STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–17.
Published online in International Academic Press (www.IAPress.org)

On the Construction of Schauder Bases in Hilbert Spaces via Unitary Representations

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Abstract This article develops a unified framework for constructing Schauder bases in Hilbert spaces from unitary representations of locally compact groups, with emphasis on the affine action (the $ax + b$ group) and its wavelet realization. We begin with the contrast between Hamel bases (algebraic existence) and Schauder bases (topological reconstruction), and show how topology—via continuity of coordinate functionals and convergence in norm—guides the validity of expansions useful in functional analysis.
At an abstract level, we review Haar measure, regular representations, and the notion of a cyclic vector, and we state Schauder-type criteria for systems generated by orbits $\{\pi(g)f\}_{g\in G}$. For the affine group, we recall the continuous wavelet transform, admissibility, and the reproduction formula; we then discretize on a dyadic lattice to obtain orthonormal (hence Schauder) systems in $L^2(\mathbb{R})$ via multiresolution and quadrature mirror filter (QMF) conditions. The Haar wavelet appears as a prototypical case: its discrete orbit under dilations and translations generates a complete orthonormal basis.
Beyond this expository part, we make explicit the link between group-generated systems and Schauder decompositions in Banach/coorbit settings, formulating a sufficient condition under which atomic decompositions arising from coorbit theory yield Schauder bases in their natural Banach spaces.
On the computational side, we implement simulations comparing Haar approximations with Fourier series on $[-3, 3]$. We consider three representative functions: t^2 (nonperiodic), rectangular wave with $T = 1$, and triangular wave with $T = 1$. We show that, for periodic functions, the Fourier series must be computed with the natural period (an indispensable correction), and that Haar offers localization advantages and robustness near discontinuities (mitigating Gibbs phenomena). For nonperiodic functions, the implicit periodization in Fourier introduces global artifacts that Haar partially avoids. We quantify the approximation errors in L^2- and L^∞-norms as functions of the number of terms and provide a reproducible numerical setup, together with public code.
We conclude by pointing to two directions for extension: (i) more regular wavelets (Daubechies, Riesz bases) and extensions to Banach spaces via coorbit theory and its discretization; and (ii) more general group actions (e.g., anisotropic semidirect products) tailored to specific geometries. The results strengthen the bridge between algebraic generation by group actions and stable reconstruction in functional analysis.

Keywords Group action, Harmonic analysis, Orthonormal bases, Schauder bases, Coorbit, Affine group, Multiresolution, Unitary representations, Fourier series, Wavelet transform

AMS 2010 subject classifications 42C40, 46B15, 22D10, 43A65

DOI: 10.19139/soic-2310-5070-3187

1. Introduction

The problem of constructing bases in infinite-dimensional spaces lies at the intersection of abstract algebra and functional analysis. On the algebraic side, the existence of a Hamel basis for any vector space follows from Zorn's lemma and thus from the Axiom of Choice; this guarantees existence but not an operative description for

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

analytic purposes. On the analytic side, Banach and Hilbert spaces require expansions that converge in norm, which naturally leads to Schauder bases [1, 2, 3]. In many concrete function spaces, such as $L^p(\mathbb{R}^d)$, $C(K)$, or Sobolev spaces, explicit Schauder bases are known only in special situations; in other cases, bases are replaced by frames or atomic decompositions [4, 5]. This gap between abstract existence and constructive analytic representations motivates the present work. The paper is partly expository, in that it collects and systematizes several results that are usually scattered in the literature, and partly original, in that it makes explicit some connections between group representations, coorbit spaces and Schauder decompositions, and complements them with a detailed numerical study.

A powerful way to obtain structured systems is to let a (locally compact or discrete) group act on a Banach or Hilbert space and consider the orbit of a single generating function. This philosophy underlies classical wavelet and Gabor analysis, where translations, dilations, or modulations of a window generate systems that are complete and often stable [5, 6]. Recent work extends this viewpoint to Banach spaces and to more general group actions, introducing group-frames, Banach frames, and Schauder frames generated by isometric representations of discrete groups [7, 8, 9]. These results show that algebraic information carried by the group (regular representations, convolution operators, double commutants) can be exploited to control completeness, stability, and reconstruction in the underlying space. In parallel, sampling and interpolation on unimodular groups and coorbit-type constructions confirm that group invariance is a robust source of bases and frames for function spaces [10, 11, 12].

However, a group-generated system is not automatically a Schauder basis: one must ensure (i) that the orbit of the generator spans a dense subspace whose closure is the whole space, (ii) that the associated coefficient functionals are continuous, and (iii) that the reconstruction series converges in the norm of the space. Recent contributions on besselian Schauder frames and group-frames for Banach spaces provide necessary and sufficient conditions of this type, often expressed in terms of Gram operators, matrix representations of the group, and weak sequential completeness [**?**, 7, 8]. These criteria are close in spirit to the classical characterization of near-Schauder bases via the kernel of the reconstruction operator [13], but they are adapted to the presence of a group action and allow us to move from a purely algebraic generator to a topologically meaningful Schauder decomposition. This gives precisely the bridge we seek between abstract generation of vectors and analytic bases.

In this paper we develop a group-based constructive scheme for building Schauder bases (and, when necessary, Schauder frames) in separable Banach spaces of functions. We start from a single function (the "mother atom"), act on it by a suitably chosen discrete subgroup of a Lie or affine group, and impose verifiable conditions on the representation so that the resulting orbit becomes a Schauder system. The analysis relies on three pillars: (a) classical functional-analytic tools (Banach–Steinhaus, closed graph, density arguments); (b) structural results on group-generated frames and coorbit spaces; and (c) algebraic constraints on the underlying group to control redundancy. In several model situations (wavelet-type actions, translation–dilation systems, shift-invariant spaces) our scheme recovers known constructions and moderately generalizes them to Banach settings. At the same time, we explicitly emphasize the limitations of our approach: most of the concrete constructions in this article live in Hilbert spaces, while extensions to Banach/coorbit settings and to more general groups are discussed at a structural level and pointed out as directions for further research.

Main contributions and structure of the paper

The present work is partly expository and partly original. On the expository side, we provide a coherent account of Schauder bases generated by unitary group representations, starting from basic notions on Hamel and Schauder bases and moving towards wavelet constructions on the affine group. On the more original side, we:

- make explicit a sufficient condition under which atomic decompositions arising from coorbit theory yield Schauder bases in the corresponding Banach spaces, thus clarifying the connection between group frames, coorbit spaces and Schauder decompositions;
- discuss how the geometry of more general groups (non-unimodular and anisotropic semidirect products) affects the possibility of constructing Schauder bases from orbits, and identify several open problems in non-Euclidean settings;

- perform a reproducible numerical comparison between Haar wavelet expansions and Fourier series for several test functions of different regularity, reporting error curves in L^2- and L^∞-norms and illustrating the practical strengths and limitations of group-generated Schauder systems.

The paper is organized as follows. In Section 2 we recall basic notions on Hamel bases, Schauder bases, Banach and Hilbert spaces, and introduce the language of representations and cyclic vectors needed later. Section 3 is devoted to unitary representations of locally compact groups and to group-generated systems; there we also discuss Schauder bases in Banach/coorbit spaces and briefly indicate how more general group actions fit into the picture. In Section 4 we specialize to the affine group and develop the construction of orthonormal (hence Schauder) wavelet bases in $L^2(\mathbb{R})$ via multiresolution analysis and QMF conditions. Section 5 contains the numerical experiments comparing Haar and Fourier expansions, with a detailed description of the numerical setup, error measures and convergence behaviour. Finally, Section 6 collects our conclusions and outlines several directions for future work.

