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Hyperfuzzy and SuperHyperfuzzy Weighted Averages

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Abstract Uncertainty modeling is fundamental to decision-making across diverse domains, and numerous frameworks, such as Fuzzy Sets, Rough Sets, Hesitant Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets, have been developed to capture different facets of imprecision. Among these, Hyperfuzzy Sets and their recursive generalization, SuperHyperfuzzy Sets, assign set-valued membership degrees at multiple hierarchical levels to represent uncertainty more richly. A Fuzzy Weighted Average computes a weighted mean of fuzzy numbers by applying the extension principle to their membership functions. In this paper, we extend this concept by defining the Hyperfuzzy Weighted Average and the SuperHyperfuzzy Weighted Average based on Hyperfuzzy and SuperHyperfuzzy Sets. We present formal definitions, prove key properties, such as well-definedness, reduction to classical cases, and idempotency, and illustrate their application through examples, demonstrating enhanced aggregation of multi-level uncertainty.

Keywords Fuzzy set, HyperFuzzy Set, SuperHyperFuzzy Set, HyperFuzzy Weighted Average, SuperHyperFuzzy Weighted Average

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1. Introduction

Uncertainty-aware decision making has become a central theme in modern soft computing, where fuzzy and neuro-fuzzy frameworks provide flexible tools for modeling complex systems and optimizing real-world performance. Recent studies have demonstrated the effectiveness of bi-level neuro-fuzzy structures in operational planning problems such as reservoir management [12], while fuzzy logic controllers combined with evolutionary optimization (e.g., genetic algorithms) have shown strong capability in enhancing stability and robustness in power systems [13]. In intelligent security applications, hybrid fuzzy learning schemes integrated with swarm optimization have also produced promising results for practical malware detection tasks [14]. Moreover, fuzzy optimization techniques continue to play an essential role in the design of reliable autonomous systems and their underlying decision algorithms [15]. Motivated by these advances, and by the growing interest in new uncertainty formalisms such as linguistic superhypersoft sets [16], the present work contributes to the development of aggregation mechanisms within the hyperfuzzy and superhyperfuzzy setting.

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The study of uncertainty has been a central theme in mathematics, computer science, and engineering for several decades. The seminal work of Zadeh on fuzzy sets established a rigorous foundation for handling vagueness by allowing partial membership between 0 and 1 [1]. This idea was further enriched by applications of fuzzy sets in diverse domains such as biology, systems theory, and control [17, 18]. Later developments introduced Z-numbers as a means of capturing both uncertainty and reliability within a unified framework, broadening the scope of fuzzy modeling [2].

Parallel to fuzzy set theory, Pawlak developed the concept of rough sets to deal with uncertainty arising from indiscernibility relations [3]. Rough set theory has since been hybridized with fuzzy and neutrosophic paradigms, leading to rough neutrosophic sets that provide stronger tools for approximating uncertain knowledge [4]. Complementary approaches such as hesitant fuzzy sets were proposed to represent situations where membership is not a single value but rather a set of possible degrees, reflecting hesitation in decision-making [5, 6].

More recently, neutrosophic and plithogenic frameworks have gained prominence. Neutrosophic sets extend fuzzy logic by incorporating degrees of truth, indeterminacy, and falsity, with applications in algebraic structures, networks, and systems modeling [7, 8]. Plithogenic sets and graphs provide another powerful extension, enabling the modeling of attributes with contradictory, neutral, or indeterminate interactions. Such models have been applied, for example, in analyzing the global spread of COVID-19 and in multi-criteria decision-making [9, 10].

Within this broader landscape, hyperfuzzy sets and their recursive generalizations, superhyperfuzzy sets, offer a natural and powerful extension of classical fuzzy sets. In hyperfuzzy sets, each element is assigned a set of possible membership values, thus encoding multi-valued uncertainty rather than a single degree. Superhyperfuzzy sets further incorporate hierarchical layers of membership sets, enabling the modeling of uncertainty at multiple levels simultaneously [11]. These extensions provide a unifying framework for integrating various uncertainty models and pave the way for advanced aggregation tools.

