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Advanced Partitioned Neutrosophic Offsets, Oversets, and Undersets: Modeling "Neither Agree nor Disagree" and Beyond

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Abstract Modern uncertainty-modeling frameworks-fuzzy sets, intuitionistic fuzzy sets, hyperfuzzy sets, neutrosophic sets, and plithogenic sets-provide powerful tools for capturing vagueness and imprecision. In particular, neutrosophic sets characterize each element by three independent degrees: truth, indeterminacy, and falsity. Classical neutrosophic sets have been refined by partitioning the membership degrees into additional components. Recently, the *Hexapartitioned Neutrosophic Set*, *Octapartitioned Neutrosophic Set*, *Nonapartitioned Neutrosophic Set*, and *Decapartitioned Neutrosophic Set* have been defined. In this paper, we introduce four novel partitioned models-*hexapartitioned*, *octapartitioned*, *nonapartitioned*, and *decapartitioned* neutrosophic offsets/oversets/undersets-and show how each can be seamlessly embedded into the plithogenic-set framework.

Keywords Neutrosophic Set, Hexapartitioned Neutrosophic Set, Octapartitioned Neutrosophic Set, Nonapartitioned Neutrosophic Set, Decapartitioned neutrosophic set, Neutrosophic Offset

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1. Introduction

1.1. Fuzzy and Neutrosophic Set Theory

Real-world problems often involve uncertainty, which has driven the creation of numerous mathematical tools to represent imprecision. Foundational models include *fuzzy sets* [1] and *intuitionistic fuzzy sets* [2, 3]. Building on these, researchers have proposed a variety of extensions-such as *vague sets* [4], *bipolar fuzzy sets* [5], *hesitant*

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fuzzy sets [6], *picture fuzzy sets* [7], Pythagorean fuzzy sets [8], hyperfuzzy sets [9, 10], hyperrough sets [11], neutrosophic sets [12, 13], q-Rung Orthopair fuzzy sets [14], and m*-polar fuzzy sets* [15].

Extending beyond these constructs, *neutrosophic sets* [16] assign three independent membership values-truth, indeterminacy, and falsity-each ranging over $[0, 1]$ and collectively bounded by 3. Further generalizations, including *hyperneutrosophic sets* [17], *bipolar neutrosophic set*[18], hesitant neutrosophic sets[19], and *pythagorean Neutrosophic Set*[20] have expanded the modeling capacity of neutrosophic theory. Moreover, several further extensions have been developed in the literature, including Plithogenic Sets[21], Uncertain Sets[22], and Functorial Sets[23, 22]. For readers who may not be familiar with neutrosophic sets, a reference comparison between fuzzy sets and neutrosophic sets is provided in Table 1.

Aspect	**Fuzzy Set (FS)**	**Neutrosophic Set (NS)**
Basic descriptor	One membership degree.	Three membership degrees: truth, indeterminacy, falsity.
Formal data on X	$\mu_A : X \to [0, 1]$.	$T_A, I_A, F_A : X \to [0, 1]$.
Per-element representation	$A = \{(x, \mu_A(x)) \mid x \in X\}$.	$A = \{(x, \langle T_A(x), I_A(x), F_A(x)\rangle) \mid x \in X\}$.
Constraints	$\forall x \in X,\ 0 \le \mu_A(x) \le 1$.	$\forall x \in X,\ 0 \le T_A(x), I_A(x), F_A(x) \le 1,\ 0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.
Role of indeterminacy	Not explicit; ambiguity is encoded indirectly (e.g., intermediate μ).	Explicitly modeled by $I_A(x)$ (separates "unknown/undecided" from truth/falsity).
Independence of components	Not applicable (single value).	$T_A(x), I_A(x), F_A(x)$ are modeled as independent degrees (subject only to the sum bound).
Recovery of FS as a special case	—	If $I_A(x) = 0$ and $F_A(x) = 1 - T_A(x)$ for all x, then NS reduces to an FS via $\mu_A(x) := T_A(x)$.

Table 1. Concise comparison between fuzzy sets and neutrosophic sets.

These uncertain techniques have found applications in diverse areas such as graph theory, control theory, chemistry, topology, algebra, and decision science [24, 25]. Consequently, ongoing investigation into fuzzy, neutrosophic, and related set-based frameworks remains of great significance.

1.2. Partitioned Neutrosophic Frameworks

Classical neutrosophic sets have been refined by partitioning the membership degrees into additional components. In a *quadripartitioned set*, a dedicated contradiction parameter is introduced so that the sum of all four degrees does not exceed 4 [26, 27]. A *pentapartitioned set* further incorporates an unknown component, capping the total of five membership measures at 5 [28]. Extending this idea, *heptapartitioned sets* subdivide truth and falsity into relative truth and relative falsity, resulting in seven degrees whose sum is bounded by 7 [29, 30].

Beyond these, researchers have also introduced *double-valued neutrosophic sets* [31], *triple-valued neutrosophic sets* [32], *quadruple-valued neutrosophic sets* [32], and *quintuple-valued neutrosophic sets* [32]. These models focus specifically on refining the *indeterminacy* component of neutrosophic sets. Table 2 summarizes single-, double-, triple-, quadruple-, quintuple-, and multi-valued neutrosophic sets formulated via indeterminacy partition.

Model name	Membership tuple at $x \in X$	Constraint (per x)
Single-valued NS (standard)	$\big(T(x),\ I(x),\ F(x)\big)$	$0 \le T(x) + I(x) + F(x) \le 3$
Double-valued (Indeterminacy-2)	$\big(T(x),\ I_1(x),\ I_2(x),\ F(x)\big)$	$0 \le T(x) + I_1(x) + I_2(x) + F(x) \le 4$
Triple-valued (Indeterminacy-3)	$\big(T(x),\ I_1(x),\ I_2(x),\ I_3(x),\ F(x)\big)$	$0 \le T(x) + \sum_{i=1}^{3} I_i(x) + F(x) \le 5$
Quadruple-valued (Indeterminacy-4)	$\big(T(x),\ I_1(x),\ I_2(x),\ I_3(x),\ I_4(x),\ F(x)\big)$	$0 \le T(x) + \sum_{i=1}^{4} I_i(x) + F(x) \le 6$
Quintuple-valued (Indeterminacy-5)	$\big(T(x),\ I_1(x),\ I_2(x),\ I_3(x),\ I_4(x),\ I_5(x),\ F(x)\big)$	$0 \le T(x) + \sum_{i=1}^{5} I_i(x) + F(x) \le 7$
Multi-Valued NS with Indeterminacy Partition ($k \ge 1$)	$\big(T(x),\ I_1(x), \dots, I_k(x),\ F(x)\big)$	$0 \le T(x) + \sum_{i=1}^{k} I_i(x) + F(x) \le k+2$

Table 2. Single-/Double-/Triple-/Quadruple-/Quintuple-/Multi-Valued Neutrosophic Sets via Indeterminacy Partition (truth and falsity are single-valued).

More recently, the *Hexapartitioned Neutrosophic Set*, *Octapartitioned Neutrosophic Set*, *Nonapartitioned Neutrosophic Set*, and *Decapartitioned Neutrosophic Set* have been defined[33]. *Octapartitioned Neutrosophic Set*, *Nonapartitioned Neutrosophic Set*, and A *Decapartitioned Neutrosophic Set* is a concept that incorporates the structural features of double-valued neutrosophic sets, triple-valued neutrosophic sets, quadruple-valued neutrosophic sets, and quintuple-valued neutrosophic sets. Each of these advanced structures can be viewed as a special case of the general *Plithogenic Set* and *Refined Neutrosophic Set* [34] framework. As a reference, Table 3 presents a summary of fuzzy, intuitionistic fuzzy, neutrosophic, and partitioned neutrosophic models.

Model	**Degrees**	**Constraint**	**Note**
Fuzzy Set	μ	$\mu \in [0,1]$	Single membership.
Intuitionistic Fuzzy Set	(μ,ν)	$\mu,\nu \in [0,1],\ \mu+\nu \le 1$	Hesitation $\pi = 1-\mu-\nu$.
Neutrosophic Set	(T,I,F)	$T,I,F \in [0,1],\ \sum \le 3$	Three independent degrees.
Quadripartitioned NS	(T,I,C,F)	all $\in [0,1],\ \sum \le 4$	Adds contradiction C.
Pentapartitioned NS	(T,I,C,U,F)	all $\in [0,1],\ \sum \le 5$	Adds unknown U.
Hexapartitioned NS	(T,C,G,U,H,F)	all $\in [0,1],\ \sum \le 6$	Adds ignorance G, hesitation H.
Heptapartitioned NS	$(T,\mathrm{RT},C,U,I,\mathrm{RF},F)$	all $\in [0,1],\ \sum \le 7$	Relative truth/falsity.
Octapartitioned NS	$(T,\mathrm{RT},C,U,I,H,\mathrm{RF},F)$	all $\in [0,1],\ \sum \le 8$	Adds H.
Nonapartitioned NS	$(T,\mathrm{SRT},\mathrm{WRT},C,U,I,\mathrm{SRF},\mathrm{WRF},F)$	all $\in [0,1],\ \sum \le 9$	Strong/weak RT/RF.
Decapartitioned NS	$(T,\mathrm{SRT},\mathrm{WRT},C,U,I,H,\mathrm{SRF},\mathrm{WRF},F)$	all $\in [0,1],\ \sum \le 10$	Adds H to nona-partition.

Table 3. Compact summary of fuzzy, intuitionistic fuzzy, neutrosophic, and partitioned neutrosophic models.

As one possible application of the concepts of Partitioned Neutrosophic and Multi-Valued (Double/Triple) Neutrosophic sets, survey measurement scales (cf.[35, 36]) can be considered. In survey research, the treatment of responses such as “neither” and closely related notions has long been debated, and a substantial body of literature has been published on this topic (e.g.,[37, 38, 39, 40, 41, 42, 43]). As related concepts, *Fuzzy Likert scales* [44, 45] and *Neutrosophic Likert scales* [46, 47] have also been investigated. Although various types of survey scales have been extensively studied, it is expected that research on surveys explicitly based on Partitioned Neutrosophic or Multi-Valued (Double/Triple) Neutrosophic frameworks—namely, Neutrosophic surveys—will become increasingly active in the future. Examples of application methods (illustrative examples of Neutrosophic surveys) are provided in the Appendix A.

1.3. *Offset, Overset, and Underset Extensions*

Conventional uncertain-set frameworks confine membership degrees to the interval $[0, 1]$. To accommodate values outside this range, the notions of *offset*, *overset*, and *underset* have been proposed [48]. In an *offset* extension, membership values may drop below 0 or rise above 1, thereby capturing negative or excessive degrees [49, 50, 51]. An *overset* allows membership to exceed 1 while remaining nonnegative[52], whereas an *underset* permits membership to fall below 0 but never exceed 1[53, 51]. Such extensions are crucial when raw measurements or expert judgments initially lie outside the normalized range, since they allow models to incorporate these out-of-band values without prior rescaling. Directly handling these extended degrees leads to more faithful representations in applications like risk assessment and information fusion. As a reference, Table 4 presents a summary of the Offset, Overset, and Underset extensions. Due to their scientific and applied significance, OffSet, OverSet, and UnderSet have continued to be actively studied in many recent research works [54, 55].

Extension	Range of $\mu(x)$	Brief note
Offset	$\mathbb{R}$	Allows $\mu \notin [0, 1]$; e.g., -0.20, 1.35. Reduction: $\mathrm{clip}(\mu) = \min\{1, \max\{0, \mu\}\}$ or affine rescale.
Overset	$[0, \infty)$	Nonnegative; may exceed 1 (e.g., 1.15). Reduction: $\min\{1, \mu\}$ or scale $\mu/(1 + \beta)$.
Underset	$(-\infty, 1]$	Upper-bounded by 1; may be negative (e.g., -0.30). Reduction: $\max\{0, \mu\}$ or shift-scale $(\mu + \alpha)/(1 + \alpha)$.

Table 4. Compact summary of Offset, Overset, and Underset extensions.

Moreover, recent work has introduced the *Quadripartitioned Offset*, *Pentapartitioned Offset*, and *Heptapartitioned Offset* as the offset analogues of the Quadripartitioned, Pentapartitioned, and Heptapartitioned sets, respectively[56].

1.4. *Our Contribution*

Neutrosophic offsets have grown increasingly important because of their wide applicability, and many extended set-theoretic frameworks have been developed. However, research on higher-order, multi-partition neutrosophic offsets remains incomplete. Moreover, there exist practical situations—such as questionnaire scales frequently used in Japan—where one needs a finer level of granularity than what a Heptapartitioned Offset can represent, and where the evaluation cannot be restricted to the binary values 0 and 1. Although the notions of a *Hexapartitioned Neutrosophic Set*, *Octapartitioned Neutrosophic Set*, and *Nonapartitioned Neutrosophic Set* have been introduced, it is natural to define the corresponding offset versions as well.

Looking ahead, a deeper investigation of generalized neutrosophic-set structures is needed. In this paper, we introduce four novel partitioned models—*hexapartitioned*, *octapartitioned*, *nonapartitioned*, and *decapartitioned* neutrosophic offsets/oversets/undersets—and show that each of them embeds seamlessly into the plithogenic-set framework. And we further examine their relationship with the Plithogenic Offset framework, while also considering the design of algorithms for structures such as the decapartitioned neutrosophic model. Table 5 provides a compact overview of these offset variants, including their under-limit and over-limit behaviours.

Highly partitioned offsets disentangle multiple sources of uncertainty—strong and weak support, contradiction, ignorance, and hesitation—thereby enabling a more faithful representation of expert judgment, the design of tailored aggregation rules, and detailed sensitivity analyses that cannot be achieved by three- or four-component neutrosophic models. Furthermore, because highly partitioned offsets naturally capture intermediate responses such as "neither agree nor disagree," which commonly appear in questionnaires, they should be viewed as an important and necessary direction of ongoing research. This paper focuses primarily on the theoretical foundations. Studies incorporating computational experiments and empirical evaluations are left for future work.

Offset Model	**Degrees**	**Offset rule (range)**
Fuzzy Offset	μ	$\mu \in [\Psi, \Omega]$; offset if $\mu \notin [0,1]$.
Intuitionistic Fuzzy Offset	(μ, ν)	$\mu, \nu \in [\Psi, \Omega]$; offset if any $\notin [0,1]$; typically enforce $\mu + \nu \leq 1$ after normalization.
Neutrosophic Offset	(T, I, F)	all $\in [\Psi, \Omega]$; offset if any $\notin [0,1]$.
Quadripartitioned NS Offset	(T, I, C, F)	all $\in [\Psi, \Omega]$; offset if any $\notin [0,1]$.
Pentapartitioned NS Offset	(T, I, C, U, F)	all $\in [\Psi, \Omega]$; offset if any $\notin [0,1]$.
Hexapartitioned NS Offset	(T, C, G, U, H, F)	all $\in [\Psi, \Omega]$; offset if any $\notin [0,1]$.
Heptapartitioned NS Offset	$(T, \mathrm{RT}, C, U, I, \mathrm{RF}, F)$	all $\in [\Psi, \Omega]$; offset if any $\notin [0,1]$.
Octapartitioned NS Offset	$(T, \mathrm{RT}, C, U, I, H, \mathrm{RF}, F)$	all $\in [\Psi, \Omega]$; offset if any $\notin [0,1]$.
Nonapartitioned NS Offset	$(T, \mathrm{SRT}, \mathrm{WRT}, C, U, I, \mathrm{SRF}, \mathrm{WRF}, F)$	all $\in [\Psi, \Omega]$; offset if any $\notin [0,1]$.
Decapartitioned NS Offset	$(T, \mathrm{SRT}, \mathrm{WRT}, C, U, I, H, \mathrm{SRF}, \mathrm{WRF}, F)$	all $\in [\Psi, \Omega]$; offset if any $\notin [0,1]$.

Table 5. Offset variants with under/over-limits $\Psi < 0 < 1 < \Omega$.

1.5. Structure of this paper

This subsection outlines the structure of the paper. Section 2 reviews several notions already defined in the existing literature, including Multipartitioned Neutrosophic Sets. Section 3 introduces the notion of Offsets, which constitutes one of the main contributions of this work. Section 4 discusses how these Offsets can be generalized within the framework of Plithogenic OffSets. Section 5 presents algorithmic formulations for the proposed Offsets. Section 6 concludes the paper and describes future research directions.

2. Preliminaries

Throughout this paper, all sets are assumed to be finite. For the basic operations associated with each concept, the reader is referred to the respective references.

2.1. Neutrosophic Sets

Neutrosophic Sets extend classical fuzzy sets by introducing an explicit *indeterminacy* degree, thereby accommodating propositions that are not wholly true nor wholly false. Each element in a neutrosophic set is characterized by three independent membership values: truth, indeterminacy, and falsity [16, 57]. Building on this framework, several generalized variants have been proposed, including *bipolar neutrosophic sets*[58], *interval-valued neutrosophic sets* [59], and *complex neutrosophic sets* [60].