2. Mathematical Preliminaries

In this section we gather the minimal algebraic and functional notions needed to construct bases in finite- and infinite-dimensional vector spaces. For functional analysis we follow Brezis [14], Folland [15], and Rudin [16]; for purely algebraic aspects we use the classical viewpoint of Banach [1] and the later formulation in terms of Schauder bases [2].

Definition 2.1. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A *vector space* over $\mathbb{K}$ is a set V endowed with two operations, addition $V \times V \to V$ and scalar multiplication $\mathbb{K} \times V \to V$, that satisfy the usual axioms (commutativity, associativity, existence of identity and inverse, distributivity). A subset $W \subset V$ is a *subspace* if it is closed under addition and scalar multiplication. [15, Ch. 1]

Definition 2.2. Let V be a vector space. A subset $E = \{v_i\}_{i \in I} \subset V$ is *linearly independent* if every finite combination

$$\sum_{j=1}^{n} \alpha_j v_{i_j} = 0, \qquad \alpha_j \in \mathbb{K},$$

implies $\alpha_1 = \cdots = \alpha_n = 0$. We say that E *spans* V if every $v \in V$ is a finite linear combination of elements of E. [16, Sec. 1.2]

Definition 2.3. A subset $B \subset V$ is a *Hamel basis* of V if:

1. B is linearly independent;
2. every $v \in V$ admits a *finite* representation

$$v = \sum_{j=1}^{n} \alpha_j b_j, \qquad b_j \in B,\ \alpha_j \in \mathbb{K}.$$

In this case, B is necessarily maximal among linearly independent subsets. [1, Ch. 1]

The above notion is *purely algebraic*. By Zorn's lemma one proves that *every* (nonzero) vector space has a Hamel basis; the proof is standard and uses the Axiom of Choice.

Theorem 2.4. *Let V be a vector space over $\mathbb{K}$. Then there exists a subset $B \subset V$ that is a Hamel basis.*

Proof
Consider the partially ordered set of linearly independent families of V, apply Zorn's lemma, and conclude that a maximal family is necessarily spanning. [16, Sec. 1.3] □

In finite dimensions, all Hamel bases have the same number of elements and this number is the *dimension* of the space.

Theorem 2.5. *Let V be a vector space, $\{v_1, \dots, v_n\}$ a linearly independent set, and $\{w_1, \dots, w_m\}$ a spanning set. Then $n \le m$ and, after reordering, one can replace some of the w_j by the v_i and still span V. As a consequence, if V has a finite basis, any other basis has the same number of elements. [15, Thm. 1.12]*

When moving to functional analysis, the setting changes: we work not only with linear structure but also with a *topology* or a *norm*. Recall the basic definitions.

Definition 2.6. A *normed space* is a pair $(X, \|\cdot\|)$ where X is a vector space over $\mathbb{K}$ and $\|\cdot\| : X \to [0, \infty)$ satisfies: (i) $\|x\| = 0 \Leftrightarrow x = 0$; (ii) $\|\alpha x\| = |\alpha| \, \|x\|$; (iii) $\|x + y\| \le \|x\| + \|y\|$. If $(X, \|\cdot\|)$ is complete with respect to the induced metric, then X is a *Banach space*. [14, Ch. 1]

Definition 2.7. A *Hilbert space* is a complex (or real) vector space endowed with an inner product $\langle \cdot, \cdot \rangle$ and complete with respect to the norm $\|x\| = \sqrt{\langle x, x \rangle}$. [14, Ch. 3]

In these spaces it is natural to require that vector expansions not only be finite but *converge in norm*. This leads to the central notion of this section.

Definition 2.8. Let X be a Banach space. A sequence $(x_n)_{n \ge 1} \subset X$ is a *Schauder basis* of X if for every $x \in X$ there exist unique scalars $(a_n)_{n \ge 1}$ such that

$$x = \sum_{n=1}^{\infty} a_n x_n$$

with convergence in the norm of X. Moreover, the *coordinate* functionals $x \mapsto a_n$ are necessarily continuous. [14, Sec. 1.5], [16, Sec. 3.13]

Remark 2.9.

1. In a Hamel basis every representation is *finite*. No norm or continuity is required.
2. In a Schauder basis the representation is generally *infinite*, but the series *must* converge in the norm of the space.
3. Every Schauder basis is automatically a dense spanning set, but it is rarely a Hamel basis (in an infinite-dimensional Banach space, a Hamel basis is necessarily uncountable, whereas a Schauder basis is countable). [16, Sec. 3.13]
4. The existence of Hamel bases is universal (requires choice); that of Schauder bases is *not*: there exist separable Banach spaces with no Schauder basis (a classical result in Banach space theory). [14, Ch. 4]

This contrast shows the essential role of topology: in an infinite-dimensional Banach space, any Hamel basis is necessarily uncountable and yields only finite representations, making it of limited use for studying continuity or limits; in contrast, a Schauder basis is countable and its expansions converge in norm, so it is compatible with continuous operators, limits, and duality.

In separable Hilbert spaces the situation is more favorable: every separable Hilbert space admits a countable orthonormal basis, and every orthonormal basis is a Schauder basis in the above sense.

Theorem 2.10. *Let H be a separable Hilbert space. Then there exists a countable orthonormal family $\{e_n\}_{n \ge 1} \subset H$ such that*

$$H = \overline{\operatorname{span}}\{e_n : n \ge 1\}$$

and for every $x \in H$ one has the Fourier expansion

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n,$$

converging in the norm of H. [16, Thm. 3.13], [14, Prop. 3.14]

Corollary 2.11. *In a separable Hilbert space every orthonormal basis is a Schauder basis. Moreover, the coordinate functionals are given by the inner products $x \mapsto \langle x, e_n \rangle$, which are continuous. [14, Ch. 3]*

Finally, for our goal—constructing bases from a group action—we need a minimum of language on representations.

Definition 2.12. Let G be a group and X a Banach space. A *representation* (or linear action) of G on X is a homomorphism

$$\pi : G \longrightarrow \mathcal{B}(X)$$

into the group of bounded linear operators on X, such that $\pi(e) = I$ and $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ for all $g_1, g_2 \in G$. If each $\pi(g)$ is an isometry, the action preserves the norm. [15, Sec. 7.1]

Definition 2.13. Let $\pi : G \to \mathcal{B}(X)$ be a linear action and $f \in X$. The *system generated by the orbit of f* is

$$\mathcal{O}(f) := \{\pi(g)f : g \in G\}.$$

If $\operatorname{span} \mathcal{O}(f)$ is dense in X, we say that f is a *cyclic vector*. The central question of this article is: under which conditions on G, on π, and on f can the indexed set $\{\pi(g)f\}_{g \in G}$ be turned into a Schauder basis? This directly links the algebraic preliminaries (generation, independence) with the functional ones (norm convergence, continuity of coordinates). [15, Sec. 8.2], [16, Ch. 3]

3. Unitary Representations of Locally Compact Groups

In this section we recall the elements of representation theory needed to relate group actions to the construction of systems generated by orbits and, later, to wavelet-type bases in Hilbert spaces. Our starting point is the notion of a locally compact group with Haar measure, the natural setting of abstract harmonic analysis [15, 16, 4, **?**].

