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S-Index Computation for Various Corona Product Graphs

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Abstract Degree based topological indices are excellent tools for distinguishing structural isomers. The S-index, which is calculated using the fifth power of the degree of each vertex, performs well in molecules where branching strongly affects activity. It is particularly powerful when applied to complex molecular graphs, such as nanotubes, polymer structures, and different corona product graph construction. In this paper, we analyze the S-index in various corona product graphs.

Keywords Zagreb indices: F-index; S-index corona product graphs; graph operations.

AMS 2010 subject classifications 05C09, 05C76

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1. Introduction

Graph theory is the branch of mathematics which represent the connection between objects. Graph theory has many applications in various fields such as computer science(Network flow), Engineering (Electrical circuit design), Chemistry (Molecular graph, topological indices), Data science (Graph based machine learning), etc. The principle of graph theory plays a crucial role in the field of mathematical chemistry. Chemical graph theory is a specialized branch of mathematical chemistry that represents chemical compounds as graph, where atoms as vertices and chemical bonds as edges.
A topological index is a numerical parameter obtained from a graph, which has an important role in theoretical chemistry. Topological indices characterize the physiochemical properties like estimate boiling point, melting point, heat formation, density, drug design. These indices are widely used in QSPR/QSAR. Topological indices compute complex graph operations such as corona products which helps in understanding and analyzing large or complex molecular structures. Chemical graph theory provides a formal mathematical framework for computing topological indices. In practical, Zagreb Indices are one among the best applications to recognize the physical properties and chemical reactions. First Zagreb index $M_1(H)$ and Second Zagreb index $M_2(H)$ were introduced by I Gutman and N Trinajstric[4] in 1972.

$$\begin{aligned} M_1(H) &= \sum_{v\in V(H)} d_H(v)^2 \\ &= \sum_{uv\in E(H)} (d_H(u)+d_H(v)) \\ M_2(H) &= \sum_{uv\in E(H)} d_H(u)d_H(v) \end{aligned}$$

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In 2019, X. Liu and Z. Zhang[7] introduced the first general Zagreb index as:

$$M_1^{\alpha+1}(H) = \sum_{v\in V(H)} d_H^{\alpha+1}(v)$$
$$= \sum_{uv\in E(H)} (d_H^{\alpha}(u) + d_H^{\alpha}(v))$$

The first and second hyper geographic index of a connected graph H is defined by MH. Khalifeha, H. Yousefi-Azaria and AR. Ashraf [6] as

$$HM_1(H) = \sum_{uv\in E(H)} (d_H(u) + d_H(v))^2$$
$$HM_2(H) = \sum_{uv\in E(H)} (d_H(u)d_H(v))^2$$

some graph operations based on second geographic index can refer [10] and [13].
B. Furtula and I. Gutman[3] in 2015 introduced forgotten index [F-index]

$$F(G) = \sum_{v\in V(H)} d_H(v)^3$$
$$= \sum_{uv\in E(H)} (d_H(u)^2 + d_H(v)^2)$$

Few more details can refer from [2] [5] [8].
In 2020, A. Alameri , et al.[1] introduced Y-index as

$$Y(H) = \sum_{v\in V(H)} d_H(v)^4$$
$$= \sum_{uv\in E(H)} (d_H(u)^3 + d_H(v)^3)$$

Some special graph operation in Y-index can refer from [12] and [14].
In 2021, S. Nagarajan, G. Kayalvizhi and G. Priyadharsini[11] introduced S-index as

$$S(H) = \sum_{v\in V(H)} d_H(v)^5$$
$$= \sum_{uv\in E(H)} (d_H(u)^4 + d_H(v)^4)$$

2. Preliminaries

Let $H = (V(H), E(H))$ be a simple connected graph with vertex set $V(H)$, and edge set $E(H)$. The number of vertices and edges in H be n and m respectively. For any vertex $v \in V(H)$ the degree of v is the number of edges incident on the vertex v and it is denoted by $d_H(v)$.

Let H_1 and H_2 be two connected simple graphs. Number of vertices and edges of H_1 be n_1 and m_1 respectively and n_2 and m_2 are denoted as number of vertices and edges of H_2.

Corona product of two graphs H_1 and H_2 denoted by $H_1 \circ H_2$ is obtained by taking one copy of H_1 and n_1 copies of H_2 and joining each vertex i^{th} copy of H_2 to i^{th} vertex of H_1 where $1 \le i \le n$. The corona product of H_1 and H_2 has total number of $(n_1n_2 + n_1)$ vertices and $(m_1 + n_1m_2 + n_1n_2)$ edges. These are referred from [2],[9], [15] In this work we propose S-index in various corona product graphs based on subdivision vertex corona, subdivision edge corona, subdivision vertex neighbourhood corona, subdivision edge neighbourhood corona, the vertex-edge corona.

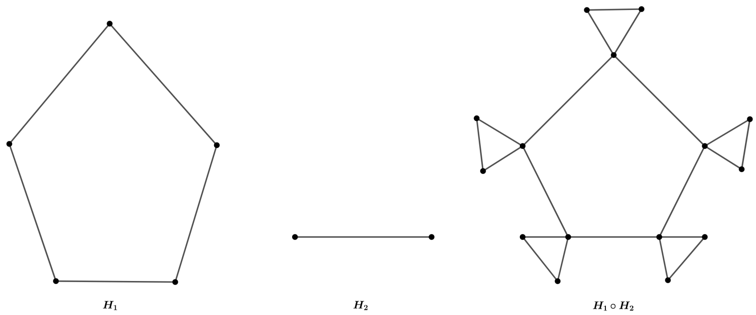

Figure 1

Here we consider the S-index[11] of some special graph as complete graph K_n, cycle graph C_n, path graph P_n, Star graph S_n,Wheel graph W_n, Ladder graph L_n

1. $S(K_n) = n(n-1)^5, n \geq 3$
2. $S(C_n) = 32n, n \geq 3$
3. $S(P_n) = 32n - 62, n \geq 3$
4. $S(S_n) = (n-1)^5 + (n-1), n \geq 3$
5. $S(W_n) = 243n + n^5, n \geq 3$
6. $S(L_n) = 128 + 243(2n-4), n \geq 2$

