = \frac{\mathbf{a}\left(\left(\frac{t}{p}\right)^2 + 2\frac{t}{p} + 2\right)}{t^3 e^{t/p}} + \frac{2\beta}{t^3} + A.$$

Since $0 < t \leq 1$, it follows that $1 < e^{t/p} \leq e^{1/p}$, $\dfrac{1}{t^3} \geq 1$, and $0 < \dfrac{t}{p} \leq \dfrac{1}{p}$. Hence,

$$\psi''(t) \leq \frac{\mathbf{a}\left(\frac{1}{p^2}+\frac{2}{p}+2\right)}{t^3} + \frac{2\beta}{t^3} + \frac{A}{t^3} = \frac{\mathbf{a}\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A}{t^3}.$$

Since $0 < \eta(s) \leq 1$ for all $s \geq 0$, we obtain

$$\psi''(\eta(s)) \leq \frac{\mathbf{a}\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A}{(\eta(s))^3}.$$

Using Lemma 6, for all $s \geq 0$,

$$\eta(s) \geq \sqrt[3]{\frac{\beta}{q(s+A)}},$$

which implies

$$\psi''(\eta(s)) \leq \frac{\mathbf{a}\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A}{\beta}\, q(s+A).$$

Since $\delta \geq 0$, setting $s = 2\delta$ yields

$$\psi''(\eta(2\delta)) \leq \frac{\mathbf{a}\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A}{\beta}\, q(2\delta+A).$$

By Remark 2, we further obtain

$$\psi''(\eta(2\delta)) \leq \frac{\mathbf{a}\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A}{\beta}\, q(2\delta+\sqrt{2A}\,\delta).$$

Consequently,

$$\frac{1}{\psi''(\eta(2\delta))} \geq \frac{\beta}{\left(\mathbf{a}\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A\right)\, q(2\delta+\sqrt{2A}\,\delta)}.$$

Therefore,

$$\alpha_2 \geq \frac{\beta}{\left(\mathbf{a}\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A\right)\, q(2\delta+\sqrt{2A}\,\delta)}.$$

Step 2: Reduction of the proximity measure.

From Lemma 15, for $\alpha = \alpha_2 \leq \alpha_1$, we have

$$\mathbb{F}(\alpha_2) \leq -\alpha_2\,\delta^2.$$

Using the lower bound on α_2, it follows that

$$\mathbb{F}(\alpha_2) \leq -\frac{\beta\,\delta^2}{\left(\alpha\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A\right)\, q(2+\sqrt{2A})\delta} = -\frac{\beta\,\delta}{\left(\alpha\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A\right)\, q(2+\sqrt{2A})}.$$

Step 3: Relation between δ and Ψ.

From Lemma 7 (1), we have

$$\delta \geq \sqrt{\frac{A}{2}}\,\Psi.$$

Substituting this estimate into the previous inequality yields

$$\mathbb{F}(\alpha_2) \leq -\frac{\beta\sqrt{\frac{A}{2}}\,\Psi^{1/2}}{\left(\mathbf{a}\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A\right)\, q(2+\sqrt{2A})}.$$

Finally, defining

$$C_{\text{red}} := \frac{\beta\sqrt{\frac{A}{2}}}{\left(\mathbf{a}\left(\frac{1}{p^2}+\frac{2}{p}+2\right)+2\beta+A\right)\, q(2+\sqrt{2A})},$$

the desired result follows. □

Lemma 17. *For any $\alpha \in [0,1]$ and $t \in [0,1]$, we have*

$$(1-t)^\alpha \leq 1-\alpha t.$$

Proof

Define $\mathbb{H}(t) = (1-t)^\alpha - (1-\alpha t)$. Then:

1. $\mathbb{H}(0) = 0$;
2. $\mathbb{H}'(t) = -\alpha(1-t)^{\alpha-1} + \alpha$, so $\mathbb{H}'(0) = 0$;
3. $\mathbb{H}''(t) = \alpha(\alpha-1)(1-t)^{\alpha-2} \leq 0$ for $t \in [0,1)$, since $\alpha \in [0,1]$ implies $\alpha - 1 \leq 0$.

Hence $\mathbb{H}'$ is non-increasing with $\mathbb{H}'(0) = 0$, so $\mathbb{H}'(t) \leq 0$ for all $t \in [0,1]$. Therefore $\mathbb{H}$ is non-increasing and, since $\mathbb{H}(0) = 0$, we conclude $\mathbb{H}(t) \leq 0$ for all $t \in [0,1]$. □

Lemma 18. *Let Ψ_0 denote the value of $\Psi(v)$ after the first outer update, and let Ψ_j, $j = 1, 2, \ldots, K$, be the sequence of values of $\Psi(v)$ at the start of each outer iteration. The number of outer iterations K satisfies*

$$K \leq \left\lceil C_{\text{outer}}\, \Psi_0^{1/2} \right\rceil,$$

where

$$C_{\text{outer}} = \frac{2\left(\left(\mathbf{a}\left(\frac{1}{p^2} + \frac{2}{p} + 2\right) + 2\beta + A\right) q(2 + \sqrt{2A})\right)}{\beta\sqrt{\frac{A}{2}}},$$

and

$$\Psi_0 = \frac{1}{1-\theta}\left(\theta\sqrt{\frac{Bm}{2}} + \sqrt{\tau}\right)^2,$$

as given in Lemma 10. In particular, $\Psi_0 = O(m)$, so $K = O(m^{1/2})$.

Proof
From Theorem 2, for each outer iteration j such that $\Psi_j \geq 1$, we have

$$\Psi_{j+1} \leq \Psi_j - C_{\text{red}}\, \Psi_j^{1/2} = \Psi_j - C_{\text{red}}\, \Psi_j^{1-\frac{1}{2}}.$$

This inequality is of the standard form

$$t_{j+1} \leq t_j - \beta\, t_j^{1-\gamma},$$

with the identifications $t_j = \Psi_j$, $\beta = C_{\text{red}}$, and $\gamma = \dfrac{1}{2}$.

Raising both sides to the power γ and applying Lemma 17 with $t = \beta t_j^{-\gamma}$, we obtain

$$t_{j+1}^{\gamma} \leq t_j^{\gamma}(1 - \beta t_j^{-\gamma})^{\gamma} \leq t_j^{\gamma}\left(1 - \gamma\beta t_j^{-\gamma}\right) = t_j^{\gamma} - \beta\gamma.$$

Iterating this inequality yields, for all $K \geq 1$,

$$0 < t_K^{\gamma} \leq t_0^{\gamma} - K\beta\gamma.$$

Consequently,

$$K \leq \left\lceil \frac{t_0^{\gamma}}{\beta\gamma} \right\rceil = \left\lceil \frac{\Psi_0^{1/2}}{C_{\text{red}} \cdot (1/2)} \right\rceil = \left\lceil \frac{2}{C_{\text{red}}}\, \Psi_0^{1/2} \right\rceil.$$

Substituting the expression of C_{red}, we obtain

$$K \leq \left\lceil \frac{2\left(\left(\mathbf{a}\left(\frac{1}{p^2} + \frac{2}{p} + 2\right) + 2\beta + A\right) q(2 + \sqrt{2A})\right)}{\beta\sqrt{\frac{A}{2}}}\, \Psi_0^{1/2} \right\rceil.$$

Defining

$$C_{\text{outer}} := \frac{2\left(\left(\mathbf{a}\left(\frac{1}{p^2} + \frac{2}{p} + 2\right) + 2\beta + A\right) q(2 + \sqrt{2A})\right)}{\beta\sqrt{\frac{A}{2}}},$$

and using the fact that $\Psi_0 = O(m)$, it follows that

$$K = O(m^{1/2}).$$

□

Remark 4. In the infeasible primal–dual setting, the bound Ψ_0 in Lemma 10 remains valid as an upper bound for $\Psi(v_+)$ after each outer update, provided that the infeasibility residuals satisfy

$$\|\mathbf{r}_d^k\| \le \|\mathbf{r}_d^0\|(1-\theta)^k \to 0, \qquad \|\mathbf{r}_p^k\| \le \|\mathbf{r}_p^0\|(1-\theta)^k \to 0,$$

which is guaranteed by the geometric decrease of the target residuals in Algorithm 1. Hence the outer iteration bound $K = O(m^{1/2})$ holds in the infeasible case as well.