Definition 3.1. Let G be a topological group such that every point has a compact neighborhood; then G is called a *locally compact* (l.c.) group. A *left Haar measure* on G is a positive Borel measure μ such that

$$\mu(gE) = \mu(E) \qquad \text{for all } g \in G,\ E \subset G \text{ Borel},$$

and which is regular. Every l.c. group admits a Haar measure, unique up to a positive scalar factor. [15, Thm. 2.9]

From Haar measure one defines $L^2(G) = L^2(G, \mu)$, a separable Hilbert space when G is second countable. On $L^2(G)$ the *regular representation* arises naturally.

Definition 3.2. Let G be an l.c. group with left Haar measure μ. Define, for $g \in G$,

$$(\lambda(g)f)(x) := f(g^{-1}x), \qquad f \in L^2(G),\ x \in G.$$

Then:

1. $\lambda(g)$ is a linear unitary operator on $L^2(G)$;
2. $\lambda(g_1 g_2) = \lambda(g_1)\lambda(g_2)$;
3. the map $g \mapsto \lambda(g)$ is strongly continuous.

Thus $\lambda : G \to \mathcal{U}(L^2(G))$ is a *unitary representation* of G. [15, Sec. 7.1], [16, Sec. 13.4]

Definition 3.3. Let G be an l.c. group and H a Hilbert space. A map

$$\pi : G \longrightarrow \mathcal{U}(H)$$

is a *strongly continuous unitary representation* if: (i) $\pi(e) = I$, (ii) $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$, (iii) for each $h \in H$ the map $g \mapsto \pi(g)h$ is continuous in the Hilbert topology. [16, Sec. 13.4]

Such representations appear implicitly in harmonic analysis because the structure of $L^2(G)$ can be studied by looking at how G acts by translations. Whenever we take a single vector $f \in H$ and generate its orbit

$$\mathcal{O}(f) := \{\pi(g)f : g \in G\},$$

we are performing harmonic analysis "from a representation": we aim to describe H from translations (or dilations, or both) of an atom. This is precisely the pattern of continuous and discrete wavelets [6, ?].

Definition 3.4. Let (π, H) be a unitary representation of G. A vector $f \in H$ is called *cyclic* if

$$\overline{\operatorname{span}}\{\pi(g)f : g \in G\} = H.$$

If there exists a cyclic vector, we say that the representation is *cyclic*. [16, Sec. 13.9]

Cyclic vectors are important for two reasons: (1) they guarantee that the orbit of a single vector is dense; (2) they allow one to *order* that orbit (or a discrete suborbit) to turn it into a Schauder basis. The former is purely harmonic analysis; the latter is functional (Schauder) and requires additional hypotheses.

Theorem 3.5 (Peter–Weyl decomposition in the compact case)**.** *If G is a compact group, every continuous unitary representation of G is an (orthogonal) sum of finite-dimensional irreducible representations; in particular,*

$$L^2(G) \cong \bigoplus_{\widehat{G}} H_\pi^{(\dim \pi)}$$

where $\widehat{G}$ is the unitary dual of G. This implies that $L^2(G)$ possesses an orthonormal basis formed by matrix coefficients of representations. [15, Sec. 7.6]

This result shows that, for compact groups, representation theory itself *builds* orthonormal (hence Schauder) bases in $L^2(G)$. The noncompact case requires finer tools (induced representations, coorbit theory), but the idea is the same: use the action of G to generate complete systems.

3.1. Group-generated systems and Schauder bases

Now let G be discrete (or a discrete subgroup coarsely dense in an l.c. group) and let (π, H) be a unitary representation. Fix a sequence $(g_n)_{n\geq 1}$ in G and a cyclic vector $f \in H$, and consider

$$x_n := \pi(g_n)f, \qquad n \geq 1.$$

The goal is: when is the sequence (x_n) a Schauder basis of H? The following statement summarizes the typical conditions appearing in the Schauder literature (see [4, 8, 7]).

Theorem 3.6. *Let (π, H) be a unitary representation and $f \in H$ a cyclic vector. Let $(g_n) \subset G$ and $x_n = \pi(g_n)f$. Suppose:*

(i) (density) $\overline{\operatorname{span}}\{x_n : n \geq 1\} = H$;
(ii) (bounded coefficient functionals) there exist continuous linear functionals $(\varphi_n) \subset H'$ such that

$$\varphi_n(x_m) = \delta_{nm} \quad \textit{and} \quad \sup_n \|\varphi_n\| < \infty;$$

(iii) (convergence) for every $h \in H$,

$$h = \lim_{N\to\infty} \sum_{n=1}^{N} \varphi_n(h)\, x_n \quad \textit{in the norm of } H.$$

Then (x_n) is a Schauder basis of H. [4, Sec. 3]

Proof
(i) guarantees completeness; (ii) guarantees uniqueness of coefficients (the family (φ_n) is biorthogonal to (x_n) and uniformly bounded); (iii) states that the reconstruction series converges in norm. These are precisely the three ingredients in the definition of a Schauder basis in Banach/Hilbert spaces. Unitarity ensures the norms $\|x_n\| = \|f\|$ are controlled. □

In many concrete examples—for instance, the affine group in the wavelet case—hypotheses (ii)–(iii) are verified using additional properties of the representation: decomposition into closed orbits on the dual, existence of an "admissibility function," and a reproduction formula. This is done in the theory of continuous wavelets and in Feichtinger–Gröchenig's coorbit theory, and is systematized in [**?**, 4, 6].

Definition 3.7. Let G be an l.c. group, H a Hilbert space, and $\pi : G \to \mathcal{U}(H)$ a suitable irreducible unitary representation (e.g., square-integrable). If there exists $f \in H$ such that

$$\int_G |\langle h, \pi(g)f\rangle|^2 \, d\mu(g) = c\,\|h\|^2 \qquad \forall h \in H,$$

then each $h \in H$ can be reconstructed via

$$h = \frac{1}{c} \int_G \langle h, \pi(g)f\rangle \, \pi(g)f \, d\mu(g),$$

with strong convergence in H. This is the prototype of a resolution of the identity underlying continuous wavelets. [6, Ch. 2], [**?**, Ch. 5]

This formula is still "continuous." To use it in the construction of a *basis* (or at least a Schauder system) one must *discretize* the group: choose a lattice $\Gamma \subset G$ so that $\{\pi(\gamma)f\}_{\gamma\in\Gamma}$ remains complete and stable. Christensen's frame theory and Führ's coorbit discretization provide sufficient conditions for this [4, **?**]. This is the gateway to the next section on wavelets.

3.2. Schauder bases in Banach spaces and coorbit theory

Although most of the concrete examples in this paper live in Hilbert spaces, many constructions naturally extend to Banach settings through coorbit theory. Let G be a locally compact group, π an irreducible unitary representation on a Hilbert space $\mathcal{H}$, and $\psi \in \mathcal{H}$ an admissible vector. Given a solid Banach function space Y on G one can associate a coorbit space $\mathrm{Co}Y$ consisting of all distributions f whose voice transform

$$V_\psi f(g) = \langle f, \pi(g)\psi\rangle$$

belongs to Y. Under suitable assumptions on Y and on a well-spread discrete subset $\Gamma \subset G$, the discretized system

$$\mathcal{G}(\psi, \Gamma) = \{\pi(\gamma)\psi : \gamma \in \Gamma\}$$

provides an atomic decomposition of $\mathrm{Co}Y$, i.e., every $f \in \mathrm{Co}Y$ can be written as

$$f = \sum_{\gamma\in\Gamma} c_\gamma \, \pi(\gamma)\psi,$$

with unconditional convergence in $\mathrm{Co}Y$ and stable reconstruction from the coefficient sequence $(c_\gamma)_{\gamma\in\Gamma}$ in a Banach sequence space Y_d; see [**?**, 4] for details.