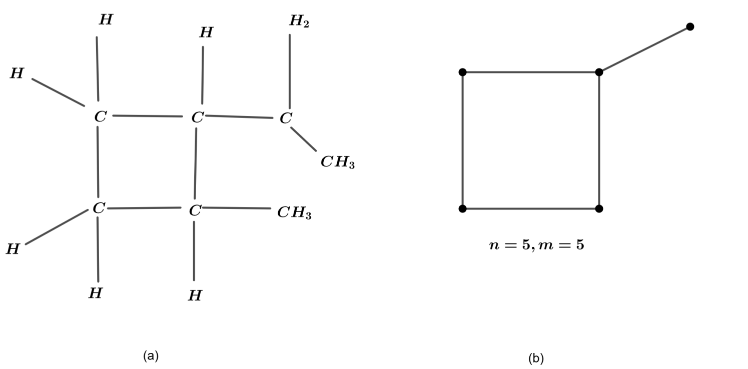

Figure 2. (a) - Methylcyclobutane C_5H_{10} and (b) - Carbon skeleton C_5H_{10} as Graph H_1

$$
\begin{aligned}
S(H_1) &= \sum_{v \in V(H_1)} d_{H_1}(v)^5 \\
&= 1^5 + 3^5 + 3(2)^5 \\
&= 340
\end{aligned}
$$

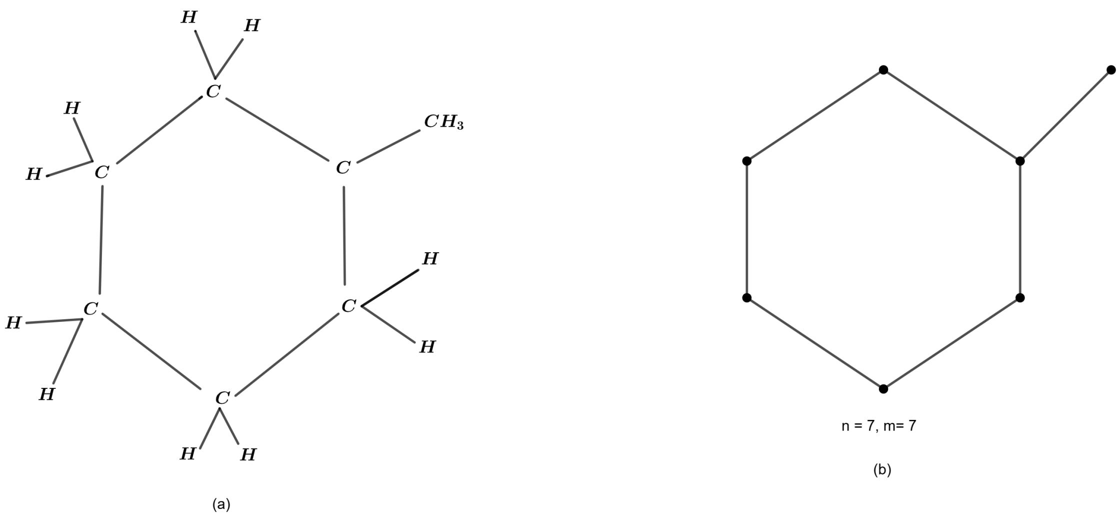

Figure 3. (a)- Ethylcyclohexane C_7H_{14} and (b)- Carbon skeleton C_7H_{14} as Graph H_2

$$
\begin{aligned}
S(H_2) &= 1^5 + 3^5 + 5(2)^5 \\
&= 404
\end{aligned}
$$

The subdivision graph of a graph H is denoted by SD(H) is obtained by inserting a new vertex in each edge of H by a path of length 2. It contains $V(SD(H)) = V(H) \cup E(H)$ and $E(SD(H)) = 2E(H)$. In Figure 4. $V(H) = \{a, b, c\}$, $E(H) = \{1, 2, 3\}$, $V(SD(H)) = \{a, b, c, d, e, f\}$ and $E(SD(H)) = \{1, 2, 3, 4, 5, 6\}$]Basic

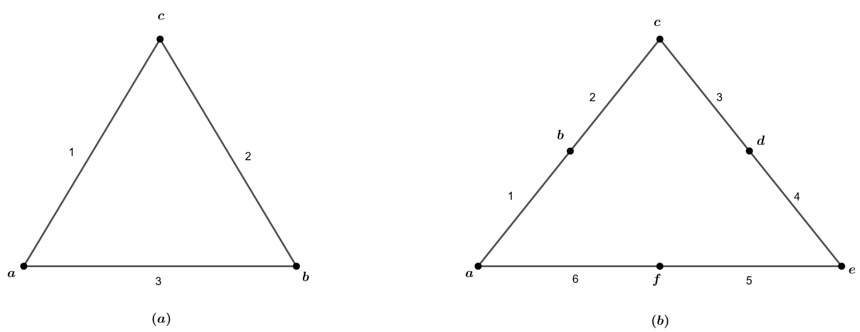

Figure 4. (a) - Graph H and (b) - Subdivision graph H

Properties of the S-index

Here we consider the S-index[11] of some special graph as complete graph K_n, cycle graph C_n, path graph P_n, Star graph S_n,Wheel graph W_n, Ladder graph L_n

1. $S(K_n) = n(n-1)^5, n \geq 3$
2. $S(C_n) = 32n, n \geq 3$
3. $S(P_n) = 32n - 62, n \geq 3$
4. $S(S_n) = (n-1)^5 + (n-1), n \geq 3$
5. $S(W_n) = 243n + n^5, n \geq 3$
6. $S(L_n) = 128 + 243(2n-4), n \geq 2$