4.2.2. Inner iterations

Lemma 19. *In the infeasible primal–dual IPM, the stopping criterion requires*

$$m\mu^k \le \varepsilon, \qquad \|\mathbf{r}_d^k\| \le \varepsilon, \qquad \|\mathbf{r}_p^k\| \le \varepsilon.$$

Since μ^k, $\|\mathbf{r}_d^k\|$, and $\|\mathbf{r}_p^k\|$ all decrease at rate $(1-\theta)^{J_k}$, where J_k depends on the update strategy, the binding constraint is

$$(1-\theta)^{J_k} \le \frac{\varepsilon}{M_0}, \quad M_0 := \max\bigl(m\mu^0, \|\mathbf{r}_d^0\|, \|\mathbf{r}_p^0\|\bigr).$$

The number of inner iterations required to satisfy this criterion for each strategy is as follows:

(S1) $j = 1$*:*

$$\mu^k = (1-\theta)^k \mu^0, \qquad k \ge \frac{1}{\theta}\log\frac{M_0}{\varepsilon} = \Theta\Bigl(\log\frac{m}{\varepsilon}\Bigr).$$

(S2) $j = 2$*:*

$$\mu^k = (1-\theta)^{2k} \mu^0, \qquad k \ge \frac{1}{2\theta}\log\frac{M_0}{\varepsilon} = \Theta\Bigl(\log\frac{m}{\varepsilon}\Bigr).$$

(S3) $j = \lceil \log(k+1) \rceil$*:*

$$\mu^k = \mu^0 \prod_{\ell=0}^{k-1} (1-\theta)^{\lceil \log(\ell+1) \rceil}, \qquad k = \Theta\left(\frac{\log(M_0/\varepsilon)}{\theta \log k}\right) = \Theta\left(\frac{\log(m/\varepsilon)}{\theta \log k}\right).$$

(S4) $j = k$*:*

$$\mu^k = (1-\theta)^{k(k-1)/2} \mu^0, \qquad k = \Theta\left(\sqrt{\frac{2}{\theta}\log\frac{M_0}{\varepsilon}}\right) = \Theta\left(\sqrt{\log\frac{m}{\varepsilon}}\right).$$

In all four cases, assuming $M_0 = O(m)$, we have $k = O\bigl(\log \frac{m}{\varepsilon}\bigr)$.

Proof
The stopping criterion in the infeasible case requires all three quantities $m\mu^k$, $\|\mathbf{r}_d^k\|$, $\|\mathbf{r}_p^k\|$ to fall below ε. Since

$$m\mu^k = m\mu^0 \cdot (1-\theta)^{J_k}, \quad \|\mathbf{r}_d^k\| \le \|\mathbf{r}_d^0\| \cdot (1-\theta)^{J_k}, \quad \|\mathbf{r}_p^k\| \le \|\mathbf{r}_p^0\| \cdot (1-\theta)^{J_k},$$

all three conditions reduce to

$$M_0 \cdot (1-\theta)^{J_k} \le \varepsilon, \quad M_0 = \max(m\mu^0, \|\mathbf{r}_d^0\|, \|\mathbf{r}_p^0\|).$$

Taking logarithms and using $-\log(1-\theta) \ge \theta$:

$$J_k \ge \frac{\log(M_0/\varepsilon)}{-\log(1-\theta)} \ge \frac{1}{\theta}\log\frac{M_0}{\varepsilon}.$$

Case (S1): $j = 1$**.** $J_k = k$. The condition gives $k \ge \frac{1}{\theta}\log\frac{M_0}{\varepsilon}$.

Case (S2): $j = 2$. $J_k = 2k$. The condition gives $k \geq \frac{1}{2\theta} \log \frac{M_0}{\varepsilon}$, requiring half as many iterations as (S1).

Case (S3): $j = \lceil \log(k+1) \rceil$. The cumulative exponent is

$$J_k = \sum_{\ell=0}^{k-1} \lceil \log(\ell+1) \rceil \sim k \log k \quad \text{as } k \to \infty.$$

The condition $J_k \geq \frac{1}{\theta} \log \frac{M_0}{\varepsilon}$ combined with $J_k \sim k \log k$ gives

$$k = \Theta\left(\frac{\log(M_0/\varepsilon)}{\theta \log k} \right).$$

Case (S4): $j = k$. The cumulative exponent is

$$J_k = \sum_{\ell=0}^{k-1} \ell = \frac{k(k-1)}{2}.$$

The condition $\frac{k(k-1)}{2} \geq \frac{1}{\theta} \log \frac{M_0}{\varepsilon}$ gives

$$k = \Theta\left(\sqrt{\frac{2}{\theta} \log \frac{M_0}{\varepsilon}} \right) = \Theta\left(\sqrt{\log \frac{m}{\varepsilon}} \right).$$

Since $\sqrt{\log(m/\varepsilon)} = O(\log(m/\varepsilon))$, all four strategies satisfy $k = O\left(\log \frac{m}{\varepsilon}\right)$ under the assumption $M_0 = O(m)$. □

Remark 5. In the infeasible primal–dual setting, the total iteration complexity corresponds to the total number of Newton steps, which is obtained by summing the number of inner iterations over all outer iterations.

More precisely, if $K = O(m^{1/2})$ denotes the number of outer iterations (Lemma 18) and $k(\mu)$ the number of inner iterations required for a given barrier parameter μ (Lemma 19), then the total number of Newton steps satisfies

$$\mathcal{N}_{\text{total}} = \sum_{\text{outer iterations}} k(\mu) = K \times k(\mu).$$

Under the assumption $M_0 = O(m)$, where $M_0 := \max\left(m\mu^0, \|\mathbf{r}_d^0\|, \|\mathbf{r}_p^0\|\right)$, this leads to the following complexity bounds for each barrier update strategy. The infeasibility of the starting point does not degrade the complexity bound. Indeed, since the infeasibility residuals satisfy

$$\|\mathbf{r}_d^k\| \leq \|\mathbf{r}_d^0\|(1-\theta)^{J_k} \to 0, \qquad \|\mathbf{r}_p^k\| \leq \|\mathbf{r}_p^0\|(1-\theta)^{J_k} \to 0,$$

the outer iteration bound $K = O(m^{1/2})$ from Lemma 18 remains valid throughout, and the inner iteration counts from Lemma 19 are unaffected.

Strategies (S1)–(S3) therefore achieve the best-known complexity bound for kernel-based *IPMs*:

$$\mathcal{N}_{\text{total}} = O\left(m^{1/2} \log \frac{m}{\varepsilon} \right),$$

matching that of feasible kernel-based *IPMs* [17].

Strategy (S4), with $j_k = k$, achieves the strictly improved total complexity

$$\mathcal{N}_{\text{total}} = O\left(m^{1/2} \sqrt{\log \frac{m}{\varepsilon}} \right),$$

since $\sqrt{\log(m/\varepsilon)} = o(\log(m/\varepsilon))$ as $m/\varepsilon \to \infty$. This improvement comes from the superlinear decrease of μ under strategy (S4), which reduces the number of inner iterations required per outer step. However, the aggressive reduction of μ for large k may affect numerical stability in practice and should be monitored carefully during implementation.

Table 1. Iteration complexity for each barrier update strategy (infeasible primal–dual *IPM*).

Strategy	Update rule j_k	K (outer)	k (inner)	Total complexity
(S1)	1	$O(m^{1/2})$	$O\left(\log \frac{m}{\varepsilon}\right)$	$O\left(m^{1/2} \log \frac{m}{\varepsilon}\right)$
(S2)	2	$O(m^{1/2})$	$O\left(\log \frac{m}{\varepsilon}\right)$	$O\left(m^{1/2} \log \frac{m}{\varepsilon}\right)$
(S3)	$\lceil \log(k+1) \rceil$	$O(m^{1/2})$	$O\left(\log \frac{m}{\varepsilon}\right)$	$O\left(m^{1/2} \log \frac{m}{\varepsilon}\right)$
(S4)	k	$O(m^{1/2})$	$O\left(\sqrt{\log \frac{m}{\varepsilon}}\right)$	$O\left(m^{1/2} \sqrt{\log \frac{m}{\varepsilon}}\right)$

5. Numerical Experiments and Performance Analysis

This section assesses the computational performance of the proposed infeasible primal–dual interior-point method on several classes of convex *QCQP* problems, evaluating convergence behavior, numerical accuracy, and overall efficiency.

Numerical experiments were conducted on a Dell Latitude 5400 laptop equipped with an Intel® Core™ i5-8365U processor (8th Generation, 4 cores / 8 threads, base frequency 1.60 GHz, up to 4.10 GHz Turbo Boost) and 16 GB of RAM, running Windows 11 Professional. All experiments were executed in a single-threaded environment without parallelization. The algorithms were implemented in Python 3.10, using `NumPy` for numerical computations and `SciPy` for sparse linear algebra operations. Each reported execution time is the arithmetic mean over 20, 10, and 5 independent runs for small-, medium-, and large-scale problems, respectively. The maximum number of iterations was set to $100,000$(outer iterations) and $700,000$ (inner iterations). The algorithm terminates when the following conditions are simultaneously satisfied:

$$m\mu \le \varepsilon_1, \qquad \|\mathbf{r}_d^k\|_\infty \le \varepsilon_2, \qquad \|\mathbf{r}_p^k\|_\infty \le \varepsilon_2, \qquad \|g(x) \odot s\|_\infty \le \varepsilon_2,$$

where

$$\mathbf{r}_d^k = \nabla_x \mathcal{L}(x,\lambda) - \mathbf{r}_d^0(1-\theta)^k, \qquad \mathbf{r}_p^k = g(x) + s - \mathbf{r}_p^0(1-\theta)^k,$$

are the dual and primal infeasibility residuals at iteration k, $\odot$ denotes the Hadamard product. These conditions are consistent with the theoretical stopping criterion of Lemma 19: $m\mu \le \varepsilon_1$ controls the duality gap, $\|\mathbf{r}_d^k\|_\infty$ and $\|\mathbf{r}_p^k\|_\infty$ ensure asymptotic recovery of dual and primal feasibility ($\varepsilon_1 = 10^{-8}$), and $\|g(x) \odot s\|_\infty \le \varepsilon_2$ verifies approximate *KKT* complementarity ($\varepsilon_2 = 10^{-6}$). Together they guarantee that the output $(x^\star, \lambda^\star, s^\star)$ is an approximate *KKT* point of problem (P1).