Proposition 3.8. *Let* $\mathrm{Co}Y$ *be a coorbit space associated with* (π, ψ) *as above, and assume that* $\mathcal{G}(\psi, \Gamma)$ *yields an atomic decomposition of* $\mathrm{Co}Y$ *with unique coefficients* $c(f) = (c_\gamma(f))_{\gamma\in\Gamma} \in Y_d$ *for every* $f \in \mathrm{Co}Y$*. Then the family* $\mathcal{G}(\psi, \Gamma)$ *forms a Schauder basis of the closed linear span of* $\mathcal{G}(\psi, \Gamma)$ *in* $\mathrm{Co}Y$*.*

Proof
By uniqueness of the atomic decomposition, the maps $\lambda_\gamma : \mathrm{Co}Y \to \mathbb{C}, \quad \lambda_\gamma(f) = c_\gamma(f)$, are well-defined. Stability of the decomposition implies that $f \mapsto c(f)$ is bounded from $\mathrm{Co}Y$ into Y_d, hence each coordinate functional λ_γ is continuous. Moreover, the atomic decomposition ensures that the partial sums $S_F(f) = \sum_{\gamma \in F} c_\gamma(f)\,\pi(\gamma)\psi$ converge in $\mathrm{Co}Y$ to f as F runs over finite subsets of Γ. Thus $\mathcal{G}(\psi, \Gamma)$ satisfies the definition of a Schauder basis for its closed linear span in the Banach space $\mathrm{Co}Y$. □

This proposition makes explicit a connection that is often only implicit in the coorbit literature: under natural conditions, atomic decompositions obtained from group representations do not only provide stable expansions but in fact yield Schauder bases in the appropriate Banach spaces. This observation complements the Hilbert-space viewpoint developed in the rest of the paper and clarifies in which sense our results extend beyond L^2-settings.

3.3. Beyond the affine group: more general group actions

Although the affine group $\mathbb{R} \rtimes \mathbb{R}_+$ is the standard framework for wavelet constructions on the real line, the group-based approach to Schauder bases is not limited to this case. Two classes of examples are particularly relevant.

First, anisotropic semidirect products such as the shearlet group provide systems that are better adapted to directional features and anisotropic singularities. In this setting, the orbits of a suitable generator under the representation include dilations, shears and translations, leading to redundant systems with strong approximation properties in function spaces designed for images or higher-dimensional signals. Identifying conditions under which appropriate subfamilies form Schauder bases, rather than merely frames, is an open and non-trivial problem.

Second, one can consider non-unimodular groups acting on manifolds or graphs. Here the lack of translation invariance and the presence of curvature or irregular geometry impose additional constraints on the existence of group-based bases. For instance, wavelet-type systems constructed on compact Riemannian manifolds or on graphs often yield frames or Riesz bases in certain Hilbert spaces, but proving that they form Schauder bases in Banach spaces associated with Besov- or Triebel–Lizorkin-type scales remains challenging.

These examples show that the geometry of the underlying group and space has a direct impact on the possibility of constructing Schauder bases from group orbits. We regard a systematic study of such phenomena, beyond the classical affine case treated in Section 4, as an interesting direction for future work.

4. Wavelets and Orthonormal Bases Generated by the Action of the Affine Group

In this section we develop the construction of orthonormal bases in $L^2(\mathbb{R})$ from the action of the affine group (the $ax + b$ group). The guiding thread is harmonic analysis on locally compact groups: Haar measure, unitary representations, and discretization of systems generated by orbits. For the harmonic analysis framework and transforms on groups we rely on [15, 16, 6, 4, **?**], and we follow [17] for the abstract part and [18, 19] for constructive details of orthogonal wavelets.

Definition 4.1. The *affine group* is the group of affine transformations of the real line given by $T_{a,b}(x) = ax + b$, identified by pairs $G = \{(a, b) : a \in \mathbb{R}^*,\ b \in \mathbb{R}\}$ with composition $(a, b) \cdot (a', b') = (aa',\ b + ab')$. The group G is locally compact and nonunimodular; a left Haar measure is $d\mu_L(a, b) = |a|^{-2}\, db\, da$. [17, Ch. 12, Ex. 12.1.1]

Definition 4.2. Define $\pi : G \to \mathcal{U}(L^2(\mathbb{R}))$ by

$$\big(\pi(a, b)f\big)(x) = |a|^{-1/2}\, f\left(\frac{x - b}{a}\right), \qquad f \in L^2(\mathbb{R}).$$

Then π is a strongly continuous unitary representation. [18, Def. 6.1, Secs. 6–7]

Definition 4.3. Let $\psi \in L^2(\mathbb{R})$. The *continuous wavelet transform* of $f \in L^2(\mathbb{R})$ with respect to ψ is

$$W_\psi f(a, b) \;=\; \langle f,\, \pi(a, b)\psi \rangle, \qquad (a, b) \in G,$$

understood as a map $W_\psi : L^2(\mathbb{R}) \to L^2(G, d\mu_L)$. We say that ψ is *admissible* if

$$C_\psi \;=\; \int_{\mathbb{R}^*} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\, d\xi \;<\; \infty.$$

[17, Sec. 12.3, (admissibility)]

Theorem 4.4. *If ψ is admissible, then W_ψ is, up to the factor C_ψ, an isometry in L^2, and for every $f \in L^2(\mathbb{R})$ one has the reproduction formula*

$$f(x) \;=\; \frac{1}{C_\psi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} W_\psi f(a,b)\, \pi(a,b)\psi(x)\, \frac{db\, da}{|a|^2},$$

where the integral is understood weakly and, under standard assumptions, pointwise. [17, Sec. 12.3, Eqs. (12.3)–(12.4)]

Proof
Nonunimodularity of G leads to the Duflo–Moore operator C_π and to the above admissibility condition; applying Fubini and Haar invariance on $\mathbb{R}^*$ gives the Plancherel identity for W_ψ and, by adjunction of W_ψ, the reconstruction formula. [17, Sec. 12.3] □

Theorem 4.5. *If $\psi \in L^2(\mathbb{R})$ and $x\psi(x) \in L^1(\mathbb{R})$, then ψ is admissible if and only if $\widehat{\psi}(0) = \int_{\mathbb{R}} \psi(x)\, dx = 0$. [17, Lemma 12.3.4]*

Definition 4.6. For $j, k \in \mathbb{Z}$ define $\psi_{j,k}(x) = 2^{j/2}\, \psi(2^j x - k) = U(2^{-j}, k2^{-j})\psi(x)$. The *discrete affine system* generated by ψ is $\mathcal{W}(\psi) = \{\psi_{j,k} : j, k \in \mathbb{Z}\}$. If $\mathcal{W}(\psi)$ is orthonormal and complete in $L^2(\mathbb{R})$, we call it an *orthonormal wavelet basis*. [19, Defs. 2.2–2.3]

The structural characterization of orthonormal bases via the affine group is formulated through multiresolution analysis (MRA). Recall:

