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Numerical Solutions to Nonlinear Fractional Differential Equations: New Results and Comparative Study in Biological and Engineering Sciences

Eman A. A. Ziada [1], Noura Roushdy [2], Mohamed F. Abouelenein [2,*], Monica Botros [3,4]

[1] *Basic Science Department, Nile Higher Institute for Engineering and Technology, Mansoura, Egypt*
[2] *Department of Insurance and Risk Management, College of Business, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Riyadh, Saudi Arabia*
[3] *Basic Science Department, Faculty of Engineering, Delta University for Science and Technology, Gamasa 11152, Egypt*
[4] *Faculty of Artificial Intelligence, Delta University for Science and Technology, Gamasa 11152, Egypt*

Abstract Mathematical models that extend beyond classical differential equations are necessary for understanding complicated dynamical systems, such as circadian rhythm-regulated brain activity or intricate feedback loops in control engineering. Nonlinear fractional differential equations (NFDEs), especially those that are constructed using the Caputo derivative (CD), have become more common in recent years because of their ability to represent memory-dependent phenomena in a variety of biological and engineering fields. Motivated by practical applications like the circadian variation of brain metabolites and the behavior of relaxation-oscillation systems, the present work performs a thorough comparative examination of these advanced models. Three different approaches to solving the NFDEs are used: the Picard Method (PM), the Adomian Decomposition Method (ADM), and the Proposed Numerical Method (PNM). Proposed Numerical Method (PNM) extends the earlier approach by providing a systematic framework capable of handling non-zero initial conditions, incorporating adaptive step-sizing. Each approach is rigorously analyzed in terms of convergence, accompanied by detailed error estimates that confirm the existence and uniqueness of the solutions.

Keywords Nonlinear fractional models, Caputo derivative, Adomian decomposition method, Picard method, Proposed Numerical Method, Convergence analysis, Existence and uniqueness, Brain metabolites and circadian rhythm, Relaxation-oscillation equation.

AMS 2010 subject classifications 34A12, 34A34, 45G05, 92C50

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1. Introduction

Calculus stands as a cornerstone in applied mathematics, serving as a fundamental tool for modelling a vast array of physical systems [1, 2, 3]. Yet, the challenges of capturing the memory, history, and nonlocal effects inherent in many practical problems often elude traditional calculus. This is where fractional calculus emerges as a transformative and essential extension, introducing derivatives and integrals of noninteger order. This innovative approach excels at articulating the complexities of physical systems and phenomena related to memory and hereditary characteristics, making it a powerful lens through which to view intricate behaviors [4, 5]. Fractional derivatives enabled numerous diffusion processes, providing a further benefit.

Over the decades, fractional calculus has captivated the attention of researchers across multiple disciplines, from science and finance to the nuances of human behavior, especially in the modelling and control of complex systems within engineering. Its ability to bridge gaps that conventional methods cannot make fractional calculus an exciting frontier of discovery and application.

*Correspondence to: Mohamed F. Abouelenein (Email: MABOUELENEIN@IMAMU.EDU.SA).

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The significance of fractional models, defined by fractional derivatives, has captured the attention of mathematicians and scientists alike, prompting an extensive investigation into numerical and approximate solutions for fractional differential equations (FDEs). Given the absence of exact analytical solutions for the majority of these equations, researchers have turned to a variety of sophisticated numerical techniques. Among these are the homotopy perturbation method [7], the finite difference method [6], spectral methods [8], and the renowned ADM [9], and several others [10].

The applications of these fractional equations are not merely academic; they permeate our everyday lives and are crucial in diverse fields. From fluid mechanics that govern the flow of liquids and gases [11], to image processing techniques [12] that enhance and analyze visual data, and from biological models that capture the dynamics of living systems [13], physics [14], and even the design of electrical circuits and filters [15, 16], the reach of fractional calculus is vast. Its versatility illustrates how fractional derivatives can illuminate and solve complex challenges, making it an indispensable tool in the modern scientific toolkit.

The NFDEs have a lot of applications in the real world, which appear in engineering [17]-[21] and also in the different fields of science [22]-[25].

In this research, three methods are used to solve an NFDE containing CD, which has a lot of important applications, such as the relaxation-oscillation equation and brain metabolite variations under circadian rhythms. These three methods are the PM [26], the ADM [27]-[29], and the PNM [28]. The two analytical methods, ADM and PM, have many benefits: they are employed for solving different types of equations [30] in stochastic or deterministic fields, in case they are linear or nonlinear, also free from linearization and discretization. The ADM has demonstrated effectiveness and has been successfully applicable for a variety of fractional order problems. By employing the ADM, we derive a series solution; however, in practice, we often approximate this solution using a truncated series. In certain cases, this series aligns with the Taylor expansion of the correct solution in the vicinity of the initial point for initial value problems. In contrast, EM in [28], is a numerical method specifically designed to handle non-zero initial conditions, overcoming limitations of traditional techniques by Igor in [24] that is only deal with zero initial conditions. It uses an efficient discretization scheme and delivers highly accurate results with significantly reduced computational time. Comparative analysis shows that EM matches the accuracy of ADM and PM but is far more efficient, especially in solving complex models like the fractional brain metabolite system. Thus, while all methods are effective, EM is particularly advantageous for practical applications due to its speed, flexibility, and compatibility with real data.

The following is the continuation of this research: Section 2 is a display of the NFDE with its key definitions and properties. Section 3 provides the solution ADM with the convergence analysis, which contains the existence and uniqueness of the solution, and error estimation. Section 4 and 5 give the steps of the PM and PNM techniques. Section 6 obtained the numerical solution of the two important applications: the fractional brain model (FBM) and the relaxation-oscillation equation. In section 7, we give a comment on each result.

2. Nonlinear fractional differential equations

Consider the following NFDE:

$$\begin{aligned} {}^{C}D^{\gamma}y\left(\nu\right)-\omega_{1}\left(\nu\right)\phi\left(\nu,y\left(\nu\right)\right) &= \omega_{2}\left(\nu\right), \quad \nu\in\left(0,T\right],\ \gamma\in\left(n,n+1\right), && (2.1)\\ y^{(j)}(0) &= c_{j}, \qquad j=0,1,2,\cdots,n. && (2.2)\end{aligned}$$

where ${}^{C}D^{\gamma}\left(\cdot\right)$ denotes the CD.

According to the following hypothesis:

(i) $\phi : Q\times R\rightarrow R$ is continuous and measurable in ν for any $y\in R$.

(ii) A bounded measurable function exists $h\left(\nu\right)\in L_{1}(Q)$ and $\exists$ a positive constant m where $|\phi(\nu,y)|\leq h(\nu)+m|y|$.

(iii) Define the set $\mathbb{Q}_{\check{r}}=\{y\in\mathbb{C}(J):|y(\nu)|<\check{r}\}$, and $|h\left(\nu\right)|<H$.

(iv) $\omega_{1},\omega_{2}:J\times J$, and $|\omega_{i}\left(\nu\right)|<\omega_{i}, i=1,2$.

(v) Define a mapping $\Omega:\hat{E}\rightarrow\hat{E}$, where $\hat{E}$ is Banach space of continuous functions on J, and $\|y\|=\max\limits_{\nu\in J}|y(\nu)|$.

Definition 2.1. The definition of the CD of order γ is [24]

$$ {}_{0}^{C}D_{t}^{\gamma}\phi(t)=\frac{1}{\Gamma(n-\gamma)}\int_{0}^{t}\frac{\phi^{\,(n)}(\nu)d\nu}{(t-\nu)^{\gamma-n+1}},\quad n-1<\gamma<n, \tag{2.3}$$

and its corresponding FI is [24]

$$ {}^{C}I^{\gamma}\phi\left(t\right)=\frac{1}{\Gamma\left(\gamma\right)}\int_{0}^{t}\phi(\nu)\left(t-\nu\right)^{\gamma-1}d\nu,\quad 0<\gamma<1, \tag{2.4}$$

moreover

$$ \left({}^{C}I^{\gamma}\right)\left({}^{C}D^{\gamma}\right)\phi\left(\nu\right)=\phi\left(\nu\right)-\phi\left(a\right). \tag{2.5}$$

The key benefits of CF are summarized in the following points:

- No need to mention fractional initial value.
- ${}^{C}D^{\gamma}\,w=0$ for any constant w.
- It starts with the similar conditions as integer-order of differential equations.