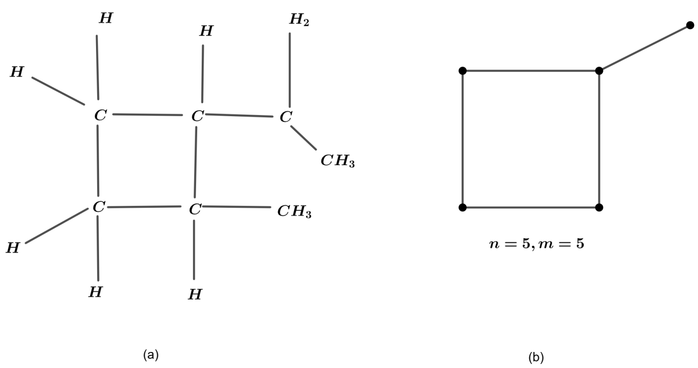

Figure 5. (a) - Methylcyclobutane C_5H_{10} and (b) - Carbon skeleton C_5H_{10} as Graph H_1

$$
\begin{aligned}
S(H_1) &= \sum_{v \in V(H_1)} d_{H_1}(v)^5 \\
&= 1^5 + 3^5 + 3(2)^5 \\
&= 340
\end{aligned}
$$

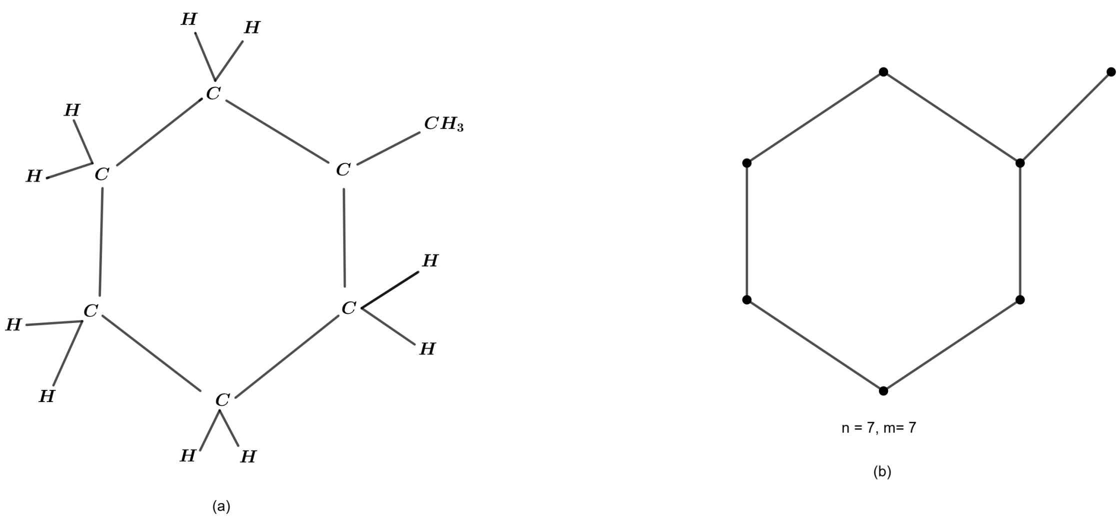

Figure 6. (a)- Ethylcyclohexane C_7H_{14} and (b)- Carbon skeleton C_7H_{14} as Graph H_2

$$S(H_2) = 1^5 + 3^5 + 5(2)^5$$
$$= 404$$

The subdivision graph of a graph H is denoted by SD(H) is obtained by inserting a new vertex in each edge of H by a path of length 2. It contains $V(SD(H)) = V(H) \cup E(H)$ and $E(SD(H)) = 2E(H)$. In Figure 4. $V(H) = \{a, b, c\}$, $E(H) = \{1, 2, 3\}$, $V(SD(H)) = \{a, b, c, d, e, f\}$ and $E(SD(H)) = \{1, 2, 3, 4, 5, 6\}$

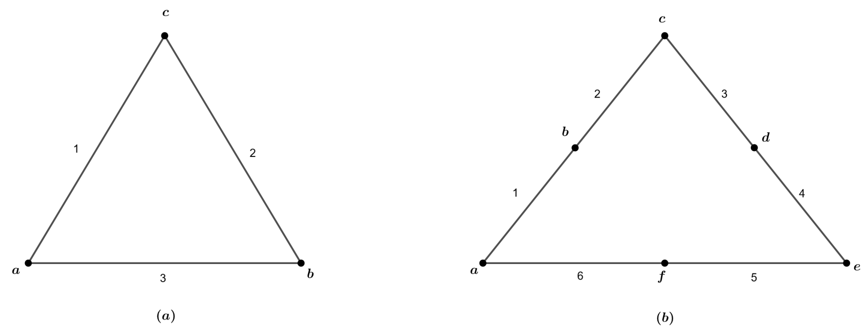

Figure 7. (a) - Graph H and (b) - Subdivision graph H

3. Main Result

3.1. Subdivision-Vertex Corona

Definition 3.1

Let H_1 and H_2 be two vertex disjoint graphs. The subdivision - vertex corona of H_1 and H_2 is denoted by $H_1 \boxdot H_2$ and obtained from $SD(H_1)$ and n_1 copies of H_2, all vertex-disjoint by joining the i^{th} vertex of $V(H_1)$ to every vertex in the i^{th} copy of H_2. Above definition $H_1 \boxdot H_2$ has $n_1(1 + n_2) + m_1$ vertices and $2m_1 + n_1(n_2 + m_2)$ edges.