Definition 4.7. A *multiresolution* is a family $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ such that: (i) $V_j \subset V_{j+1}$; (ii) $f \in V_j \iff f(2\cdot) \in V_{j+1}$; (iii) $\bigcap_j V_j = \{0\}$ and $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$; (iv) there exists $\varphi \in V_0$ (the *scaling function*) with $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ orthonormal in V_0. [19, Ch. 2], [6, Ch. 6]

Theorem 4.8. *Let $\{V_j\}$ be an MRA with scaling function φ satisfying the two-scale equation*

$$\varphi(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} p_k\, \varphi(2x - k),$$

with filter $p = \{p_k\}$. Define the wavelet *by $\psi(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} q_k\, \varphi(2x - k)$ with $q_k = (-1)^k p_{1-k}$. Then the quadrature mirror filter (QMF) conditions*

$$\sum_k p_k\, \overline{p_{k-2m}} = \delta_{m,0}, \qquad q_k = (-1)^k p_{1-k}$$

are equivalent to $\mathcal{W}(\psi)$ being an orthonormal basis of $L^2(\mathbb{R})$. [6, Thm. 6.8 & Ch. 7], [19, Sec. 2.2]

Proof
Orthonormality of the translations of φ yields frequency identities for the filter p_k; the QMF relations guarantee (via the symbol $P(\omega)$) that $\widehat{\psi}$ is orthogonal to its translations and that $\{V_{j+1}\} = V_j \oplus W_j$, where $W_j = \mathrm{span}\{\psi_{j,k}\}_k$. By recursion, $\bigoplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{R})$, and orthonormality of $\mathcal{W}(\psi)$ follows from the filter relations. [19, Secs. 2.2–2.4], [6, Chs. 6–7] □

Corollary 4.9. *If $\mathcal{W}(\psi)$ is an orthonormal basis, then for every $f \in L^2(\mathbb{R})$,*

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle\, \psi_{j,k}, \qquad \|f\|_2^2 = \sum_{j,k} |\langle f, \psi_{j,k}\rangle|^2.$$

Since every orthonormal basis in a Hilbert space is a Schauder basis, $\mathcal{W}(\psi)$ is also a Schauder basis in $L^2(\mathbb{R})$ (see Section 2).

Connection with the unitary representation. The basis $\mathcal{W}(\psi)$ is obtained by *sampling* the orbit $G \cdot \psi$ on the discrete lattice $\Gamma = \{(2^{-j}, k2^{-j}) : j, k \in \mathbb{Z}\} \subset G$ and ordering it as a sequence. At the continuous level, W_ψ is interpreted as the "coefficient map" $f \mapsto \langle f, \pi(\cdot)\psi\rangle$; admissibility yields a *resolution of the identity*, and choosing the appropriate Γ (the dyadic case) leads to orthonormality and completeness as above. See [17, Ch. 12] for square-integrable representations and [18, Chs. 5–7] for strong continuity and reconstruction in $L^2(\mathbb{R})$.

Theorem 4.10. *With the above notation, if $f \in L^2(\mathbb{R})$ and ψ is admissible, then*

$$f(x) = \frac{1}{C_\psi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} \langle f, \pi(a,b)\psi\rangle\, \pi(a,b)\psi(x)\, \frac{db\, da}{|a|^2},$$

with convergence in L^2 and, under standard assumptions, almost everywhere. [17, Sec. 12.3], [18, Ch. 6]

Proof
This is the adjoint formula $W_\psi^* W_\psi = C_\psi\, \mathrm{Id}$ using $d\mu_L = |a|^{-2}\, db\, da$ and the condition $\int_{\mathbb{R}^*} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\, d\xi < \infty$. [17, Sec. 12.3] □

5. Numerical Results

In this section we close the conceptual thread of the article by showing, step by step, how the action of the affine group on the *scaling function* and the *Haar wavelet* produces an orthonormal (hence Schauder) basis of $L^2(\mathbb{R})$. We also present simulations that compare reconstruction by Haar wavelets with Fourier series, including error curves. The result in $L^2(\mathbb{R})$ is obtained first in $L^2([0,1])$ and extended by translation to $\mathbb{R}$ (or via compact windows), as is standard in the literature [6, 4, **?**, 15, 16, 17].

5.1. Numerical setup and implementation details

All experiments are carried out on the interval $[-3, 3]$ using a uniform grid of $M = 2^{12}$ points. We consider the following test functions:

$$\begin{aligned} f_1(t) &= t^2, && t \in [-3,3],\\ f_2(t) &= \text{rectangular wave of period } T = 1, &&\\ f_3(t) &= \text{triangular wave of period } T = 1. && \end{aligned}$$

The first function is nonperiodic, whereas f_2 and f_3 are genuinely periodic with period $T = 1$.
For each function and each truncation level N we compute two approximations:

- a Haar approximation $S_N^{\mathrm{Haar}} f$ obtained by truncating the Haar expansion at a finest scale J such that the total number of wavelet functions is comparable to N;
- a Fourier approximation $S_N^{\mathrm{Fourier}} f$ obtained by truncating the trigonometric series after N modes.

In the Fourier case we distinguish nonperiodic and periodic situations. For f_1 we expand the periodization of f_1 on $[-3, 3]$, which has period 6. For f_2 and f_3, we compute Fourier coefficients on a single period $[0, 1]$ and extend the resulting series periodically to $[-3, 3]$. This corrects the frequency mismatch that would arise if one forced an artificial period.

To assess the quality of the approximations we use discrete versions of the L^2- and L^∞-errors:

$$E_j^{(2)}(N) = \|f_j - S_N^{\text{Haar}} f_j\|_2, \qquad \widetilde{E}_j^{(2)}(N) = \|f_j - S_N^{\text{Fourier}} f_j\|_2,$$

$$E_j^{(\infty)}(N) = \|f_j - S_N^{\text{Haar}} f_j\|_\infty, \qquad \widetilde{E}_j^{(\infty)}(N) = \|f_j - S_N^{\text{Fourier}} f_j\|_\infty,$$

where norms are approximated on the grid. For each f_j we record these errors as functions of N and represent them on log–log plots (not reproduced here).

All computations were implemented in Python using standard fast wavelet transforms and FFT-based routines. The complete source code used to generate the figures and error curves has been uploaded to a public repository:

`https://github.com/afkamelo/Schauder-Bases-Unitary-Representations.git`

and is also provided as supplementary material with the submission, so that all experiments are reproducible.