3. Numerical methods

In this section, we present three different methods for finding numerical solutions to the equation under consideration.

3.1. First method: ADM

3.1.1. The formula of the solution

Our equation (2.1) is reduced to the integral equation (IE)

$$ y\left(\nu\right)=\sum_{j=0}^{n}\frac{c_{j}}{j!}\nu^{j}+\ {}^{C}I^{\gamma}\left[\omega_{1}\left(\nu\right)\phi\left(\nu,y\left(\nu\right)\right)+\omega_{2}\left(\nu\right)\right], \tag{3.1}$$

where $\nu\in J=(0,T],\nu\in R^{+},\phi\left(\nu,y\right)$ is continuous function satisfies Lipschitz condition

$$ \left|\phi\left(\nu,y\right)-\phi\left(\nu,z\right)\right|\leq\Re\left|y-z\right|, \tag{3.2}$$

where $\Re$ is the Lipschitz constant.
The ADM approach of the IE (3.1) is

$$ y_{0}\left(\nu\right)=\sum_{j=0}^{n}\frac{c_{j}}{j!}\nu^{j}+\ {}^{C}I^{\gamma}\left[\omega_{2}\left(\nu\right)\right],\ y_{\kappa}\left(\nu\right)=\ {}^{C}I^{\gamma}\omega_{1}\left(\nu\right)\Lambda_{\kappa-1}\left(\nu\right), \tag{3.3}$$

where $\kappa\geq 1$, and Λ_{κ} are Adomian polynomials (APs) of the nonlinear term $\phi\left(\nu,y\right)$ which is described as

$$ \Lambda_{\kappa}=\frac{1}{\kappa!}\left[\frac{d^{\kappa}}{d\lambda^{\kappa}}\phi\left(\sum_{j=0}^{\infty}\lambda^{j}y_{j}\right)\right]_{\lambda=0}. \tag{3.4}$$

The ADM solution becomes

$$ y(\nu)=\sum_{\kappa=0}^{\infty}y_{\kappa}(\nu). \tag{3.5}$$

3.1.2. Convergence analysis

Existence and uniqueness

Theorem 1
Assume that $\phi(\nu, y)$ meet the Lipschitz condition (3.2). If $T^{\gamma} < \frac{\Gamma(\gamma+1)}{\Re\omega_1}$, then the IE (3.1) has a unique solution $y \in \mathbb{C}(J)$.

Proof
From (3.1), the mapping Ω is defined as

$$\Omega y = \sum_{j=0}^{n} \frac{c_j}{j!}\nu^j + {}^{C}I^{\gamma}\left[\omega_1(\nu)\,\phi(\nu, y(\nu)) + \omega_2(\nu)\right].$$

Let $y, z \in \hat{E}$, then

$$\begin{aligned}
\Omega y - \Omega z &= {}^{C}I^{\gamma}\omega_1(\nu)\,\phi(\nu, y(\nu)) - {}^{C}I^{\gamma}\omega_1(\nu)\,\phi(\nu, z(\nu)) \\
\|\Omega y - \Omega z\| &\leq \max_{\nu \in J} {}^{C}I^{\gamma}\left|\omega_1(\nu)\,\phi(\nu, y(\nu)) - \omega_1(\nu)\,\phi(\nu, z(\nu))\right| \\
&\leq \Re\omega_1 \|y - z\| \frac{1}{\Gamma(\gamma)} \int_0^{\nu} (\nu - \hat{s})^{\gamma-1}\, d\hat{s} \\
&\leq \frac{\Re\omega_1 T^{\gamma}}{\Gamma(\gamma+1)} \|y - z\| \\
&\leq \Upsilon_1 \|y - z\|.
\end{aligned}$$

with the condition $0 < \Upsilon_1 < 1$, the mapping Ω is contraction, consequently for $T^{\gamma} < \frac{\Gamma(\gamma+1)}{\Re\omega_1}$ there exists a unique solution $y \in \mathbb{C}(Q)$. □

Corollary 2
Consider the NDE

$$\frac{dy(\nu)}{d\nu} - \omega_1(\nu)\,\phi(\nu, y(\nu)) = \omega_2(\nu), \; y(0) = c_0. \tag{3.6}$$

Operating with $I(.) = \int_0^{\nu}(.)\,d\nu$ to both sides of equation (3.6), we obtain

$$\begin{aligned}
y(\nu) &= c_0 + \int_0^{\nu} \left[\omega_1(\hat{s})\,\phi(\hat{s}, y(\hat{s})) + \omega_2(\hat{s})\right] d\hat{s} \\
&= c_0 + I\left[\omega_1(\nu)\,\phi(\nu, y(\nu)) + \omega_2(\nu)\right].
\end{aligned} \tag{3.7}$$

In problem (2.1), let $\gamma \to 1$, so (3.7) which is equivalent to (3.6) has a unique solution $y \in \mathbb{C}(Q)$, if $T < \frac{1}{\Re\omega_1}$.

Solution convergence

Theorem 3
Let the solution (3.1) exist. If $|y_1(\nu)| < \Bbbk$, where *$\Bbbk$ is a positive constant, hence the series solution (3.5) of IE (3.1) using ADM converges.*

Proof
Define a sequence $\Phi_{\breve{n}}$ where, $\Phi_{\breve{n}} = \sum_{\kappa=0}^{\breve{n}} y_{\kappa}(\nu)$ is the series of partial sums from the series solution $\sum_{\kappa=0}^{\infty} y_{\kappa}(\nu)$, and we get

$$\phi(\nu, y) = \sum_{\kappa=0}^{\infty} \Lambda_{\kappa}.$$

Taking two partial sums; $\Phi_{\breve{n}}$ and Φ_{υ} from the ADM series solution, such as $\breve{n} > \upsilon$. Now, we aim to display that $\Phi_{\breve{n}}$ is a Cauchy sequence in this Banach space $\hat{E}$.

$$\begin{aligned}
\Phi_{\breve{n}} - \Phi_{\upsilon} &= \sum_{\kappa=0}^{\breve{n}} y_{\kappa} - \sum_{\kappa=0}^{\upsilon} y_{\kappa} \\
&= {}^{C}I^{\gamma} \sum_{\kappa=0}^{\breve{n}} \omega_1(\nu) \Lambda_{\kappa-1}(\nu) - {}^{C}I^{\gamma} \sum_{\kappa=0}^{\upsilon} \omega_1(\nu) \Lambda_{\kappa-1}(\nu) \\
&= {}^{C}I^{\gamma} \left[\sum_{\kappa=\upsilon}^{\breve{n}-1} \omega_1(\nu) \Lambda_{\kappa-1}(\nu) \right] \\
&= {}^{C}I^{\gamma} \omega_1(\nu) \left[\phi(\nu, \Phi_{\breve{n}-1}) - \phi(\nu, \Phi_{\upsilon-1}) \right],
\end{aligned} \tag{3.8}$$

where $\kappa \geq 1$.

$$\begin{aligned}
\|\Phi_{\breve{n}} - \Phi_{\upsilon}\| &\leq \max_{\nu \in J} {}^{C}I^{\gamma} \omega_1(\nu) \left| \phi(\nu, \Phi_{\breve{n}-1}) - \phi(\nu, \Phi_{\upsilon-1}) \right| \\
&\leq \frac{\Re \omega_1 T^{\gamma}}{\Gamma(\gamma+1)} \|\Phi_{\breve{n}-1} - \Phi_{\upsilon-1}\| \\
&\leq \Upsilon_1 \|\Phi_{\breve{n}-1} - \Phi_{\upsilon-1}\|.
\end{aligned}$$