Definition 2.1 (Neutrosophic Set)**.** [16] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

2.2. Single-Valued Neutrosophic Offset

A *Single-Valued Neutrosophic Offset* relaxes the usual neutrosophic-set constraints by permitting the truth, indeterminacy, or falsity degrees of some elements to lie outside the unit interval (cf.[61]). This models situations of extreme uncertainty where membership can be "underset" (below 0) or "overset" (above 1) without prior normalization.

Definition 2.2 (Single-Valued Neutrosophic Offset)**.** [51] Let U_{off} be a universe of discourse and choose real bounds $\Psi < 0 < 1 < \Omega$. A *Single-Valued Neutrosophic Offset* is a collection

$$A_{\text{off}} = \Big\{ (x,\ \langle T(x),\ I(x),\ F(x)\rangle) \ \Big|\ x \in U_{\text{off}},$$

$$T(x), I(x), F(x) \in [\Psi, \Omega],\ \exists\, \mu \in \{T, I, F\} :\ \mu(x) \notin [0,1] \Big\}.$$

Here $T(x)$, $I(x)$, and $F(x)$ denote the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively, each allowed to range over $[\Psi, \Omega]$ so that some degrees may exceed 1 or fall below 0.

Example 2.3 (Weather Forecast Confidence as a Single-Valued Neutrosophic Offset)**.** Let $U_{\text{off}} = \{\text{Day}_1, \text{Day}_2\}$ be two consecutive days for which a meteorologist assigns probabilistic forecasts with an explicit indeterminacy component. Fix under- and over-limits $\Psi = -0.1$ and $\Omega = 1.1$. For each day $x \in U_{\text{off}}$, define:

$$\begin{aligned} T(x) &: \text{degree of confidence that it will rain,} \\ I(x) &: \text{degree of indeterminacy due to model ambiguity,} \\ F(x) &: \text{degree of confidence that it will not rain.} \end{aligned}$$

Each of $T(x), I(x), F(x)$ lies in $[\Psi, \Omega]$, and at least one lies outside $[0, 1]$.

Day	T	I	F
Day_1	1.05	0.10	-0.05
Day_2	0.80	-0.10	0.40

On Day_1, the rain–confidence $T = 1.05$ exceeds 1 (overset) and the no-rain confidence $F = -0.05$ is below 0 (underset), capturing exceptionally strong but conflicting model signals. On Day_2, the indeterminacy $I = -0.10$ falls below 0, indicating overconfidence in the prediction. In both cases, the triple $\langle T, I, F\rangle \in [\Psi, \Omega]^3$ with at least one component outside $[0, 1]$ constitutes a valid single-valued neutrosophic offset.

2.3. Hexapartitioned Neutrosophic Set

A *Hexapartitioned Neutrosophic Set* refines the classical neutrosophic set by introducing six independent membership degrees-truth, contradiction, ignorance, unknown, hesitation, and falsity-whose sum is bounded by 6 [33].

Definition 2.4 (Hexapartitioned Neutrosophic Set)**.** [62, 33] Let U be a nonempty universe. A *hexapartitioned neutrosophic set* on U is given by

$$N \ = \ \big\{ \langle x,\ T(x),\ C(x),\ G(x),\ U(x),\ H(x),\ F(x)\rangle \ \big|\ x \in U \big\},$$

where each function

$$T,\ C,\ G,\ U,\ H,\ F : U \longrightarrow [0,1]$$

assigns the degrees of truth, contradiction, ignorance, unknown, hesitation, and falsity, respectively, and satisfies

$$0 \leq T(x) + C(x) + G(x) + U(x) + H(x) + F(x) \leq 6, \quad \forall x \in U.$$

In this formulation, the "ambiguous" component of earlier six-valued schemes is replaced by the *ignorance* degree $G(x)$.

Remark 2.5 (On the choice of sum constraints)**.** The global bounds of the form

$$\Psi \leq \sum_{i=1}^{k} d_i(x) \leq \Omega + (k-1)$$

for a k-partitioned (offset) profile $d_1(x), \ldots, d_k(x)$ are not chosen arbitrarily, but to enforce a specific calibration principle.

First, in the classical non-offset case $\Psi = 0, \ \Omega = 1$ we recover the usual neutrosophic normalisation: each component lies in $[0, 1]$, so $\sum_{i=1}^{k} d_i(x) \in [0, k]$. Thus the upper bound reduces to k, and the proposed constraint is exactly the standard one.

Second, in the offset setting $\Psi < 0 < 1 < \Omega$ we want to allow "extreme" but still interpretable configurations where at most one component attains the overlimit Ω while the remaining $k - 1$ components take their classical maximum 1. In that case the sum is

$$\Omega + (k-1),$$

so any upper bound smaller than $\Omega + (k-1)$ would exclude such maximally informative profiles. Symmetrically, the lower bound Ψ ensures that one component may reach the underlimit Ψ while the others are as low as 0, without allowing all components to drift arbitrarily far into the negative range.

Third, fixing the envelope to $[\Psi, \Omega + (k-1)]$ keeps the total scale of a k-partitioned profile comparable to that of the underlying three-component or four-component model: increasing the number of subcomponents refines the description of a single neutrosophic state, rather than artificially inflating its total "mass". This is important for cross-model comparisons and for aggregation operators that mix profiles with different partition sizes.

Finally, none of our structural results relies on the specific expression $\Omega + (k-1)$ beyond the existence of a finite affine upper bound. A practitioner could replace $\Omega + (k-1)$ by any function B_k with $B_k \geq \Omega + (k-1)$ without affecting the validity of the theory; we adopt $\Omega + (k-1)$ because it is the minimal linear bound that (i) reproduces the classical case when $\Psi = 0, \Omega = 1$ and (ii) keeps all single-component-saturated offset profiles feasible.

Example 2.6 (Restaurant Quality Assessment as a Hexapartitioned Neutrosophic Set)**.** Let $U = \{\mathrm{R}_1, \mathrm{R}_2\}$ be two candidate restaurants. We evaluate each on six criteria, yielding a hexapartitioned neutrosophic set:

$$N = \left\{ \langle x, \ T(x), \ C(x), \ G(x), \ U(x), \ H(x), \ F(x) \rangle \ \middle| \ x \in U \right\},$$

where

$T(x)$: degree of genuine culinary excellence,
$C(x)$: degree of contradictory reviews,
$G(x)$: degree of missing information,
$U(x)$: degree of novel or unexpected features,
$H(x)$: degree of reviewer hesitation,
$F(x)$: degree of genuine shortcomings.

Each value lies in $[0, 1]$ and their sum is bounded by 6. A possible assessment is:

Restaurant	T	C	G	U	H	F
R_1	0.80	0.10	0.30	0.20	0.15	0.05
R_2	0.60	0.05	0.40	0.10	0.25	0.20

For R_1, the sum is $0.80 + 0.10 + 0.30 + 0.20 + 0.15 + 0.05 = 1.60 \leq 6$. For R_2, the sum is $0.60 + 0.05 + 0.40 + 0.10 + 0.25 + 0.20 = 1.60 \leq 6$. Thus both tuples satisfy the hexapartitioned neutrosophic-set constraints.

2.4. Octapartitioned Neutrosophic Set

An *octapartitioned neutrosophic set* refines the classical neutrosophic set by assigning eight independent membership degrees-truth, relative truth, contradiction, unknown, ignorance, hesitancy, relative falsity, and falsity-to each element.

Definition 2.7 (Octapartitioned Neutrosophic Set)**.** [33] Let U be a nonempty universe. An *octapartitioned neutrosophic set* on U is given by

$$O = \left\{ \langle x,\ T_O(x),\ M_O(x),\ C_O(x),\ U_O(x),\ I_O(x),\ H_O(x),\ K_O(x),\ F_O(x) \rangle \ \middle|\ x \in U \right\},$$

where

$$T_O,\ M_O,\ C_O,\ U_O,\ I_O,\ H_O,\ K_O,\ F_O : U \longrightarrow [0,1]$$

are the membership functions for truth, relative truth, contradiction, unknown, ignorance, hesitancy, relative falsity, and falsity, respectively. These satisfy, for every $x \in U$,

$$0 \ \leq\ T_O(x) + M_O(x) + C_O(x) + U_O(x) + I_O(x) + H_O(x) + K_O(x) + F_O(x) \ \leq\ 8.$$

Example 2.8 (Smartphone Review Assessment as an Octapartitioned Neutrosophic Set)**.** Let $U = \{\mathrm{Phone}_A, \mathrm{Phone}_B\}$ be two smartphone models under consideration. We evaluate each on eight neutrosophic membership degrees:

$$\begin{aligned}
T_O(x) &: \text{genuine performance quality,}\\
M_O(x) &: \text{relative performance (compared to peers),}\\
C_O(x) &: \text{conflicting user feedback,}\\
U_O(x) &: \text{unknown or emergent features,}\\
I_O(x) &: \text{missing data (ignorance),}\\
H_O(x) &: \text{reviewer hesitation,}\\
K_O(x) &: \text{relative falsity (overhyped claims),}\\
F_O(x) &: \text{clear shortcomings (falsity).}
\end{aligned}$$

Each degree lies in $[0,1]$ and their sum does not exceed 8. A sample assessment is:

Model	T_O	M_O	C_O	U_O	I_O	H_O	K_O	F_O
Phone_A	0.80	0.60	0.10	0.20	0.30	0.15	0.05	0.10
Phone_B	0.70	0.50	0.20	0.25	0.10	0.20	0.15	0.05

For each model,

$$0 \ \leq\ T_O + M_O + C_O + U_O + I_O + H_O + K_O + F_O \ \leq\ 8,$$

so this table provides a valid octapartitioned neutrosophic set representation of the smartphone evaluations.

2.5. Nonapartitioned Neutrosophic Set

A *nonapartitioned neutrosophic set* further generalizes this idea by introducing nine membership degrees-truth, strongly relative truth, weakly relative truth, contradiction, unknown, ignorance, strongly relative falsity, weakly relative falsity, and falsity.

Definition 2.9 (Nonapartitioned Neutrosophic Set)**.** [33] Let U be a nonempty universe. A *nonapartitioned neutrosophic set* on U is defined as

$$N = \Big\{ \langle x,\ T_N(x),\ ST_N(x),\ WT_N(x),\ C_N(x),\ U_N(x),\ I_N(x),\ SF_N(x),\ WF_N(x),\ F_N(x)\rangle \ \Big|\ x \in U \Big\},$$

where

$$T_N,\, ST_N,\, WT_N,\, C_N,\, U_N,\, I_N,\, SF_N,\, WF_N,\, F_N : U \longrightarrow [0,1]$$

are the membership functions corresponding to truth, strongly relative truth, weakly relative truth, contradiction, unknown, ignorance, strongly relative falsity, weakly relative falsity, and falsity. They satisfy, for every $x \in U$,

$$0 \le T_N(x) + ST_N(x) + WT_N(x) + C_N(x) + U_N(x) + I_N(x) + SF_N(x) + WF_N(x) + F_N(x) \le 9.$$

Remark 2.10. The functions $ST_N(x)$ and $WT_N(x)$ (and similarly $SF_N(x)$ and $WF_N(x)$) are used to distinguish *strong* from *weak* evidence in favour of, or against, a proposition. For instance, $ST_N(x)$ may collect clearly positive information (such as "strongly agree"), while $WT_N(x)$ records more tentative or borderline support (such as "somewhat agree"). This separation allows aggregation rules that weight strong and weak components differently and makes it possible to analyse how conclusions change when weak evidence is discounted or reclassified.

Example 2.11 (University Applicant Evaluation as a Nonapartitioned Neutrosophic Set)**.** Let $U = \{\text{Alice}, \text{Bob}\}$ be two applicants for a graduate program. We assess each on nine neutrosophic membership degrees, all valued in $[0, 1]$ and summing to at most 9:

$T_N(x)$: baseline suitability (truth),
$ST_N(x)$: strong relative suitability,
$WT_N(x)$: weak relative suitability,
$C_N(x)$: conflicting indicators,
$U_N(x)$: unknown or novel qualifications,
$I_N(x)$: data gaps (ignorance),
$SF_N(x)$: strong relative unsuitability,
$WF_N(x)$: weak relative unsuitability,
$F_N(x)$: baseline unsuitability (falsity).

A possible evaluation is:

Applicant	T_N	ST_N	WT_N	C_N	U_N	I_N	SF_N	WF_N	F_N
Alice	0.80	0.50	0.10	0.05	0.20	0.10	0.05	0.08	0.12
Bob	0.70	0.30	0.20	0.10	0.15	0.05	0.10	0.12	0.10

For Alice,

$$T_N + ST_N + WT_N + C_N + U_N + I_N + SF_N + WF_N + F_N = 2.00 \le 9,$$

and similarly for Bob. Thus this table constitutes a valid nonapartitioned neutrosophic set representation of the applicants' evaluations.

2.6. Decapartitioned Neutrosophic Set

A *Decapartitioned Neutrosophic Set* refines the neutrosophic framework by assigning ten independent membership degrees-two levels of truth, contradiction, unknown, ignorance, hesitation, two levels of falsity, and standard truth and falsity-to each element.

Definition 2.12 (Decapartitioned Neutrosophic Set)**.** [33] Let U be a nonempty universe. A *decapartitioned neutrosophic set* on U is a collection

$$D = \{ \langle x, T(x), \mathrm{SRT}(x), \mathrm{WRT}(x), C(x), U(x), I(x), H(x), \mathrm{SRF}(x),$$

$$\mathrm{WRF}(x), F(x) \rangle \mid x \in U \},$$

where the ten functions

$$T, \mathrm{SRT}, \mathrm{WRT}, C, U, I, H, \mathrm{SRF}, \mathrm{WRF}, F :$$

$$U \longrightarrow [0,1]$$

denote, respectively:

- $T(x)$: truth,
- $\mathrm{SRT}(x)$: strongly relative truth,
- $\mathrm{WRT}(x)$: weakly relative truth,
- $C(x)$: contradiction,
- $U(x)$: unknown,
- $I(x)$: ignorance,
- $H(x)$: hesitation,
- $\mathrm{SRF}(x)$: strongly relative falsity,
- $\mathrm{WRF}(x)$: weakly relative falsity,
- $F(x)$: falsity.

These satisfy, for every $x \in U$,

$$0 \leq T(x) + \mathrm{SRT}(x) + \mathrm{WRT}(x) + C(x) + U(x) + I(x) + H(x) + \mathrm{SRF}(x) + \mathrm{WRF}(x) + F(x) \leq 10.$$

Example 2.13 (Investment Opportunity Evaluation as a Decapartitioned Neutrosophic Set)**.** Let $U = \{\mathrm{Inv}_A, \mathrm{Inv}_B\}$ be two investment projects under consideration. We assign ten membership degrees-truth (T), strongly relative truth (SRT), weakly relative truth (WRT), contradiction (C), unknown (U), ignorance (I), hesitation (H), strongly relative falsity (SRF), weakly relative falsity (WRF), and falsity (F)-to each project, all valued in $[0,1]$. These degrees satisfy

$$0 \leq T(x) + \mathrm{SRT}(x) + \mathrm{WRT}(x) + C(x) + U(x) + I(x)$$

$$+H(x) + \mathrm{SRF}(x) + \mathrm{WRF}(x) + F(x) \leq 10, \quad x \in U.$$

Project	T	SRT	WRT	C	U	I	H	SRF	WRF	F
Inv_A	0.80	0.70	0.50	0.20	0.30	0.10	0.20	0.15	0.10	0.05
Inv_B	0.60	0.50	0.40	0.30	0.20	0.20	0.10	0.20	0.10	0.40

Here, for example, Inv_A has strong baseline confidence ($T = 0.80$), significant positive signals ($\mathrm{SRT} = 0.70$), moderate uncertainty ($U = 0.30$), and low outright rejection ($F = 0.05$). The total for each project is well below 10, satisfying the decapartitioned neutrosophic set constraints.

3. Results: Partitioned Neutrosophic Offset

As the main outcome of this work, we introduce and formally define the concept of a *Partitioned Neutrosophic Offset*.

3.1. Hexapartitioned Neutrosophic Offset

We begin by defining a six–component offset model that subsumes both the single-valued neutrosophic offset and the classical hexapartitioned neutrosophic set.

Definition 3.1 (Hexapartitioned Neutrosophic OffSet (H-NOS))**.** Let U be a nonempty set and fix real constants

$$\Psi < 0 < 1 < \Omega \quad (\text{the } \textit{UnderLimit} \text{ and } \textit{OverLimit}).$$

A *hexapartitioned neutrosophic offset* is a collection

$$N_{\text{off}} = \big\{ \langle x, T(x), C(x), G(x), U(x), H(x), F(x)\rangle \;\big|\; x \in U \big\},$$

where the six functions

$$T, C, G, U, H, F : U \longrightarrow [\Psi, \Omega]$$

are called, respectively, the *truth*, *contradiction*, *ignorance*, *unknown*, *hesitation*, and *falsity* degrees. They satisfy, for every $x \in U$,

$$\Psi \;\le\; T(x) + C(x) + G(x) + U(x) + H(x) + F(x) \;\le\; \Omega + 5.$$

Moreover, at least one of the six values must lie outside the unit interval $[0, 1]$ (i.e. exhibit *overset* or *underset* behavior); otherwise N_{off} reduces to the ordinary hexapartitioned neutrosophic set.