Scaling and Haar wavelet functions (constructive formulation)

Let $X = L^2([a,b])$ with $[a,b] = [0,1]$. The *scaling function* is

$$\phi(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

For $k \in \mathbb{Z}$ set $\phi_k(t) = \phi(t-k)$. The subspace

$$M = \operatorname{span}\{\phi_k : k \in \mathbb{Z}\}, \qquad V_0 = \overline{M},$$

is generated by translations of ϕ. Since the supports $[k, k+1)$ are disjoint and $\|\phi_k\|_2 = 1$, the family $\{\phi_k\}_{k\in\mathbb{Z}}$ is orthonormal in $L^2(\mathbb{R})$, and the projection coefficients are

$$c_k = \langle f, \phi_k \rangle = \int_k^{k+1} f(t)\, dt. \tag{2}$$

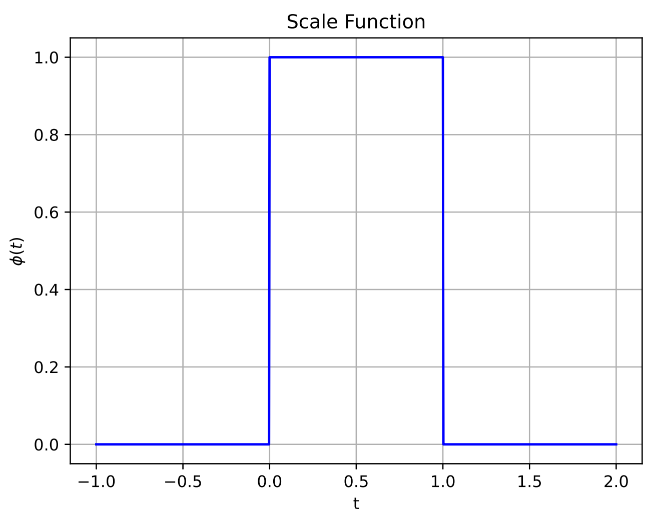

Figure 1. Scale Function

To increase resolution we dilate and translate ϕ:

$$\phi_{j,k}(t) = 2^{j/2}\,\phi(2^j t - k), \qquad j \in \mathbb{N},\ k \in \mathbb{Z}, \tag{3}$$

and define

$$V_j = \overline{\operatorname{span}}\{\phi_{j,k} : k \in \mathbb{Z}\}, \qquad V_j \subset V_{j+1}.$$

The scaling coefficients are

$$c_{j,k} = \langle f, \phi_{j,k}\rangle = \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} f(t)\cdot 2^{j/2}\,dt. \tag{4}$$

The *Haar wavelet* is

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

and satisfies $\langle \phi, \psi\rangle = 0$ on $[0,1]$. Its translations $\psi_k(t) = \psi(t-k)$ generate

$$W_0 = \overline{\operatorname{span}}\{\psi_k : k \in \mathbb{Z}\},$$

and, more generally,

$$\psi_{j,k}(t) = 2^{j/2}\,\psi(2^j t - k), \qquad W_j = \overline{\operatorname{span}}\{\psi_{j,k} : k \in \mathbb{Z}\}.$$

We have the orthogonal multiresolution decomposition

$$V_{j+1} = V_j \oplus W_j, \qquad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \quad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}). \tag{6}$$

In particular, every $g \in V_1$ admits

$$g(t) = \sum_{k\in\mathbb{Z}} c_{0,k}\,\phi_{0,k}(t) + \sum_{k\in\mathbb{Z}} d_{0,k}\,\psi_{0,k}(t), \tag{7}$$

and, in general, for $f \in L^2(\mathbb{R})$,

$$f(t) = \sum_{k\in\mathbb{Z}} c_{j,k}\,\phi_{j,k}(t) + \sum_{k\in\mathbb{Z}} d_{j,k}\,\psi_{j,k}(t), \qquad d_{j,k} = \langle f, \psi_{j,k}\rangle. \tag{8}$$

Theorem 5.1 (Orthonormality and completeness of Haar)**.** *The system*

$$\mathcal{H} = \{\phi_{0,0}\} \cup \{\psi_{j,k} : j \in \mathbb{N}_0,\ 0 \le k < 2^j\}$$

is orthonormal and complete in $L^2([0,1])$. Therefore it is an orthonormal basis of $L^2([0,1])$; consequently, it is also a Schauder basis. [6, Chs. 6–7], [19, Sec. 2.2], [17, Sec. 12.3]

Proof
Orthonormality of $\{\phi(\cdot - k)\}_k$ is immediate from disjoint supports. For ψ, the structure $(+1,-1)$ on $[0,1)$ guarantees orthogonality within and across levels, while the factors $2^{j/2}$ normalize the norm. Completeness follows from multiresolution: $V_j = \overline{\operatorname{span}}\{\phi_{j,k}\}_k$ with $V_j \subset V_{j+1}$, $V_{j+1} = V_j \oplus W_j$, and $W_j = \overline{\operatorname{span}}\{\psi_{j,k}\}_k$; moreover $\overline{\bigcup_j V_j} = L^2([0,1])$ and $\bigcap_j V_j = \{0\}$, implying $L^2([0,1]) = \bigoplus_{j\ge 0} W_j \oplus V_0$. See [6, Ch. 6], [19, Ch. 2]. □

Corollary 5.2. *For every $f \in L^2([0,1])$ there exist coefficients*

$$c_{0,0} = \langle f, \phi_{0,0}\rangle, \qquad d_{j,k} = \langle f, \psi_{j,k}\rangle$$

such that

$$f = c_{0,0}\,\phi_{0,0} + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}\,\psi_{j,k} \quad \text{with convergence in the } L^2 \text{ norm.}$$

Thus, $\mathcal{H}$ is a Schauder basis in the sense of Section 2. [14, 16]

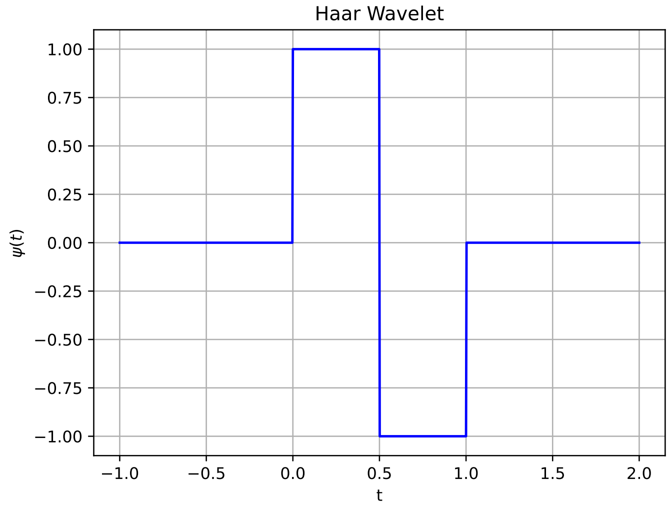

Figure 2. Haar Wavelet Function

From the affine action to the basis. Note that $\psi_{j,k} = \pi(2^{-j}, k2^{-j})\,\psi$; i.e., each $\psi_{j,k}$ is a vector in the discrete orbit of ψ under the affine representation π. The key point is that sampling the orbit on the dyadic grid $\Gamma = \{(2^{-j}, k2^{-j})\}$ produces a complete orthonormal system. In the continuous setting, the *admissibility* of ψ (integral condition $C_\psi = \int_{\mathbb{R}^*} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\, d\xi < \infty$) yields the reproduction formula

$$f(x) = \frac{1}{C_\psi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} \langle f, \pi(a,b)\psi\rangle\, \pi(a,b)\psi(x)\, \frac{db\, da}{|a|^2},$$

which is discretized using Γ to obtain bases or orthobases according to the filter (QMF) conditions [17, Sec. 12.3], [6, Chs. 6–7], [4, **?**].

Constructive procedure for the Schauder basis

1. **Generators and orbit.** Fix $\phi = \mathbf{1}_{[0,1)}$ and $\psi = \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}$. Generate the discrete orbit $\{\phi_{0,0}\} \cup \{\psi_{j,k}\}$ via dyadic dilations and translations.
2. **Trivial biorthogonality.** Use disjoint supports and signs to prove $\langle \psi_{j,k}, \psi_{j',k'}\rangle = \delta_{jj'}\delta_{kk'}$ and $\langle \psi_{j,k}, \phi_{0,0}\rangle = 0$.
3. **Density.** Let $V_j = \mathrm{span}\{\phi_{j,k}\}$. Verify $V_j \subset V_{j+1}$ and $V_{j+1} = V_j \oplus W_j$, with $W_j = \mathrm{span}\{\psi_{j,k}\}$. Then $\overline{\bigcup_j V_j} = L^2([0,1])$.
4. **Schauder.** Every orthonormal basis of a separable Hilbert space is a Schauder basis: the coordinate functionals $f \mapsto \langle f, \psi_{j,k}\rangle$ are continuous and the series converges in norm (Parseval). See [14, 16].