Let $\breve{n} = \upsilon + 1$, then

$$\|\Phi_{\upsilon+1} - \Phi_{\upsilon}\| \leq \Upsilon_1 \|\Phi_{\upsilon} - \Phi_{\upsilon-1}\| \leq \Upsilon_1^2 \|\Phi_{\upsilon-1} - \Phi_{\upsilon-2}\| \leq \cdots \leq \Upsilon_1^{\upsilon} \|\Phi_1 - \Phi_0\|.$$

From the triangle inequality, we have

$$\begin{aligned}
\|\Phi_{\breve{n}} - \Phi_{\upsilon}\| &\leq \|\Phi_{\upsilon+1} - \Phi_{\upsilon}\| + \|\Phi_{\upsilon+2} - \Phi_{\upsilon+1}\| + \cdots + \|\Phi_{\breve{n}} - \Phi_{\breve{n}-1}\| \\
&\leq \left[\Upsilon_1^{\upsilon} + \Upsilon_1^{\upsilon+1} + \cdots + \Upsilon_1^{\breve{n}-1} \right] \|\Phi_1 - \Phi_0\| \\
&\leq \Upsilon_1^{\upsilon} \left[1 + \Upsilon_1 + \cdots + \Upsilon_1^{\breve{n}-\upsilon-1} \right] \|\Phi_1 - \Phi_0\| \\
&\leq \Upsilon_1 \left[\frac{1 - \Upsilon_1^{\breve{n}-\upsilon}}{1 - \Upsilon_1} \right] \|y_1\|.
\end{aligned}$$

Now $0 < \Upsilon_1 < 1$, and $\breve{n} > \upsilon$ implies that $(1 - \Upsilon_1^{\breve{n}-\upsilon}) \leq 1$. Hence,

$$\|\Phi_{\breve{n}} - \Phi_{\upsilon}\| \leq \frac{\Upsilon_1^{\upsilon}}{1 - \Upsilon_1} \|y_1\| \leq \frac{\Upsilon_1^{\upsilon}}{1 - \Upsilon_1} \max_{\nu \in J} |y_1(\nu)|.$$

But, $|y_1(\nu)| < \Bbbk$ and as $\upsilon \to \infty$ then, $\|\Phi_{\breve{n}} - \Phi_{\upsilon}\| \to 0$ so, $\Phi_{\breve{n}}$ is a Cauchy sequence in this Banach space $\hat{E}$, then the series $\sum_{\kappa=0}^{\infty} y_{\kappa}(\nu)$ converges. □

Error analysis

Theorem 4
The maximum absolute error (MAE) of (3.5) is estimated as

$$\max_{\nu \in J} \left| y(\nu) - \sum_{\kappa=0}^{\upsilon} y_{\kappa}(\nu) \right| \leq \frac{\Upsilon_1^{\upsilon}}{1 - N_1} \max_{\nu \in J} |y_1(\nu)|. \tag{3.9}$$

Proof
From Theorem 2, we obtain $\|\Phi_{\breve{n}} - \Phi_{\upsilon}\| \leq \frac{\Upsilon_1^{\upsilon}}{1-\Upsilon_1} \max_{\nu \in J} |y_1(\nu)|$.

But, $\Phi_{\check{n}} = \sum\limits_{\kappa=0}^{\check{n}} y_\kappa(\nu)$ as $\check{n} \to \infty$ then, $\Phi_{\check{n}} \to y(\nu)$, so

$$\|y(\nu) - \Phi_\upsilon\| \leq \frac{\Upsilon_1^\upsilon}{1-\Upsilon_1} \max_{\nu \in J} |y_1(\nu)|,$$

then the MAE in the interval J is

$$\max_{\nu \in J} \left| y(\nu) - \sum_{\kappa=0}^{\upsilon} y_\kappa(\nu) \right| \leq \frac{\Upsilon_1^\upsilon}{1-\Upsilon_1} \max_{\nu \in J} |y_1(\nu)|.$$

□

3.2. Second method: PM

Applying PM to (3.1), the solution would be

$$y_0(\nu) = \sum_{j=0}^{n} \frac{c_j}{j!} \nu^j + {}^C I^\gamma \omega_2(\nu), \tag{3.10}$$

$$y_\kappa(\nu) = y_0(\nu) + {}^C I^\gamma \omega_1(\nu) \phi\left(\nu, y_{(\kappa-1)}(\nu)\right), \tag{3.11}$$

where the functions $y_\kappa(\nu)$ are continuous functions as

$$y_\kappa(\nu) = y_0(\nu) + \sum_{\kappa=1}^{n} (y_\kappa - y_{\kappa-1}), \tag{3.12}$$

so the sequence y_κ is convergent and comparable to the infinite series $\sum (y_\kappa - y_{\kappa-1})$. Hence PM solution takes the form

$$y(\nu) = \lim_{\kappa \to \infty} y_\kappa(\nu), \tag{3.13}$$

lastly, we can presume that if the series $\sum (y_\kappa - y_{\kappa-1})$ is convergent, further the sequence $y_\kappa(\nu)$ is convergent to $y(\nu)$. To show that $\{y_\kappa(\nu)\}$ is informally convergent, assume the associated series

$$\sum_{\kappa=1}^{\infty} [y_\kappa(\nu) - y_{\kappa-1}(\nu)]. \tag{3.14}$$

From (3.10) for $\kappa = 1$, we get

$$\begin{aligned}
y_1(\nu) - y_0(\nu) &= {}^C I^\gamma \omega_1(\nu) \phi(\nu, y_0(\nu)) \\
|y_1(\nu) - y_0(\nu)| &= \left| {}^C I^\gamma \omega_1(\nu) \phi(\nu, y_0(\nu)) \right| \\
&\leq \frac{\omega_1}{\Gamma(\gamma)} \int_0^\nu (\nu - \hat{s})^{\gamma-1} [|h(\nu)| + m |y_0(\hat{s})|] \, d\hat{s} \\
&\leq \frac{\omega_1}{\Gamma(\gamma)} \left[\int_0^\nu (\nu - \hat{s})^{\gamma-1} |h(\nu)| \, d\hat{s} + m \int_0^\nu (\nu - \hat{s})^{\gamma-1} |y_0(\hat{s})| \, d\hat{s} \right] \\
&\leq \frac{\omega_1 T^\gamma}{\Gamma(\gamma+1)} [H + \check{r} m] \\
&\leq \psi_1,
\end{aligned} \tag{3.15}$$

where $\psi_1 = \frac{\omega_1 T^\gamma}{\Gamma(\gamma+1)} [H + \check{r} m]$. Now, we get an estimate for $y_\kappa(\nu) - y_{\kappa-1}(\nu)$, $\kappa \geqslant 2$,

$$\begin{aligned}
y_\kappa(\nu) - y_{\kappa-1}(\nu) &= {}^C I^\gamma \omega_1(\nu) \phi(\nu, y_{\kappa-1}(\nu)) - {}^C I^\gamma \omega_1(\nu) \phi(\nu, y_{\kappa-2}(\nu)) \\
|y_\kappa(\nu) - y_{\kappa-1}(\nu)| &\leq \omega_1 \, {}^C I^\gamma |\phi(\nu, y_{\kappa-1}(\nu)) - \phi(\nu, y_{\kappa-2}(\nu))| \\
&\leq \frac{\omega_1 \Re T^\gamma}{\Gamma(\gamma+1)} |y_{\kappa-1}(\nu) - y_{\kappa-2}(\nu)| \\
\|y_\kappa - y_{\kappa-1}\| &\leq \Upsilon_1 \|y_{\kappa-1} - y_{\kappa-2}\|.
\end{aligned} \tag{3.16}$$

In the relationship mentioned above, if we put $\kappa = 2$, and use (3.15) we get

$$\begin{aligned} \|y_2 - y_1\| &\leq \Upsilon_1 \, \|y_1(\nu) - y_0(\nu)\| \\ \|y_2 - y_1\| &\leq \Upsilon_1 \, \psi_1. \end{aligned}$$

Doing the same for $\kappa = 3, 4, \cdots$,

$$\begin{aligned} \|y_3 - y_2\| &\leq \Upsilon_1 \, \|y_2 - y_1\| \leq \Upsilon_1^2 \, \psi \\ |y_4 - y_3| &\leq \Upsilon_1 \, \|y_3 - y_2\| \leq \Upsilon_1^3 \, \psi_1 \\ &\vdots \end{aligned}$$