The focus of this paper is to develop and analyze *Hyperfuzzy Weighted Averages* and *Superhyperfuzzy Weighted Averages*. By generalizing the classical fuzzy weighted average through hyperfuzzy and superhyperfuzzy constructions, we aim to capture complex, multi-tiered uncertainties inherent in decision-making and information processing. This work contributes both theoretical results—such as well-definedness, reduction to classical cases, and idempotency—and practical illustrations, demonstrating the enhanced expressiveness and applicability of the proposed framework.

2. Preliminaries

We collect here the basic notions and notation used throughout the paper. Unless stated otherwise, all underlying sets are finite.

2.1. Basic Set Constructions

A *fuzzy set* associates to each element a degree of membership in the unit interval, allowing partial inclusion rather than a strict binary classification [1, 17, 18, 19]. A *hyperfuzzy set* refines this idea by assigning each element a nonempty set of values in the unit interval, thereby accommodating variability and imprecision in membership [20, 21, 22, 23]. More generally, an (m, n)-*superhyperfuzzy set* assigns to each m-level subset a family of n-level membership collections, modeling multi-tiered uncertainty (cf.[24, 25]).

Definition 2.1 (Universe)
Let U be a nonempty finite set, called the *universe* or *base set*. All further constructions (powersets, hyperstructures, etc.) are formed from U.

Definition 2.2 (Powerset)
(cf.[26, 27]) The *powerset* of U is

$$\mathcal{P}(U) \;=\; \{\, A \mid A \subseteq U\}.$$

Definition 2.3 (n-fold Powerset)
(cf.[28, 29, 30, 31, 32]) For each integer $n \geq 1$, define the n-fold iterated powerset of U by

$$\mathcal{P}^1(U) = \mathcal{P}(U), \quad \mathcal{P}^{n+1}(U) = \mathcal{P}\big(\mathcal{P}^n(U)\big).$$

If one wishes to exclude the empty set at each stage, replace $\mathcal{P}$ by $\mathcal{P}^*(\cdot) = \mathcal{P}(\cdot) \setminus \{\emptyset\}$.

Example 2.4 (Iterated Powersets of $U = \{a, b\}$)
Let $U = \{a, b\}$. Then

$$\mathcal{P}^1(U) = \big\{\emptyset, \{a\}, \{b\}, \{a, b\}\big\},$$

with $\big|\mathcal{P}^1(U)\big| = 4$. Next,

$$\mathcal{P}^2(U) = \mathcal{P}\big(\mathcal{P}^1(U)\big) = \big\{\, X \mid X \subseteq \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\big\},$$

a collection of $2^4 = 16$ subsets. For instance,

$$\{\emptyset\}, \quad \{\{a\}, \{b\}\}, \quad \{\emptyset, \{a\}, \{b\}, \{a, b\}\},$$

are three typical elements of $\mathcal{P}^2(U)$. Finally,

$$\mathcal{P}^3(U) = \mathcal{P}\big(\mathcal{P}^2(U)\big),$$

which has $2^{16} = 65{,}536$ elements.

If one excludes the empty set at each stage, replace $\mathcal{P}$ by $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$, yielding $\mathcal{P}^{*1}(U) = \{\{a\}, \{b\}, \{a, b\}\}$, and so on.

Definition 2.5 (Fuzzy Set)
[1, 19] A *fuzzy set* F on U is a function

$$\mu_F : U \longrightarrow [0, 1],$$

assigning to each element $x \in U$ a *membership degree* $\mu_F(x)$.

Definition 2.6 (Fuzzy Relation)
[33] Given a fuzzy set F on U, a *fuzzy relation* R on U is a fuzzy subset of $U \times U$, i.e. a map

$$R : U \times U \longrightarrow [0, 1],$$

such that

$$R(x, y) \;\leq\; \min\{\mu_F(x),\, \mu_F(y)\} \quad \text{for all } x, y \in U.$$

Definition 2.7 (Hyperfuzzy Set)
[20, 34, 35, 36, 37] A *hyperfuzzy set* $\tilde{F}$ on U is given by

$$\tilde{\mu} : U \longrightarrow \mathcal{P}([0, 1]) \setminus \{\emptyset\},$$

where for each $x \in U$, the set $\tilde{\mu}(x) \subseteq [0, 1]$ represents the *possible membership grades* of x.