Example 3.2 (Medical Diagnosis as a Hexapartitioned Neutrosophic Offset)**.** Consider a clinical setting where two patients, $U = \{\mathrm{P}_1, \mathrm{P}_2\}$, are evaluated for a complex syndrome. We fix under- and overlimits $\Psi = -0.2$ and $\Omega = 1.2$. For each patient x, clinicians assign six "degrees":

$$\begin{aligned} T(x) &: \text{evidence supporting the diagnosis,} \\ C(x) &: \text{contradictory findings,} \\ G(x) &: \text{missing information (ignorance),} \\ U(x) &: \text{novel or atypical features (unknown),} \\ H(x) &: \text{clinical hesitation,} \\ F(x) &: \text{evidence against the diagnosis.} \end{aligned}$$

These values lie in $[\Psi, \Omega]$ and must satisfy $\Psi \le T + C + G + U + H + F \le \Omega + 5$, with at least one component outside $[0, 1]$.

Patient	T	C	G	U	H	F
P_1	1.10	0.05	0.15	0.10	0.08	0.02
P_2	0.80	0.10	0.05	0.20	0.15	−0.05

For P_1, the truth-evidence degree $T = 1.10$ slightly exceeds 1 (overset), reflecting exceptionally strong supporting data, while all other values remain within $[0, 1]$. For P_2, the falsity-evidence degree $F = -0.05$ falls below 0 (underset), capturing negative confidence in the diagnosis. In both cases,

$$-0.2 \; \leq \; T + C + G + U + H + F \; \leq \; 1.2 + 5,$$

so the table constitutes a valid hexapartitioned neutrosophic offset.

The allowance of $[\Psi, \Omega]$ enables each degree to take *offset* values: strictly negative (underset) or exceeding 1 (overset).

Theorem 3.3
The hexapartitioned neutrosophic offset (H-NOS) strictly generalises both

1. the single-valued neutrosophic offset (SV-NOS), and
2. the classical hexapartitioned neutrosophic set (H-NS).

Proof
We show that SV-NOS and H-NS appear as special cases of H-NOS, and that H-NOS contains elements that do not belong to either of these subclasses.

(1) Embedding SV-NOS into H-NOS.

Let U be a nonempty universe and choose real bounds $\Psi < 0 < 1 < \Omega$. A single-valued neutrosophic offset (SV-NOS) on U is given by

$$A_{\text{off}} = \left\{ (x, \langle T_{\text{SV}}(x), I_{\text{SV}}(x), F_{\text{SV}}(x) \rangle) \;\middle|\; x \in U \right\},$$

where

$$T_{\text{SV}}, I_{\text{SV}}, F_{\text{SV}} : U \to [\Psi, \Omega]$$

and, by the usual neutrosophic convention,

$$0 \; \leq \; T_{\text{SV}}(x) + I_{\text{SV}}(x) + F_{\text{SV}}(x) \; \leq \; 3, \qquad \forall x \in U. \tag{1}$$

Moreover, there exists at least one element $x_0 \in U$ for which at least one of $T_{\text{SV}}(x_0), I_{\text{SV}}(x_0), F_{\text{SV}}(x_0)$ lies outside the unit interval $[0, 1]$, so that A_{off} is genuinely an *offset*.

Define six functions

$$T, C, G, U, H, F : U \to [\Psi, \Omega]$$

by

$$T(x) := T_{\text{SV}}(x), \quad C(x) := 0, \quad G(x) := I_{\text{SV}}(x), \quad U(x) := 0, \quad H(x) := 0, \quad F(x) := F_{\text{SV}}(x). \tag{2}$$

Since $\Psi < 0 < 1 < \Omega$, we have $0 \in [\Psi, \Omega]$, hence each of C, U, H indeed maps into $[\Psi, \Omega]$, and of course $T, G, F \in [\Psi, \Omega]$ because $T_{\text{SV}}, I_{\text{SV}}, F_{\text{SV}}$ do. Thus the range condition in Definition 3.1 is satisfied.

For every $x \in U$, the sum of the six degrees in (2) is

$$T(x) + C(x) + G(x) + U(x) + H(x) + F(x) = T_{\mathrm{SV}}(x) + I_{\mathrm{SV}}(x) + F_{\mathrm{SV}}(x).$$

Using (1) and the bounds on Ψ and Ω, we obtain

$$0 \ \leq \ T_{\mathrm{SV}}(x) + I_{\mathrm{SV}}(x) + F_{\mathrm{SV}}(x) \ \leq \ 3.$$

Since $\Psi < 0$, we have

$$\Psi \ < \ 0 \ \leq \ T_{\mathrm{SV}}(x) + I_{\mathrm{SV}}(x) + F_{\mathrm{SV}}(x),$$

so the lower H-NOS bound $\Psi \leq T + C + G + U + H + F$ holds. Similarly, because $\Omega > 1$,

$$\Omega + 5 \ \geq \ 1 + 5 = 6 \ > \ 3,$$

hence

$$T(x) + C(x) + G(x) + U(x) + H(x) + F(x) = T_{\mathrm{SV}}(x) + I_{\mathrm{SV}}(x) + F_{\mathrm{SV}}(x) \ \leq \ 3 \ \leq \ \Omega + 5,$$

so the upper H-NOS bound $T + C + G + U + H + F \leq \Omega + 5$ is also satisfied.

Next, we check the *offset* condition. By assumption on A_{off}, there exists $x_0 \in U$ and some $\mu \in \{T_{\mathrm{SV}}, I_{\mathrm{SV}}, F_{\mathrm{SV}}\}$ such that $\mu(x_0) \notin [0, 1]$. From (2) we see that

$$T(x_0) = T_{\mathrm{SV}}(x_0), \quad G(x_0) = I_{\mathrm{SV}}(x_0), \quad F(x_0) = F_{\mathrm{SV}}(x_0),$$

so the same value $\mu(x_0)$ appears unchanged in $\{T(x_0), G(x_0), F(x_0)\}$. Thus at least one of the six degrees $T(x_0), C(x_0), G(x_0), U(x_0), H(x_0), F(x_0)$ lies outside $[0, 1]$, proving that

$$N_{\mathrm{off}} := \left\{ \langle x, T(x), C(x), G(x), U(x), H(x), F(x) \rangle \;\middle|\; x \in U \right\}$$

is a valid hexapartitioned neutrosophic offset.

Finally, the map

$$\iota_{\mathrm{SV}} \colon A_{\mathrm{off}} \longmapsto N_{\mathrm{off}}$$

is injective: if two SV-NOS structures $(T_{\mathrm{SV}}, I_{\mathrm{SV}}, F_{\mathrm{SV}})$ and $(T'_{\mathrm{SV}}, I'_{\mathrm{SV}}, F'_{\mathrm{SV}})$ produce the same H-NOS via (2), then for all $x \in U$,

$$T_{\mathrm{SV}}(x) = T'(x) = T'_{\mathrm{SV}}(x), \quad I_{\mathrm{SV}}(x) = G(x) = G'(x) = I'_{\mathrm{SV}}(x), \quad F_{\mathrm{SV}}(x) = F(x) = F'(x) = F'_{\mathrm{SV}}(x),$$

so the triples coincide. Hence every SV-NOS can be seen as an H-NOS with $C \equiv U \equiv H \equiv 0$, and this embedding is faithful.

(2) Embedding H-NS into H-NOS.

A classical hexapartitioned neutrosophic set (H-NS) on a universe U is given by

$$N = \left\{ \langle x, T(x), C(x), G(x), U(x), H(x), F(x) \rangle \;\middle|\; x \in U \right\},$$

where

$$T, C, G, U, H, F : U \to [0, 1]$$

and, for every $x \in U$,

$$0 \ \leq \ T(x) + C(x) + G(x) + U(x) + H(x) + F(x) \ \leq \ 6. \tag{3}$$

To view such an N as a special case of H-NOS, we simply choose the offset parameters

$$\Psi := 0, \qquad \Omega := 1.$$

Then the range condition in Definition 3.1,

$$T, C, G, U, H, F : U \to [\Psi, \Omega] = [0, 1],$$

is exactly the same as in H-NS, and the H-NOS sum constraint

$$\Psi \ \le\ T(x) + C(x) + G(x) + U(x) + H(x) + F(x) \ \le\ \Omega + 5 = 1 + 5 = 6$$

is identical to (3). Thus every hexapartitioned neutrosophic set is an H-NOS with parameters $\Psi = 0$ and $\Omega = 1$. Equivalently, H-NS is the subfamily of H-NOS for which all six membership degrees stay inside the unit interval $[0, 1]$ (no offset behaviour occurs).

(3) Strictness of the generalisation.

It remains to show that H-NOS is strictly more general than both SV-NOS and H-NS, in the sense that there exist H-NOS structures which are not in the image of ι_{SV} and are not H-NS.

Consider the one–element universe $U = \{x_0\}$ and choose $\Psi = -0.2$, $\Omega = 1.2$. Define

$$T(x_0) = 0.9, \quad C(x_0) = 1.1, \quad G(x_0) = 0.2, \quad U(x_0) = 0, \quad H(x_0) = 0, \quad F(x_0) = 0.$$

First, check the range requirement:

$$T(x_0), C(x_0), G(x_0), U(x_0), H(x_0), F(x_0) \in [-0.2, 1.2] = [\Psi, \Omega],$$

so the component-wise bounds hold. Next, compute the sum:

$$T(x_0) + C(x_0) + G(x_0) + U(x_0) + H(x_0) + F(x_0) = 0.9 + 1.1 + 0.2 + 0 + 0 + 0 = 2.2.$$

Since

$$\Psi = -0.2 \le 2.2 \le 1.2 + 5 = 6.2 = \Omega + 5,$$

the H-NOS sum constraint is satisfied. Moreover,

$$C(x_0) = 1.1 > 1,$$

so at x_0 at least one membership degree lies outside $[0, 1]$, and the structure is genuinely an *offset*. Hence

$$N^*_{\mathrm{off}} = \big\{ \langle x_0, 0.9, 1.1, 0.2, 0, 0, 0 \rangle \big\}$$

is a valid hexapartitioned neutrosophic offset.

Now observe:

- N^*_{off} is *not* a classical H-NS, because for an H-NS all six degrees must lie in $[0, 1]$, whereas $C(x_0) = 1.1 > 1$.
- N^*_{off} is *not* in the image of the embedding ι_{SV}. Indeed, for any SV-NOS embedded via (2), one always has $C(x) \equiv U(x) \equiv H(x) \equiv 0$ for all $x \in U$. In N^*_{off} we have $C(x_0) = 1.1 \neq 0$; hence there is no SV-NOS triple $(T_{\mathrm{SV}}, I_{\mathrm{SV}}, F_{\mathrm{SV}})$ such that $\iota_{\mathrm{SV}}(T_{\mathrm{SV}}, I_{\mathrm{SV}}, F_{\mathrm{SV}})$ coincides with N^*_{off}.

Thus H-NOS properly contains the embedded copy of SV-NOS and properly contains the embedded copy of H-NS. In other words, the class of hexapartitioned neutrosophic offsets is strictly larger and therefore strictly generalises both the single-valued neutrosophic offsets and the classical hexapartitioned neutrosophic sets. □

3.2. Octapartitioned Neutrosophic Offset

We generalize the offset concept to an eight–component framework, unifying several earlier models.

Definition 3.4 (Octapartitioned Neutrosophic OffSet (O-NOS))**.** Let U be a nonempty universe and fix real bounds

$$\Psi < 0 < 1 < \Omega \quad \text{(UnderLimit and OverLimit)}.$$

An *octapartitioned neutrosophic offset* is a function

$$\mathbf{n}\colon U \longrightarrow [\Psi, \Omega]^8, \quad x \mapsto \big(T(x),\, M(x),\, C(x),\, U(x),\, I(x),\, K(x),\, H(x),\, F(x)\big),$$

where the eight components represent:

$$\begin{array}{ll} T: \text{truth}, & M: \text{relative truth}, \\ C: \text{contradiction}, & U: \text{unknown}, \\ I: \text{ignorance}, & K: \text{relative falsity}, \\ H: \text{hesitation}, & F: \text{falsity}, \end{array}$$

and for each $x \in U$ the normalization condition

$$\Psi \;\leq\; T(x) + M(x) + C(x) + U(x) + I(x) + K(x) + H(x) + F(x) \;\leq\; \Omega + 7$$

holds. Moreover, there must exist at least one $x \in U$ for which one of these eight values lies outside the unit interval $[0, 1]$, otherwise the structure reduces to the classical octapartitioned neutrosophic set.

Example 3.5 (Corporate Credit Assessment as an Octapartitioned Neutrosophic Offset)**.** Consider two firms, $U = \{\mathrm{Firm}_A, \mathrm{Firm}_B\}$, whose bond default risk is evaluated using eight neutrosophic offset degrees. We set the under- and overlimits to $\Psi = -0.1$ and $\Omega = 1.2$. For each firm x, analysts assign:

$$\begin{array}{ll} T(x): \text{baseline creditworthiness}, & M(x): \text{relative credit strength}, \\ C(x): \text{contradictory indicators}, & U(x): \text{unknown market factors}, \\ I(x): \text{data gaps (ignorance)}, & K(x): \text{relative falsity (overstated strength)}, \\ H(x): \text{analyst hesitation}, & F(x): \text{negative signals (falsity)}. \end{array}$$

All eight values lie in $[\Psi, \Omega]$ and satisfy $\Psi \leq T + M + C + U + I + K + H + F \leq \Omega + 7$, with at least one entry outside $[0, 1]$.

Firm	T	M	C	U	I	K	H	F
Firm_A	1.15	0.10	0.05	0.08	0.07	0.00	0.03	0.02
Firm_B	0.85	0.20	0.10	0.05	0.02	−0.05	0.04	0.03

For Firm_A, the baseline credit degree $T = 1.15$ exceeds 1 (overset), indicating exceptionally strong fundamentals. For Firm_B, the relative falsity degree $K = -0.05$ falls below 0 (underset), reflecting slight negative bias in reported strengths. Both rows satisfy

$$-0.1 \;\leq\; T + M + C + U + I + K + H + F \;\leq\; 1.2 + 7,$$

validating this as an octapartitioned neutrosophic offset.

When $\Psi = 0$ and $\Omega = 1$, Definition 3.4 specializes exactly to the ordinary octapartitioned neutrosophic set.

Theorem 3.6
The octapartitioned neutrosophic offset (O-NOS) strictly generalizes:

1. the single-valued neutrosophic offset (SV-NOS),
2. the hexapartitioned neutrosophic offset (H-NOS), and
3. the classical octapartitioned neutrosophic set (O-NS).

Proof
We work on a fixed nonempty universe U. Recall that an octapartitioned neutrosophic offset (O-NOS) is given by

$$N_{\text{off}}^{(8)} = \left\{ \langle x,\, T(x), M(x), C(x), U(x), I(x), K(x), H(x), F(x) \rangle \;\middle|\; x \in U \right\},$$

where

$$T, M, C, U, I, K, H, F : U \longrightarrow [\Psi, \Omega]$$

for some real interval $[\Psi, \Omega]$ that contains the unit interval $[0, 1]$, and for every $x \in U$ the sum

$$S(x) := T(x) + M(x) + C(x) + U(x) + I(x) + K(x) + H(x) + F(x)$$

satisfies

$$\Psi \;\leq\; S(x) \;\leq\; \Omega + 7. \tag{4}$$

In addition, there exists at least one $x \in U$ and one component among T, M, C, U, I, K, H, F whose value lies outside the unit interval $[0, 1]$; this is the *offset* condition.

We now treat each item in turn and finally show strictness.

(1) Embedding SV-NOS into O-NOS.

Let

$$A_{\text{off}} = \left\{ (x, \langle T_{\text{SV}}(x), I_{\text{SV}}(x), F_{\text{SV}}(x) \rangle) \;\middle|\; x \in U \right\}$$

be a single-valued neutrosophic offset (SV-NOS) on U. Thus

$$T_{\text{SV}},\; I_{\text{SV}},\; F_{\text{SV}} : U \longrightarrow [\Psi, \Omega],$$

and for every $x \in U$ we have the usual neutrosophic constraint

$$0 \;\leq\; T_{\text{SV}}(x) + I_{\text{SV}}(x) + F_{\text{SV}}(x) \;\leq\; 3. \tag{5}$$

Moreover, there exists at least one point $x_0 \in U$ such that at that point at least one of $T_{\text{SV}}(x_0), I_{\text{SV}}(x_0), F_{\text{SV}}(x_0)$ lies outside $[0, 1]$, so that A_{off} is genuinely an *offset*.