Subsequently, this relation's general form ought to be

$$|y_\kappa - y_{\kappa-1}| \leq \Upsilon_1^{\kappa-1} \, \psi_1. \tag{3.17}$$

Since $\Upsilon_1 < 1$, then the series

$$\sum_{\kappa=1}^{\infty} \left[y_\kappa(\nu) - y_{\kappa-1}(\nu) \right],$$

is convergent. Thus, the sequence $\{y_\kappa(\nu)\}$ can be convergent, as $\phi(\nu, y(\nu))$ is continuous in y, then

$$y(\nu) = \lim_{\kappa \to \infty} {}^C I^\gamma \omega_1(\nu) \, \phi\left(\nu, y_{(\kappa-1)}(\nu)\right) \; = \; {}^C I^\gamma \omega_1(\nu) \, \phi(\nu, y(\nu)).$$

3.3. *Third method: PNM (El-Sayed method)*

This technique was applied for the first time in [28] and used to overcome the shortage in the numerical method given in [24]. It has a drawback in that it solves only FDE with zero initial conditions (IC), but here we solve this drawback.

IC are changed to be homogeneous through substitution, so the substitution is

$$X = y - \sum_{p=0}^{n-1} c_j \frac{\nu^p}{p!}, \tag{3.18}$$

use the following relations to generate the solution algorithm:

$$\begin{aligned} D^{\gamma_i} X &= h^{-\gamma_i} \sum_{p=0}^{m} w_p^{(\gamma_i)} X_{m-p}, \quad w_p^{(\gamma_i)} = (-1)^p \frac{\Gamma(\gamma_i + 1)}{\Gamma(\gamma_i + 1 - p)}, \\ \nu_m &= mh \ (m = 0, 1, 2, \ldots), \quad X(\nu) = X_m(\nu_m), \end{aligned} \tag{3.19}$$

where $w_p^{(\gamma_i)}$ represents the fractional binomial coefficients or weights derived from the fractional derivative approximation, computed as $w_p^{(\gamma_i)} = (-1)^p \binom{\gamma_i}{p}$ and X_m are the discrete numerical approximation of the unknown function y at the m-th time
then get the values of X_m. Go back to the initial variables, from (3.18), we have

$$y_m = \sum_{p=0}^{n-1} c_j \frac{(\nu_m)^p}{p!} + X_m. \tag{3.20}$$

Therefore, the PNM utilizes convolution weights to successfully handle non-zero initial conditions, resulting in enhanced accuracy and stability.

4. Stability

Theorem 5
There exists a unique solution of the FDE (2.1)-(2.2) is consistently stable if the hypothesis $(i)-(v)$ is satisfied.

Proof
Let $\widetilde{y}$ is also a solution to the FDE (2.1)-(2.2)

$$\begin{aligned}
|y(v)-\widetilde{y}(v)| &= \left|\left[\sum_{j=0}^{n}\frac{c_j}{j!}\nu^j + {}^{C}I^{\gamma}\left[\omega_1(\nu)\phi(\nu,y(\nu))+\omega_2(\nu)\right]\right]\right. \\
&\quad \left. -\left[\sum_{j=0}^{n}\frac{\widetilde{c}_j}{j!}\nu^j + {}^{C}I^{\gamma}\left[\omega_1(\nu)\phi(\nu,\widetilde{y}(\nu))+\omega_2(\nu)\right]\right]\right| \\
&\le \left|\sum_{j=0}^{n}\frac{c_j}{j!}\nu^j-\sum_{j=0}^{n}\frac{\widetilde{c}_j}{j!}\nu^j\right| + {}^{C}I^{\gamma}\left|\omega_1(\nu)\phi(\nu,y(\nu))-\omega_1(\nu)\phi(\nu,\widetilde{y}(\nu))\right| \\
&\le \left|\sum_{j=0}^{n}\frac{[c_j-\widetilde{c}_j]}{j!}\nu^j\right| + {}^{C}I^{\gamma}\left|\omega_1(\nu)\left[\phi(\nu,y(\nu))-\phi(\nu,\widetilde{y}(\nu))\right]\right| \\
&\le \sum_{j=0}^{n}\frac{|c_j-\widetilde{c}_j|}{j!}\nu^j + {}^{C}I^{\gamma}\left|\omega_1(\nu)\right|\left|\phi(\nu,y(\nu))-\phi(\nu,\widetilde{y}(\nu))\right| \\
&\le \sum_{j=0}^{n}\frac{|c_j-\widetilde{c}_j|}{j!}\nu^j + \Re\omega_1\left|y(\nu)-\widetilde{y}(\nu)\right|\frac{1}{\Gamma(\gamma)}\int_0^{\nu}(\nu-\hat{s})^{\gamma-1}\,d\hat{s} \\
&\le \sum_{j=0}^{n}\frac{|c_j-\widetilde{c}_j|}{j!}\nu^j + \frac{\Re\omega_1T^{\gamma}}{\Gamma(\gamma+1)}\left|y(\nu)-\widetilde{y}(\nu)\right|
\end{aligned}$$

so,

$$\begin{aligned}
|y(\nu)-\widetilde{y}(\nu)|-\frac{\Re\omega_1T^{\gamma}}{\Gamma(\gamma+1)}|y(\nu)-\widetilde{y}(\nu)| &\le \sum_{j=0}^{n}\frac{|c_j-\widetilde{c}_j|}{j!}\nu^j \\
\left[1-\frac{\Re\omega_1T^{\gamma}}{\Gamma(\gamma+1)}\right]|y(\nu)-\widetilde{y}(\nu)| &\le \sum_{j=0}^{n}\frac{|c_j-\widetilde{c}_j|}{j!}\nu^j \\
|y(\nu)-\widetilde{y}(\nu)| &\le \left[1-\frac{\Re\omega_1T^{\gamma}}{\Gamma(\gamma+1)}\right]^{-1}\left(\sum_{j=0}^{n}\frac{\nu^j}{j!}\right)\left|C-\widetilde{C}\right|.
\end{aligned}$$

where C denotes the vector c_j and $\widetilde{C}$ denotes the vector $\widetilde{c}_j$.

If $\left[1-\frac{\Re\omega_1T^{\gamma}}{\Gamma(\gamma+1)}\right]^{-1}\left(\sum_{j=0}^{n}\frac{\nu^j}{j!}\right)<\delta(\varepsilon)$, then $|y(\nu)-\widetilde{y}(\nu)|<\varepsilon$, which implies that the solution of the FDE (2.1)-(2.2) is consistently stable. □

5. Numerical applications

5.1. Application 1

The relaxation-oscillation equation [24]

$$\begin{aligned}
{}^{C}D^{\gamma}y(\nu)+Ay(\nu) &= g(v), \quad \gamma\in n\,[n,n+1], \\
y^{(j)}(0) &= 0, \quad j=0,1,\cdots,n. \qquad (5.1)
\end{aligned}$$

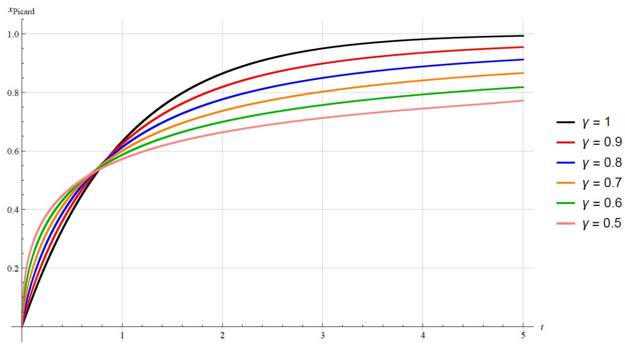

Figure 1. PM solution of (5.2) at many values of γ for case (1)

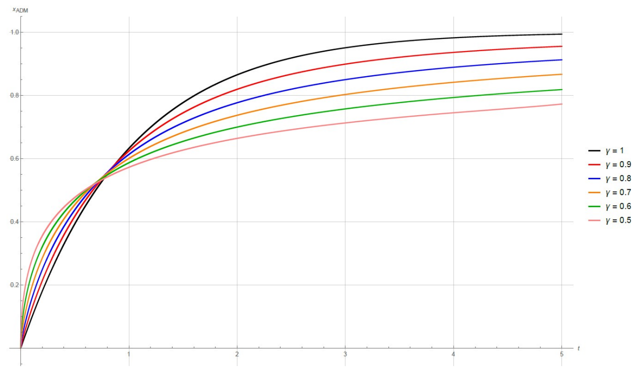

Figure 2. ADM solution of (5.2) at many values of γ for case (1)

It is an important application in control theory [taking $A = 1$, $g(v) = H(\tau)$ and $H(\tau)$ is the unit step function].