Example 2.8 (Temperature Comfort Hyperfuzzy Set)
(cf.[38, 39]) Consider the universe U of typical indoor temperatures measured in degrees Celsius:

$$U = \{18, 20, 22, 24, 26\}.$$

We define a hyperfuzzy set $\tilde{C}$ on U representing the *perceived comfort level* under varying humidity and clothing conditions. For each $t \in U$, we assign a nonempty set of possible comfort-membership degrees:

- $\tilde{\mu}_C(18) = \{0.10, 0.15, 0.20\}$, reflecting that 18 °C feels somewhat cool to most people, but some may find it just tolerable.

- $\tilde{\mu}_C(20) = \{0.40, 0.50, 0.60\}$, since at 20 °C, comfort varies moderately with individual preferences.
- $\tilde{\mu}_C(22) = \{0.75, 0.80, 0.85\}$, as 22 °C is generally regarded as comfortably warm.
- $\tilde{\mu}_C(24) = \{0.60, 0.65, 0.70\}$, because 24 °C may feel slightly warm for some but still acceptable.
- $\tilde{\mu}_C(26) = \{0.30, 0.35, 0.40\}$, indicating that 26 °C is often perceived as too warm, though a few may still feel comfortable.

This hyperfuzzy set captures the inherent variability and uncertainty in human comfort perception, assigning each temperature a set of plausible membership values rather than a single crisp degree.

Definition 2.9 ((m, n)-SuperHyperfuzzy Set)
[11, 40, 41] Fix nonnegative integers m, n. Let

$$\mathcal{P}_m^*(U) = (\underbrace{\mathcal{P}^* \circ \cdots \circ \mathcal{P}^*}_{m \text{ times}})(U), \quad \mathcal{P}_n^*([0,1]) = (\underbrace{\mathcal{P}^* \circ \cdots \circ \mathcal{P}^*}_{n \text{ times}})([0,1]),$$

where $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. An (m, n)-*superhyperfuzzy set* on U is a map

$$\tilde{\mu}_{m,n} \colon \mathcal{P}_m^*(U) \longrightarrow \mathcal{P}\big(\mathcal{P}_n([0,1])\big) \setminus \{\emptyset\},$$

assigning each nonempty m-level subset of U a nonempty family of n-level membership sets, thereby modeling hierarchical uncertainty.

Example 2.10 (Smartphone Feature Satisfaction)
(cf.[42, 43]) Let U be the set of key smartphone features:

$$U = \{\text{Battery Life}, \ \text{Camera Quality}, \ \text{Screen Resolution}\}.$$

We take $m = 1$ and $n = 2$. Then $\mathcal{P}_1^*(U) = \{\{\text{Battery}\}, \{\text{Camera}\}, \{\text{Screen}\}, \{\text{Battery,Camera}\}, \dots\}$ and $\mathcal{P}_2^*([0,1])$ is the collection of nonempty sets of subsets of $[0,1]$. Define $\tilde{\mu}_{1,2} \colon \mathcal{P}_1^*(U) \to \mathcal{P}(\mathcal{P}([0,1])) \setminus \{\emptyset\}$ by assigning to each feature-subset a family of two possible membership-degree sets:

$$\begin{aligned}
\tilde{\mu}_{1,2}(\{\text{Battery Life}\}) &= \big\{\{0.60, 0.70\}, \ \{0.65, 0.75\}\big\}, \\
\tilde{\mu}_{1,2}(\{\text{Camera Quality}\}) &= \big\{\{0.80, 0.90\}, \ \{0.85, 0.95\}\big\}, \\
\tilde{\mu}_{1,2}(\{\text{Screen Resolution}\}) &= \big\{\{0.70, 0.80\}, \ \{0.75, 0.85\}\big\}, \\
\tilde{\mu}_{1,2}(\{\text{Battery Life, Camera Quality}\}) &= \big\{\{0.55, 0.65\}, \ \{0.60, 0.70\}\big\}, \\
&\dots
\end{aligned}$$

Here each inner set (e.g. $\{0.60, 0.70\}$) is a possible range of satisfaction degrees, and the outer set collects multiple such ranges, modeling two hierarchical uncertainty levels.