For benchmarking purposes, the performance of the proposed method is systematically compared with `IPOPT` (version 3.14.11), which is widely recognized as a state-of-the-art solver for large-scale nonlinear optimization problems.

Table 2. IPOPT solver configurations used in the experiments

Full Name	**Abbreviation**
IPOPT — Default-Adaptive (10^{-7})	IPOPT-DA
IPOPT — Tight-Adaptive (10^{-8})	IPOPT-TA
IPOPT — Loose-Adaptive (10^{-6})	IPOPT-LA
IPOPT — Default-Monotone (10^{-7})	IPOPT-DM
IPOPT — VeryTight-Adaptive (10^{-10})	IPOPT-VTA
IPOPT — Fast-Adaptive (10^{-6})	IPOPT-FA

Table 3. Kernel functions used in the benchmark. Here c_1 and c_2 are constants depending on parameters $\mathbf{a}$, β, p as defined in row 5.

i	The kernel function $\psi_i(t)$	Ref
1	$\psi_1(t) = t^2 + \frac{2}{t} - 3$	[13]
2	$\psi_2(t) = \frac{t^2-1}{2} - \log(t)$	[18]
3	$\psi_3(t) = t^2 - 1 - \log(t) + \frac{t^{-p}-1}{p}, \quad p \geq 1$	[6]
4	$\psi_4(t) = t^2 - 1 + \frac{1}{p}\left(\frac{\cosh^p(t^{-1}) - \cosh^p(1)}{\tanh(1)\cosh^p(1)\, t^p} - \log(t^p)\right), \quad p \geq 4$	[4]
5	$\psi_{\text{new}}(t) = \dfrac{\mathbf{a}}{t\, e^{t/p}} + \dfrac{\beta}{t} + c_1\, t^2 - c_2$ where $c_1 = \dfrac{\mathbf{a}\left(\frac{1}{p}+1\right)}{2\, e^{1/p}} + \dfrac{\beta}{2}, \quad c_2 = \dfrac{\mathbf{a}\left(\frac{1}{p}+3\right)}{2\, e^{1/p}} + \dfrac{3\beta}{2}$ **V1:** p=2, β=2, $\mathbf{a}$=0.5 ; **V2:** p=1.1, β=1, $\mathbf{a}$=0.1	New

Example 1. For $m = 1$ and $n = 2$, the functions f and g are defined as follows :

$$\begin{cases} \min f(x_1, x_2) = (x_1 + x_2 - \pi)^2 - x_1 x_2 \\ g(x_1, x_2) = x_1 + x_2 - \pi \leq 0. \end{cases}$$

We test the algorithm starting from the following infeasible initial points:

x^0	$g(x^0)$	s^0
$(\frac{\pi}{2}+1,\ \frac{\pi}{2}+1)$	2.0000	5.00×10^{-2}
$(\pi,\ 1)$	1.0000	1.00×10^{-1}
$(4,\ 4)$	4.8584	2.06×10^{-2}

Remark 6. For this test problem, IPOPT shows superior runtime performance, while the proposed kernels maintain comparable robustness and convergence behavior.

Example 2. For $n = m = 3$, the convex problem is defined as follows :

$$\begin{cases} \min \dfrac{1}{2} x^T P x + q^T x + r \\ x_i - 1 \leq 0, \quad i = 1, 2, 3, \end{cases}$$

where :

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad P = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix}, \quad q = \begin{pmatrix} -22 \\ -14.5 \\ 13 \end{pmatrix} \quad \text{and} \quad r = 1.$$

We test the algorithm starting from the following infeasible initial points:

x^0	$g(x^0)$	s^0
(2, 2, 2)	(1, 1, 1)	0.100
(1.5, 0.5, 2)	(0.5, −0.5, 1)	0.100
(1.1, 1.1, 1.1)	(0.1, 0.1, 0.1)	1.000
(5, 5, 5)	(4, 4, 4)	0.025

Example 3. For $n = m = 5$, the functions f and g are defined as follows :

$$\begin{cases} \min f(x_1,x_2,x_3,x_4,x_5) = x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_1x_2+x_3x_4+x_4x_5 \\ g(x_1,x_2,x_3,x_4,x_5) = \begin{pmatrix} x_1+x_2+x_3+x_4+x_5-10 \\ x_1^2+x_2^2-4 \\ -x_3-x_4-1 \\ -x_5 \\ x_5+x_4-2x_2 \end{pmatrix} \leq 0. \end{cases}$$

We test the algorithm starting from the following infeasible initial points:

x^0	$g(x^0)$	$s^0_{\max}$
$(3,3,1,1,2)$	$(0,\ 14,\ -3,\ -2,\ -3)$	1.00×10^{5}
$(1,1,-0.4,-0.4,-2)$	$(-10.8,\ -2,\ -0.2,\ 2,\ -4.4)$	5.00×10^{-1}
$(0,0,-0.5,2,2)$	$(-6.5,\ -4,\ -2.5,\ -2,\ 4)$	5.00×10^{-2}
$(4,4,4,4,4)$	$(10,\ 28,\ -9,\ -4,\ 0)$	1.00×10^{5}

Example 4. For arbitrary dimensions n (variables) and m (constraints), the *QCQP* problem is:

$$\begin{cases} \min\limits_{x\in\mathbb{R}^n} \quad f(x) = \dfrac{1}{2}x^\top Q_0 x + c_0^\top x + d_0 \\ \text{s.t.} \quad g_i(x) = \dfrac{1}{2}x^\top Q_i x + c_i^\top x + d_i \leq 0, \quad \forall i \in \{1,\ldots,m\}, \end{cases}$$

where all matrices $Q_0, Q_1, \ldots, Q_m$ are diagonal with:

$$Q_0 = \operatorname{diag}\left(0.5+\tfrac{j}{n}\right)_{j=1}^{n}, \quad Q_i = \operatorname{diag}\left(0.1+0.5\tfrac{j}{n}\right)_{j=1}^{n},$$

$$c_0 = 0.1\cdot\left(\cos\left(\tfrac{2\pi j}{n}\right)\right)_{j=1}^{n}, \quad c_i = 0.1\cdot\sin\left(\tfrac{\pi i}{m}\right)\cdot\left(\tfrac{j}{n}\right)_{j=1}^{n},$$

$$d_0 = 0, \quad d_i = -1-\tfrac{i}{2m}.$$

We test the algorithm starting from the following infeasible initial points:

Table 4. Initial conditions and constraints analysis (Part I: $n \leq 1000$).

(n,m)	x^0	$\max_i g_i(x^0)$	$s^0_{\max}$
$(100,20)$	$(2,\ldots,2)$	2.1906	4.57×10^{-2}
	$(2,-0.1,2,-0.1,\ldots)$	2.1906	9.99×10^{-2}
	$(3,\ldots,3)$	8.3974	1.19×10^{-2}
	$(1.8,\ldots,1.8,-0.1,\ldots)$	1.3786	7.25×10^{-2}
$(500,100)$	$(2,\ldots,2)$	2.1906	4.57×10^{-2}
	$(2,-0.1,2,-0.1,\ldots)$	2.1906	9.99×10^{-2}
	$(3,\ldots,3)$	8.3974	1.19×10^{-2}
	$(1.8,\ldots,1.8,-0.1,\ldots)$	1.3786	7.25×10^{-2}
$(1000,200)$	$(2,\ldots,2)$	2.1906	4.57×10^{-2}
	$(2,-0.1,2,-0.1,\ldots)$	2.1906	9.99×10^{-2}
	$(3,\ldots,3)$	8.3974	1.19×10^{-2}
	$(1.8,\ldots,1.8,-0.1,\ldots)$	1.3786	7.25×10^{-2}

Table 5. Initial conditions and constraints analysis (Part II: $n \geq 5000$).