The reduced IE of FDE (5.1) is

$$y(\nu) = {}^{C}I^{\gamma}\left[H\left(v\right) - y(\nu)\right]. \tag{5.2}$$

ADM solution: applying ADM to equation (5.2), we get

$$\begin{aligned} y_0(v) &= {}^{C}I^{\gamma}\left[H\left(v\right)\right], \\ y_{\kappa}(v) &= -\,{}^{C}I^{\gamma}\left[y(\nu)\right], \quad \kappa \geq 1. \end{aligned} \tag{5.3}$$

PM solution: using PM to equation (5.2), the solution algorithm is

$$\begin{aligned} y_0(v) &= {}^{C}I^{\gamma}\left[H\left(v\right)\right], \\ y_{\kappa}(v) &= y_0(v) - {}^{C}I^{\gamma}\left[y_{\kappa-1}\left(v\right)\right], \quad \kappa \geq 1. \end{aligned} \tag{5.4}$$

EM solution: using EM to equation (5.2), the solution algorithm is

$$X_m = \frac{1}{h^{-\gamma}}\left(H\left(v_m\right) - \left(X_{m-1} + y_0\right) - h^{-\gamma}\sum_{p=1}^{m} w_p^{(\gamma)} X_{m-p}\right), \quad m = 1, 2, 3, \cdots. \tag{5.5}$$

From the above solution algorithms we find the following figures: Figures 1-3 show ADM, PM, and EM solutions at differing values of γ ($\gamma = 0.5$, 0.6, 0.7, 0.8, 0.9, and 1), while Figures 4-6 show ADM, PM, and EM solutions at differing values of γ ($\gamma = 1.2$, 1.4, 1.6, 1.8, and 2).

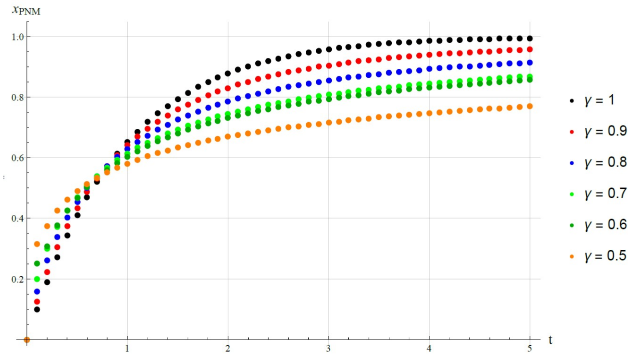

Figure 3. EM solution of (5.2) at many values of γ for case (1)

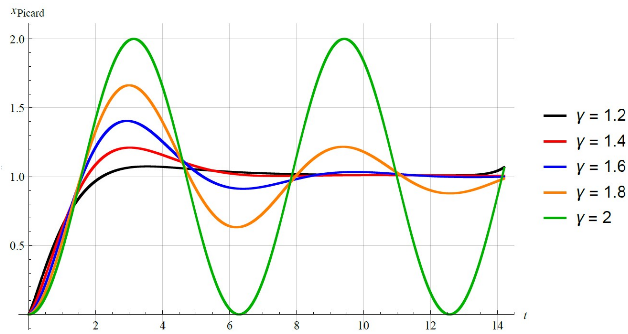

Figure 4. PM solution of (5.2) at many values of γ for case (2)

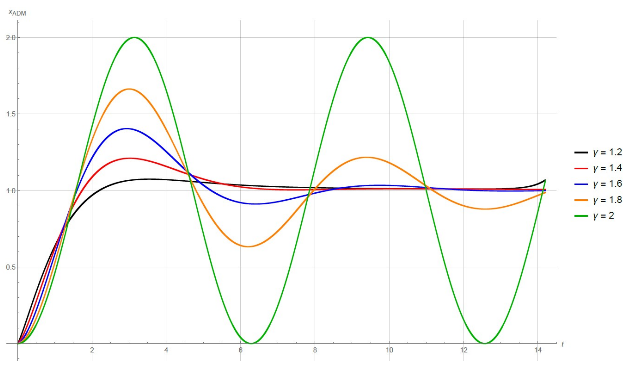

Figure 5. ADM solution of (5.2) at many values of γ for case (2)

Table 1, shows the absolute difference (AD) between $(ADM, PM, \text{and } EM)$**.** And Table 2, gives the time used to get $(ADM, PM, \text{and } EM)$ solutions**.** We see that the ADM and PM solutions match with each other.

From Table 2, we see that EM takes less time than ADM and PM. Also, ADM takes less time than PM.

If we take a simple case at $(\alpha = 1)$, the exact solution will be $(1 - e^{-t})$. Table 3, shows the absolute difference (AD) between the exact solution and $(ADM, PM, \text{and } EM)$ solutions**.**

Also, we add plot error of ADM, Picad and EM approches.

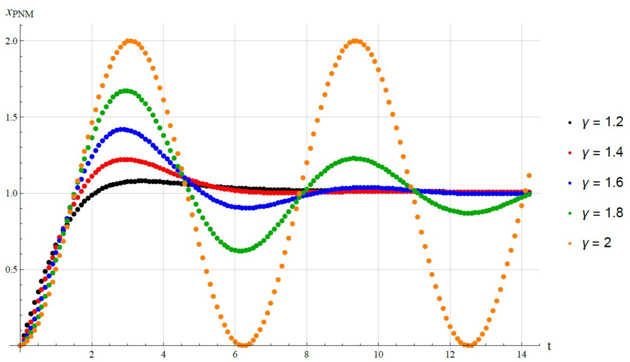

Figure 6. EM solution of (5.2) at many values of γ for case (2)

Table 1. AD between (ADM, PM, and EM) of (5.2) at $\gamma = 0.5$

v	$\lvert y_{ADM} - y_{EM}\rvert$	$\lvert y_{Picard} - y_{EM}\rvert$	$\lvert y_{ADM} - y_{Picard}\rvert$
0.5	0.01288530	0.01288530	0.0000000000
1	0.00842845	0.00842845	0.0000000000
1.5	0.00620493	0.00620493	0.0000000000
2	0.00485923	0.00485923	2.22045×10^{-16}
2.5	0.00395817	0.00395817	4.44089×10^{-16}
3	0.00331341	0.00331341	1.11022×10^{-16}
3.5	0.00281746	0.00281746	7.77156×10^{-16}
4	0.00232473	0.00232473	1.77636×10^{-15}
4.5	0.00127888	0.00127888	2.44249×10^{-16}
5	0.00277928	0.00277928	8.43769×10^{-15}

Table 2. Time used to (ADM, PM, and EM) solutions of (5.2)

Case $no.$	ADM Time ($v = 30$)	PM Time ($\kappa = 30$)	EM Time ($h = 0.1$)
$Case$ (1) ($\gamma = 0.5, 0.6, 0.7, 0.8, 0.9,$ and 1)	118.282 $Sec.$	823.297 Sec.	0.328 Sec.
$Case$ (2) ($\gamma = 1.2, 1.4, 1.6, 1.8,$ and 2)	125.343 $Sec.$	323.828 Sec.	0.687 Sec.