Example 2.11 (Smartphone Pairwise Feature Evaluation)
Let U be the set of three key smartphone features:

$$U = \{\text{Battery Life}, \ \text{Camera Quality}, \ \text{Screen Resolution}\}.$$

Then $\mathcal{P}_1^*(U)$ is the collection of all nonempty subsets of U, and $\mathcal{P}_2^*(U)$ consists of all nonempty sets of those subsets. For instance, consider two elements of $\mathcal{P}_2^*(U)$:

$$B_1 = \big\{\{\text{Battery Life, Camera Quality}\}, \ \{\text{Camera Quality, Screen Resolution}\}\big\}$$

and

$$B_2 = \big\{\{\text{Battery Life, Screen Resolution}\}\big\}.$$

We define a $(2, 2)$-superhyperfuzzy set $\tilde{\mu}_{2,2} \colon \mathcal{P}_2^*(U) \to \mathcal{P}(\mathcal{P}_2([0,1])) \setminus \{\emptyset\}$ by assigning two possible "membership-grade collections" to each B_k:

$$\tilde{\mu}_{2,2}(B_1) = \Big\{M_1^{(1)}, \ M_1^{(2)}\Big\}, \quad \tilde{\mu}_{2,2}(B_2) = \Big\{M_2^{(1)}, \ M_2^{(2)}\Big\},$$

where

$$M_1^{(1)} = \{\{0.60,\ 0.70\},\ \{0.65,\ 0.75\}\}, \quad M_1^{(2)} = \{\{0.70,\ 0.80\},\ \{0.75,\ 0.85\}\},$$
$$M_2^{(1)} = \{\{0.55,\ 0.65\},\ \{0.60,\ 0.70\}\}, \quad M_2^{(2)} = \{\{0.65,\ 0.75\},\ \{0.70,\ 0.80\}\}.$$

Here each inner set like $\{0.60, 0.70\}$ is a possible range of satisfaction at the lowest level, grouped into two-element collections (e.g. $M_1^{(1)}$), and the outer braces collect these into two distinct second-level possibilities. This structure captures hierarchical uncertainty in evaluating combined smartphone features under varying user preferences and environmental conditions.

Example 2.12 (Pairwise Feature Bundles)
Let U be the set of three central smartphone features:

$$U = \{\text{Battery Life},\ \text{Camera Quality},\ \text{Screen Resolution}\}.$$

Then the collection of nonempty 2-level subsets is

$$\mathcal{P}_2^*(U) = \{\ \{\{\text{Battery Life}, \text{Camera Quality}\},$$
$$\{\text{Camera Quality}, \text{Screen Resolution}\}\},\ \{\{\text{Battery Life}, \text{Screen Resolution}\}\},\ \dots\ \}.$$

We define a $(2, 3)$-superhyperfuzzy set

$$\tilde{\mu}_{2,3} \colon \mathcal{P}_2^*(U) \to \mathcal{P}(\mathcal{P}_2([0, 1])) \setminus \{\emptyset\}$$

by specifying two hierarchical "uncertainty bundles" for each feature bundle. For example:

$$B_1 = \{\{\text{Battery Life}, \text{Camera Quality}\}, \{\text{Camera Quality}, \text{Screen Resolution}\}\},$$
$$B_2 = \{\{\text{Battery Life}, \text{Screen Resolution}\}\}.$$

$$\tilde{\mu}_{2,3}(B_1) = \Big\{ \underbrace{\{\{0.60, 0.70\},\ \{0.65, 0.75\}\}}_{S_1}, \underbrace{\{\{0.70, 0.80\},\ \{0.75, 0.85\}\}}_{S_2} \Big\},$$
$$\tilde{\mu}_{2,3}(B_2) = \Big\{ \underbrace{\{\{0.55, 0.65\},\ \{0.60, 0.70\}\}}_{S_3}, \underbrace{\{\{0.65, 0.75\},\ \{0.70, 0.80\}\}}_{S_4} \Big\}.$$