Define eight membership functions

$$T, M, C, U, I, K, H, F : U \longrightarrow [\Psi, \Omega]$$

by

$$\begin{aligned} T(x) &:= T_{\text{SV}}(x), & M(x) &:= 0, \\ C(x) &:= 0, & U(x) &:= 0, \\ I(x) &:= I_{\text{SV}}(x), & K(x) &:= 0, \\ H(x) &:= 0, & F(x) &:= F_{\text{SV}}(x). \end{aligned} \tag{6}$$

Since $[0,1] \subseteq [\Psi,\Omega]$, we have $0 \in [\Psi,\Omega]$, and hence each of M, C, U, K, H indeed maps into $[\Psi,\Omega]$. The functions T, I, F also take values in $[\Psi,\Omega]$ by assumption on $T_{\mathrm{SV}}, I_{\mathrm{SV}}, F_{\mathrm{SV}}$. Thus the component-wise range condition for O-NOS is satisfied.

For every $x \in U$, the O-NOS sum $S(x)$ is

$$\begin{aligned} S(x) &= T(x) + M(x) + C(x) + U(x) + I(x) + K(x) + H(x) + F(x) \\ &= T_{\mathrm{SV}}(x) + 0 + 0 + 0 + I_{\mathrm{SV}}(x) + 0 + 0 + F_{\mathrm{SV}}(x) \\ &= T_{\mathrm{SV}}(x) + I_{\mathrm{SV}}(x) + F_{\mathrm{SV}}(x). \end{aligned}$$

Combining this identity with (11) gives

$$0 \;\le\; S(x) \;\le\; 3.$$

Since $\Psi \le 0$ (because $[0,1] \subseteq [\Psi,\Omega]$) and $\Omega \ge 1$, we have

$$\Psi \;\le\; 0 \;\le\; S(x) \quad \text{and} \quad S(x) \;\le\; 3 \;<\; 8 \;\le\; \Omega + 7,$$

so the O-NOS sum constraint (13) holds for all $x \in U$.

The offset condition is inherited: by assumption there exists $x_0 \in U$ and some $\mu \in \{T_{\mathrm{SV}}, I_{\mathrm{SV}}, F_{\mathrm{SV}}\}$ with $\mu(x_0) \notin [0,1]$. From (6) we see that the same value $\mu(x_0)$ appears unchanged as one of $T(x_0), I(x_0), F(x_0)$, so at least one of the eight O-NOS components at x_0 lies outside $[0,1]$. Thus the structure defined by (6) is a valid O-NOS.

Finally, the assignment

$$\iota_{\mathrm{SV}} : (T_{\mathrm{SV}}, I_{\mathrm{SV}}, F_{\mathrm{SV}}) \longmapsto (T, M, C, U, I, K, H, F)$$

given by (6) is injective: if two SV-NOS structures produce the same O-NOS, then, for every $x \in U$,

$$T_{\mathrm{SV}}(x) = T(x) = T'_{\mathrm{SV}}(x), \qquad I_{\mathrm{SV}}(x) = I(x) = I'_{\mathrm{SV}}(x), \qquad F_{\mathrm{SV}}(x) = F(x) = F'_{\mathrm{SV}}(x),$$

so the triples coincide. Hence SV-NOS embeds faithfully into O-NOS.

(2) Embedding H-NOS into O-NOS.

Next, let

$$N^{(6)}_{\mathrm{off}} = \left\{ \langle x, T_H(x), C_H(x), G_H(x), U_H(x), H_H(x), F_H(x) \rangle \;\middle|\; x \in U \right\}$$

be a hexapartitioned neutrosophic offset (H-NOS) on U. Thus

$$T_H, C_H, G_H, U_H, H_H, F_H : U \to [\Psi,\Omega],$$

and for every $x \in U$ the H-NOS sum

$$S_H(x) := T_H(x) + C_H(x) + G_H(x) + U_H(x) + H_H(x) + F_H(x)$$

satisfies

$$\Psi \;\le\; S_H(x) \;\le\; \Omega + 5. \tag{7}$$

There is also an offset condition: for at least one $x_0 \in U$, some value among $T_H(x_0), C_H(x_0), G_H(x_0), U_H(x_0), H_H(x_0), F_H(x_0)$ lies outside $[0,1]$.

We embed this H-NOS into an O-NOS by defining

$$T, M, C, U, I, K, H, F : U \to [\Psi,\Omega]$$

via

$$\begin{aligned} T(x) &:= T_H(x), \quad M(x) := 0, \\ C(x) &:= C_H(x), \quad U(x) := U_H(x), \\ I(x) &:= G_H(x), \quad K(x) := 0, \\ H(x) &:= H_H(x), \quad F(x) := F_H(x). \end{aligned} \tag{8}$$

Again, because $0 \in [\Psi, \Omega]$, all eight functions take values in $[\Psi, \Omega]$.

For every $x \in U$, the O-NOS sum is

$$\begin{aligned} S(x) &= T(x) + M(x) + C(x) + U(x) + I(x) + K(x) + H(x) + F(x) \\ &= T_H(x) + 0 + C_H(x) + U_H(x) + G_H(x) + 0 + H_H(x) + F_H(x) \\ &= T_H(x) + C_H(x) + G_H(x) + U_H(x) + H_H(x) + F_H(x) \\ &= S_H(x). \end{aligned}$$

Combining this equality with (7) shows that

$$\Psi \ \le\ S(x) \ \le\ \Omega + 5 \ \le\ \Omega + 7,$$

so the O-NOS sum constraint (13) is satisfied.

The offset condition is also preserved. By assumption there is some $x_0 \in U$ such that at least one of the values $T_H(x_0), C_H(x_0), G_H(x_0), U_H(x_0), H_H(x_0), F_H(x_0)$ belongs to $(-\infty, 0) \cup (1, \infty)$. From (8), the same value appears (unchanged) among $\{T(x_0), C(x_0), I(x_0), U(x_0), H(x_0), F(x_0)$, so at x_0 the O-NOS has at least one component outside $[0, 1]$.

Finally, if two H-NOS structures $(T_H, C_H, G_H, U_H, H_H, F_H)$ and $(T'_H, C'_H, G'_H, U'_H, H'_H, F'_H)$ produce the same O-NOS via (8), then, for all $x \in U$,

$$T_H(x) = T(x) = T'_H(x), \quad C_H(x) = C(x) = C'_H(x), \quad G_H(x) = I(x) = I'_H(x),$$

$$U_H(x) = U(x) = U'_H(x), \quad H_H(x) = H(x) = H'_H(x), \quad F_H(x) = F(x) = F'_H(x),$$

so the two six-tuples coincide. Therefore H-NOS embeds injectively as the subclass of O-NOS characterized by $M \equiv 0$ and $K \equiv 0$.

(3) Recovering the classical O-NS.

A classical octapartitioned neutrosophic set (O-NS) on U is given by

$$N^{(8)} = \left\{ \langle x,\, T_{\mathrm{NS}}(x), M_{\mathrm{NS}}(x), C_{\mathrm{NS}}(x), U_{\mathrm{NS}}(x), I_{\mathrm{NS}}(x), K_{\mathrm{NS}}(x), H_{\mathrm{NS}}(x), F_{\mathrm{NS}}(x) \rangle \;\middle|\; x \in U \right\},$$

where

$$T_{\mathrm{NS}}, M_{\mathrm{NS}}, C_{\mathrm{NS}}, U_{\mathrm{NS}}, I_{\mathrm{NS}}, K_{\mathrm{NS}}, H_{\mathrm{NS}}, F_{\mathrm{NS}} : U \to [0, 1],$$

and, for each $x \in U$,

$$0 \ \le\ T_{\mathrm{NS}}(x) + M_{\mathrm{NS}}(x) + C_{\mathrm{NS}}(x) + U_{\mathrm{NS}}(x) + I_{\mathrm{NS}}(x) + K_{\mathrm{NS}}(x) + H_{\mathrm{NS}}(x) + F_{\mathrm{NS}}(x) \ \le\ 8. \tag{9}$$

No offset behaviour is allowed, i.e., all values lie in $[0, 1]$.

To see O-NS as a special case of O-NOS, we simply interpret it as an O-NOS with the specific bounds

$$\Psi := 0, \qquad \Omega := 1.$$

The component-wise range condition for O-NOS,

$$T, M, C, U, I, K, H, F : U \to [\Psi, \Omega] = [0, 1],$$

is exactly the same as in the classical O-NS definition. The O-NOS sum constraint (13) then becomes

$$0 \;\leq\; S(x) \;\leq\; 1 + 7 = 8,$$

which coincides with (9). The only difference is that O-NOS *allows* some components to move outside $[0, 1]$, whereas O-NS *forbids* this. Hence every O-NS is an O-NOS whose components all remain in $[0, 1]$; that is, O-NS is exactly the subfamily of O-NOS with $\Psi = 0$, $\Omega = 1$, and no offset behaviour.

(4) Strictness of the generalisation.

It remains to show that O-NOS is strictly more general than each of SV-NOS, H-NOS, and O-NS; in other words, there exist O-NOS structures that do not lie in the embedded images of these three classes.

Consider the one-element universe $U = \{x_0\}$ and choose bounds $\Psi = -0.2$, $\Omega = 1.2$. Define an eight-tuple of membership degrees at x_0 by

$$T(x_0) = 1.1, \quad M(x_0) = 0.1, \quad C(x_0) = 0.2, \quad U(x_0) = 0, \quad I(x_0) = 0, \quad K(x_0) = 0, \quad H(x_0) = 0, \quad F(x_0) = 0.$$

First we check that this satisfies the O-NOS axioms.

Component-wise bounds. Each of the eight values belongs to $[\Psi, \Omega] = [-0.2, 1.2]$:

$$-0.2 \leq 0 \leq 0.1 \leq 0.2 \leq 1.1 \leq 1.2,$$

so the range condition holds.

Sum constraint. The sum at x_0 is

$$S(x_0) = 1.1 + 0.1 + 0.2 + 0 + 0 + 0 + 0 + 0 = 1.4.$$

We have

$$\Psi = -0.2 \leq 1.4 \leq 1.2 + 7 = 8.2 = \Omega + 7,$$

so the inequality (13) is satisfied.

Offset behaviour. We clearly have

$$T(x_0) = 1.1 > 1,$$

so at x_0 at least one O-NOS component lies outside the unit interval $[0, 1]$, and the structure is indeed an offset.

Thus

$$N^*_{\text{off}} := \left\{ \langle x_0,\, 1.1,\, 0.1,\, 0.2,\, 0,\, 0,\, 0,\, 0,\, 0 \rangle \right\}$$

is a valid O-NOS.

We now verify that N^*_{off} does *not* belong to any of the three embedded subclasses.

(a) *Not in the image of SV-NOS.* Every O-NOS arising from the embedding ι_{SV} in (6) satisfies

$$M(x) \equiv 0, \quad C(x) \equiv 0, \quad U(x) \equiv 0, \quad K(x) \equiv 0, \quad H(x) \equiv 0$$

for all $x \in U$. In N^*_{off} we have

$$M(x_0) = 0.1 \neq 0 \quad \text{and} \quad C(x_0) = 0.2 \neq 0,$$

so N^*_{off} cannot be of the form (6); hence it is not in the image of SV-NOS.

(b) *Not in the image of H-NOS.* Every O-NOS obtained from the embedding (8) satisfies

$$M(x) \equiv 0 \quad \text{and} \quad K(x) \equiv 0$$

for all $x \in U$. In N^*_{off} we again have $M(x_0) = 0.1 \neq 0$, so N^*_{off} cannot arise from any H-NOS via (8).

(c) *Not a classical O-NS.* In a classical O-NS all membership degrees must lie in $[0, 1]$. However, N^*_{off} has

$$T(x_0) = 1.1 > 1,$$

so it cannot be an octapartitioned neutrosophic set.

We have constructed a concrete O-NOS N^*_{off} that is not in the embedded image of SV-NOS, not in the embedded image of H-NOS, and not a classical O-NS. Together with the injective embeddings of SV-NOS and H-NOS and the identification of O-NS as a special case of O-NOS, this shows that the class of octapartitioned neutrosophic offsets properly contains each of SV-NOS, H-NOS, and O-NS. Hence O-NOS *strictly* generalizes all three. □

3.3. Nonapartitioned Neutrosophic Offset

We conclude the hierarchy of neutrosophic offsets with a nine–component model that simultaneously subsumes all previously defined offset and classical partitioned structures.

Definition 3.7 (Nonapartitioned Neutrosophic OffSet (N-NOS))**.** Let U be a nonempty universe and fix real bounds

$$\Psi < 0 < 1 < \Omega \quad \text{(underlimit and overlimit)}.$$

A *nonapartitioned neutrosophic offset* is a mapping

$$\mathbf{n} \colon U \to [\Psi, \Omega]^9, \qquad x \mapsto \big(T(x),\, ST(x),\, WT(x),\, C(x),\, U(x),\, I(x),\, SF(x),\, WF(x),\, F(x)\big),$$

where the nine components are interpreted as

$$\begin{aligned} T &: \text{truth}, & ST &: \text{strong relative truth},\\ WT &: \text{weak relative truth}, & C &: \text{contradiction},\\ U &: \text{unknown}, & I &: \text{ignorance},\\ SF &: \text{strong relative falsity}, & WF &: \text{weak relative falsity},\\ F &: \text{falsity}, \end{aligned}$$

and satisfy the normalization condition

$$\Psi \;\le\; T(x) + ST(x) + WT(x) + C(x) + U(x) + I(x) + SF(x) + WF(x) + F(x) \;\le$$
$$\Omega + 8 \quad \text{for all } x \in U.$$

Moreover, for at least one $x \in U$, at least one of these nine values must lie outside the unit interval $[0, 1]$, else the model reduces to the ordinary (non-offset) nonapartitioned neutrosophic set.

Example 3.8 (Fraud Detection Scores as a Nonapartitioned Neutrosophic Offset)**.** Let $U = \{\mathrm{Txn}_1, \mathrm{Txn}_2\}$ be two financial transactions under review for potential fraud. Fix underlimit $\Psi = -0.1$ and overlimit $\Omega = 1.1$. For each transaction $x \in U$, an automated system assigns nine "offset" degrees:

$$\begin{aligned} T(x) &: \text{baseline fraud likelihood,} \\ ST(x) &: \text{strong relative likelihood,} \\ WT(x) &: \text{weak relative likelihood,} \\ C(x) &: \text{contradictory signals,} \\ U(x) &: \text{unknown/unseen patterns,} \\ I(x) &: \text{data gaps (ignorance),} \\ SF(x) &: \text{strong relative non-fraud,} \\ WF(x) &: \text{weak relative non-fraud,} \\ F(x) &: \text{baseline non-fraud likelihood.} \end{aligned}$$

These values lie in $[\Psi, \Omega]$ and satisfy $\Psi \leq \sum T + ST + WT + C + U + I + SF + WF + F \leq \Omega + 8$, with at least one component outside $[0, 1]$.

Transaction	T	ST	WT	C	U	I	SF	WF	F
Txn_1	1.05	0.10	0.05	0.07	0.03	0.04	0.06	0.02	0.08
Txn_2	0.90	0.25	0.05	0.03	0.02	0.01	0.04	−0.03	0.10

For Txn_1, the baseline fraud score $T = 1.05$ exceeds 1 (overset), reflecting an unusually strong risk signal. For Txn_2, the weak non-fraud score $WF = -0.03$ falls below 0 (underset), indicating slight negative confidence in non-fraud. In both cases

$$-0.1 \;\leq\; \sum_{d \in \{T, ST, \ldots, F\}} d(x) \;\leq\; 1.1 + 8,$$

so these assignments form a valid nonapartitioned neutrosophic offset.

Theorem 3.9
The nonapartitioned neutrosophic offset (N-NOS) strictly generalises:

1. the single-valued neutrosophic offset (SV-NOS),
2. the octapartitioned neutrosophic offset (O-NOS),
3. the classical nonapartitioned neutrosophic set (N-NS).

Proof
We work on a fixed nonempty universe U. By Definition 3.7, a nonapartitioned neutrosophic offset (N-NOS) is given by

$$N^{(9)}_{\text{off}} = \big\{ \langle x,\, T(x), ST(x), WT(x), C(x), U(x), I(x), SF(x), WF(x), F(x) \rangle \;\big|\; x \in U \big\},$$

where

$$T, ST, WT, C, U, I, SF, WF, F : U \longrightarrow [\Psi, \Omega]$$

for some real interval $[\Psi, \Omega]$ containing $[0, 1]$, and where, for every $x \in U$, the sum

$$S_9(x) := T(x) + ST(x) + WT(x) + C(x) + U(x) + I(x) + SF(x) + WF(x) + F(x)$$

satisfies

$$\Psi \;\le\; S_9(x) \;\le\; \Omega + 8. \tag{10}$$

Moreover, there exists at least one point $x_0 \in U$ such that at x_0 at least one of the nine components lies outside the unit interval $[0, 1]$; this is the offset condition.