The degree of vertices $H_1 \boxdot H_2$ is given by

$$d_{H_1 \boxdot H_2}(v_i) = d_{H_1}(v_i) + n_2 \qquad \text{for } i = 1, 2, ...n_1$$
$$d_{H_1 \boxdot H_2}(e_i) = 2 \qquad \text{for } i = 1, 2, ..., m_1$$
$$d_{H_1 \boxdot H_2}(u_j^i) = d_{H_2}(u_j) + 1 \qquad \text{for } i = 1, 2, ..., n_1$$
$$j = 1, 2, ..., n_2$$

Theorem 3.1. *The S-index of the subdivision-vertex corona $H_1 \boxdot H_2$ is given by*

$$S(H_1 \boxdot H_2) = S(H_1) + 5n_2 Y(H_1) + 10n_2^2 F(H_1) + 10n_2^3 M_1(H_1) + 10n_2^4 m_1 + n_1 n_2^5 + 32m_1$$
$$+ n_1 S(H_2) + 5n_1 Y(H_2) + 10n_2 F(H_2) + 10n_1 M_1(H_2) + 10n_1 m_2 + n_1 n_2$$

Proof

From the definition of subdivision-vertex corona $H_1 \boxdot H_2$ is divided as

$v_i \in V(H_1)$	$e_i \in E(H_1)$	$u_j \in V(H_2)$
$d_{H_1}(v_i) + n_2$	2	$d_{H_2}(u_j) + n_1$

Table 1. Degree of the vertices of $H_1 \boxdot H_2$

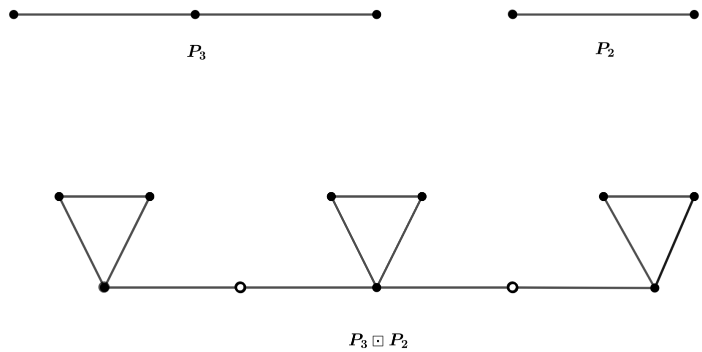

Figure 8. Subdivision - vertex corona

$$
\begin{aligned}
S(H_1 \boxdot H_2) &= \sum_{i=1}^{n_1}[d_{H_1}(v_i)+n_2]^5 + \sum_{i=1}^{m_1} 2^5 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}[d_{H_2}(u_j)+1]^5 \\
&= \sum_{i=1}^{n_1} d_{H_1}(v_i)^5 + 5n_2\sum_{i=1}^{n_1} d_{H_1}(v_i)^4 + 10n_2^2\sum_{i=1}^{n_1} d_{H_1}(v_i)^3 + 10n_2^3\sum_{i=1}^{n_1} d_{H_1}(v_i)^2 + 5n_2^4\sum_{i=1}^{n_1} d_{H_1}(v_i) \\
&\quad + n_2^5(n_1) + \sum_{i=1}^{m_1} 2^5 + n_1\sum_{j=1}^{n_2} d_{H_2}(u_j)^5 + 5n_1\sum_{j=1}^{n_2} d_{H_2}(u_j)^4 + 10n_1\sum_{j=1}^{n_2} d_{H_2}(u_j)^3 \\
&\quad + 10n_1\sum_{j=1}^{n_2} d_{H_2}(u_j)^2 + 5n_1\sum_{j=1}^{n_2} d_{H_2}(u_j) + n_1\sum_{j=1}^{n_2} 1^5. \\
S(H_1 \boxdot H_2) &= S(H_1) + 5n_2Y(H_1) + 10n_2^2F(H_1) + 10n_2^3M_1(H_1) + 10n_2^4m_1 + n_1n_2^5 + 32m_1 + n_1S(H_2) \\
&\quad + 5n_1Y(H_2) + 10n_2F(H_2) + 10n_1M_1(H_2) + 10n_1m_2 + n_1n_2
\end{aligned}
$$

□

3.2. Subdivision - Edge Corona

Definition 3.2
The subdivision-edge corona product $H_1 \oplus H_2$ of H_1 and H_2 is obtained from the $SD(H_1)$ and n_1 copies of H_2 such that for all disjoint vertices joining the i^{th} vertex of $SD(H_1)$ to every vertex in the i^{th} copy of H_2. $H_1 \oplus H_2$ has $m_1(1+n_2)+n_1$ vertices and $m_1(n_2+m_2+2)$ edges.

The degree of vertices of $H_1 \oplus H_2$ is given by

$$
\begin{aligned}
d_{H_1\oplus H_2}(v_i) &= d_{H_1}(v_i), && \text{for i= 1,2,...} n_1 \\
d_{H_1\oplus H_2}(e_i) &= 2+n_2, && \text{for i= 1,2,...} m_1 \\
d_{H_1\oplus H_2}(u_j^i) &= d_{H_2}(u_j)+1, && \text{for i= 1,2,...} n_1 \\
& && \text{j = 1,2,...,} n_2
\end{aligned}
$$

Theorem 3.2. *The S-index of subdivision - edge corona $H_1 \oplus H_2$ is given by*
$S(H_1 \oplus H_2) = S(H_1) + m_1(2+n_2)^5 + m_1S(H_2) + 5m_1Y(H_2) + 10m_1F(H_2) + 10m_1M_1(H_2) + 11m_1n_2.$

Proof
From the definition of subdivision - edge corona $H_1 \oplus H_2$ is given by

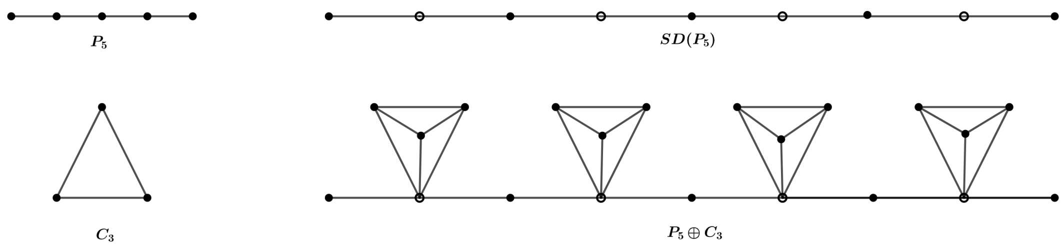

Figure 9. Subdivision - edge corona

$v_i \in V(H_1)$	$e_i \in E(H_1)$	$u_j \in V(H_2)$
$d_{H_1}(v)$	$2+n_2$	$d_{H_2}(u_j)+1$