(n, m)	x^0	$\max_i g_i(x^0)$	$s^0_{\max}$
$(5000, 5000)$	$(2, \ldots, 2)$	2.1906	4.57×10^{-2}
	$(2, -0.1, 2, -0.1, \ldots)$	2.1906	9.99×10^{-2}
	$(3, \ldots, 3)$	8.3974	1.19×10^{-2}
	$(1.8, \ldots, 1.8, -0.1, \ldots)$	1.3786	7.25×10^{-2}
$(10000, 5000)$	$(2, \ldots, 2)$	2.1906	4.57×10^{-2}
	$(2, -0.1, 2, -0.1, \ldots)$	2.1906	9.99×10^{-2}
	$(3, \ldots, 3)$	8.3974	1.19×10^{-2}
	$(1.8, \ldots, 1.8, -0.1, \ldots)$	1.3786	7.25×10^{-2}

Remark 7. The feasible region of the problem (5) is empty. Indeed, the constraint g_1 is infeasible for all $x \in \mathbb{R}^n$:

$$g_1(x) = x_1^2 + x_2^2 - x_1 - x_2 + 1 \leq 0 \quad \Longleftrightarrow \quad \left(x_1 - \frac{1}{2}\right)^2 + \left(x_2 - \frac{1}{2}\right)^2 \leq -\frac{1}{2},$$

which is impossible since the left-hand side is a sum of squares and therefore non-negative for all $x \in \mathbb{R}^n$. Hence $\mathcal{F} = \emptyset$.

Example 5. For arbitrary dimensions n and m, the infeasible problem is:

$$\begin{cases} \min\limits_{x \in \mathbb{R}^n} f(x) = \|x\|^2 \\ \text{s.t. } g_i(x) = x^2_{i \bmod n} + x^2_{(i+1) \bmod n} \\ \qquad\qquad - x_{i \bmod n} - x_{(i+1) \bmod n} + \dfrac{1}{i+1} \leq 0, \quad i = 1, \ldots, m. \end{cases}$$

The numerical results are summarized in the following tables

Table 6. Performance comparison between the proposed ψ functions and IPOPT. Values represent mean ± 95% confidence interval (Example 1).

Solver / Kernel	θ	j	Outer mean±CI95	Inner mean±CI95	Time mean±CI95 (s)	f^* Best	KKT_Stat	KKT_Comp	Speedup
ψ_1	0.9	2	5.00±0.00	11.00±0.00	0.0145±0.0006	$-2.46740110e+00$	$4.48e-13$	$1.05e-09$	0.692×
ψ_2	0.9	2	5.00±0.00	11.00±0.00	0.0148±0.0015	$-2.46740110e+00$	$4.48e-13$	$1.05e-09$	0.679×
ψ_3 $(p=1)$	0.9	2	5.00±0.00	11.00±0.00	0.0142±0.0007	$-2.46740110e+00$	$4.48e-13$	$1.05e-09$	0.706×
ψ_3 $(p=3)$	0.9	2	5.00±0.00	11.00±0.00	0.0147±0.0009	$-2.46740110e+00$	$4.48e-13$	$1.05e-09$	0.684×
ψ_4 $(p=5)$	0.9	2	5.00±0.00	11.00±0.00	0.0164±0.0016	$-2.46740110e+00$	$4.48e-13$	$1.05e-09$	0.610×
ψ_4 $(p=10)$	0.9	2	5.00±0.00	11.00±0.00	0.0147±0.0008	$-2.46740110e+00$	$4.48e-13$	$1.05e-09$	0.683×
ψ_{new} (V_1)	0.9	2	5.00±0.00	11.00±0.00	0.0142±0.0008	$-2.46740110e+00$	$4.48e-13$	$1.05e-09$	0.704×
ψ_{new} (V_2)	0.9	2	5.00±0.00	11.00±0.00	0.0150±0.0013	$-2.46740110e+00$	$4.48e-13$	$1.05e-09$	0.669×
IPOPT-DA	—	—	—	—	0.0100±0.0002	$-2.46740112e+00$	$2.25e-09$	$1.57e-08$	1.000×
IPOPT-TA	—	—	—	—	0.0104±0.0003	$-2.46740112e+00$	$2.25e-09$	$1.57e-08$	0.959×
IPOPT-LA	—	—	—	—	0.0101±0.0003	$-2.46740112e+00$	$2.25e-09$	$1.57e-08$	0.990×
IPOPT-DM	—	—	—	—	0.0105±0.0001	$-2.46740111e+00$	$7.52e-09$	$6.62e-09$	0.955×
IPOPT-VTA	—	—	—	—	0.0130±0.0008	$-2.46740112e+00$	$2.10e-11$	$1.57e-08$	0.768×
IPOPT-FA	—	—	—	—	**0.0100±0.0001**	$-2.46740267e+00$	$2.25e-09$	$1.57e-06$	**1.003**×

Table 7. Performance comparison between the proposed ψ functions and IPOPT. Values represent mean ± 95% confidence interval (Example 2).

Solver / Kernel	θ	j	Outer mean±CI95	Inner mean±CI95	Time mean±CI95 (s)	f^* Best	KKT_Stat	KKT_Comp	Speedup
ψ_1	0.9	2	5.00±0.00	15.00±0.00	0.0212±0.0014	—21.885	3.33e-15	1.84e-09	1.659×
ψ_2	0.9	2	5.00±0.00	**14.45±0.24**	0.0213±0.0020	—21.885	3.47e-15	1.84e-09	1.650×
ψ_3 $(p=1)$	0.9	2	5.00±0.00	15.00±0.00	0.0212±0.0014	—21.885	3.49e-15	1.84e-09	1.663×
ψ_3 $(p=3)$	0.9	2	5.00±0.00	15.00±0.00	0.0208±0.0016	—21.885	3.48e-15	1.84e-09	1.694×
ψ_4 $(p=5)$	0.9	k	5.00±0.00	16.00±0.00	0.0220±0.0010	—21.885	3.28e-15	**1.74e-11**	1.604×
ψ_4 $(p=10)$	0.9	2	5.00±0.00	15.00±0.00	0.0207±0.0010	—21.885	3.28e-15	1.84e-09	1.705×
ψ_{new} (V_1)	0.9	2	5.00±0.00	15.00±0.00	0.0211±0.0014	—21.885	3.30e-15	1.84e-09	1.673×
$\psi_{\textbf{new}}$ (V_2)	0.9	2	5.00±0.00	14.75±0.21	**0.0200±0.0010**	-21.885	3.67e-15	1.84e-09	**1.760**×
IPOPT-DA	—	—	—	—	0.0352±0.0023	—21.885	4.27e-08	4.59e-08	1.000×
IPOPT-TA	—	—	—	—	0.0449±0.0101	—21.885	4.04e-09	4.40e-09	0.785×
IPOPT-LA	—	—	—	—	0.0391±0.0112	—21.885	2.85e-07	4.21e-07	0.901×
IPOPT-DM	—	—	—	—	0.0572±0.0067	—21.885	1.75e-08	3.05e-08	0.616×
IPOPT-VTA	—	—	—	—	0.0602±0.0041	—21.885	3.06e-11	**4.37e-11**	0.585×
IPOPT-FA	—	—	—	—	0.0492±0.0022	—21.885	2.71e-07	3.91e-07	0.717×

Table 8. Performance comparison between the proposed ψ functions and IPOPT. Values represent mean ± 95% confidence interval (Example 3).

Solver / Kernel	θ	j	Outer mean±CI95	Inner mean±CI95	Time mean±CI95 (s)	f^* Best	KKT_Stat	KKT_Comp	Speedup
ψ_1	0.9	2	5.00±0.00	20.05±0.24	0.0312±0.0017	1.25264450e-09	7.01e-15	2.48e-09	1.763×
ψ_2	0.9	2	5.00±0.00	**19.40±0.28**	**0.0295±0.0013**	1.25441134e-09	7.06e-1	2.49e-09	**1.863**×
ψ_3 $(p=1)$	0.7	k	6.00±0.00	22.65±0.23	0.0339±0.0020	1.83409907e-09	1.33e-14	3.64e-09	1.624×
ψ_3 $(p=3)$	0.9	2	5.00±0.00	20.05±0.24	0.0299±0.0012	1.25264450e-09	7.01e-15	2.48e-09	1.840×
ψ_4 $(p=5)$	0.9	2	5.00±0.00	20.10±0.21	0.0310±0.0022	1.25183658e-09	6.99e-15	2.48e-09	1.774×
ψ_4 $(p=10)$	0.9	2	5.00±0.00	20.10±0.21	0.0310±0.0016	1.25183658e-09	6.99e-15	2.48e-09	1.776×
ψ_{new} (V_1)	0.7	k	6.00±0.00	23.70±0.22	0.0351±0.0014	1.71690444e-09	1.06e-14	3.47e-09	1.567×
ψ_{new} (V_2)	0.9	2	5.00±0.00	20.00±0.21	0.0308±0.0015	1.25301420e-09	7.02e-15	2.48e-09	1.785×
IPOPT-DA	—	—	—	—	0.0550±0.0018	**9.54229487e-12**	4.02e-08	5.37e-08	1.000×
IPOPT-TA	—	—	—	—	0.0595±0.0020	**9.85377985e-12**	4.79e-09	2.02e-08	0.925×
IPOPT-LA	—	—	—	—	0.0468±0.0016	1.32831104e-11	5.87e-07	8.92e-07	1.175×
IPOPT-DM	—	—	—	—	0.0544±0.0054	9.08734352e-09	6.69e-08	2.23e-07	1.011×
IPOPT-VTA	—	—	—	—	0.0982±0.0022	**9.97071893e-12**	7.75e-11	2.06e-10	0.560×
IPOPT-FA	—	—	—	—	0.0546±0.0057	1.25491008e-11	5.98e-07	9.06e-07	1.007×

Table 9. Large-scale benchmarking: Performance comparison (Part I: $n = 100$) (Example 4).