Numerical experiments

We use a high-resolution uniform grid on $[-3,3]$ to approximate coefficient integrals for both Haar (via projections onto $\psi_{j,k}$ mapped from $[0,1)$) and Fourier. For Fourier we distinguish two cases: (i) nonperiodic functions (quadratic), where we use the natural periodization of the interval $[-3,3]$ (period 6); (ii) periodic functions with $T = 1$ (rectangular and triangular), where coefficients are computed exactly over one period and the reconstruction is extended periodically to $[-3,3]$. This distinction corrects the frequency error that would occur if a period different from the true one of the function were forced. In all cases the number of Haar and Fourier modes is chosen so that the total number of degrees of freedom is comparable, making the error curves directly comparable.

Example 1: quadratic function $f(t) = t^2$ ***on*** $[-3, 3]$

This function is nonperiodic; its Fourier series on $[-3, 3]$ represents the periodization of the segment. The Haar approximation uses $N = 2^J$ functions (level J) and captures local changes without introducing global oscillations. Figure 3 shows the reconstructed signals, while Figure 4 depicts the pointwise errors. In our tests, the L^2-error of the Fourier approximation decays faster than that of Haar, reflecting the smoothness of t^2 and the well-known efficiency of trigonometric polynomials for smooth data; however, the Haar approximation remains competitive and exhibits very localized oscillations near the boundaries induced by periodization.

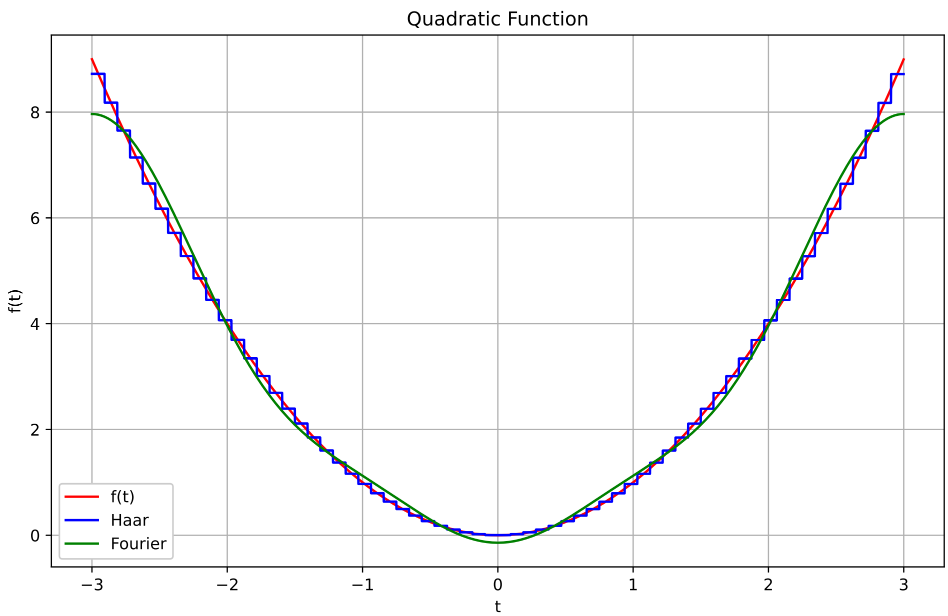

Figure 3. Representation of $f(t) = t^2$ on $[-3, 3]$ via Haar (blue) and Fourier (green).

Example 2: rectangular wave $T = 1$ ***on*** $[-3, 3]$

For the rectangular wave, Fourier coefficients are computed with period $T = 1$ and extended periodically. The Haar approximation, being localized, robustly handles discontinuities (fronts) and mitigates the global overshoots typical of Fourier (Gibbs effect). Figure 5 illustrates the reconstructions. In this case the L^∞-error of Haar is significantly smaller than that of Fourier for moderate N, since the Gibbs oscillations of trigonometric polynomials cannot be removed by simply increasing the truncation level. The L^2-errors of both methods are comparable, but Haar achieves them with highly localized basis functions.

Example 3: triangular wave $T = 1$ ***on*** $[-3, 3]$

For the triangular wave, also $T = 1$, the Fourier basis converges more smoothly than in the rectangular case due to continuity, while Haar performs well through localization and piecewise linear representation at increasing resolutions. As shown in Figure 6, both approximations exhibit small L^2-errors for moderate N, but Haar retains the advantage of local support and sparse representation of singularities in the derivative of the signal.

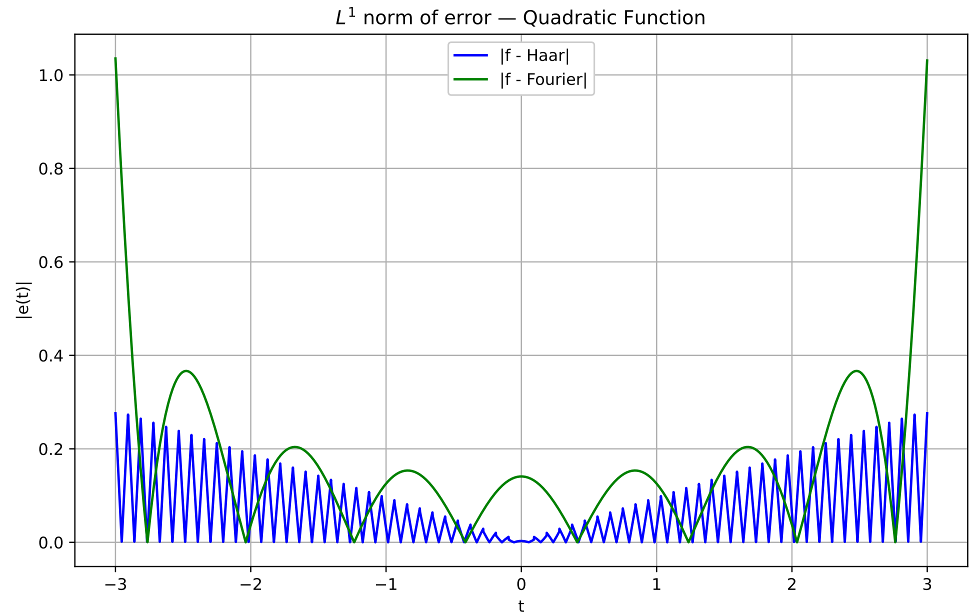

Figure 4. Pointwise errors $|f - \text{Haar}|$ and $|f - \text{Fourier}|$ for t^2 on $[-3, 3]$.