Table 2. Degree of the vertices of $H_1 \oplus H_2$

$$S(H_1 \oplus H_2) = \sum_{i=1}^{n_1} d_{H_1}(v_i)^5 + \sum_{i=1}^{m_1}(2+n_2)^5 + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2}[d_{H_2}(u_j)+1]^5$$

$$= \sum_{i=1}^{n_1} d_{H_1}(v_i)^5 + m_1(2+n_2)^5 + m_1\sum_{j=1}^{n_2} d_{H_2}(u_j)^5 + 5m_1\sum_{j=1}^{n_2} d_{H_2}(u_j)^4 + 10m_1\sum_{j=1}^{n_2} d_{H_2}(u_j)^3$$

$$+ 10m_1\sum_{j=1}^{n_2} d_{H_1}(u_j)^2 + 5m_1\sum_{j=1}^{n_2} d_{H_2}(u_j) + m_1\sum_{j=1}^{n_2} 1$$

$$S(H_1 \oplus H_2) = S(H_1) + m_1(2+n_2)^5 + m_1 S(H_2) + 5m_1 Y(H_2) + 10m_1 F(H_2) + 10m_1 M_1(H_2) + 11m_1 n_2$$

□

3.3. Subdivision - vertex neighbourhood corona

Definition 3.3

The subdivision - vertex neighbourhood corona product $H_1 \odot H_2$ of H_1 and H_2 is obtained from $SD(H_1)$ and n_1 copies of H_2 all disjoint by joining the neighbours of the i^{th} vertex of $V(H_1)$ to every vertex in the i^{th} copy of H_2. The degree of the vertices of $H_1 \odot H_2$ is given by

$$d_{H_1 \odot H_2}(v_i) = d_{H_1}(v_i) \qquad \text{for i = 1,2,...} n_1$$

$$d_{H_1 \odot H_2}(e_i) = 2n_2 + 2 \qquad \text{for i = 1,2,...} m_1$$

$$d_{H_1 \odot H_2}(u_j^i) = d_{H_2}(u_j) + d_{H_1}(v_i) \qquad \text{for i = 1,2,...} n_1, \ \text{j = 1,2,...} n_2$$

Theorem 3.3. *The S-index of $H_1 \odot H_2$ is given by*

$$S(H_1 \odot H_2) = (n_2+1)S(H_1) + n_1 S(H_2) + 10m_1 Y(H_2) + 10m_2 Y(H_1) + 10F(H_2)M_1(H_1)$$
$$+ 10M_1(H_2)F(H_1) + m_1(2n_2+2)^5$$

Proof

From the definition of $H_1 \odot H_2$ we have

$v_i \in V(H_1)$	$e_i \in E(H_1)$	$v_i \in V(H_1)$ $u_j \in V(H_2)$
$d_{H_1}(v_i)$	$2n_2+2$	$d_{H_2}(u_j)+d_{H_1}(v_i)$

Table 3. Degree of the vertices of $H_1 \odot H_2$

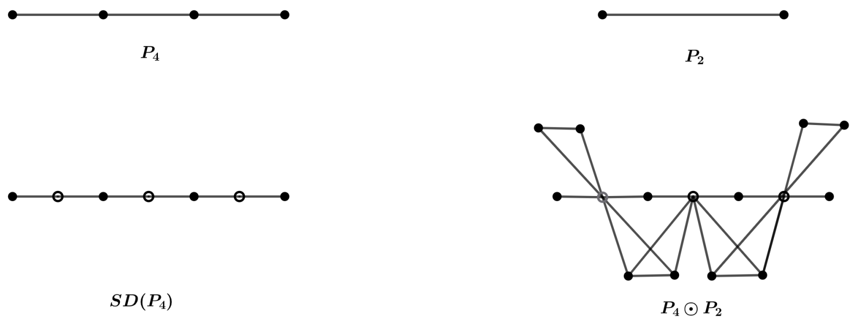

Figure 10. Subdivision - vertex neighbourhood corona

$$\begin{aligned}
S(H_1 \odot H_2) &= \sum_{i=1}^{n_1} d_{H_1}(v_i)^5 + \sum_{i=1}^{m_1} (2n_2+2)^5 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} [d_{H_2}(u_j) + d_{H_1}(v_i)]^5 \\
&= \sum_{i=1}^{n_1} d_{H_1}(v_i)^5 + m_1(2n_2+2)^5 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} [d_{H_2}(u_j)^5 + 5d_{H_2}(u_j)^4 d_{H_1}(v_i) + 10d_{H_2}(u_j)^3 d_{H_1}(v_i)^2 \\
&\quad + 10d_{H_2}(u_j)^2 d_{H_1}(v_i)^3 + 5d_{H_2}(u_j) d_{H_1}(v_i)^4 + d_{H_1}(v_i)^5] \\
&= S(H_1) + m_1(2n_2+2)^5 + n_1 S(H_2) + 5Y(H_2)(2m_1) + 10F(H_2)M_1(H_1) + 10M_1(H_2)F(H_1) \\
&\quad + 5(2m_2)Y(H_1) + n_2 S(H_1) \\
&= (n_2+1)S(H_1) + n_1 S(H_2) + 10m_1 Y(H_2) + 10m_2 Y(H_1) \\
&\quad + 10F(H_2)M_1(H_1) + 10M_1(H_2)F(H_1) + m_1(2n_2+2)^5
\end{aligned}$$

□

3.4. Subdivision - edge neighbourhood corona

Definition 3.4

The subdivision - edge neighbourhood corona H_1 and H_2 is denoted by $H_1 \square H_2$ and obtained from $SD(H_1)$ and n_1 copies of H_2, all vertex disjoint, by joining the neighbours of the i^{th} vertex of $V(H_1)$ to every vertex in the i^{th} copy of H_2. The degree of the vertices of $H_1 \square H_2$ are given by

$$\begin{aligned}
d_{H_1 \square H_2}(v_i) &= (n_2+1)d_{H_1}(v_i) && \text{for i = 1,2,...}n_1 \\
d_{H_1 \square H_2}(e_i) &= 2 && \text{for i = 1,2,...}m_1 \\
d_{H_1 \square H_2}(u_j^i) &= d_{H_2}(u_j) + 2 && \text{for i = 1,2,...}n_1 \\
& && \text{j = 1,2,...}n_2
\end{aligned}$$