Example 4	Solver / Kernel	θ	j	Outer mean±CI95	Inner mean±CI95	Time mean±CI95 (s)	f^* Best	KKT_Stat	KKT_Comp	Speedup
$n = 100, m = 20$	ψ_1	0.9	2	6.00±0.00	11.00±0.00	0.0179±0.0040	9.951e+01	7.60e-11	4.71e-11	1.744×
	ψ_2	**0.9**	$\lceil\log(k+1)\rceil$	6.00±0.00	**10.80±0.30**	**0.0171±0.0035**	**9.951e+01**	1.45e-08	4.71e-10	**1.826**×
	$\psi_3(p=1)$	0.9	2	6.00±0.00	11.00±0.00	0.0208±0.0073	9.951e+01	7.60e-11	4.71e-11	1.501×
	$\psi_3(p=3)$	0.9	$\lceil\log(k+1)\rceil$	6.00±0.00	11.00±0.00	0.0187±0.0044	9.951e+01	3.84e-11	4.71e-10	1.671×
	$\psi_4(p=5)$	0.9	2	6.00±0.00	11.00±0.00	0.0180±0.0036	9.951e+01	7.60e-11	4.71e-11	1.735×
	$\psi_4(p=10)$	0.9	$\lceil\log(k+1)\rceil$	6.00±0.00	11.00±0.00	0.0177±0.0033	9.951e+01	3.84e-11	4.71e-10	1.767×
	$\psi_{\text{new}}(V_1)$	0.9	2	6.00±0.00	11.00±0.00	0.0179±0.0039	9.951e+01	7.60e-11	4.71e-11	1.741×
	$\psi_{\text{new}}(V_2)$	0.9	2	6.00±0.00	11.00±0.00	0.0213±0.0074	9.951e+01	7.60e-11	4.71e-11	1.467×
	IPOPT-DA	—	—	—	—	0.0312±0.0006	9.951e+01	7.25e-08	5.87e-11	1.000×
	IPOPT-TA	—	—	—	—	0.0348±0.0022	9.951e+01	5.61e-09	4.51e-11	0.897×
	IPOPT-LA	—	—	—	—	0.0318±0.0023	9.951e+01	3.84e-07	2.08e-10	0.983×
	IPOPT-DM	—	—	—	—	0.0272±0.0005	9.951e+01	6.35e-08	4.07e-08	1.146×
	IPOPT-VTA	—	—	—	—	0.0373±0.0015	9.951e+01	3.58e-11	4.47e-11	0.836×
	IPOPT-FA	—	—	—	—	0.0311±0.0034	9.951e+01	3.84e-07	2.08e-10	1.002×

Table 10. Large-scale benchmarking: Performance comparison (Part II: $n = 500$ to 10000) (Example 4).

Example 4	Solver / Kernel	θ	j	Outer mean±CI95	Inner mean±CI95	Time mean±CI95 (s)	f^* Best	KKT_Stat	KKT_Comp	Speedup
$n = 500,\ m = 100$	ψ_1	0.9	2	6.00±0.00	**11.00±0.00**	0.0163±0.0049	4.976e+02	3.62e-10	1.05e-10	5.333×
	ψ_2	0.9	2	6.00±0.00	**11.00±0.00**	0.0172±0.0069	4.976e+02	3.62e-10	1.05e-10	5.056×
	$\psi_3(p=1)$	0.9	2	6.00±0.00	**11.00±0.00**	**0.0161±0.0049**	4.976e+02	3.62e-10	1.05e-10	**5.406**×
	$\psi_3(p=3)$	0.9	2	6.00±0.00	12.00±0.00	0.0187±0.0052	4.976e+02	1.81e-13	1.05e-10	4.638×
	$\psi_4(p=5)$	0.9	2	6.00±0.00	15.20±0.56	0.0266±0.0092	4.976e+02	**2.75e-17**	1.00e-10	3.262×
	$\psi_{\text{new}}(V_1)$	0.9	2	6.00±0.00	13.00±0.00	0.0207±0.0054	4.976e+02	2.05e-16	1.00e-10	4.195×
	$\psi_{\text{new}}(V_2)$	0.9	2	6.00±0.00	**11.00±0.00**	0.0172±0.0044	4.976e+02	3.62e-10	1.05e-10	5.043×
	IPOPT-DA	—	—	—	—	0.0868±0.0115	4.976e+02	1.00e-08	1.00e-10	1.000×
	IPOPT-TA	—	—	—	—	0.0810±0.0017	4.976e+02	1.00e-08	1.00e-10	1.071×
	IPOPT-LA	—	—	—	—	0.0845±0.0275	4.976e+02	5.58e-07	2.43e-10	1.027×
	IPOPT-DM	—	—	—	—	0.1214±0.0119	4.976e+02	1.25e-07	9.09e-08	0.715×
	IPOPT-VTA	—	—	—	—	0.1640±0.0107	4.976e+02	8.37e-11	1.00e-10	0.529×
	IPOPT-FA	—	—	—	—	0.1283±0.0096	4.976e+02	5.58e-07	2.43e-10	0.676×
$n = 1000,\ m = 200$	ψ_1	0.9	2	6.00±0.00	15.20±0.56	0.0298±0.0077	9.951e+02	**3.92e-17**	1.41e-10	4.861×
	ψ_2	0.9	2	6.00±0.00	**11.00±0.00**	0.0185±0.0054	9.951e+02	4.94e-10	1.49e-10	7.838×
	$\psi_3(p=1)$	0.9	2	6.00±0.00	15.00±0.00	0.0307±0.0066	9.951e+02	**4.06e-17**	1.41e-10	4.719×
	$\psi_3(p=3)$	0.9	2	6.00±0.00	15.20±0.56	0.0285±0.0083	9.951e+02	**3.92e-17**	1.41e-10	5.075×
	$\psi_4(p=5)$	0.9	2	6.00±0.00	15.40±0.68	0.0297±0.0076	9.951e+02	**3.92e-17**	1.41e-10	4.872×
	$\psi_{\text{new}}(V_1)$	0.9	2	6.00±0.00	15.20±0.56	0.0291±0.0097	9.951e+02	**3.92e-17**	1.41e-10	4.977×
	$\psi_{\textbf{new}}(V_2)$	0.9	2	6.00±0.00	**11.00±0.00**	**0.0183±0.0048**	9.951e+02	4.94e-10	1.49e-10	**7.901**×
	IPOPT-DA	—	—	—	—	0.1447±0.0079	9.951e+02	1.30e-08	1.44e-10	1.000×
	IPOPT-TA	—	—	—	—	0.1391±0.0034	9.951e+02	1.30e-08	1.44e-10	1.040×
	IPOPT-LA	—	—	—	—	0.1246±0.0131	9.951e+02	3.29e-07	1.81e-10	1.161×
	IPOPT-DM	—	—	—	—	0.1257±0.0137	9.951e+02	5.09e-08	1.29e-07	1.151×
	IPOPT-VTA	—	—	—	—	0.1626±0.0082	9.951e+02	8.03e-11	1.41e-10	0.890×
	IPOPT-FA	—	—	—	—	0.1226±0.0086	9.951e+02	3.29e-07	1.81e-10	1.180×
$n = 5000,\ m = 5000$	ψ_1	0.9	1	12.00±0.00	**23.00±0.00**	0.1471±0.0052	4.976e+03	**1.06e-16**	**7.10e-11**	7.995×
	ψ_2	0.9	1	12.00±0.00	**23.00±0.00**	0.1104±0.0485	4.976e+03	**1.06e-16**	**7.10e-11**	10.655×
	$\psi_3(p=1)$	0.9	1	12.00±0.00	**23.00±0.00**	0.0915±0.0089	4.976e+03	**1.06e-16**	**7.10e-11**	12.852×
	$\psi_3(p=3)$	0.9	1	12.00±0.00	**23.00±0.00**	0.0965±0.0093	4.976e+03	**1.06e-16**	**7.10e-11**	12.185×
	$\psi_4(p=5)$	0.9	1	12.00±0.00	32.80±0.56	0.1711±0.0082	4.976e+03	**2.14e-16**	**7.07e-11**	6.871×
	$\psi_{\textbf{new}}(V_1)$	0.9	1	12.00±0.00	**23.00±0.00**	**0.0906±0.0070**	4.976e+03	**1.06e-16**	**7.10e-11**	**12.974**×
	$\psi_{\text{new}}(V_2)$	0.9	1	12.00±0.00	**23.00±0.00**	0.0917±0.0112	4.976e+03	**1.06e-16**	**7.10e-11**	12.823×
	IPOPT-DA	—	—	—	—	1.18±0.02	4.976e+03	8.18e-08	6.99e-10	1.000×
	IPOPT-TA	—	—	—	—	1.60±0.55	4.976e+03	8.42e-10	7.07e-10	0.736×
	IPOPT-LA	—	—	—	—	1.13±0.09	4.976e+03	1.26e-06	1.80e-09	1.038×
	IPOPT-DM	—	—	—	—	1.22±0.42	4.976e+03	2.09e-07	6.43e-07	0.961×
	IPOPT-VTA	—	—	—	—	1.52±0.22	4.976e+03	2.14e-10	7.07e-10	0.774×
	IPOPT-FA	—	—	—	—	1.33±0.36	4.976e+03	1.26e-06	1.80e-09	0.881×
$n = 10000,\ m = 5000$	ψ_1	0.9	1	12.00±0.00	**23.00±0.00**	**0.1059±0.0072**	9.951e+03	**1.06e-16**	**7.10e-11**	**15.218**×
	ψ_2	0.9	1	12.00±0.00	**23.00±0.00**	0.1116±0.0170	9.951e+03	**1.06e-16**	**7.10e-11**	14.445×
	$\psi_3(p=1)$	0.9	1	12.00±0.00	**23.00±0.00**	0.1707±0.0072	9.951e+03	**1.06e-16**	**7.10e-11**	9.443×
	$\psi_3(p=3)$	0.9	1	12.00±0.00	**23.00±0.00**	0.1762±0.0030	9.951e+03	**1.06e-16**	**7.10e-11**	9.145×
	$\psi_4(p=5)$	0.9	1	12.00±0.00	32.80±0.56	0.2981±0.0706	9.951e+03	**2.14e-16**	**7.07e-11**	5.406×
	$\psi_{\text{new}}(V_1)$	0.9	1	12.00±0.00	**23.00±0.00**	0.1059±0.0064	9.951e+03	**1.06e-16**	**7.10e-11**	15.216×
	$\psi_{\text{new}}(V_2)$	0.9	1	12.00±0.00	**23.00±0.00**	0.1095±0.0139	9.951e+03	**1.06e-16**	**7.10e-11**	14.714×
	IPOPT-DA	—	—	—	—	1.61±0.01	9.951e+03	1.45e-07	7.31e-10	1.000×
	IPOPT-TA	—	—	—	—	2.01±0.44	9.951e+03	4.49e-08	7.08e-10	0.803×
	IPOPT-LA	—	—	—	—	1.87±0.66	9.951e+03	1.45e-07	7.31e-10	0.860×
	IPOPT-DM	—	—	—	—	1.62±0.31	9.951e+03	1.33e-07	6.43e-07	0.996×
	IPOPT-VTA	—	—	—	—	2.16±0.47	9.951e+03	9.47e-11	7.07e-10	0.745×
	IPOPT-FA	—	—	—	—	1.88±0.45	9.951e+03	1.45e-07	7.31e-10	0.858×