We now treat each item of the theorem in turn and finally prove strictness.

(1) Embedding SV-NOS into N-NOS.

Let

$$N_{\mathrm{SV}} = \left\{ \langle x, T_{\mathrm{SV}}(x), I_{\mathrm{SV}}(x), F_{\mathrm{SV}}(x) \rangle \;\middle|\; x \in U \right\}$$

be a single-valued neutrosophic offset (SV-NOS) on U. Thus

$$T_{\mathrm{SV}}, I_{\mathrm{SV}}, F_{\mathrm{SV}} : U \longrightarrow [\Psi, \Omega],$$

and for each $x \in U$ the usual neutrosophic offset constraint holds:

$$0 \;\le\; T_{\mathrm{SV}}(x) + I_{\mathrm{SV}}(x) + F_{\mathrm{SV}}(x) \;\le\; \Omega + 2. \tag{11}$$

In addition, there exists some $x_0 \in U$ such that at least one of $T_{\mathrm{SV}}(x_0), I_{\mathrm{SV}}(x_0), F_{\mathrm{SV}}(x_0)$ lies outside $[0, 1]$, so that N_{SV} is a genuine offset.

We embed N_{SV} into an N-NOS by defining nine membership functions

$$T, ST, WT, C, U, I, SF, WF, F : U \to [\Psi, \Omega]$$

via

$$\begin{aligned} T(x) &:= T_{\mathrm{SV}}(x), & ST(x) &:= 0, & WT(x) &:= 0, \\ C(x) &:= I_{\mathrm{SV}}(x), & U(x) &:= 0, & I(x) &:= 0, \\ SF(x) &:= F_{\mathrm{SV}}(x), & WF(x) &:= 0, & F(x) &:= 0. \end{aligned} \tag{12}$$

Because $[0, 1] \subseteq [\Psi, \Omega]$, we have $0 \in [\Psi, \Omega]$, so all nine functions indeed map into $[\Psi, \Omega]$.

For each $x \in U$, the N-NOS sum is

$$\begin{aligned} S_9(x) &= T(x) + ST(x) + WT(x) + C(x) + U(x) + I(x) + SF(x) + WF(x) + F(x) \\ &= T_{\mathrm{SV}}(x) + 0 + 0 + I_{\mathrm{SV}}(x) + 0 + 0 + F_{\mathrm{SV}}(x) + 0 + 0 \\ &= T_{\mathrm{SV}}(x) + I_{\mathrm{SV}}(x) + F_{\mathrm{SV}}(x). \end{aligned}$$

Combining this identity with (11), we obtain

$$0 \;\le\; S_9(x) \;\le\; \Omega + 2 \;\le\; \Omega + 8,$$

so the N-NOS sum constraint (20) holds. Thus the structure defined by (12) is a valid N-NOS.

The offset condition is preserved as well: there exists some $x_0 \in U$ with, say, $I_{\mathrm{SV}}(x_0) \notin [0, 1]$. From (12) we see that $C(x_0) = I_{\mathrm{SV}}(x_0)$, so at x_0 at least one N-NOS component lies outside $[0, 1]$. Hence the resulting N-NOS is indeed an offset.

Finally, the assignment

$$\iota_{\mathrm{SV}} : (T_{\mathrm{SV}}, I_{\mathrm{SV}}, F_{\mathrm{SV}}) \longmapsto (T, ST, WT, C, U, I, SF, WF, F)$$

given by (12) is injective: if two SV-NOS structures produce the same N-NOS, then, for every $x \in U$,

$$T_{\mathrm{SV}}(x) = T(x), \quad I_{\mathrm{SV}}(x) = C(x), \quad F_{\mathrm{SV}}(x) = SF(x),$$

so the triples coincide. Thus SV-NOS embeds faithfully into N-NOS.

(2) Embedding O-NOS into N-NOS.

Let

$$N_{\mathrm{O}} = \left\{ \langle x, T_{\mathrm{O}}(x), M_{\mathrm{O}}(x), C_{\mathrm{O}}(x), U_{\mathrm{O}}(x), I_{\mathrm{O}}(x), K_{\mathrm{O}}(x), H_{\mathrm{O}}(x), F_{\mathrm{O}}(x) \rangle \;\middle|\; x \in U \right\}$$

be an octapartitioned neutrosophic offset (O-NOS) on U. Thus

$$T_{\mathrm{O}}, M_{\mathrm{O}}, C_{\mathrm{O}}, U_{\mathrm{O}}, I_{\mathrm{O}}, K_{\mathrm{O}}, H_{\mathrm{O}}, F_{\mathrm{O}} : U \to [\Psi, \Omega],$$

and, for each $x \in U$, the O-NOS sum

$$S_8(x) := T_{\mathrm{O}}(x) + M_{\mathrm{O}}(x) + C_{\mathrm{O}}(x) + U_{\mathrm{O}}(x) + I_{\mathrm{O}}(x) + K_{\mathrm{O}}(x) + H_{\mathrm{O}}(x) + F_{\mathrm{O}}(x)$$

satisfies

$$\Psi \;\le\; S_8(x) \;\le\; \Omega + 7. \tag{13}$$

Again, there exists at least one $x_0 \in U$ where at least one of the eight components lies outside $[0, 1]$.

We define an N-NOS by setting

$$\begin{aligned} T(x) &:= T_{\mathrm{O}}(x), & ST(x) &:= M_{\mathrm{O}}(x), & WT(x) &:= 0, \\ C(x) &:= C_{\mathrm{O}}(x), & U(x) &:= U_{\mathrm{O}}(x), & I(x) &:= I_{\mathrm{O}}(x), \\ SF(x) &:= K_{\mathrm{O}}(x), & WF(x) &:= H_{\mathrm{O}}(x), & F(x) &:= F_{\mathrm{O}}(x). \end{aligned} \tag{14}$$

Each of these functions takes values in $[\Psi, \Omega]$, because all O-NOS components do and $0 \in [\Psi, \Omega]$.

For each $x \in U$, the N-NOS sum becomes

$$\begin{aligned} S_9(x) &= T(x) + ST(x) + WT(x) + C(x) + U(x) + I(x) + SF(x) + WF(x) + F(x) \\ &= T_{\mathrm{O}}(x) + M_{\mathrm{O}}(x) + 0 + C_{\mathrm{O}}(x) + U_{\mathrm{O}}(x) + I_{\mathrm{O}}(x) + K_{\mathrm{O}}(x) + H_{\mathrm{O}}(x) + F_{\mathrm{O}}(x) \\ &= S_8(x). \end{aligned}$$

Combining this equality with (13) yields

$$\Psi \;\le\; S_9(x) = S_8(x) \;\le\; \Omega + 7 \;\le\; \Omega + 8,$$

so the N-NOS sum constraint (20) holds. Therefore the nine-tuple defined by (14) is a valid N-NOS.

The offset condition is again preserved: there exists some $x_0 \in U$ and some O-NOS component $\mu_{\mathrm{O}} \in \{T_{\mathrm{O}}, M_{\mathrm{O}}, C_{\mathrm{O}}, U_{\mathrm{O}}, I_{\mathrm{O}}, K_{\mathrm{O}}, H_{\mathrm{O}}, F_{\mathrm{O}}\}$ such that $\mu_{\mathrm{O}}(x_0) \notin [0, 1]$. From (14) we see that the same value appears verbatim as one of $\{T(x_0), ST(x_0), C(x_0), U(x_0), I(x_0), SF(x_0), WF(x_0), F(x_0)$, so at x_0 at least one N-NOS component lies outside $[0, 1]$.

Moreover, the map

$$\iota_{\mathrm{O}} : (T_{\mathrm{O}}, M_{\mathrm{O}}, C_{\mathrm{O}}, U_{\mathrm{O}}, I_{\mathrm{O}}, K_{\mathrm{O}}, H_{\mathrm{O}}, F_{\mathrm{O}}) \longmapsto (T, ST, WT, C, U, I, SF, WF, F)$$

given by (14) is injective: from the N-NOS we can recover the O-NOS components uniquely by

$$T_O = T, \quad M_O = ST, \quad C_O = C, \quad U_O = U, \quad I_O = I, \quad K_O = SF, \quad H_O = WF, \quad F_O = F,$$

while $WT \equiv 0$ on the image. Thus O-NOS embeds as the subclass of N-NOS characterised by the constraint $WT \equiv 0$ together with the above identifications of components.

(3) Recovering the classical N-NS.

A classical nonapartitioned neutrosophic set (N-NS) on U is given by

$$N_{\mathrm{NS}} = \left\{ \langle x, T_{\mathrm{NS}}(x), ST_{\mathrm{NS}}(x), WT_{\mathrm{NS}}(x), C_{\mathrm{NS}}(x), U_{\mathrm{NS}}(x), I_{\mathrm{NS}}(x), SF_{\mathrm{NS}}(x), WF_{\mathrm{NS}}(x), F_{\mathrm{NS}}(x) \rangle \;\middle|\; x \in U \right\},$$

where

$$T_{\mathrm{NS}}, ST_{\mathrm{NS}}, WT_{\mathrm{NS}}, C_{\mathrm{NS}}, U_{\mathrm{NS}}, I_{\mathrm{NS}}, SF_{\mathrm{NS}}, WF_{\mathrm{NS}}, F_{\mathrm{NS}} : U \to [0,1],$$

and, for all $x \in U$,

$$0 \;\le\; T_{\mathrm{NS}}(x) + ST_{\mathrm{NS}}(x) + WT_{\mathrm{NS}}(x) + C_{\mathrm{NS}}(x) + U_{\mathrm{NS}}(x) + I_{\mathrm{NS}}(x) + SF_{\mathrm{NS}}(x) + WF_{\mathrm{NS}}(x) + F_{\mathrm{NS}}(x) \;\le\; 9. \tag{15}$$

Here no offset behaviour is allowed: all components must lie in $[0,1]$.

To see N-NS as a special case of N-NOS, we choose

$$\Psi := 0, \qquad \Omega := 1.$$

Then the component-wise range condition of N-NOS,

$$T, ST, WT, C, U, I, SF, WF, F : U \to [\Psi, \Omega] = [0,1],$$

coincides exactly with that of the classical N-NS. The N-NOS sum constraint (20) specialises to

$$0 \;\le\; S_9(x) \;\le\; 1 + 8 = 9,$$

which matches (15). Thus every classical N-NS is obtained by taking an N-NOS with $\Psi = 0, \Omega = 1$ and forbidding offset values (i.e., requiring all components to remain in $[0,1]$).

(4) Strictness of the generalisation.

To prove that the generalisation is *strict*, we construct an explicit N-NOS that does not lie in the embedded image of any of the three classes SV-NOS, O-NOS, or N-NS.

Let the universe be the singleton $U = \{x_0\}$ and choose bounds

$$\Psi := 0, \qquad \Omega := 1.2.$$

Clearly $[0,1] \subseteq [\Psi, \Omega]$. Define nine membership degrees at x_0 by

$$\begin{array}{rrr} T(x_0) := 1.1, & ST(x_0) := 0.1, & WT(x_0) := 0.1, \\ C(x_0) := 0, & U(x_0) := 0, & I(x_0) := 0, \\ SF(x_0) := 0, & WF(x_0) := 0, & F(x_0) := 0. \end{array} \tag{16}$$

First we check that this defines a valid N-NOS.

Component-wise bounds. All nine values in (16) lie in $[0, 1.2]$, so the range condition is satisfied.

Sum constraint. The sum at x_0 is

$$S_9(x_0) = 1.1 + 0.1 + 0.1 + 0 + 0 + 0 + 0 + 0 + 0 = 1.3.$$

We clearly have

$$0 = \Psi \ \leq \ 1.3 \ \leq \ 1.2 + 8 = 9.2 = \Omega + 8,$$

so the inequality (20) holds.

Offset behaviour. We have

$$T(x_0) = 1.1 > 1,$$

so at x_0 at least one membership degree lies outside $[0, 1]$. Thus the structure (16) is an N-NOS offset.

We now verify that this N-NOS does not lie in any of the three embedded subclasses.

(a) *Not in the image of SV-NOS.* By (12), every N-NOS arising from an SV-NOS has

$$ST(x) \equiv 0, \quad WT(x) \equiv 0, \quad U(x) \equiv 0, \quad I(x) \equiv 0, \quad WF(x) \equiv 0, \quad F(x) \equiv 0.$$

In our example (16) we have

$$ST(x_0) = 0.1 \neq 0, \qquad WT(x_0) = 0.1 \neq 0,$$

so this N-NOS cannot be of the form (12). Hence it is not in the embedded image of any SV-NOS.

(b) *Not in the image of O-NOS.* By (14), every N-NOS arising from an O-NOS satisfies

$$WT(x) \equiv 0$$

on the entire universe. In the example (16), however,

$$WT(x_0) = 0.1 \neq 0,$$

so this N-NOS cannot be obtained from any O-NOS via (14).

(c) *Not a classical N-NS.* In a classical N-NS, all nine membership degrees must lie in the unit interval $[0, 1]$. Our N-NOS has

$$T(x_0) = 1.1 > 1,$$

so it cannot be an N-NS.

We have therefore exhibited a concrete N-NOS that does not belong to the embedded image of SV-NOS, does not belong to the embedded image of O-NOS, and is not a classical N-NS. Together with the embeddings constructed in parts (1)–(3), this shows that the class of nonapartitioned neutrosophic offsets properly contains each of SV-NOS, O-NOS, and N-NS. Hence N-NOS *strictly* generalises all three structures. □

3.4. Decapartitioned Neutrosophic Offset

We now introduce the most expressive offset model to date, featuring ten independent membership degrees.

Definition 3.10 (Decapartitioned Neutrosophic OffSet (D-NOS))**.** Let U be a nonempty universe and fix real constants

$$\Psi < 0 < 1 < \Omega$$

(called the *UnderLimit* and *OverLimit*). A *decapartitioned neutrosophic offset* is a map

$$\mathbf{n} : U \longrightarrow [\Psi, \Omega]^{10},$$

$$x \longmapsto \big(T(x),\, \mathrm{SRT}(x),\, \mathrm{WRT}(x),\, C(x),\, U(x),\, I(x),\, H(x),\, \mathrm{SRF}(x),\, \mathrm{WRF}(x),\, F(x)\big),$$

where the ten functions

$$T,\ \mathrm{SRT},\ \mathrm{WRT},\ C,\ U,\ I,\ H,\ \mathrm{SRF},\ \mathrm{WRF},\ F : U \longrightarrow [\Psi, \Omega]$$

are called respectively: truth, strongly relative truth, weakly relative truth, contradiction, unknown, ignorance, hesitation, strongly relative falsity, weakly relative falsity, and falsity. They satisfy the normalization

$$\Psi \ \le\ T(x) + \mathrm{SRT}(x) + \mathrm{WRT}(x) + C(x) + U(x)$$

$$+ I(x) + H(x) + \mathrm{SRF}(x) + \mathrm{WRF}(x) + F(x) \ \le\ \Omega + 9, \quad \forall\, x \in U,$$

and there must exist at least one $x \in U$ and one component μ among these ten such that $\mu(x) \notin [0, 1]$, ensuring genuine offset behavior. If all values lie in $[0, 1]$, the model reduces to the classical decapartitioned neutrosophic set.

Example 3.11 (Bridge Health Monitoring as a Decapartitioned Neutrosophic Offset)**.** Consider two bridges, $U = \{\mathrm{Bridge}_A, \mathrm{Bridge}_B\}$, whose structural integrity is assessed using ten neutrosophic offset degrees. Fix underlimit $\Psi = -0.2$ and overlimit $\Omega = 1.2$. For each bridge x, engineers assign:

$$\begin{aligned}
T(x) &: \text{baseline integrity (truth)},\\
\mathrm{SRT}(x) &: \text{strongly relative integrity},\\
\mathrm{WRT}(x) &: \text{weakly relative integrity},\\
C(x) &: \text{conflicting sensor readings},\\
U(x) &: \text{unknown environmental factors},\\
I(x) &: \text{data gaps (ignorance)},\\
H(x) &: \text{inspection hesitation},\\
\mathrm{SRF}(x) &: \text{strongly relative failure},\\
\mathrm{WRF}(x) &: \text{weakly relative failure},\\
F(x) &: \text{baseline failure (falsity)}.
\end{aligned}$$

Each degree lies in $[\Psi, \Omega]$, and for every x, $\Psi \le T + \mathrm{SRT} + \mathrm{WRT} + C + U + I + H + \mathrm{SRF} + \mathrm{WRF} + F \le \Omega + 9$, with at least one component outside $[0, 1]$. A possible evaluation is:

Bridge	T	SRT	WRT	C	U	I	H	SRF	WRF	F
Bridge_A	1.10	0.08	0.05	0.06	0.04	0.03	0.02	0.07	0.01	0.04
Bridge_B	0.85	0.20	0.05	0.10	0.03	0.02	0.01	0.09	−0.05	0.15

For Bridge_A, the truth degree $T = 1.10$ exceeds 1 (overset), indicating exceptionally high confidence in stability. For Bridge_B, the weak failure degree $\mathrm{WRF} = -0.05$ is below 0 (underset), reflecting slight negative confidence in failure predictions. Both satisfy

$$-0.2 \ \le\ T + \mathrm{SRT} + \mathrm{WRT} + C + U + I + H + \mathrm{SRF} + \mathrm{WRF} + F \ \le\ 1.2 + 9,$$

so this table provides a valid decapartitioned neutrosophic offset.