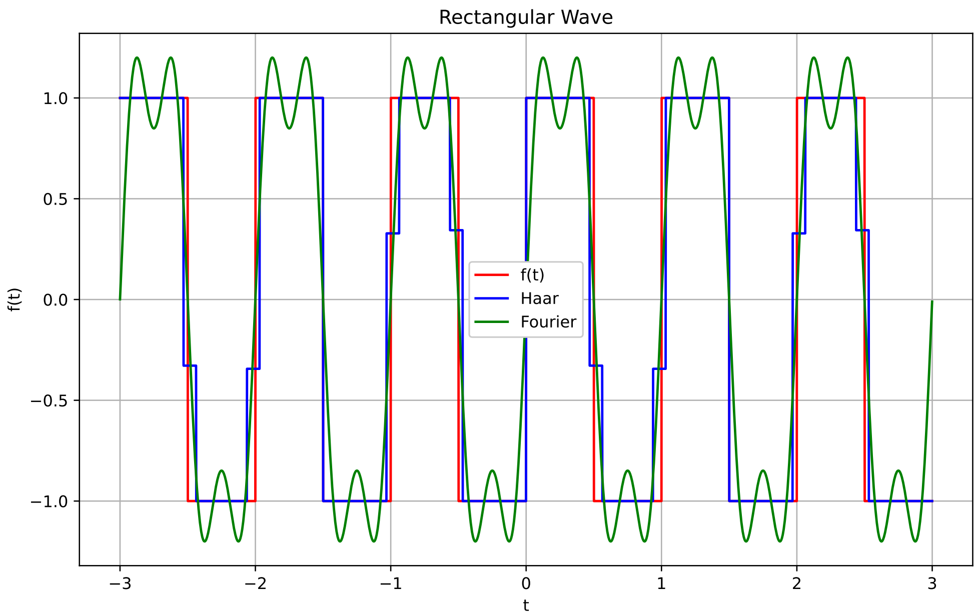

Figure 5. Representation of the rectangular wave $T = 1$ on $[-3, 3]$ via Haar (blue) and Fourier (green).

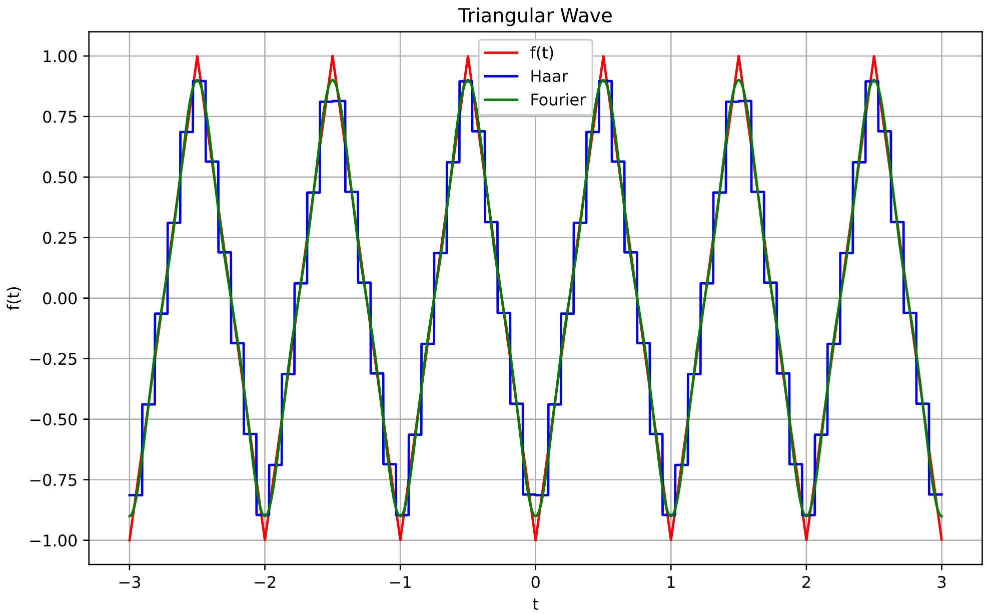

Figure 6. Representation of the triangular wave $T = 1$ on $[-3, 3]$ via Haar (blue) and Fourier (green).

5.2. Discussion and limitations

The numerical experiments confirm the qualitative picture suggested by the theory. For smooth functions such as t^2, Fourier series exhibit faster decay of the L^2-error than Haar expansions, in line with classical approximation results. For discontinuous or piecewise smooth functions, such as the rectangular and triangular waves, Haar expansions largely mitigate the Gibbs phenomenon and provide smaller L^∞-errors for comparable numbers of modes, thanks to their localization in both time and scale.

From a computational point of view, both systems admit fast transforms of complexity $O(N \log N)$, but the sparse structure of Haar coefficients for piecewise smooth signals is advantageous for thresholding and denoising tasks. On the other hand, our numerical study has several limitations: it is restricted to one-dimensional signals on a bounded interval, does not include noise or quantization effects, and does not address higher-dimensional or non-Euclidean geometries. These aspects are important in applications and will be the subject of future work.

6. Conclusion

This work articulated, from a unified viewpoint, the algebraic and functional ingredients needed to construct useful bases in function spaces: from the purely algebraic concept of a Hamel basis to the topologically meaningful notion of a Schauder basis in Banach spaces and, in particular, to orthonormal bases in Hilbert spaces. Within this framework, we showed how the action of the affine group on a generating function (Haar scaling and wavelet functions) produces, after suitable discretization, complete orthonormal systems in $L^2(\mathbb{R})$, which therefore are Schauder bases.

At an abstract level, we emphasized that topology and continuity of coordinate functionals are the elements that turn a group-generated orbit into a basis fit for analysis: density, uniform boundedness of biorthogonal functionals, and convergence in norm form the core of Schauder criteria. We saw how these criteria connect with unitary representations of locally compact groups (especially the affine representation) and with multiresolution theory, where filter conditions (QMF) capture orthonormality and the decomposition $V_{j+1} = V_j \oplus W_j$.

At a constructive level, we developed the Haar basis as a prototype: dyadic dilation and translation of the generator, sampled on the appropriate discrete lattice, yields an orthonormal basis of $L^2(\mathbb{R})$. Through numerical experiments, we showed that Haar reconstruction is competitive and, in several scenarios, preferable to Fourier series due to its localization and robustness near discontinuities (mitigation of Gibbs phenomena). We also discussed the key point for comparison with Fourier: for functions with period $T = 1$ (rectangular and triangular), the Fourier series must be computed with its natural period for a fair comparison with wavelet expansions.

Beyond the purely Hilbert-space setting, we made explicit how coorbit theory leads to Schauder-type decompositions in Banach spaces associated with a given unitary representation. In particular, we formulated a simple criterion ensuring that atomic decompositions obtained from coorbit discretizations give rise to Schauder bases in the natural Banach spaces, thereby clarifying the role of group-generated systems in non-Hilbert contexts. We also argued, at a conceptual level, that the geometry of more general groups (for instance, shearlet-type or non-unimodular groups on manifolds and graphs) has a direct impact on the existence and stability of Schauder bases built from orbits.

Several directions for future research remain open. From a theoretical perspective, it would be interesting to identify minimal conditions under which group-generated systems yield Schauder bases in Banach scales associated with coorbit spaces, and to quantify stability constants in terms of representation-theoretic parameters. From a more geometric point of view, the extension of our framework to anisotropic groups and to signal models on manifolds or graphs poses challenging questions at the interface between harmonic analysis and geometry. Finally, on the numerical side, further investigations of multi-dimensional examples, noisy data and adaptive or data-driven discretizations could provide additional insights into the role of Schauder bases in modern approximation theory and signal processing.

Acknowledgement

The authors are grateful for the support and academic environment of the Universidad Tecnológica de Pereira, whose infrastructure and guidance made this work possible. We also thank colleagues and students for valuable feedback during internal seminars.

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