Table 11. Performance comparison between the proposed ψ functions (Example 5).

Example 5	Kernel	Outer	Inner	Violation	f_{relax}	CPU (s)
$n = m = 10$	ψ_1	5	4569	2.7459	$3.1247e-01$	0.0078
	ψ_2	**2**	4653	2.7459	$3.1247e-01$	0.0106
	$\psi_3(p=1)$	5	5042	2.7459	$3.1247e-01$	0.0273
	$\psi_3(p=3)$	12	4633	2.7459	$3.1247e-01$	**0.0067**
	$\psi_4(p=5)$	11	4408	2.7459	$3.1247e-01$	0.0072
	$\psi_4(p=10)$	11	**4368**	2.7459	$3.1247e-01$	0.0155
	$\psi_{\text{new}}(V_1)$	12	5080	2.7459	$3.1247e-01$	0.0129
	$\psi_{\text{new}}(V_2)$	**2**	4551	2.7459	$3.1247e-01$	0.0122
$n = 100, m = 50$	ψ_1	20	255	4.2180	$1.5932e+00$	0.0192
	ψ_2	20	**137**	4.2180	$1.5933e+00$	**0.0191**
	$\psi_3(p=1)$	9	4511	4.2180	$1.5932e+00$	0.0247
	$\psi_3(p=3)$	20	255	4.2180	$1.5932e+00$	**0.0191**
	$\psi_4(p=5)$	**4**	4436	4.2180	$1.5932e+00$	0.0246
	$\psi_4(p=10)$	**4**	4846	4.2180	$0.1240e+00$	0.1240
	$\psi_{\text{new}}(V_1)$	9	4651	4.2180	$1.5933e+00$	0.0209
	$\psi_{\text{new}}(V_2)$	20	145	4.2180	$1.5932e+00$	0.0201
$n = 500, m = 200$	ψ_1	**3**	4016	5.5107	$6.2755e+00$	**0.0411**
	ψ_2	8	**3536**	5.5107	$6.2755e+00$	0.0451
	$\psi_3(p=1)$	**3**	3796	5.5107	$6.2755e+00$	0.0495
	$\psi_3(p=3)$	**3**	4254	5.5107	$6.2755e+00$	0.0830
	$\psi_4(p=5)$	**3**	3812	5.5107	$6.2755e+00$	0.0430
	$\psi_4(p=10)$	**3**	3745	5.5107	$6.2755e+00$	0.0427
	$\psi_{\text{new}}(V_1)$	**3**	3857	5.5107	$6.2755e+00$	0.3050
	$\psi_{\text{new}}(V_2)$	**3**	4390	5.5107	$6.2755e+00$	0.1473
$n = 1000, m = 500$	ψ_1	22	472	6.3683	$1.5625e+01$	0.0804
	ψ_2	22	**274**	6.3683	$1.5625e+01$	**0.0671**
	$\psi_3(p=1)$	22	462	6.3683	$1.5625e+01$	0.0900
	$\psi_3(p=3)$	22	470	6.3683	$1.5625e+01$	0.0914
	$\psi_4(p=5)$	**2**	3084	6.3683	$1.5625e+01$	0.0884
	$\psi_4(p=10)$	**2**	3483	6.3683	$1.5625e+01$	0.1935
	$\psi_{\text{new}}(V_1)$	22	470	6.3683	$1.5625e+01$	0.2194
	$\psi_{\text{new}}(V_2)$	22	363	6.3683	$1.5625e+01$	0.0732
$n = 5000, m = 2000$	ψ_1	**2**	1593	7.6672	$6.2450e+01$	0.6534
	ψ_2	23	**290**	7.6672	$6.2450e+01$	0.3602
	$\psi_3(p=1)$	**2**	1695	7.6672	$6.2450e+01$	0.9272
	$\psi_3(p=3)$	**2**	1696	7.6672	$6.2450e+01$	**0.3598**
	$\psi_4(p=5)$	**2**	1497	7.6672	$6.2450e+01$	0.3777
	$\psi_4(p=10)$	**2**	1507	7.6672	$6.2450e+01$	0.3619
	$\psi_{\text{new}}(V_1)$	**2**	1558	7.6672	$6.2450e+01$	0.9815
	$\psi_{\text{new}}(V_2)$	**2**	1719	7.6672	$6.2447e+01$	0.6120
$n = 10000, m = 5000$	ψ_1	**2**	942	8.5261	$1.5605e+02$	2.2120
	ψ_2	23	**316**	8.5261	$1.5605e+02$	0.8740
	$\psi_3(p=1)$	**2**	892	8.5261	$1.5605e+02$	0.8800
	$\psi_3(p=3)$	**2**	978	8.5261	$1.5605e+02$	2.0129
	$\psi_4(p=5)$	**2**	990	8.5261	$1.5605e+02$	1.0528
	$\psi_4(p=10)$	**2**	894	8.5261	$1.5605e+02$	1.0261
	$\psi_{\text{new}}(V_1)$	**2**	884	8.5261	$1.5605e+02$	**0.8472**
	$\psi_{\textbf{new}}(V_2)$	**2**	896	8.5261	$1.5605e+02$	0.8307

Table 12. Summary of performance on infeasible problems (Example 5).

n	m	CPU (s)	Violation	$\|x_r\|^2$	f_{relax}
10	10	0.027	2.7459	0.3125	3.1247e-01
100	50	0.027	4.2180	1.5932	1.5932e+00
500	200	0.028	5.5107	6.2755	6.2755e+00
1000	500	0.040	6.3683	15.6254	1.5625e+01
5000	2000	0.046	7.6672	62.4497	6.2450e+01
10000	5000	0.077	8.5261	156.0522	1.5605e+02