Theorem 3.12 (Unifying Power of D-NOS)
The decapartitioned neutrosophic offset (D-NOS) strictly generalises:

1. the single-valued neutrosophic offset (SV-NOS),
2. the nonapartitioned neutrosophic offset (N-NOS),
3. the classical decapartitioned neutrosophic set (D-NS).

Proof
We work on a fixed nonempty universe U. By Definition 3.10, a decapartitioned neutrosophic offset $\mathrm{D}-\mathrm{NOS}$ on U is given by

$$N_{\text{off}}^{(10)} = \big\{ \langle x,\, T(x), \mathrm{SRT}(x), \mathrm{WRT}(x), C(x), U(x), I(x), H(x), \mathrm{SRF}(x), \mathrm{WRF}(x), F(x)\rangle \ \big|\ x \in U \big\},$$

where

$$T, \mathrm{SRT}, \mathrm{WRT}, C, U, I, H, \mathrm{SRF}, \mathrm{WRF}, F : U \longrightarrow [\Psi, \Omega]$$

for some real interval $[\Psi, \Omega]$ with $[0, 1] \subseteq [\Psi, \Omega]$, and where, for every $x \in U$, the sum

$$S_{10}(x) := T(x) + \mathrm{SRT}(x) + \mathrm{WRT}(x) + C(x) + U(x) + I(x) + H(x) + \mathrm{SRF}(x) + \mathrm{WRF}(x) + F(x)$$

satisfies

$$\Psi \ \le\ S_{10}(x) \ \le\ \Omega + 9. \tag{17}$$

Moreover, there exists at least one point $x_0 \in U$ such that at x_0 at least one of the ten components lies outside the unit interval $[0, 1]$; this is the offset condition.

We prove each item in turn and then show strictness.

(1) Embedding SV-NOS into D-NOS.

Let

$$N_{\mathrm{SV}} = \big\{ \langle x,\, T_{\mathrm{SV}}(x), I_{\mathrm{SV}}(x), F_{\mathrm{SV}}(x)\rangle \ \big|\ x \in U \big\}$$

be a single-valued neutrosophic offset (SV-NOS). Thus

$$T_{\mathrm{SV}}, I_{\mathrm{SV}}, F_{\mathrm{SV}} : U \longrightarrow [\Psi, \Omega],$$

and, for each $x \in U$, the usual SV-NOS sum satisfies

$$0 \ \le\ T_{\mathrm{SV}}(x) + I_{\mathrm{SV}}(x) + F_{\mathrm{SV}}(x) \ \le\ \Omega + 2. \tag{18}$$

In addition, there exists $x_0 \in U$ such that at least one of $T_{\mathrm{SV}}(x_0), I_{\mathrm{SV}}(x_0), F_{\mathrm{SV}}(x_0)$ lies outside $[0, 1]$, so N_{SV} is a genuine offset.

We embed N_{SV} into D-NOS by defining

$$T, \mathrm{SRT}, \mathrm{WRT}, C, U, I, H, \mathrm{SRF}, \mathrm{WRF}, F : U \to [\Psi, \Omega]$$

via

$$\begin{array}{lll} T(x) := T_{\mathrm{SV}}(x), & \mathrm{SRT}(x) := 0, & \mathrm{WRT}(x) := 0, \\ C(x) := I_{\mathrm{SV}}(x), & U(x) := 0, & I(x) := 0, \\ H(x) := 0, & \mathrm{SRF}(x) := F_{\mathrm{SV}}(x), & \mathrm{WRF}(x) := 0, \\ F(x) := 0. & & \end{array} \tag{19}$$

Since $[0,1] \subseteq [\Psi, \Omega]$, we have $0 \in [\Psi, \Omega]$, so all ten functions indeed map into $[\Psi, \Omega]$.

For each $x \in U$, the D-NOS sum is

$$\begin{aligned} S_{10}(x) &= T(x) + \mathrm{SRT}(x) + \mathrm{WRT}(x) + C(x) + U(x) + I(x) + H(x) + \mathrm{SRF}(x) + \mathrm{WRF}(x) + F(x) \\ &= T_{\mathrm{SV}}(x) + 0 + 0 + I_{\mathrm{SV}}(x) + 0 + 0 + 0 + F_{\mathrm{SV}}(x) + 0 + 0 \\ &= T_{\mathrm{SV}}(x) + I_{\mathrm{SV}}(x) + F_{\mathrm{SV}}(x). \end{aligned}$$

Combining this with (18), we obtain

$$0 \;\le\; S_{10}(x) \;\le\; \Omega + 2 \;\le\; \Omega + 9,$$

so the D-NOS sum constraint (17) holds.

The offset condition is preserved: there exists $x_0 \in U$ such that (at least) one of $T_{\mathrm{SV}}(x_0), I_{\mathrm{SV}}(x_0), F_{\mathrm{SV}}(x_0)$ lies outside $[0,1]$. By (19) this same value appears as $T(x_0)$, $C(x_0)$, or $\mathrm{SRF}(x_0)$, so in particular at x_0 some D-NOS component lies outside $[0,1]$, and the constructed D-NOS is an offset.

Finally, the embedding

$$\iota_{\mathrm{SV}} : (T_{\mathrm{SV}}, I_{\mathrm{SV}}, F_{\mathrm{SV}}) \longmapsto (T, \mathrm{SRT}, \mathrm{WRT}, C, U, I, H, \mathrm{SRF}, \mathrm{WRF}, F)$$

is injective: if two SV-NOS structures produce the same D-NOS under (19), then for every $x \in U$ we have

$$T_{\mathrm{SV}}(x) = T(x), \quad I_{\mathrm{SV}}(x) = C(x), \quad F_{\mathrm{SV}}(x) = \mathrm{SRF}(x),$$

so the original triples coincide. Thus SV-NOS embeds faithfully into D-NOS.

(2) Embedding N-NOS into D-NOS.

By Definition 3.7, a nonapartitioned neutrosophic offset (N-NOS) on U is given by

$$N_{\mathrm{N}} = \left\{ \langle x,\, T_{\mathrm{N}}(x), ST_{\mathrm{N}}(x), WT_{\mathrm{N}}(x), C_{\mathrm{N}}(x), U_{\mathrm{N}}(x), I_{\mathrm{N}}(x), SF_{\mathrm{N}}(x), WF_{\mathrm{N}}(x), F_{\mathrm{N}}(x) \rangle \;\middle|\; x \in U \right\},$$

where

$$T_{\mathrm{N}}, ST_{\mathrm{N}}, WT_{\mathrm{N}}, C_{\mathrm{N}}, U_{\mathrm{N}}, I_{\mathrm{N}}, SF_{\mathrm{N}}, WF_{\mathrm{N}}, F_{\mathrm{N}} : U \to [\Psi, \Omega],$$

and, for each $x \in U$, the N-NOS sum

$$S_9(x) := T_{\mathrm{N}}(x) + ST_{\mathrm{N}}(x) + WT_{\mathrm{N}}(x) + C_{\mathrm{N}}(x) + U_{\mathrm{N}}(x) + I_{\mathrm{N}}(x) + SF_{\mathrm{N}}(x) + WF_{\mathrm{N}}(x) + F_{\mathrm{N}}(x)$$

satisfies

$$\Psi \;\le\; S_9(x) \;\le\; \Omega + 8. \tag{20}$$

Again, an offset condition requires that at some $x_0 \in U$ at least one of these nine components lies outside $[0,1]$.

We embed N_{N} into D-NOS by defining

$$T, \mathrm{SRT}, \mathrm{WRT}, C, U, I, H, \mathrm{SRF}, \mathrm{WRF}, F : U \to [\Psi, \Omega]$$

via

$$\begin{aligned} &T(x) := T_{\mathrm{N}}(x), \quad && \mathrm{SRT}(x) := ST_{\mathrm{N}}(x), \quad && \mathrm{WRT}(x) := WT_{\mathrm{N}}(x), \\ &C(x) := C_{\mathrm{N}}(x), && U(x) := U_{\mathrm{N}}(x), && I(x) := I_{\mathrm{N}}(x), \\ &H(x) := 0, && \mathrm{SRF}(x) := SF_{\mathrm{N}}(x), && \mathrm{WRF}(x) := WF_{\mathrm{N}}(x), \\ &F(x) := F_{\mathrm{N}}(x). \end{aligned} \tag{21}$$

All ten functions take values in $[\Psi, \Omega]$, since the nine N-NOS components do and $0 \in [\Psi, \Omega]$.

For each $x \in U$, the D-NOS sum becomes

$$\begin{aligned} S_{10}(x) &= T(x) + \mathrm{SRT}(x) + \mathrm{WRT}(x) + C(x) + U(x) + I(x) + H(x) + \mathrm{SRF}(x) + \mathrm{WRF}(x) + F(x) \\ &= T_{\mathrm{N}}(x) + ST_{\mathrm{N}}(x) + WT_{\mathrm{N}}(x) + C_{\mathrm{N}}(x) + U_{\mathrm{N}}(x) + I_{\mathrm{N}}(x) + 0 + SF_{\mathrm{N}}(x) + WF_{\mathrm{N}}(x) + F_{\mathrm{N}}(x) \\ &= S_9(x). \end{aligned}$$

Thus, combining with (20), we obtain

$$\Psi \ \le\ S_{10}(x) = S_9(x) \ \le\ \Omega + 8 \ \le\ \Omega + 9,$$

so the D-NOS sum constraint (17) holds.

The offset property is also preserved: if there exists $x_0 \in U$ and some N-NOS component $\mu_{\mathrm{N}} \in \{T_{\mathrm{N}}, ST_{\mathrm{N}}, WT_{\mathrm{N}}, C_{\mathrm{N}}, U_{\mathrm{N}}, I_{\mathrm{N}}, SF_{\mathrm{N}}, WF_{\mathrm{N}}, F_{\mathrm{N}}\}$ such that $\mu_{\mathrm{N}}(x_0) \notin [0,1]$, then by (21) the same value appears as one of $T(x_0), \mathrm{SRT}(x_0), \mathrm{WRT}(x_0), C(x_0), U(x_0), I(x_0), \mathrm{SRF}(x_0), \mathrm{WRF}(x_0), F(x_0)$, so the constructed D-NOS is also an offset.

The embedding

$$\iota_{\mathrm{N}} : (T_{\mathrm{N}}, ST_{\mathrm{N}}, WT_{\mathrm{N}}, C_{\mathrm{N}}, U_{\mathrm{N}}, I_{\mathrm{N}}, SF_{\mathrm{N}}, WF_{\mathrm{N}}, F_{\mathrm{N}}) \longmapsto (T, \mathrm{SRT}, \mathrm{WRT}, C, U, I, H, \mathrm{SRF}, \mathrm{WRF}, F)$$

given by (21) is injective: from the D-NOS we recover the N-NOS components uniquely by

$$T_{\mathrm{N}} = T, \quad ST_{\mathrm{N}} = \mathrm{SRT}, \quad WT_{\mathrm{N}} = \mathrm{WRT}, \quad C_{\mathrm{N}} = C, \quad U_{\mathrm{N}} = U, \quad I_{\mathrm{N}} = I, \quad SF_{\mathrm{N}} = \mathrm{SRF}, \quad WF_{\mathrm{N}} = \mathrm{WRF}, \quad F_{\mathrm{N}} = F,$$

with the additional constraint $H \equiv 0$ on the image. Hence N-NOS is (isomorphic to) the subclass of D-NOS characterised by $H \equiv 0$.

(3) Recovering the classical D-NS.

A classical decapartitioned neutrosophic set (D-NS) on U has the same ten-component structure

$$\langle x,\ T_{\mathrm{D}}(x), \mathrm{SRT}_{\mathrm{D}}(x), \mathrm{WRT}_{\mathrm{D}}(x), C_{\mathrm{D}}(x), U_{\mathrm{D}}(x), I_{\mathrm{D}}(x), H_{\mathrm{D}}(x), \mathrm{SRF}_{\mathrm{D}}(x), \mathrm{WRF}_{\mathrm{D}}(x), F_{\mathrm{D}}(x) \rangle$$

but with all degrees constrained to the unit interval:

$$T_{\mathrm{D}}, \mathrm{SRT}_{\mathrm{D}}, \mathrm{WRT}_{\mathrm{D}}, C_{\mathrm{D}}, U_{\mathrm{D}}, I_{\mathrm{D}}, H_{\mathrm{D}}, \mathrm{SRF}_{\mathrm{D}}, \mathrm{WRF}_{\mathrm{D}}, F_{\mathrm{D}} : U \to [0,1],$$

and, for each $x \in U$, the sum satisfies

$$0 \ \le\ T_{\mathrm{D}}(x) + \mathrm{SRT}_{\mathrm{D}}(x) + \mathrm{WRT}_{\mathrm{D}}(x) + C_{\mathrm{D}}(x) + U_{\mathrm{D}}(x) + I_{\mathrm{D}}(x) + H_{\mathrm{D}}(x) + \mathrm{SRF}_{\mathrm{D}}(x) + \mathrm{WRF}_{\mathrm{D}}(x) + F_{\mathrm{D}}(x) \ \le\ 10. \tag{22}$$

No offset behaviour is allowed: all membership degrees must lie in $[0, 1]$.

We obtain D-NS as a special case of D-NOS by choosing

$$\Psi := 0, \qquad \Omega := 1,$$

and forbidding offset values. Then the D-NOS component-wise condition

$$T, \mathrm{SRT}, \mathrm{WRT}, C, U, I, H, \mathrm{SRF}, \mathrm{WRF}, F : U \to [\Psi, \Omega] = [0, 1]$$

coincides exactly with the D-NS range requirement. The D-NOS sum constraint (17) specialises to

$$0 \le S_{10}(x) \le 1 + 9 = 10,$$

which is identical to (22). Thus every classical D-NS is obtained from a D-NOS by restricting to $(\Psi, \Omega) = (0, 1)$ and disallowing offsets. Conversely, any such D-NOS is a D-NS by definition. Hence D-NS is precisely the $(\Psi, \Omega) = (0, 1)$ slice of the D-NOS family without offset values.

(4) Strictness of the generalisation.

It remains to show that the inclusions are strict, i.e., that there exists a D-NOS which is not:

- in the embedded image of any SV-NOS,
- in the embedded image of any N-NOS,
- a classical D-NS.

Let the universe be the singleton $U = \{x_0\}$ and choose

$$\Psi := 0, \qquad \Omega := 1.2.$$

Clearly $[0, 1] \subseteq [0, 1.2] = [\Psi, \Omega]$. Define ten membership degrees at x_0 by

$$\begin{aligned} &T(x_0) := 1.1, && \mathrm{SRT}(x_0) := 0, && \mathrm{WRT}(x_0) := 0, \\ &C(x_0) := 0, && U(x_0) := 0, && I(x_0) := 0, \\ &H(x_0) := 0.1, && \mathrm{SRF}(x_0) := 0, && \mathrm{WRF}(x_0) := 0, \\ &F(x_0) := 0. \end{aligned} \tag{23}$$

(a) This is a valid D-NOS. All ten values in (23) lie in $[0, 1.2]$, so the component-wise range condition is satisfied. The sum at x_0 is

$$\begin{aligned} S_{10}(x_0) &= T(x_0) + \mathrm{SRT}(x_0) + \mathrm{WRT}(x_0) + C(x_0) + U(x_0) + I(x_0) + H(x_0) + \mathrm{SRF}(x_0) + \mathrm{WRF}(x_0) + F(x_0) \\ &= 1.1 + 0 + 0 + 0 + 0 + 0 + 0.1 + 0 + 0 + 0 \\ &= 1.2. \end{aligned}$$

We have

$$0 = \Psi \le S_{10}(x_0) = 1.2 \le 1.2 + 9 = 10.2 = \Omega + 9,$$

so the sum constraint (17) holds. Moreover,

$$T(x_0) = 1.1 > 1,$$

so at x_0 at least one component lies outside $[0,1]$; hence (23) defines a genuine D-NOS offset.