Figure 11. Subdivision - edge neighbourhood corona

Theorem 3.4. *The S-index of $H_1 \Box H_2$ is given by*

$$S(H_1 \Box H_2) = (n_2+1)^5 S(H_1) + 32m_1 + n_1 S(H_2) + 10n_1 Y(H_2) + 40n_1 F(H_2) + 80n_1 M_1(H_2) + 160m_2 n_1 + 32n_1 n_2$$

Proof

From the definition of Subdivision-edge neighbourhood corona product of two graphs we have

$v_i \in V(H_1)$	$e_i \in E(H_1)$	$u_j \in V(H_2)$
$(n_2+1)d_{H_1}(v_i)$	2	$d_{H_2}(u_j)+2$

Table 4. Degree of the vertices of $H_1 \Box H_2$

$$S(H_1 \Box H_2) = \sum_{i=1}^{n_1}(n_2+1)^5 d_{H_1}(v_i)^5 + \sum_{i=1}^{m_1}(2)^5 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}[d_{H_2}(u_j)+2]^5$$

$$= (n_2+1)^5 \sum_{i=1}^{n_1} d_{H_1}(v_i)^5 + 32m_1 + n_1 \sum_{j=1}^{n_2}[d_{H_2}(u_j)^5 + 10d_{H_2}(u_j)^4 + 40d_{H_2}(u_j)^3 + 80d_{H_2}(u_j)^2 + 80d_{H_2}(u_j) + 32]$$

$$= (n_2+1)^5 S(H_1) + 32m_1 + n_1 S(H_2) + 10n_1 Y(H_2) + 40n_1 F(H_2) + 80n_1 M_1(H_2) + 160m_2 n_1 + 32n_1 n_2$$

□

3.5. The Vertex-edge corona

Definition 3.5

The Vertex-edge corona of two graphs H_1 and H_2 is denoted by $H_1 \boxtimes H_2$ is the graph obtained by taking one copy of H_1, n_1 copies of H_2 and also m_1 copies of H_2 then joining the i^{th} vertex of H_1 to every in the i^{th} vertex copy of H_2 and also joining the end vertices of j^{th} edge copy of H_1 to every vertex in the j^{th} edge copy of H_2 where $1 \leq i \leq n_2$ and $1 \leq j \leq m_1$. The degree of the vertices of $H_1 \boxtimes H_2$ are given by

$$d_{H_1 \boxtimes H_2}(v_i) = (n_2+1)d_{H_1}(v_i) + n_2 \qquad \forall v_i \in V(H_1)$$
$$d_{H_1 \boxtimes H_2}(u_{ij}) = d_{H_2}(u_j) + 2 \qquad \forall u_{ij} \in V_{ie}(H_2)$$
$$d_{H_1 \boxtimes H_2}(w_{ij}) = d_{H_2}(w_j) + 1 \qquad \forall w_{ij} \in V_{ie}(H_2)$$

Where, the notation and the sets are

- $V_{je}(H_2)$ is the vertex set of the j^{th} edge copy of H_2
- $V_{iv}(H_2)$ is the vertex set of the i^{th} vertex copy of H_2
- $E_{je}(H_2)$ is the edge set of the j^{th} edge copy of H_2
- $E_{iv}(H_2)$ is the edge set of the i^{th} vertex copy of H_2

Theorem 3.5. *The S-index of $H_1 \boxtimes H_2$ is given by*

$$S(H_1 \boxtimes H_2) = (n_2+1)^5 S(H_1) + (m_1+n_1)S(H_2) + 5n_2(n_2+1)^4 Y(H_1) + 5(2m_1+n_1)Y(H_2) + 10(n_2+1)^3 n_2^2 F(H_1) + 10(4m_1+n_1)F(H_2) + 10n_2^3(n_2+1)^2 M_1(H_1) + 10(8m_1+n_1)M_1(H_2) + 10n_2^4(n_2+1)m_1 + 160m_1m_2 + n_1n_2^5 + 10n_1m_2 + n_1 + 32$$

Proof

From the definition of vertex-edge corona product of two graphs we have

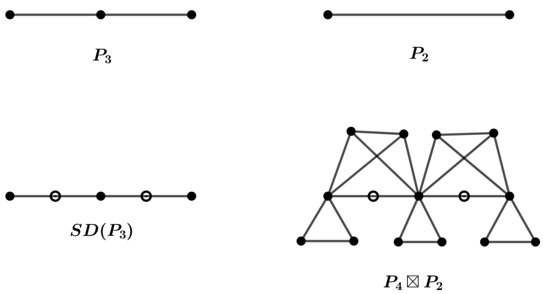

Figure 12. The Vertex-edge corona graph

$v_i \in V(H_1)$	$u_{ij} \in V_{ie}(H_2)$	$w_{ij} \in V_{ie}(H_2)$
$(n_2+1)d_{H_1}(v_i)+n_2$	$d_{H_2}(u_j)+2$	$d_{H_2}(w_j)+1$