Here each inner set like $\{0.60, 0.70\}$ is a possible first-level membership range, grouped into second-level sets S_k, and the outer braces collect those into two distinct third-level possibilities. This multi-tiered structure reflects the complex, hierarchical uncertainty in how users evaluate combined smartphone features under varying contexts.

3. Main Results: HyperFuzzy Weighted Average

3.1. Fuzzy weighted average

A Fuzzy Weighted Average computes a weighted mean of fuzzy numbers by applying the extension principle to their membership functions [44, 45, 46, 47].

Definition 3.1 (Fuzzy Weighted Average)
(cf.[48, 49, 50]) Let $A_1, \dots, A_n$ be fuzzy numbers on $\mathbb{R}$ with membership functions

$$\mu_{A_i} \colon \mathbb{R} \to [0, 1], \quad i = 1, \dots, n,$$

and let $W_1, \dots, W_n$ be fuzzy numbers on $\mathbb{R}_{\geq 0}$ with membership functions

$$\mu_{W_i} \colon \mathbb{R}_{\geq 0} \to [0, 1], \quad i = 1, \dots, n.$$

The *fuzzy weighted average* of $A_1, \ldots, A_n$ with weights $W_1, \ldots, W_n$ is the fuzzy number Y whose membership function is given by the extension principle:

$$\mu_Y(y) = \sup_{\substack{x_i \in \mathbb{R},\, w_i \in \mathbb{R}_{\geq 0} \\ \sum_{i=1}^n w_i \neq 0}} \Big\{ \min\big(\min_{1 \leq i \leq n} \mu_{A_i}(x_i),\, \min_{1 \leq i \leq n} \mu_{W_i}(w_i) \big) \;\Big|\; y = \frac{\sum_{i=1}^n w_i\, x_i}{\sum_{i=1}^n w_i} \Big\}.$$

Equivalently, Y can be characterized by its α-cuts

$$Y_\alpha \;=\; [\, Y_\alpha^L,\, Y_\alpha^U\,], \quad 0 \leq \alpha \leq 1,$$

where

$$Y_\alpha^L = \min_{\substack{x_i \in (A_i)_\alpha,\, w_i \in (W_i)_\alpha \\ \sum_{i=1}^n w_i \neq 0}} \frac{\sum_{i=1}^n w_i\, x_i}{\sum_{i=1}^n w_i}, \qquad Y_\alpha^U = \max_{\substack{x_i \in (A_i)_\alpha,\, w_i \in (W_i)_\alpha \\ \sum_{i=1}^n w_i \neq 0}} \frac{\sum_{i=1}^n w_i\, x_i}{\sum_{i=1}^n w_i},$$

and

$$(A_i)_\alpha = \big\{ x \mid \mu_{A_i}(x) \geq \alpha \big\}, \quad (W_i)_\alpha = \big\{ w \mid \mu_{W_i}(w) \geq \alpha \big\}.$$

Example 3.2 (Overall Hotel Rating)
Suppose we evaluate a hotel on three criteria—service quality, room comfort, and cleanliness—each rated on a scale from 0 to 10 but with imprecision. We model each criterion by a triangular fuzzy number:

$$A_1 = (6, 8, 10), \quad A_2 = (5, 7, 9), \quad A_3 = (7, 9, 10),$$

where, for example, $A_1 = (6, 8, 10)$ means service quality is "about 8" with lower support 6 and upper support 10. We assign fuzzy importance weights (on $[0, 1]$) also as triangular fuzzy numbers:

$$W_1 = (0.2, 0.4, 0.6), \quad W_2 = (0.1, 0.3, 0.5), \quad W_3 = (0.3, 0.5, 0.7).$$

The fuzzy weighted average combines these by the extension principle to produce a fuzzy overall score $\tilde{Y}$, whose membership function reflects both the rating uncertainty in each criterion and the uncertainty in their relative importance.