(b) This D-NOS is not in the image of SV-NOS. By (19), any D-NOS arising from an SV-NOS must satisfy

$$\mathrm{SRT}(x) \equiv 0, \quad \mathrm{WRT}(x) \equiv 0, \quad U(x) \equiv 0, \quad I(x) \equiv 0, \quad H(x) \equiv 0, \quad \mathrm{WRF}(x) \equiv 0, \quad F(x) \equiv 0.$$

In the example (23), however,

$$H(x_0) = 0.1 \neq 0.$$

Therefore this D-NOS cannot be obtained from any SV-NOS via (19), and hence does not belong to the embedded SV-NOS subclass.

(c) This D-NOS is not in the image of N-NOS. By (21), any D-NOS arising from an N-NOS satisfies

$$H(x) \equiv 0$$

on the whole universe. Our example (23) has

$$H(x_0) = 0.1 \neq 0,$$

so it cannot be in the image of ι_{N}. Thus this D-NOS does not belong to the embedded N-NOS subclass.

(d) This D-NOS is not a classical D-NS. In a classical D-NS, all ten membership degrees must lie in $[0,1]$. In (23) we have

$$T(x_0) = 1.1 > 1,$$

so the example violates the D-NS range condition and hence is not a decapartitioned neutrosophic set.

We have constructed an explicit D-NOS (with parameters $\Psi = 0, \Omega = 1.2$) that:

- is not in the embedded image of any SV-NOS,
- is not in the embedded image of any N-NOS,
- is not a classical D-NS.

Together with the injective embeddings established in parts (1) and (2) and the identification in part (3), this shows that:

$$\mathrm{SV\!-\!NOS} \subsetneq \mathrm{D\!-\!NOS}, \qquad \mathrm{N\!-\!NOS} \subsetneq \mathrm{D\!-\!NOS}, \qquad \mathrm{D\!-\!NS} \subsetneq \mathrm{D\!-\!NOS},$$

i.e., the decapartitioned neutrosophic offset strictly generalises all three structures. □

4. Additional Result: Representing Partitioned Neutrosophic OffSets as Plithogenic OffSets

A *plithogenic set* enriches each element with multiple attribute-based membership degrees alongside a measure of contradiction, offering a highly flexible framework for uncertainty modeling [63].

Definition 4.1 (Plithogenic Set)**.** [21] Let S be a universe and $P \subseteq S$ a subset. A *plithogenic set* is the quintuple

$$PS = \big(P,\, v,\, P_v,\, \mathrm{pdf},\, \mathrm{pCF}\big),$$

where:

- v is an attribute;
- P_v is the set of all possible values of v;
- $\mathrm{pdf} : P \times P_v \to [0,1]^s$ is the *degree of appurtenance function*;
- $\mathrm{pCF} : P_v \times P_v \to [0,1]^t$ is the *degree of contradiction function*.

These satisfy, for all $a, b \in P_v$:

$$\mathrm{pCF}(a,a) = 0, \qquad \text{(Reflexivity)}$$
$$\mathrm{pCF}(a,b) = \mathrm{pCF}(b,a). \qquad \text{(Symmetry)}$$

Example 4.2 (House-Hunting with a Plithogenic Set)**.** Suppose S is the set of all houses in a city and $P = \{\mathrm{H}_1, \mathrm{H}_2\} \subseteq S$ are two finalists. We take the attribute

$$v = \text{“amenity category”}, \quad P_v = \{\,\text{Safety}, \text{Walkability}, \text{Affordability}, \text{SchoolQuality}\}.$$

Define the *degree of appurtenance function* $\mathrm{pdf} : P \times P_v \to [0,1]$ by the membership table:

House	Safety	Walkability	Affordability	SchoolQuality
H_1	0.90	0.70	0.60	0.80
H_2	0.40	0.90	0.80	0.50

Next, the *degree of contradiction function* $\mathrm{pCF} : P_v \times P_v \to [0,1]$ encodes pairwise conflict between amenities:

	Safety	Walk.	Afford.	SchoolQ.
Safety	0	0.20	0.30	0.15
Walkability	0.20	0	0.50	0.25
Affordability	0.30	0.50	0	0.40
SchoolQuality	0.15	0.25	0.40	0

Note that $\mathrm{pCF}(a,a) = 0$ and $\mathrm{pCF}(a,b) = \mathrm{pCF}(b,a)$.

Hence

$$PS_{\text{off}} = \big(P,\ v,\ P_v,\ \mathrm{pdf},\ \mathrm{pCF}\big)$$

is a valid plithogenic set: each house in P has a vector of membership degrees across four amenity categories, and the symmetric contradiction matrix quantifies how strongly two categories conflict.

Remarkably, every partitioned neutrosophic offset (hexapartitioned, octapartitioned, nonapartitioned, decapartitioned) can be cast as a plithogenic offset. Below we extend this construction to oversets, undersets, and offsets [64].

Definition 4.3 (Plithogenic Offset)**.** [65] Let S be a universal set and $P \subseteq S$. A *plithogenic offset* is a quintuple

$$PS_{\text{off}} = \big(P,\ v,\ P_v,\ \mathrm{pdf},\ \mathrm{pCF}\big),$$

where

- v is an attribute and P_v its set of possible values;
- $\mathrm{pdf}\colon P \times P_v \to [\Psi_v,\ \Omega_v]^s$ is the *degree of appurtenance function*, with real bounds $\Psi_v < 0 < 1 < \Omega_v$, so that membership degrees may fall below 0 (*under-membership*) or exceed 1 (*over-membership*);
- $\mathrm{pCF}\colon P_v \times P_v \to [\Psi_v,\ \Omega_v]^t$ is the *degree of contradiction function*, satisfying $\mathrm{pCF}(a,a) = 0$ and $\mathrm{pCF}(a,b) = \mathrm{pCF}(b,a)$.

When $\Psi_v = 0$ the model reduces to a *plithogenic overset*; when $\Omega_v = 1$ it becomes a *plithogenic underset*; and when $[\Psi_v, \Omega_v] = [0,1]$ one recovers the ordinary plithogenic set.

Example 4.4 (Software Module Evaluation as a Plithogenic Offset)**.** Let S be the universe of all software modules and

$$P = \{\mathrm{M}_1,\ \mathrm{M}_2\} \subseteq S$$

the two candidate modules. We choose the attribute

$$v = \text{"quality criterion"}, \quad P_v = \{\text{Performance},\ \text{Security},\ \text{Maintainability},\ \text{Usability},\ \text{Scalability}\}.$$

We fix under- and over-limits $\Psi_v = -0.1$ and $\Omega_v = 1.2$.

Degree of Appurtenance Function pdf**.** Each module $x \in P$ is assigned a vector of five membership degrees in $[\Psi_v, \Omega_v]$:

Module	Performance	Security	Maintainability	Usability	Scalability
M_1	1.20	0.80	0.50	−0.05	0.90
M_2	0.90	1.15	0.70	0.60	0.00

Here $1.20 > 1$ for *Performance* of M_1 (overset) and $-0.05 < 0$ for *Usability* of M_1 (underset).

Degree of Contradiction Function pCF**.** We encode pairwise conflicts between criteria by a symmetric matrix in $[\Psi_v, \Omega_v]$:

	Perf.	Sec.	Maint.	Usab.	Scal.
Perf.	0	0.30	0.20	0.10	0.15
Sec.	0.30	0	0.25	0.05	0.10
Maint.	0.20	0.25	0	0.15	0.05
Usab.	0.10	0.05	0.15	0	0.20
Scal.	0.15	0.10	0.05	0.20	0

Note that $\mathrm{pCF}(a,a) = 0$ and $\mathrm{pCF}(a,b) = \mathrm{pCF}(b,a)$.

Thus the quintuple

$$PS_{\text{off}} = \big(P,\ v,\ P_v,\ \mathrm{pdf},\ \mathrm{pCF}\big)$$

is a valid *plithogenic offset*, integrating five-dimensional overset/underset appurtenance with a symmetric conflict measure.

We now show that the plithogenic offset subsumes all of the partitioned neutrosophic offset models introduced above.

Theorem 4.5
Let $PS_{\text{off}} = (P, v, P_v, \text{pdf}, \text{pCF})$ be any plithogenic offset with bounds $\Psi_v < 0 < 1 < \Omega_v$. Then by choosing the attribute-value set P_v appropriately and defining pdf to enumerate the component degrees, one recovers exactly:

1. the *hexapartitioned neutrosophic offset* (H-NOS) when $|P_v| = 6$ and

$$\text{pdf}(x, a) = \begin{cases} T(x), & a = T, \\ C(x), & a = C, \\ G(x), & a = G, \\ U(x), & a = U, \\ H(x), & a = H, \\ F(x), & a = F, \end{cases}$$

2. the *octapartitioned neutrosophic offset* (O-NOS) when $|P_v| = 8$ and

$$\text{pdf}(x, a) = \begin{cases} T(x), & a = T, \\ M(x), & a = M, \\ C(x), & a = C, \\ U(x), & a = U, \\ I(x), & a = I, \\ K(x), & a = K, \\ H(x), & a = H, \\ F(x), & a = F, \end{cases}$$

3. the *nonapartitioned neutrosophic offset* (N-NOS) when $|P_v| = 9$ and

$$\text{pdf}(x, a) = \begin{cases} T(x), & a = T, \\ ST(x), & a = ST, \\ WT(x), & a = WT, \\ C(x), & a = C, \\ U(x), & a = U, \\ I(x), & a = I, \\ SF(x), & a = SF, \\ WF(x), & a = WF, \\ F(x), & a = F, \end{cases}$$

4. the *decapartitioned neutrosophic offset* (D-NOS) when $|P_v| = 10$ and

$$\mathrm{pdf}(x,a) = \begin{cases} T(x), & a = T, \\ \mathrm{SRT}(x), & a = \mathrm{SRT}, \\ \mathrm{WRT}(x), & a = \mathrm{WRT}, \\ C(x), & a = C, \\ U(x), & a = U, \\ I(x), & a = I, \\ H(x), & a = H, \\ \mathrm{SRF}(x), & a = \mathrm{SRF}, \\ \mathrm{WRF}(x), & a = \mathrm{WRF}, \\ F(x), & a = F. \end{cases}$$

Moreover, in each case the contradiction function pCF may be taken identically zero or defined to reflect any desired inter-component conflicts. Hence every partitioned neutrosophic offset is a special case of the plithogenic offset.

Proof
Fix one of the four offset types and its carrier set U. Let the plithogenic attribute v range over exactly the named components of that offset (six, eight, nine, or ten values). Define

$$P_v = \{\text{component labels}\}, \quad \mathrm{pdf}(x,a) = \text{the membership degree of component } a \text{ at } x.$$

Since each component degree lies in $[\Psi_v, \Omega_v]$ and their sum satisfies the corresponding normalization bound (offset plus partition size minus one), the plithogenic degree function reproduces exactly the offset constraints. The symmetry and reflexivity axioms for pCF may be met by setting $\mathrm{pCF}(a,b) = 0$ for all a, b, or by importing any nontrivial contradiction structure without altering the underlying offset semantics.

Thus, by this straightforward identification of P_v and pdf, each hexapartitioned, octapartitioned, nonapartitioned, or decapartitioned neutrosophic offset is realized as an instance of a plithogenic offset. □

5. Algorithms of Decapartitioned Neutrosophic Offset

In this section we describe a simple decision–score algorithm for a Decapartitioned Neutrosophic OffSet (D–NOS) and analyse its correctness and complexity. We assume that for each element $x \in U$ we are given the ten degrees

$$T(x),\ \mathrm{SRT}(x),\ \mathrm{WRT}(x),\ C(x),\ U(x),\ I(x),\ H(x),\ \mathrm{SRF}(x),\ \mathrm{WRF}(x),\ F(x) \in [\Psi, \Omega],$$

with real bounds $\Psi < 0 < 1 < \Omega$ as in the definition of D–NOS.

Algorithm 1: Decision score for a Decapartitioned Neutrosophic OffSet

Input : Finite universe U; bounds $\Psi < 0 < 1 < \Omega$;
D–NOS profile $\mathbf{n} : U \to [\Psi, \Omega]^{10}$ with components
$T, \mathrm{SRT}, \mathrm{WRT}, C, U, I, H, \mathrm{SRF}, \mathrm{WRF}, F$;
nonnegative weights $w_T, w_{\mathrm{SRT}}, w_{\mathrm{WRT}} \geq 0$,
$w_{\mathrm{SRF}}, w_{\mathrm{WRF}}, w_F \geq 0$, and $\alpha_C, \alpha_U, \alpha_I, \alpha_H \geq 0$.
Output: Score function $S : U \to \mathbb{R}$.
foreach $x \in U$ **do**
  `// Normalise all ten components to the unit interval`
  foreach $\mu \in \{T, \mathrm{SRT}, \mathrm{WRT}, C, U, I, H, \mathrm{SRF}, \mathrm{WRF}, F\}$ **do**
    $\mu^*(x) \leftarrow \dfrac{\mu(x) - \Psi}{\Omega - \Psi}$
  `// Aggregate positive, negative, and penalty parts`
  $P(x) \leftarrow w_T\, T^*(x) + w_{\mathrm{SRT}}\, \mathrm{SRT}^*(x) + w_{\mathrm{WRT}}\, \mathrm{WRT}^*(x)$
  $N(x) \leftarrow w_{\mathrm{SRF}}\, \mathrm{SRF}^*(x) + w_{\mathrm{WRF}}\, \mathrm{WRF}^*(x) + w_F\, F^*(x)$
  $Q(x) \leftarrow \alpha_C\, C^*(x) + \alpha_U\, U^*(x) + \alpha_I\, I^*(x) + \alpha_H\, H^*(x)$
  `// Final decision score`
  $S(x) \leftarrow P(x) - N(x) - Q(x)$
return S

The normalisation step maps each offset degree in $[\Psi, \Omega]$ to a value in $[0, 1]$ by the affine transformation

$$\mu^*(x) \;=\; \frac{\mu(x) - \Psi}{\Omega - \Psi}.$$

Since the D–NOS axioms guarantee $\Psi \leq \mu(x) \leq \Omega$, we obtain $0 \leq \mu^*(x) \leq 1$ for every component μ and every element $x \in U$.

Theorem 5.1 (Order preservation of the D–NOS score)
Let $S : U \to \mathbb{R}$ be the score produced by Algorithm 1. Assume all weights $w_T, w_{\mathrm{SRT}}, w_{\mathrm{WRT}}, w_{\mathrm{SRF}}, w_{\mathrm{WRF}}, w_F, \alpha_C, \alpha_U, \alpha_I, \alpha_H$ are nonnegative.

Let $x, y \in U$. Suppose that the normalised components satisfy

$$\begin{aligned}
&\text{(Positive part)} && T^*(x) \geq T^*(y), \quad \mathrm{SRT}^*(x) \geq \mathrm{SRT}^*(y), \quad \mathrm{WRT}^*(x) \geq \mathrm{WRT}^*(y),\\
&\text{(Negative part)} && \mathrm{SRF}^*(x) \leq \mathrm{SRF}^*(y), \quad \mathrm{WRF}^*(x) \leq \mathrm{WRF}^*(y), \quad F^*(x) \leq F^*(y),\\
&\text{(Penalty part)} && C^*(x) \leq C^*(y), \quad U^*(x) \leq U^*(y), \quad I^*(x) \leq I^*(y), \quad H^*(x) \leq H^*(y),
\end{aligned}$$

and that there exists at least one component among these ten for which the inequality is strict and the corresponding weight is strictly positive.

Then

$$S(x) \;>\; S(y).$$

In particular, the ranking induced by S is consistent with the componentwise ordering that treats larger positive degrees and smaller negative/penalty degrees as preferable.

Proof
By construction of Algorithm 1, we have for every $z \in U$

$$S(z) \;=\; P(z) - N(z) - Q(z),$$

where

$$\begin{aligned}
P(z) &= w_T\, T^*(z) + w_{\mathrm{SRT}}\, \mathrm{SRT}^*(z) + w_{\mathrm{WRT}}\, \mathrm{WRT}^*(z),\\
N(z) &= w_{\mathrm{SRF}}\, \mathrm{SRF}^*(z) + w_{\mathrm{WRF}}\, \mathrm{WRF}^*(z) + w_F\, F^*(z),\\
Q(z) &= \alpha_C\, C^*(z) + \alpha_U\, U^*(z) + \alpha_I\, I^*(z) + \alpha_H\, H^*(z).
\end{aligned}$$

Consider the difference

$$\Delta \;=\; S(x) - S(y).$$

Substituting the above expressions yields

$$\begin{aligned}
\Delta &= \Big(P(x) - P(y)\Big) - \Big(N(x) - N(y)\Big) - \Big(Q(x) - Q(y)\Big)\\
&= w_T\big(T^*(x) - T^*(y)\big) + w_{\mathrm{SRT}}\big(\mathrm{SRT}^*(x) - \mathrm{SRT}^*(y)\big)\\
&\quad + w_{\mathrm{WRT}}\big(\mathrm{WRT}^*(x) - \mathrm{WRT}^*(y)\big)\\
&\quad - w_{\mathrm{SRF}}\big(\mathrm{SRF}^*(x) - \mathrm{SRF}^*(y)\big) - w_{\mathrm{WRF}}\big(\mathrm{WRF}^*(x) - \mathrm{WRF}^*(y)\big) - w_F\big(F^*(x) - F^*(y)\big)\\
&\quad - \alpha_C\big(C^*(x) - C^*(y)\big) - \alpha_U\big(U^*(x) - U^*(y)\big)\\
&\quad - \alpha_I\big(I^*(x) - I^*(y)\big) - \alpha_H\big(H^*(x) - H^*(y)\big).
\end{aligned}$$

We now analyse each group of terms.