Table 3. AD between exact sol. and (ADM, PM, and EM) solutions of (5.2) at $\gamma = 1$

v	$\lvert y_{Exact} - y_{EM}\rvert$	$\lvert y_{Exact} - y_{ADM}\rvert$	$\lvert y_{Exact} - y_{Picard}\rvert$
1	0.019201	1.11022×10^{-16}	1.11022×10^{-16}
2	0.0137586	2.22045×10^{-16}	2.22045×10^{-16}
3	0.00739591	0.0000000000	0.0000000000
4	0.00353476	3.33067×10^{-16}	3.33067×10^{-16}
5	0.00158417	7.93809×10^{-14}	7.93809×10^{-14}

5.2. Application 2

The **FBM**, The circadian rhythm model of variations in brain metabolites are introduced for integer orders in [31]. They consider this model as the ODE [31], here we discuss it with fractional orders in two cases as follows:

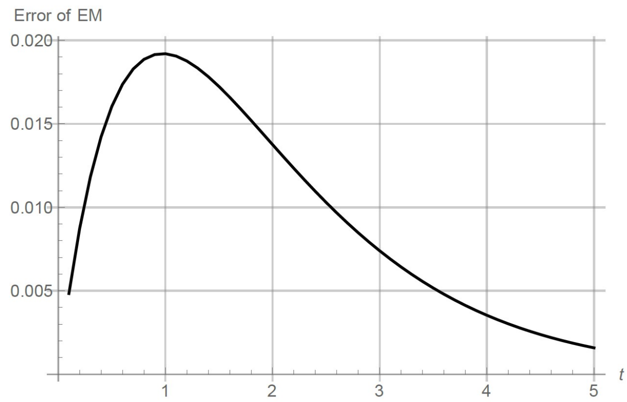

Figure 7. Error of EM solution of (5.2)

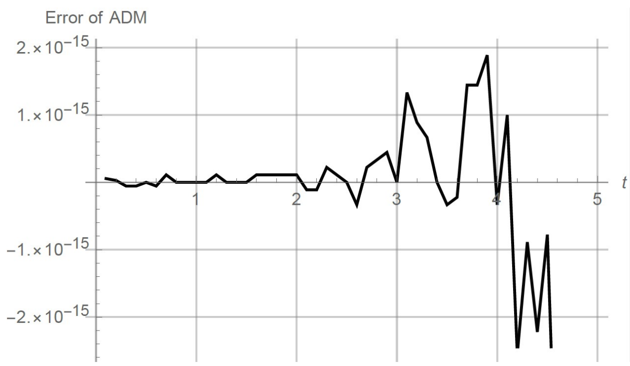

Figure 8. Error of ADM solution of (5.2)

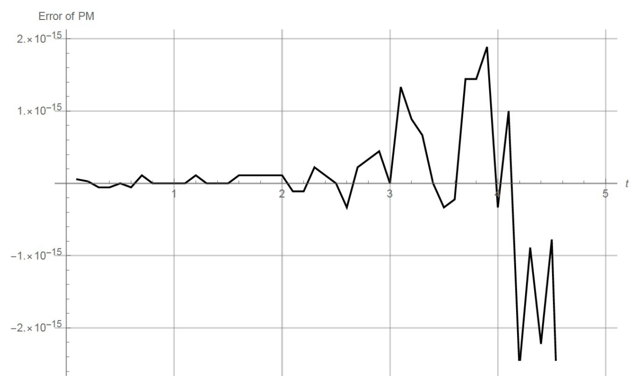

Figure 9. Error of PM solution of (5.2)

Case (1):

$$\begin{aligned} {}^{C}D^{\gamma}y+\frac{2y}{5.5+y} &= 0.9\sin^{2}\left(0.265v+0.9\right), \\ y\left(0\right) &= 0.8 \end{aligned} \tag{5.6}$$

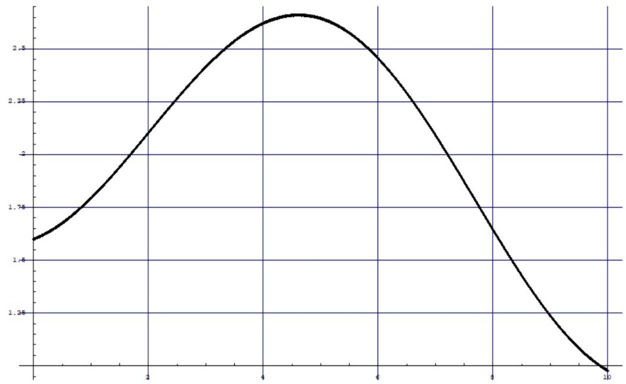

Figure 10. EM solution of (5.7) at specific values of $\gamma = 1$ for case (1)

The reduced IE of FBM (5.6) is

$$y(\nu) = 0.8 + {}^{C}I^{\gamma}\left[0.9\sin^{2}(0.265v + 0.9) - \frac{2y}{5.5 + y}\right]. \tag{5.7}$$

ADM solution: applying ADM to equation (5.7), we get

$$\begin{aligned} y_0(v) &= 0.8 + {}^{C}I^{\gamma}\left[0.9\sin^{2}(0.265v + 0.9)\right], \\ y_\kappa(v) &= -{}^{C}I^{\gamma}\left[\Lambda_{\kappa-1}(v)\right], \quad \kappa \geq 1. \end{aligned} \tag{5.8}$$

PM solution: using PM to equation (5.7), the solution algorithm is

$$\begin{aligned} y_0(v) &= 0.8 + {}^{C}I^{\gamma}\left[0.9\sin^{2}(0.265v + 0.9)\right], \\ y_\kappa(v) &= y_0(v) - {}^{C}I^{\gamma}\left[\frac{2y_{\kappa-1}}{5.5 + y_{\kappa-1}}\right], \quad \kappa \geq 1. \end{aligned} \tag{5.9}$$

EM solution: the solution algorithm of FBM (5.6) using EM is

$$\begin{aligned} X_m &= \frac{1}{h^{-\gamma}}\Bigg(0.9\sin^{2}(0.265v_m + 0.9) - \frac{2\left(X_{m-1} + y_0\right)}{5.5 + \left(X_{m-1} + y_0\right)} \\ &\quad - h^{-\gamma}\sum_{j=1}^{m}\mathfrak{W}_j^{(\gamma)}X_{m-j}\Bigg), \qquad m = 1, 2, 3, \cdots. \end{aligned} \tag{5.10}$$

Figures 7-12 show EM solutions at specific values of γ ($\gamma = 1, 0.9, 0.8, 0.7, 0.6$, and 0.5), respectively.
Case (2):

$$\begin{aligned} {}^{C}D^{\gamma}y + \frac{0.7y}{10.7 + y} &= 0.8\sin^{2}(0.1325v + 1.4), \\ y(0) &= 6.4 \end{aligned} \tag{5.11}$$

The solution algorithm of FBM (5.11) using EM is

$$\begin{aligned} X_m &= \frac{1}{h^{-\gamma}}\Bigg(0.8\sin^{2}(0.1325v_m + 1.4) - \frac{0.7\left(X_{m-1} + y_0\right)}{10.7 + \left(X_{m-1} + y_0\right)} \\ &\quad - h^{-\gamma}\sum_{j=1}^{m}\mathfrak{W}_j^{(\gamma)}X_{m-j}\Bigg), \qquad m = 1, 2, 3, \cdots. \end{aligned} \tag{5.12}$$

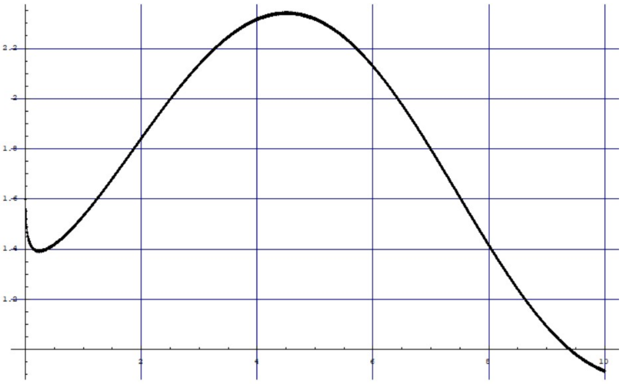

Figure 11. EM solution of (5.7) at specific values of $\gamma = 0.9$ for case (1)