Table 5. Degree of the vertices of $H_1 \boxtimes H_2$

$$S(H_1 \boxtimes H_2) = \sum_{v_i \in V(H_1)} ((n_2+1)d_{H_1}(v_i)+n_2)^5 + \sum_{e_i \in E(H_1)} \sum_{u_{ij} \in V(H_2)} [d_{H_2}(u_j)+2]^5 + \sum_{v_i \in V(H_1)} \sum_{w_{ij} \in V(H_2)} [d_{H_2}(w_j)+1]^5$$

$$\begin{aligned}
S(H_1 \boxtimes H_2) &= (n_2+1)^5 \sum_{v_i \in V(H_1)} d_{H_1}(v_i)^5 + 5n_2(n_2+1)^4 \sum_{v_i \in V(H_1)} d_{H_1}(v_i)^4 + 10n_2^2(n_2+1)^3 \sum_{v_i \in V(H_1)} d_{H_1}(v_i)^3 \\
&+ 10n_2^3(n_2+1)^2 \sum_{v_i \in V(H_1)} d_{H_1}(v_i)^2 + 5(n_2+1)n_2^4 \sum_{v_i \in V(H_1)} d_{H_1}(v_i) + n_2^5 \sum_{v_i \in V(H_1)} 1 \\
&+ m_1 \sum_{u_{ij} \in V(H_2)} d_{H_2}(u_j)^5 + 10m_1 \sum_{u_{ij} \in V(H_2)} d_{H_2}(u_j)^4 + 40m_1 \sum_{u_{ij} \in V(H_2)} d_{H_2}(u_j)^3 \\
&+ 80m_1 \sum_{u_{ij} \in V(H_2)} d_{H_2}(u_j)^2 + 80m_1 \sum_{u_{ij} \in V(H_2)} d_{H_2}(u_j) + m_1 \sum_{u_{ij} \in V(H_2)} 32 + n_1 \sum_{w_{ij} \in V(H_2)} d_{H_2}(w_j)^5 \\
&+ 5n_1 \sum_{w_{ij} \in V(H_2)} d_{H_2}(w_j)^4 + 10n_1 \sum_{w_{ij} \in V(H_2)} d_{H_2}(w_j)^3 + 10n_1 \sum_{w_{ij} \in V(H_2)} d_{H_2}(w_j)^2 \\
&+ 5n_1 \sum_{w_{ij} \in V(H_2)} d_{H_2}(w_j) + n_1 \sum_{w_{ij} \in V(H_2)} 1 \\
&= (n_2+1)^5 S(H_1) + (m_1+n_1)S(H_2) + 5n_2(n_2+1)^4 Y(H_1) + 5(2m_1+n_1)Y(H_2) \\
&+ 10(n_2+1)^3 n_2^2 F(H_1) + 10(4m_1+n_1)F(H_2) + 10n_2^3(n_2+1)^2 M_1(H_1) \\
&+ 10(8m_1+n_1)M_1(H_2) + 10n_2^4(n_2+1)m_1 + 160m_1m_2 + n_1n_2^5 + 10n_1m_2 \\
&+ n_1 + 32
\end{aligned}$$

□

4. Application

1. $S(P_n \boxdot P_m) = nm^5 + 10(n-1)m^4 + 20(n-1)m^3 + 20(4n-3)m^2 + 242nm - 290m - 208n - 94$
2. $S(C_n \boxdot C_m) = nm^5 + 10nm^4 + 40nm^3 + 80(n-1)m^2 + 243nm + 64n$
3. $S(C_n \boxdot P_m) = nm^5 + 10nm^4 + 40nm^3 + 80(n-1)m^2 + 243nm - 218n - 140m$
4. $S(P_n \oplus P_m) = (n-1)m^5 + 10(n-1)m^4 + 40(n-1)m^3 + 80(n-1)m^2 + 323(n-1)m - 348n + 318$

5. $S(C_n \oplus C_m) = nm^5 + 10nm^4 + 40nm^3 + 80nm^2 + 323nm + 62n$
6. $S(C_n \oplus P_m) = nm^5 + 10nm^4 + 40nm^3 + 80nm^2 + 323nm - 348n$
7. $S(P_n \odot P_m) = 32(n-1)m^5 + 160(n-1)m^4 + 320(n-1)m^3 + 320(n-1)m^2 + 1584nm - 2322m - 1498n + 2186$
8. $S(C_n \odot C_m) = 32nm^5 + 160nm^4 + 320nm^3 + 320nm^2 + 1184nm + 64n$
9. $S(C_n \odot P_m) = 32nm^5 + 160nm^4 + 320nm^3 + 320nm^2 + 1184nm - 1498n$
10. $S(P_n \boxminus P_m) = 32nm^5 - 62m^5 + 160nm^4 - 310m^4 + 320nm^3 - 620m^3 + 320nm^2 - 620m^2 + 1184nm - 1498n - 310m - 94$
11. $S(C_n \boxminus C_m) = 32nm^5 + 160nm^4 + 320nm^5 + 320nm^2 + 1184nm + 32n + 32m$
12. $S(C_n \boxminus P_m) = 32nm^5 + 160nm^4 + 320nm^3 + 320nm^2 + 1184nm - 1498n$
13. $S(P_n \boxtimes P_m) = 243nm^5 - 1422m^5 + 810nm^4 - 1460m^4 + 1080nm^3 - 2000m^3 + 720nm^2 - 1360m^2 + 1378nm - 1420m - 32n^2m - 62n^2 - 1797n + 902$
14. $S(C_n \boxtimes C_m) = 243nm^5 + 810nm^4 + 1080nm^3 + 720nm^2 + 1474nm + 33n + 32$
15. $S(C_n \boxtimes P_m) = 243nm^5 + 810nm^4 + 1080nm^3 + 720nm^2 + 1474nm - 1983n + 64$

Example 4.1. *Topological index calculation:*

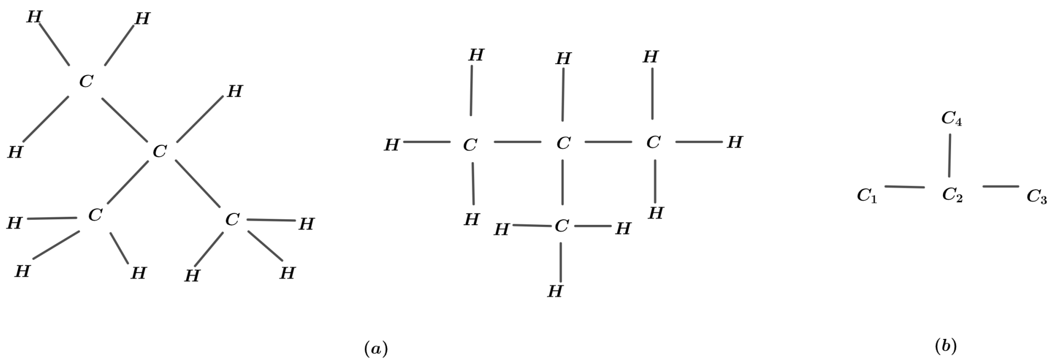

Figure 13. (a) - Isopentane[C_6H_{10}] and (b) Carbon Skeleton of C_6H_{10} as Graph H_3