1. For the positive components we have

$$T^*(x) - T^*(y) \ge 0, \quad \mathrm{SRT}^*(x) - \mathrm{SRT}^*(y) \ge 0, \quad \mathrm{WRT}^*(x) - \mathrm{WRT}^*(y) \ge 0,$$

and the corresponding weights are nonnegative. Therefore

$$\begin{aligned}
&w_T\big(T^*(x) - T^*(y)\big) \ge 0,\\
&w_{\mathrm{SRT}}\big(\mathrm{SRT}^*(x) - \mathrm{SRT}^*(y)\big) \ge 0,\\
&w_{\mathrm{WRT}}\big(\mathrm{WRT}^*(x) - \mathrm{WRT}^*(y)\big) \ge 0.
\end{aligned}$$

2. For the negative components we have

$$\mathrm{SRF}^*(x) - \mathrm{SRF}^*(y) \le 0, \quad \mathrm{WRF}^*(x) - \mathrm{WRF}^*(y) \le 0, \quad F^*(x) - F^*(y) \le 0.$$

Hence

$$\begin{aligned}
-w_{\mathrm{SRF}}\big(\mathrm{SRF}^*(x) - \mathrm{SRF}^*(y)\big) &= w_{\mathrm{SRF}}\big(\mathrm{SRF}^*(y) - \mathrm{SRF}^*(x)\big) \ge 0,\\
-w_{\mathrm{WRF}}\big(\mathrm{WRF}^*(x) - \mathrm{WRF}^*(y)\big) &= w_{\mathrm{WRF}}\big(\mathrm{WRF}^*(y) - \mathrm{WRF}^*(x)\big) \ge 0,\\
-w_F\big(F^*(x) - F^*(y)\big) &= w_F\big(F^*(y) - F^*(x)\big) \ge 0,
\end{aligned}$$

because each weight is nonnegative and each parenthesis on the right-hand side is nonnegative.

3. For the penalty components we have

$$C^*(x) - C^*(y) \le 0, \quad U^*(x) - U^*(y) \le 0, \quad I^*(x) - I^*(y) \le 0, \quad H^*(x) - H^*(y) \le 0,$$

so that

$$\begin{aligned}
-\alpha_C\big(C^*(x)-C^*(y)\big) &= \alpha_C\big(C^*(y)-C^*(x)\big) \geq 0,\\
-\alpha_U\big(U^*(x)-U^*(y)\big) &= \alpha_U\big(U^*(y)-U^*(x)\big) \geq 0,\\
-\alpha_I\big(I^*(x)-I^*(y)\big) &= \alpha_I\big(I^*(y)-I^*(x)\big) \geq 0,\\
-\alpha_H\big(H^*(x)-H^*(y)\big) &= \alpha_H\big(H^*(y)-H^*(x)\big) \geq 0.
\end{aligned}$$

Summing all contributions, each term in the expression for Δ is greater than or equal to 0. Therefore

$$\Delta \;=\; S(x)-S(y) \;\geq\; 0.$$

By assumption, there exists at least one component with a strict inequality (for example $T^*(x) > T^*(y)$ or $\mathrm{SRF}^*(x) < \mathrm{SRF}^*(y)$ or $C^*(x) < C^*(y)$) whose associated weight is strictly positive. For this component, the corresponding term in the expression for Δ is strictly positive. All other terms are nonnegative, so their sum is strictly positive:

$$\Delta \;>\; 0.$$

Hence $S(x) > S(y)$, which proves the claim. □

Theorem 5.2 (Time and space complexity)
Let U be finite with $|U| = n$. Suppose that Algorithm 1 is executed to compute $S(x)$ for all $x \in U$. Then:

1. The worst–case running time of the algorithm is $\Theta(n)$.
2. The additional memory used, beyond storage of the input D–NOS profile and the output scores, is $O(1)$. Storing the scores for all elements requires $\Theta(n)$ memory.

Proof
We count arithmetic and assignment operations.

For a fixed element $x \in U$, the algorithm performs:

- Exactly 10 normalisations of the form $\mu^*(x) = (\mu(x) - \Psi)/(\Omega - \Psi)$. Each normalisation consists of one subtraction and one division; we treat this as a constant number of operations.
- Computation of $P(x)$, which uses 3 multiplications and 2 additions.
- Computation of $N(x)$, which uses 3 multiplications and 2 additions.
- Computation of $Q(x)$, which uses 4 multiplications and 3 additions.
- Computation of $S(x) = P(x) - N(x) - Q(x)$, which uses 2 subtractions.

The total number of primitive operations for one element is therefore bounded above by a fixed constant $c > 0$ that does not depend on n. Since the algorithm processes each $x \in U$ once, the total running time is at most cn. Hence the time complexity is $O(n)$.

Conversely, the algorithm must read the ten input degrees for each element $x \in U$. This already requires $\Omega(n)$ operations, because there are $10n$ values to access. Therefore the running time is bounded below by a positive constant multiple of n, so it is $\Omega(n)$. Combining the upper and lower bounds, we conclude that the time complexity is $\Theta(n)$.

For the space bound, the algorithm can overwrite the normalised values $\mu^*(x)$ in place or compute each of $P(x)$, $N(x)$, $Q(x)$, and $S(x)$ using a fixed number of scalar variables. Thus, apart from the storage needed for the input degrees and the output score $S(x)$, the algorithm uses only a constant amount of extra memory, independent of n. This gives $O(1)$ additional space.

If the scores $S(x)$ are stored for all $x \in U$, then we need one real number per element. Hence the memory required for the output is proportional to n, which is $\Theta(n)$. This completes the proof. □

6. Conclusion and Future Work

This paper introduced four new partitioned neutrosophic offset families-*hexapartitioned*, *octapartitioned*, *nonapartitioned*, and *decapartitioned*-and demonstrated that each of them can be embedded naturally within the plithogenic–offset framework. This makes it easier to apply offset-based reasoning to concepts that involve more sophisticated and diverse uncertainty parameters.

Looking ahead, we hope that the ideas developed here will be explored further in a variety of settings, including *neutrosophic graph theory* [66, 67], *neutrosophic algebra*[68], neutrosophic Probability[69], neutrosophic statistics[70], *neutrosophic topology*[71], *neutrosophic control theory* [72], and *neutrosophic decision science*[73]. Investigating concrete applications and computational techniques in these domains remains an open and promising direction for future research. I also hope that domain experts will conduct numerical experiments on real datasets such as questionnaire responses and Likert scales, in order to compare how each partitioned offset behaves under practical conditions.

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Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

A. Appendix: Multi-Valued Neutrosophic Survey with Indeterminacy Partition

A wide variety of survey methodologies have been developed and studied in order to investigate social trends and real-world phenomena [74, 75, 76, 77]. In this Appendix, we define a *Multi-Valued Neutrosophic Survey*. We then present an illustrative example of a neutrosophic survey scale grounded in the Partitioned Neutrosophic and Multi-Valued Neutrosophic frameworks. The questionnaire consists of items evaluated on a neutrosophic scale that explicitly captures agreement, disagreement, and indeterminacy.

Definition A.1 (Triple-Valued Neutrosophic Set (TVNS) [32])**.** Let X be a nonempty set. A *triple-valued neutrosophic set* A on X is

$$A = \big\{(x,\, T_A(x),\, I_T(x),\, I_N(x),\, I_F(x),\, F_A(x)) :\; x \in X\big\},$$

where $T_A(x), I_T(x), I_N(x), I_F(x), F_A(x) \in [0,1]$ denote, respectively, the truth degree, the indeterminacy leaning to truth, the neutral indeterminacy, the indeterminacy leaning to falsity, and the falsity degree, and they satisfy

$$0 \le T_A(x) + I_T(x) + I_N(x) + I_F(x) + F_A(x) \le 5 \quad (x \in X).$$

Definition A.2 (Multi-Valued Neutrosophic Set with Indeterminacy Partition)**.** Fix an integer $k \ge 1$. A *k-valued indeterminacy-partitioned neutrosophic set* (briefly, an *Indeterminacy-k Multi-Valued Neutrosophic Set*, or *Ik-MVNS*) A on a universe X is defined by the data

$$A = \Big\{\big(x,\; T_A(x),\; I_{A,1}(x), \dots, I_{A,k}(x),\; F_A(x)\big) : x \in X\Big\},$$

where

$$T_A : X \to [0,1], \qquad I_{A,i} : X \to [0,1]\ (i = 1, \dots, k), \qquad F_A : X \to [0,1],$$

and for each $x \in X$ we require the bounded-sum condition

$$0 \le T_A(x) + \sum_{i=1}^{k} I_{A,i}(x) + F_A(x) \le k + 2.$$

The components $I_{A,1}(x), \dots, I_{A,k}(x)$ are called the *refined (partitioned) indeterminacy degrees*. Semantically, each $I_{A,i}$ corresponds to a distinct source/type of indeterminacy (e.g., missing data, conflicting experts, sensor noise, model mismatch), while truth and falsity remain single-valued.

Remark A.3 (Compatibility with triple-valued and double-valued models)**.** When $k = 1$ and we set $I_{A,1} = I_A$, we recover the standard neutrosophic set. When $k = 2$, one may interpret $I_{A,1}$ as indeterminacy leaning toward truth and $I_{A,2}$ as indeterminacy leaning toward falsity (a double-valued split of indeterminacy). When $k = 3$, one may interpret $(I_{A,1}, I_{A,2}, I_{A,3})$ as (indeterminacy leaning toward truth, neutral indeterminacy, indeterminacy leaning toward falsity), matching the usual triple-valued split of indeterminacy.

Proposition A.4 (Embedding of TVNS into I3-MVNS (indeterminacy-only refinement))
Let A be a triple-valued neutrosophic set presented as

$$\Big\{(x,\ T_A(x),\ I_T(x),\ I_N(x),\ I_F(x),\ F_A(x)) : x \in X\Big\}.$$

Define an I3-MVNS $A^\star$ by

$$T_{A^\star}(x) = T_A(x), \quad I_{A^\star,1}(x) = I_T(x), \quad I_{A^\star,2}(x) = I_N(x), \quad I_{A^\star,3}(x) = I_F(x), \quad F_{A^\star}(x) = F_A(x).$$

Then $A^\star$ is an I3-MVNS, and the construction is faithful (it preserves all membership degrees componentwise).

Proof
For each $x \in X$, all components are in $[0, 1]$ by assumption. Moreover, the triple-valued bounded-sum condition implies

$$0 \le T_A(x) + I_T(x) + I_N(x) + I_F(x) + F_A(x) \le 5.$$

Since $k = 3$, the Ik-MVNS bound is $k + 2 = 5$, hence

$$0 \le T_{A^\star}(x) + \sum_{i=1}^{3} I_{A^\star,i}(x) + F_{A^\star}(x) \le 5,$$

so $A^\star$ satisfies the I3-MVNS axioms. □

Example A.5 (Indeterminacy-partitioned evaluation of cloud-service options)**.** Let $X = \{\mathsf{AWS}, \mathsf{Azure}, \mathsf{GCP}\}$ be three alternatives. We model a single decision-maker's uncertain evaluation by an I3-MVNS A, where

$$I_{A,1} = \text{(indeterminacy due to missing benchmark data)},$$

$$I_{A,2} = \text{(indeterminacy due to conflicting reports)},$$

$$I_{A,3} = \text{(indeterminacy due to unstable requirements)}.$$

Assign (for illustration):

$$\begin{aligned}
\mathsf{AWS}&: \ (T, I_1, I_2, I_3, F) = (0.70,\ 0.10,\ 0.05,\ 0.05,\ 0.10),\\
\mathsf{Azure}&: \ (T, I_1, I_2, I_3, F) = (0.60,\ 0.15,\ 0.10,\ 0.05,\ 0.10),\\
\mathsf{GCP}&: \ (T, I_1, I_2, I_3, F) = (0.55,\ 0.10,\ 0.15,\ 0.10,\ 0.10).
\end{aligned}$$

Check the bounded-sum condition (here $k = 3$, so the upper bound is 5):

$$\text{AWS} : 0.70 + 0.10 + 0.05 + 0.05 + 0.10 = 0.90 \leq 5,$$
$$\text{Azure} : 0.60 + 0.15 + 0.10 + 0.05 + 0.10 = 1.00 \leq 5,$$
$$\text{GCP} : 0.55 + 0.10 + 0.15 + 0.10 + 0.10 = 1.00 \leq 5.$$

Thus A is a well-defined I3-MVNS. Compared with a standard NS, this model distinguishes *why* the evaluation is indeterminate, while keeping T and F single-valued.

Here, we present the definition of a *Multi-Valued Neutrosophic Survey* below.

Definition A.6 (Multi-Valued Neutrosophic Survey)**.** Let X be a set of items (alternatives), let R be a non-empty set of respondents, and fix $k \geq 1$. A *Multi-Valued Neutrosophic Survey with indeterminacy partition size* k is a mapping

$$\mathcal{S} : R \times X \longrightarrow [0,1]^{k+2},$$

such that for each respondent $r \in R$, the induced assignment

$$A_r = \Big\{ \big(x,\ T_r(x),\ I_{r,1}(x), \ldots, I_{r,k}(x),\ F_r(x)\big) : x \in X \Big\}$$

is an Ik-MVNS on X, i.e., for all $x \in X$,

$$0 \leq T_r(x) + \sum_{i=1}^{k} I_{r,i}(x) + F_r(x) \leq k + 2.$$

Definition A.7 (Simple aggregation of a Multi-Valued Neutrosophic Survey)**.** Given a survey $\mathcal{S}$ over respondents R and items X, define the *mean-aggregated Ik-MVNS* $\overline{A}$ on X by

$$\overline{T}(x) = \frac{1}{|R|} \sum_{r \in R} T_r(x), \quad \overline{I}_i(x) = \frac{1}{|R|} \sum_{r \in R} I_{r,i}(x)\ (i = 1, \ldots, k), \quad \overline{F}(x) = \frac{1}{|R|} \sum_{r \in R} F_r(x).$$

Then $\overline{A} = \{(x, \overline{T}(x), \overline{I}_1(x), \ldots, \overline{I}_k(x), \overline{F}(x)) : x \in X\}$ is an Ik-MVNS.

Proof

For each $x \in X$, all averaged values lie in $[0,1]$ because $[0,1]$ is convex. Moreover, for each respondent r and fixed x,

$$T_r(x) + \sum_{i=1}^{k} I_{r,i}(x) + F_r(x) \leq k + 2.$$

Averaging over $r \in R$ yields

$$\overline{T}(x) + \sum_{i=1}^{k} \overline{I}_i(x) + \overline{F}(x) = \frac{1}{|R|} \sum_{r \in R} \Big(T_r(x) + \sum_{i=1}^{k} I_{r,i}(x) + F_r(x) \Big) \leq k + 2,$$

and non-negativity is immediate. Hence $\overline{A}$ is an Ik-MVNS. □

Example A.8 (A small I2-MVNS survey: software-tool adoption)**.** Let $X = \{\text{ToolA}, \text{ToolB}\}$ and $R = \{r_1, r_2, r_3\}$. Take $k = 2$ where

$$I_{r,1} = \text{(indeterminacy from missing internal usage data)},$$
$$I_{r,2} = \text{(indeterminacy from conflicting stakeholder opinions)}.$$

Suppose responses are:

	ToolA	ToolB
r_1	$(T, I_1, I_2, F) = (0.65, 0.15, 0.05, 0.15)$	$(0.50, 0.10, 0.20, 0.20)$
r_2	$(0.60, 0.10, 0.10, 0.20)$	$(0.55, 0.15, 0.10, 0.20)$
r_3	$(0.70, 0.05, 0.10, 0.15)$	$(0.45, 0.10, 0.25, 0.20)$

Check one item (for $k = 2$, upper bound is $k + 2 = 4$):

$$\mathsf{ToolA}, r_1 : \ 0.65 + 0.15 + 0.05 + 0.15 = 1.00 \leq 4,$$

and similarly for the others, so each A_{r_j} is an I2-MVNS. The mean-aggregated evaluation becomes

$$\overline{A}(\mathsf{ToolA}) = (0.65,\ 0.10,\ 0.0833\ldots,\ 0.1666\ldots), \quad \overline{A}(\mathsf{ToolB}) = (0.50,\ 0.1166\ldots,\ 0.1833\ldots,\ 0.20).$$

Thus the survey distinguishes two concrete kinds of indeterminacy at the group level, while keeping truth and falsity single-valued.

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