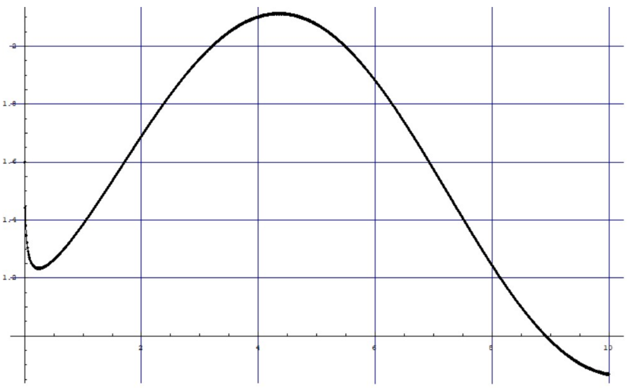

Figure 12. EM solution of (5.7) at specific values of $\gamma = 0.8$ for case (1)

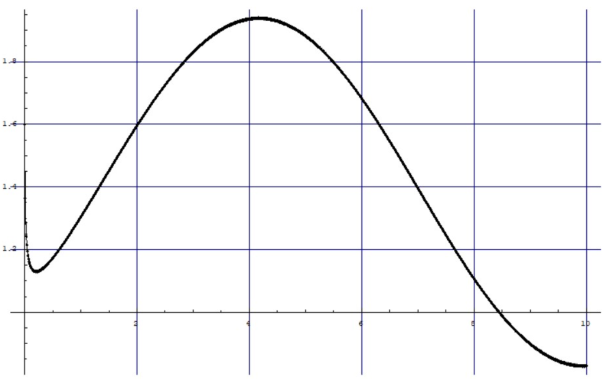

Figure 13. EM solution of (5.7) at specific values of $\gamma = 0.7$ for case (1)

Figures 13-18 show EM solutions at different values of γ ($\gamma = 1$, 0.9, 0.8, 0.7, 0.6, and 0.5), respectively.

The oscillation's damping and frequency are influenced by the fractional order γ, as the fractional order γ decreases, it affects how the system reflects past states. The solution to the relaxation-oscillation equation exhibits increased damping, characteristic of systems with stronger memory effects. Higher values of γ allow the behavior to look like standard models with faster oscillations, as lower values tend to slow down the response, resulting in more durable oscillations and less damping. In terms of mathematics, varying γ modifies the system's phase

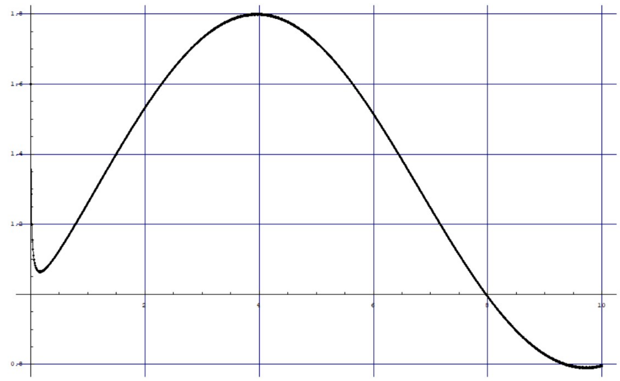

Figure 14. EM solution of (5.7) at specific values of $\gamma = 0.6$ for case (1)

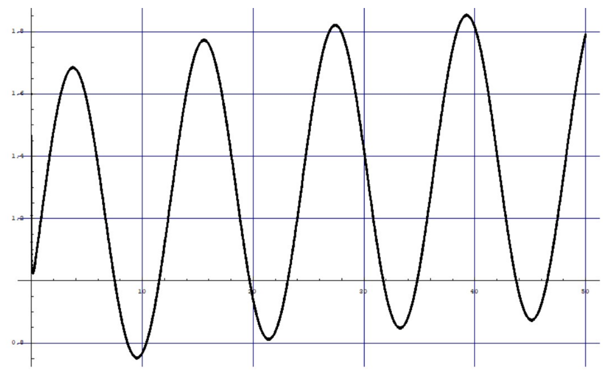

Figure 15. EM solution of (5.7) at specific values of $\gamma = 0.5$ for case (1)

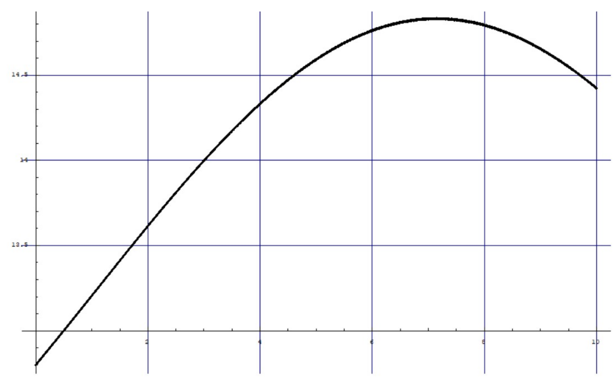

Figure 16. EM solution of (5.7) at specific values of $\gamma = 1$ for case (2)

and amplitude, which affects how it responds over time. This means that different metabolic or circadian rhythms, such as slower or longer swings, can be modeled by the fractional order. In general, changing γ makes it easier to comprehend and execute biological and physical processes that occur in the real world.

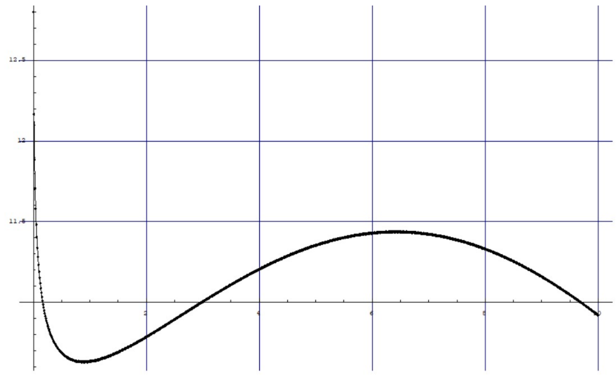

Figure 17. EM solution of (5.7) at specific values of $\gamma = 0.9$ for case (2)

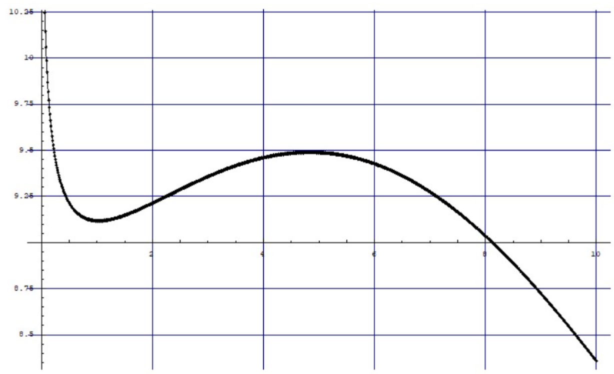

Figure 18. EM solution of (5.7) at specific values of $\gamma = 0.8$ for case (2)

Figure 19. EM solution of (5.7) at specific values of $\gamma = 0.7$ for case (2)

6. Conclusion

In this research, we introduce analytical and numerical solutions to a nonlinear FDEs with Caputo derivatives. This kind of this equations have a lot of important applications such as the circadian rhythm model of variations in brain metabolites and the relaxation-oscillation equation. The existence and uniqueness of its solution are demonstrated. Three different methods are used to get the solution: two analytical methods (ADM and PM), while the third is a numerical method (EM). We conduct a comparative analysis of the EM against both the PM and the ADM.