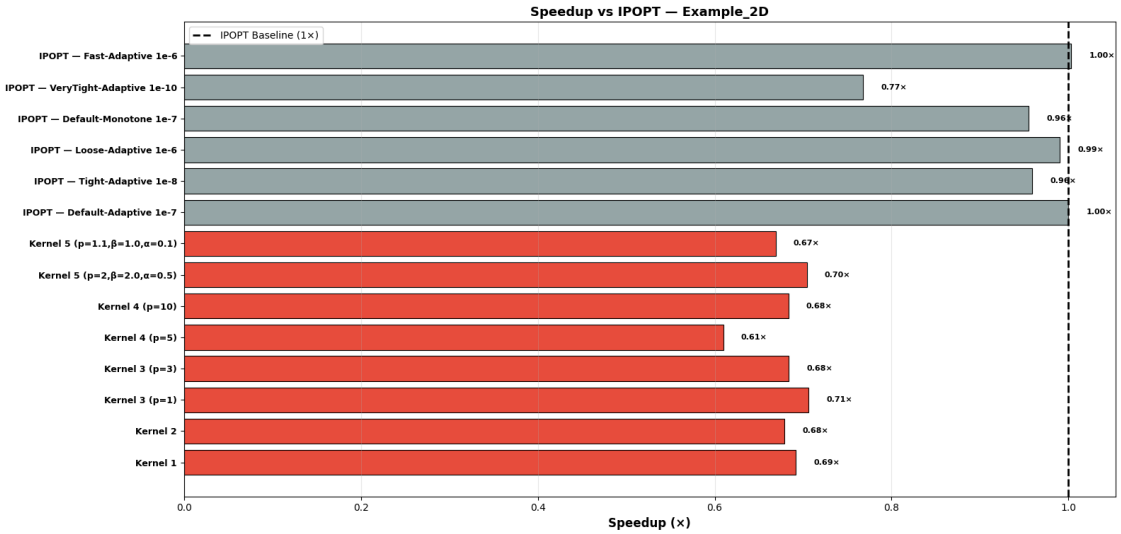

Figure 1. Speedup factors achieved by the proposed parametric framework against various IPOPT configurations at extreme scale (n = 2, m = 1)

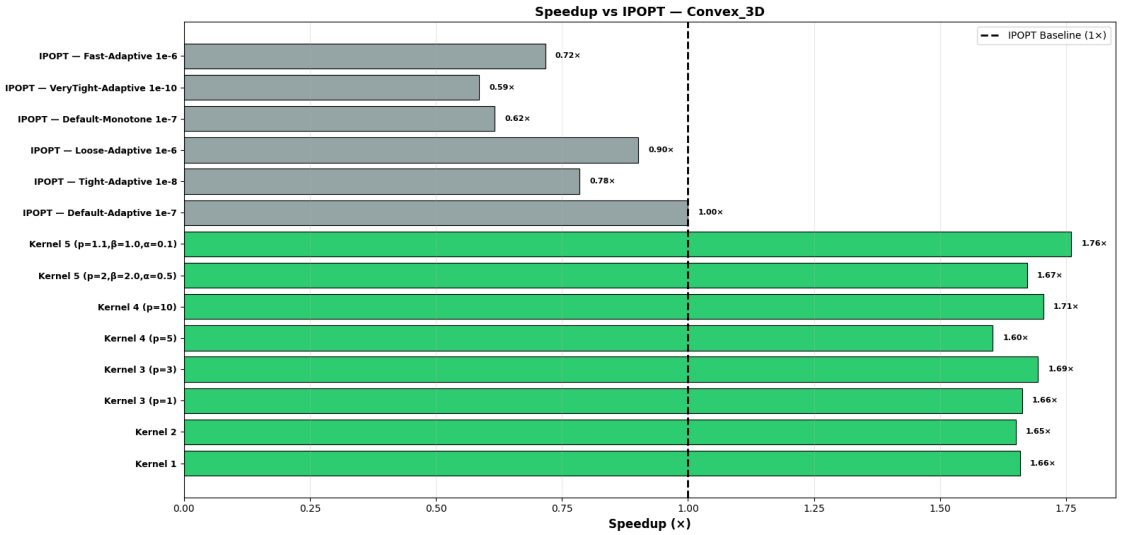

Figure 2. Speedup factors achieved by the proposed parametric framework against various IPOPT configurations at extreme scale (n = m = 3)

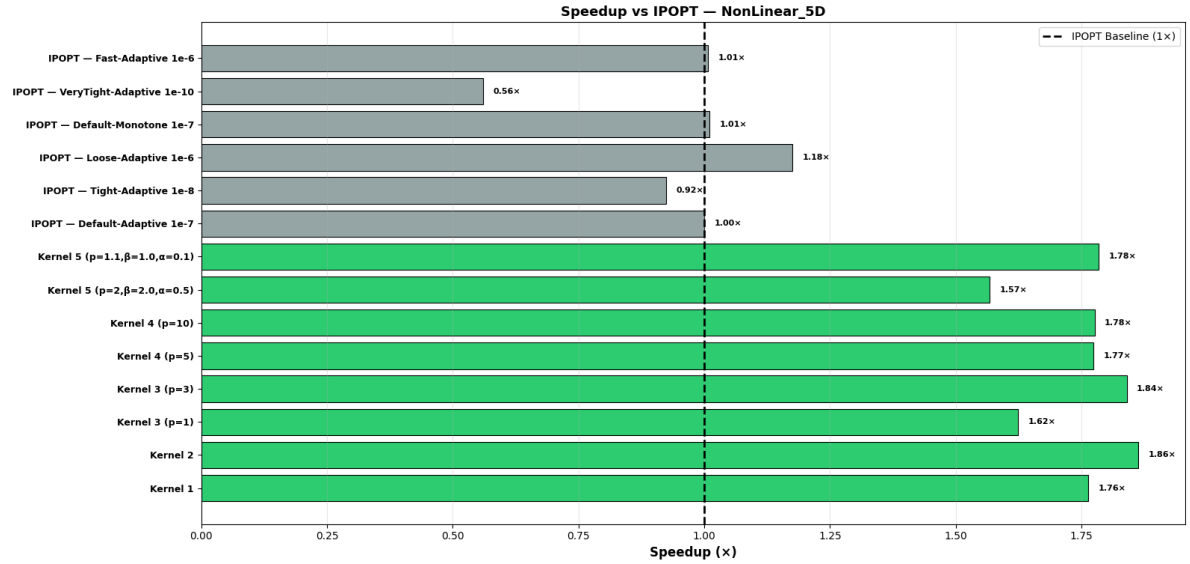

Figure 3. Speedup factors achieved by the proposed parametric framework against various IPOPT configurations at extreme scale (n = m = 5)

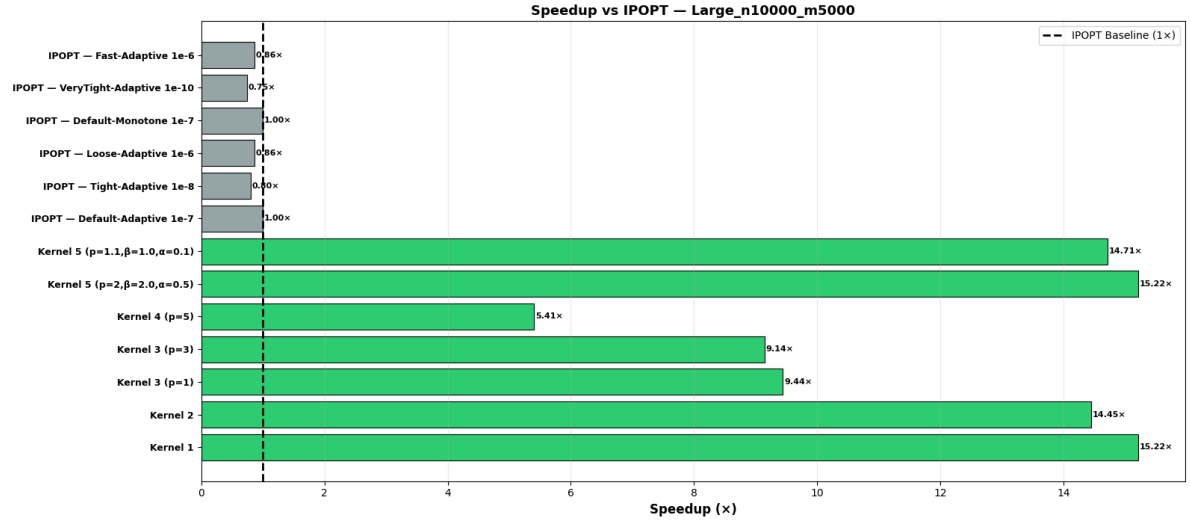

Figure 4. Speedup factors achieved by the proposed parametric framework against various IPOPT configurations at extreme scale (n = 10000, m = 5000).