C_2 is the central carbon, degree = 3
C_1, C_2, C_3 are terminal carbon, degree = 1
Degree sequence of Isobutane $(1, 3, 1, 1)$

$$F(H_3) = \sum_{v \in V(H_3)} d_{H_3}(v)^3 = 1^3 + 3^3 + 1^3 + 1^3 = 30$$

$$Y(H_3) = \sum_{v \in V(H_3)} d_{H_3}(v)^4 = 1^4 + 3^4 + 1^4 + 1^4 = 84$$

$$S(H_3) = \sum_{v \in V(H_3)} d_{H_3}(v)^5 = 1^5 + 3^5 + 1^5 + 1^5 = 246$$

Example 4.2. *Topological index calculation:*

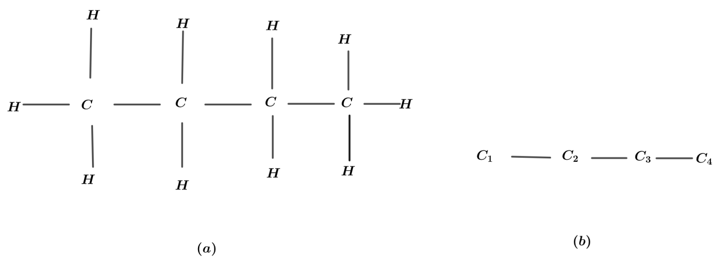

Figure 14. (a) - n- Butane[C_4H_{10} and (b) - Carbon Skeleton Graph of H_4]

n-Butane degree sequence is$(1, 2, 2, 1))$

$$F(H_4) = 18$$
$$Y(H_4) = 34$$
$$S(H_4) = 66$$

The central carbon in Isobutane implies greater steric hindrane, different reactivity (eg., Stability of carbocations, hydrogen, abstraction pathway) and influences physical properties. The indices reflect these topological connectivity difference quantitatively.

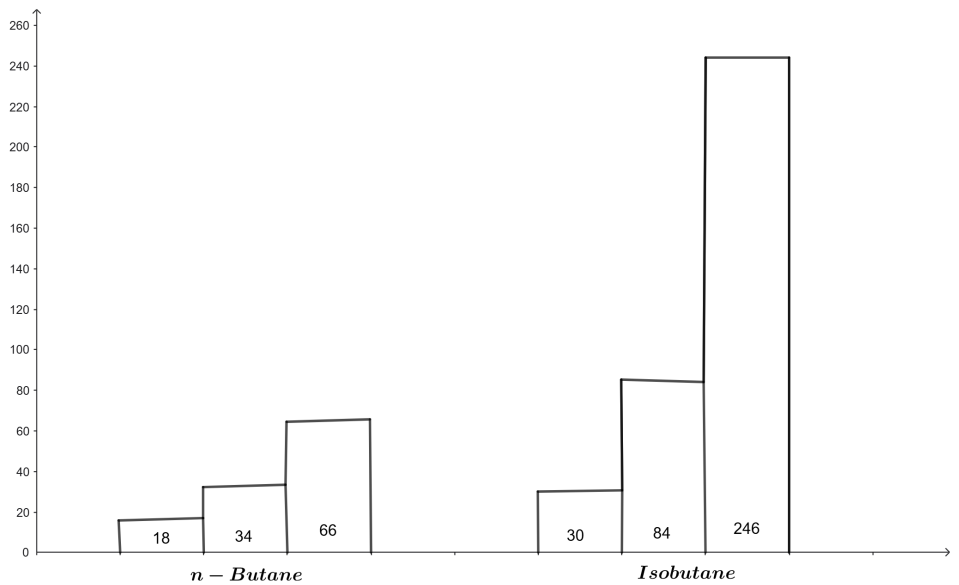

Figure 15

Based on the comparison table, degree based indices are powerful measures of molecular branching regularity and complexity . S-index depends entirely on vertex degrees, not directly on number of edges and number of vertices, but when n (or) m increases, the degree distribution changes, which affects S(H).

Adding vertices without increasing edges, S-index stays the same. If we are adding vertices with low degrees, S-index increase moderately. Adding vertices with branching structure (i.e., high degree) S-index increase and grow rapidly. For example naphthalene (degree 3) have high S-index. Even a single edge increase the value of S-index increases drastically.

Compound	Degree of Sequence	F(H)	Y(H)	S(H)
Methane	(0)	0	0	0
Ethane	(1,1)	2	2	2
Propane	(1,2,1)	10	18	34
n-Butane	(1,2,2,1)	18	34	66
Isobutane	(1,1,1,3)	30	84	246
Neopentane	(1,1,1,1,4)	68	260	1028
Toluene	(3,2,2,2,2,2,1)	68	162	404
Naphthalene	(3,3,2,2,2,2,2,2,3,3)	156	420	1164

Table 6. Comparison table for common F(H), Y(H) and S(H) chemical compounds

5. Conclusion

This paper describes a full review of S-index of five subdivisions-based corona graph operations interms of common known graph invariants (i.e., Y-index , F index and Zagreb indices) establishing a clear relationship between the structure of graph and its properties. While degree increased heterogeneity occurs due to both subdivision and addition of new neighbouring atoms and it is applicable for the construction and characterization of complex molecular structures. The derived formulas help to compute the S-index for large structured graphs. Future research could be the study of external properties, constructions of corona product graph and the behaviour of other topological indices under similar structural tranformation.

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