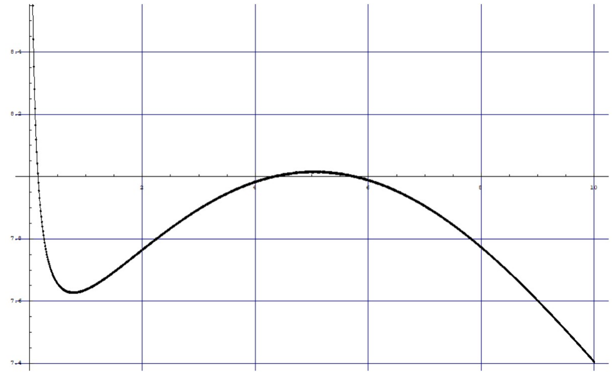

Figure 20. EM solution of (5.7) at specific values of $\gamma = 0.6$ for case (2)

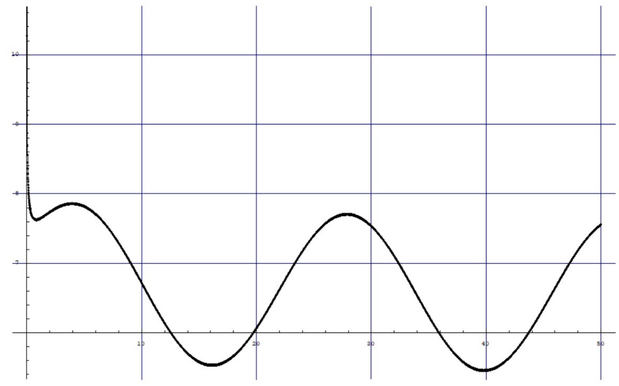

Figure 21. EM solution of (5.7) at specific values of $\gamma = 0.5$ for case (2)

Our findings demonstrate that PNM is more effectively applicable to get the solution of the FBM because of the difficulty in evaluating the integrals, while we can get the solution easily by using EM. A comparison between the three methods is given in example 1, we see that EM takes less time than ADM and PM. Also, ADM takes less time than PM (see table 2). Moreover, we see that the EM solution coincides with the solutions given in [31] for integer order ($\gamma = 1$) which coordinates perfectly with the medical data (see: [31]) and our solution in Figures 7 and 13.

Conflict of interest

This work does not have any conflicts of interest.

REFERENCES

1. Grace, S. R., Chhatria, G. N., Kaleeswari, S., Alnafisah, Y., & Moaaz, O. (2025). Forced-perturbed fractional differential equations of higher order: Asymptotic properties of non-oscillatory solutions. *Fractal and Fractional*, 9(1), 6.
2. Moaaz, O., & Abouelregal, A. E. (2023). Multi-fractional-differential operators for a thermo-elastic magnetic response in an unbounded solid with a spherical hole via the DPL model. *AIMS Math.*, 8(3), 5588–5615. https://doi.org/10.3934/math.2023282
3. Moaaz, O., Abouelregal, A. E., & Alesemi, M. (2022). Moore–Gibson–Thompson photothermal model with a proportional Caputo fractional derivative for a rotating magneto-thermoelastic semiconducting material. *Mathematics*, 10(17), 3087.
4. Elshenhab, A. M., Wang, X., Cesarano, C., Almarri, B., & Moaaz, O. (2022). Finite-time stability analysis of fractional delay systems. *Mathematics*, 10(11), 1883.

5. Ganesh, A., Deepa, S., Baleanu, D., Santra, S. S., Moaaz, O., Govindan, V., & Ali, R. (2022). Hyers–Ulam–Mittag–Leffler stability of fractional differential equations with two Caputo derivatives using fractional Fourier transform. *AIMS Mathematics*, 7(2), 1791–1810. https://doi.org/10.3934/math.2022103
6. Sweilam, N. H., Ahmed, S. M., & Adel, M. (2020). A simple numerical method for two-dimensional nonlinear fractional anomalous sub-diffusion equations. *Math. Meth. Appl. Sci.*, 12(2), 1–20.
7. Khader, M. M. (2012). Introducing an efficient modification of the homotopy perturbation method by using Chebyshev polynomials. *Arab J. Math. Sci.*, 18(1), 61–71.
8. Khader, M. M., & Adel, M. (2020). Numerical approach for solving the Riccati and logistic equations via QLM-rational Legendre collocation method. *Comput. Appl. Math.*, 39(166), 1–9.
9. Verma, P., & Kumar, M. (2022). An analytical solution of linear/nonlinear fractional-order partial differential equations and with new existence and uniqueness conditions. *Proc. Natl. Acad. Sci. India Sect. A Phys. Sci.*, 92(1), 47–55.
10. Saeed, K. (2013). An integral operational matrix based on Jacobi polynomials for solving fractional-order differential equations. *Appl. Math. Modell.*, 37, 1126–1136.
11. Inc, M., Aliyu, A., Yusuf, A., & Baleanu, D. (2018). Time-fractional Cahn–Allen and time-fractional Klein–Gordon equations: Lie symmetry analysis, explicit solutions and convergence analysis. *Physica A*, 493, 94–106.
12. Cuesta, E., & Finat, J. (2003). Image processing by means of a linear integro-differential equations. *Int. Assoc. Sci. Technol. Dev.*, 5(1), 438–442.
13. Chu, Y. M., Ali, Y. M., Khan, M. A., Islam, S., & Ullah, S. (2021). Dynamics of fractional order COVID-19 model with a case study of Saudi Arabia. *Results Phys.*, 21(103787), 1–10.
14. Hashemi, M. S., Inc, M., & Yusuf, A. (2020). On three-dimensional variable order time fractional chaotic system with nonsingular kernel. *Chaos Solitons Fractals*, 133(109628), 1–10.
15. Zhang, L., Kartci, A., Elwakil, A., Bagci, H., & Salama, K. N. (2021). Fractional-order inductor: design, simulation, and implementation. *IEEE Access*, 9, 73695–73702.
16. Mahata, S., Kar, R., & Mandal, D. (2021). Optimal analog-to-digital transformation of fractional-order Butterworth filter using binomial series expansion with Al-Alaoui operator. *Int. J. Circ. Theory Appl.*, 49(1), 44–79.
17. Hasanen, A., & De la Sen, M. (2021). Tripled fixed point techniques for solving system of tripled-fractional differential equations. *AIMS Math.*, 6(3), 2330–2343.
18. Daraghmeh, A., Qatanani, N., & Saadeh, A. (2020). Numerical solution of fractional differential equations. *Appl. Math.*, 11, 1100–1115. https://doi.org/10.4236/am.2020.1111074
19. Syam, M. I., & Al-Refai, M. (2019). Fractional differential equations with Atangana–Baleanu fractional derivative: Analysis and applications. *Chaos Solitons Fractals X*, 2, 1–5.
20. Fernand, A. A. (2021). Complex analysis approach to Atangana–Baleanu fractional calculus. *Math. Meth. Appl. Sci.*, 44, 8070–8087. https://doi.org/10.1002/mma.5754FERNANDTF8087
21. Atangana, A., & Alabaraoye, E. (2013). Solving a system of fractional partial differential equations arising in the model of HIV infection of CD4+ cells and attractor one-dimensional Keller–Segel equations. *Adv. Differ. Equ.*, 94, 1–14.
22. Rida, S. Z., & Arafa, A. A. M. (2011). New method for solving linear fractional differential equations. *Int. J. Differ. Equ.*, 1–8. https://doi.org/10.1155/2011/814132
23. Miller, K. S., & Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley-Interscience, New York.
24. Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press, New York.
25. Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier, New York.
26. El-Sayed, A. M. A., Hashem, H. H. G., & Ziada, E. A. A. (2014). Picard and Adomian decomposition methods for a quadratic integral equation of fractional order. *Comput. Appl. Math.*, 33, 95–109.
27. Zwillinger, D. (1997). *Handbook of Differential Equations*. Academic Press.
28. El-Sayed, A. M. A., El-Kalla, I. L., & Ziada, E. A. A. (2010). Analytical and numerical solutions of multi-term nonlinear fractional orders differential equations. *Appl. Numer. Math.*, 60(8), 788–797.
29. Ziada, E. (2013). Numerical solution for nonlinear quadratic integral equations. *J. Fract. Calc. Appl.*, 7(7), 1–12.
30. Adomian, G. (1995). *Solving Frontier Problems of Physics: The Decomposition Method*. Kluwer.
31. Hatchond, L., Guillevin, C., Naudin, M., Cherfils, L., Miranville, A., & Guillevin, R. (2019). Mathematical modeling of brain metabolites variations in the circadian rhythm. *AIMS Math.*, 5(1), 216–225.