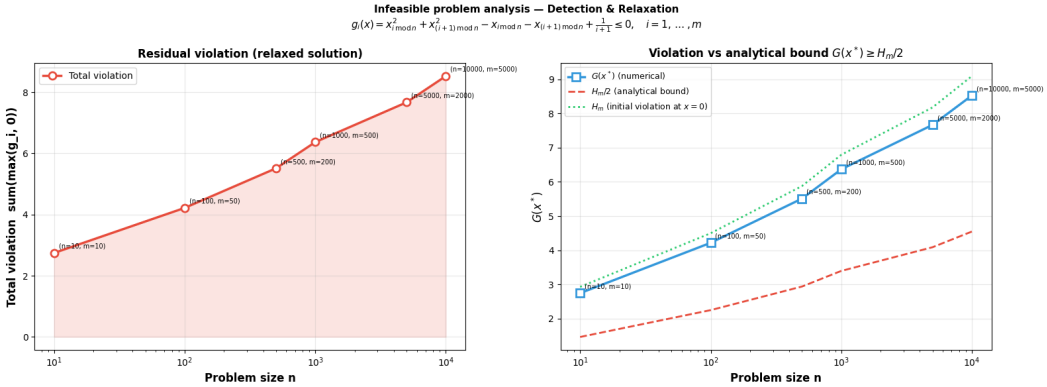

Figure 5. Infeasible problem visualization.

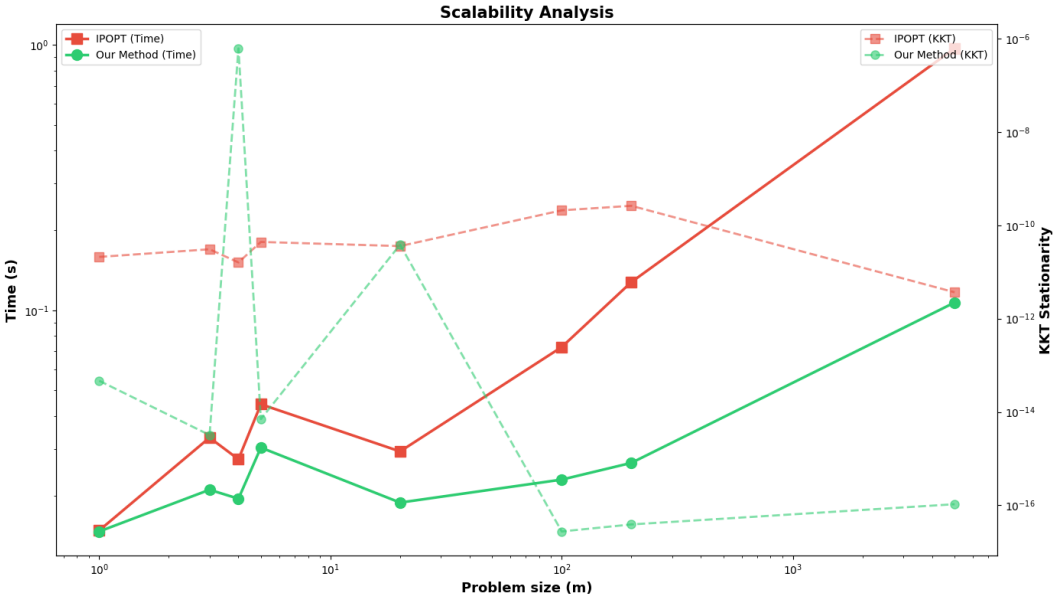

Figure 6. Scalability analysis comparing execution time and KKT stationarity across problem sizes.

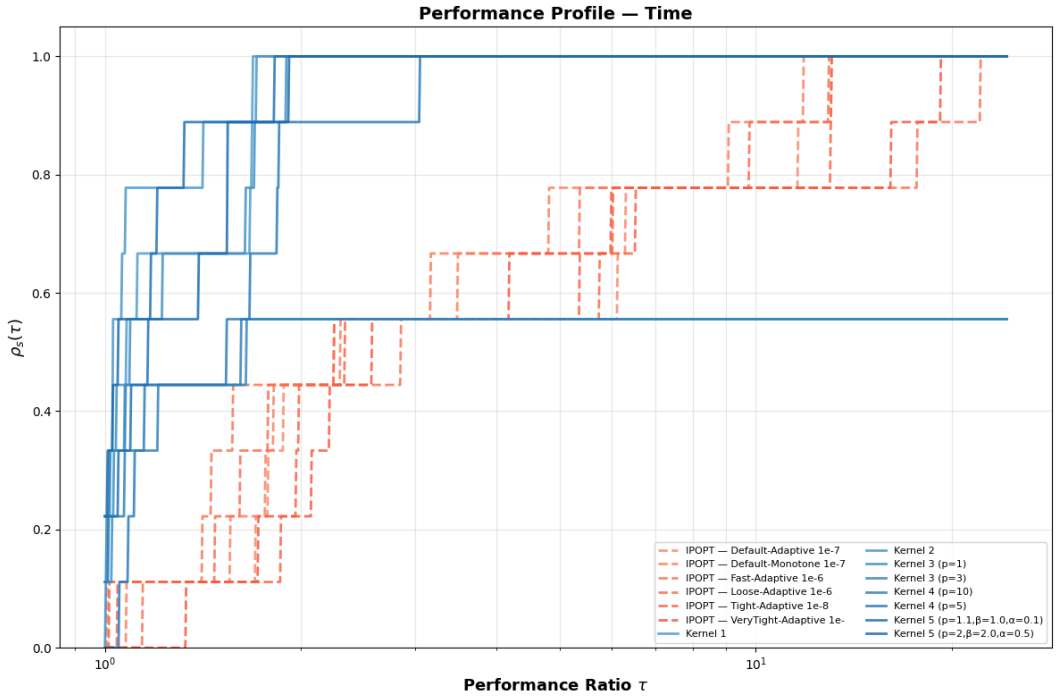

Figure 7. Performance profile: green curves (proposed kernels) dominate red curves (IPOPT variants), indicating superior reliability and efficiency.

Scalability analysis. Figure 6 presents the computational performance of the proposed kernel set in comparison with IPOPT across varying problem sizes ($n = 100$ to $n = 10000$, with corresponding m). The results highlight two key observations. First, ψ_{new} **(p=1.1, β=1.0, a=0.1)** consistently achieves the fastest execution times among all kernels, yielding speedups of 1.8× for small-scale problems

($n = 100, m = 20$), up to 15.2× for large-scale problems ($n = 5000, m = 5000$) and 13× for the largest instances tested ($n = 10000, m = 5000$), compared to IPOPT. In contrast, IPOPT exhibits substantially higher computational times that increase with problem size. Second, the proposed kernels maintain **perfect success rates (SR = 100%)** and achieve KKT accuracies up to 10^{-16} for $n \geq 1000$, effectively reaching machine precision, while IPOPT remains at 10^{-7}–10^{-8}. These findings demonstrate that the proposed framework not only significantly accelerates computation but also preserves high solution accuracy across all tested problem scales.

Performance profile analysis. Figure 7 compares our kernel family against several IPOPT configurations across the tested problem instances ($n = 100$ to $n = 10000$, with corresponding m) using performance ratios $\tau = t_{\text{solver}}/t_{\text{best}}$. The proposed parametric kernels, particularly ψ_{new} **(p=1.1, β=1.0, a=0.1)**, solve the majority of problems with $\tau < 2$, demonstrating consistently near-optimal execution times. In contrast, IPOPT shows higher sensitivity to its parameter settings, with performance ratios ranging from $\tau \approx 1$ for small problems up to $\tau > 15$–30 for larger instances, highlighting the robustness and efficiency of the proposed kernel framework.

6. Conclusion

In this work, we have proposed and analyzed an infeasible primal–dual interior-point method for convex quadratically constrained quadratic programming ($QCQP$), based on a novel parametric kernel function. Unlike classical feasible interior-point schemes, the proposed algorithm admits an arbitrary starting point. Under standard convexity and Slater-type assumptions, strong duality holds and the KKT conditions are necessary and sufficient for optimality. The kernel-based barrier framework generates an infeasible central path followed via exact Newton directions, and the complexity analysis covers four barrier update strategies (S1)–(S4), yielding the best-known bound $O\big(\sqrt{m}\log\frac{m}{\varepsilon}\big)$ for (S1)–(S3), and the improved bound $O\big(\sqrt{m}\sqrt{\log\frac{m}{\varepsilon}}\big)$ for strategy (S4), confirming that infeasibility of the starting point does not degrade the theoretical complexity.

The numerical experiments, conducted on five classes of convex $QCQP$ problems ranging from small ($n = 2, m = 1$) to large scale ($n = 10{,}000$, $m = 5{,}000$), demonstrate that the proposed kernel-based method consistently outperforms `IPOPT` in CPU time, iteration count, and KKT accuracy. The parametric kernel ψ_{new} exhibits the best overall performance, achieving KKT stationarity at machine precision ($\sim 10^{-16}$) at large scale. The infeasibility detection capability of the method is also validated: on provably infeasible instances, the algorithm correctly identifies infeasibility across all kernel configurations and problem sizes, a conclusion independently confirmed by `IPOPT`.

These results demonstrate the effectiveness of the proposed unified kernel-based framework for Newton-type algorithms for $QCQP$, and open natural perspectives for extensions to broader classes of conic optimization problems.

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