3.2. HyperFuzzy weighted average

A Hyperfuzzy Weighted Average extends fuzzy weighting by aggregating hyperfuzzy inputs and weights, combining their sets of possible membership degrees.

Definition 3.3 (Hyperfuzzy Weighted Average)
Let $\tilde{A}_1, \ldots, \tilde{A}_n$ be hyperfuzzy numbers on $\mathbb{R}$ with membership-value maps

$$\tilde{\mu}_{A_i} : \mathbb{R} \to \mathcal{P}([0, 1]) \setminus \{\emptyset\}, \quad i = 1, \ldots, n,$$

and let $\tilde{W}_1, \ldots, \tilde{W}_n$ be hyperfuzzy weights on $\mathbb{R}_{\geq 0}$ with

$$\tilde{\mu}_{W_i} : \mathbb{R}_{\geq 0} \to \mathcal{P}([0, 1]) \setminus \{\emptyset\}, \quad i = 1, \ldots, n.$$

The *hyperfuzzy weighted average* of $\{\tilde{A}_i\}$ with weights $\{\tilde{W}_i\}$ is the hyperfuzzy number $\tilde{Y}$ whose membership-value map $\tilde{\mu}_Y : \mathbb{R} \to \mathcal{P}([0, 1]) \setminus \{\emptyset\}$ is given, for each $y \in \mathbb{R}$, by

$$\tilde{\mu}_Y(y) = \bigcup_{\substack{x_i \in \mathbb{R},\, w_i \geq 0 \\ \sum_i w_i \neq 0 \\ y = \frac{\sum w_i x_i}{\sum w_i}}} \Big\{ \min\big(u_1, \ldots, u_n, v_1, \ldots, v_n\big) \;\Big|\; u_i \in \tilde{\mu}_{A_i}(x_i),\; v_i \in \tilde{\mu}_{W_i}(w_i) \Big\}.$$

Example 3.4 (Aggregated Course Satisfaction)
Let $X = \{3, 4, 5\}$ be the possible ratings for three course aspects: content quality ($\tilde{A}_1$), instructor clarity ($\tilde{A}_2$), and platform usability ($\tilde{A}_3$). Define their hyperfuzzy membership-value maps as follows:

$$\tilde{\mu}_{A_1}(3) = \{0.4, 0.5\}, \quad \tilde{\mu}_{A_1}(4) = \{0.6, 0.7\}, \quad \tilde{\mu}_{A_1}(5) = \{0.8, 0.9\},$$
$$\tilde{\mu}_{A_2}(3) = \{0.3, 0.4\}, \quad \tilde{\mu}_{A_2}(4) = \{0.5, 0.6\}, \quad \tilde{\mu}_{A_2}(5) = \{0.7, 0.8\},$$
$$\tilde{\mu}_{A_3}(3) = \{0.2, 0.3\}, \quad \tilde{\mu}_{A_3}(4) = \{0.4, 0.5\}, \quad \tilde{\mu}_{A_3}(5) = \{0.6, 0.7\}.$$

Assign hyperfuzzy weights on two possible weight levels for each criterion:

$$\tilde{\mu}_{W_1}(0.3) = \{0.5, 0.6\}, \quad \tilde{\mu}_{W_1}(0.4) = \{0.7, 0.8\},$$
$$\tilde{\mu}_{W_2}(0.2) = \{0.4, 0.5\}, \quad \tilde{\mu}_{W_2}(0.3) = \{0.6, 0.7\},$$
$$\tilde{\mu}_{W_3}(0.2) = \{0.3, 0.4\}, \quad \tilde{\mu}_{W_3}(0.3) = \{0.5, 0.6\}.$$

By Definition X, the hyperfuzzy weighted average $\tilde{Y}$ is obtained by taking all combinations of crisp ratings x_i and weights w_i, computing the crisp average $y = \dfrac{\sum_i w_i x_i}{\sum_i w_i}$, and aggregating the minima of the selected membership values. For example, choosing $(x_1, x_2, x_3) = (4, 5, 4)$ and $(w_1, w_2, w_3) = (0.4, 0.3, 0.3)$ yields $y = 4.1$ and a possible membership value $\min\{0.6, 0.8, 0.6, 0.7, 0.6, 0.5\} = 0.5$. Hence $0.5 \in \tilde{\mu}_Y(4.1)$. This construction encapsulates uncertainty in both the ratings and their relative importance.

Example 3.5 (Asset Class Return Estimation)
Consider three asset classes with possible annual returns $X = \{5,\ 7,\ 9\}\%$:

$$\tilde{\mu}_{A_1}(5) = \{0.2, 0.3\}, \quad \tilde{\mu}_{A_1}(7) = \{0.4, 0.5\}, \quad \tilde{\mu}_{A_1}(9) = \{0.6, 0.7\},$$

for stocks ($\tilde{A}_1$);

$$\tilde{\mu}_{A_2}(5) = \{0.3, 0.4\}, \quad \tilde{\mu}_{A_2}(7) = \{0.5, 0.6\}, \quad \tilde{\mu}_{A_2}(9) = \{0.7, 0.8\},$$

for bonds ($\tilde{A}_2$);

$$\tilde{\mu}_{A_3}(5) = \{0.1, 0.2\}, \quad \tilde{\mu}_{A_3}(7) = \{0.3, 0.4\}, \quad \tilde{\mu}_{A_3}(9) = \{0.5, 0.6\},$$

for real estate ($\tilde{A}_3$). Assign hyperfuzzy weights on budget proportions:

$$\tilde{\mu}_{W_1}(0.5) = \{0.6, 0.7\}, \quad \tilde{\mu}_{W_1}(0.6) = \{0.8, 0.9\},$$
$$\tilde{\mu}_{W_2}(0.2) = \{0.4, 0.5\}, \quad \tilde{\mu}_{W_2}(0.3) = \{0.6, 0.7\},$$
$$\tilde{\mu}_{W_3}(0.2) = \{0.3, 0.4\}, \quad \tilde{\mu}_{W_3}(0.3) = \{0.5, 0.6\}.$$

By the hyperfuzzy weighted average, each combination of crisp returns x_i and weights w_i yields a portfolio return $y = \dfrac{\sum_i w_i x_i}{\sum_i w_i}$ and a membership-value set $\min\{u_1, u_2, u_3, v_1, v_2, v_3\}$ for $u_i \in \tilde{\mu}_{A_i}(x_i)$, $v_i \in \tilde{\mu}_{W_i}(w_i)$. For instance, choosing $(x_1, x_2, x_3) = (9, 7, 5)$ and $(w_1, w_2, w_3) = (0.6, 0.3, 0.1)$ gives $y = 8.1\%$ and a possible membership value $\min\{0.6, 0.5, 0.2, 0.8, 0.6, 0.3\} = 0.2$, so $0.2 \in \tilde{\mu}_Y(8.1)$. This captures expert uncertainty in both expected returns and allocation preferences.

Theorem 3.6 (Hyperfuzzy Average is Hyperfuzzy)
With notation as above, the map $\tilde{\mu}_Y$ indeed assigns to each y a nonempty subset of $[0, 1]$; hence $\tilde{Y}$ is a hyperfuzzy number on $\mathbb{R}$.

Proof
For any y in the range of weighted averages there exist tuples (x_i, w_i) with $\sum w_i \neq 0$ and $y = \sum w_i x_i / \sum w_i$. Since each $\tilde{\mu}_{A_i}(x_i)$ and $\tilde{\mu}_{W_i}(w_i)$ is nonempty, we can pick u_i and v_i to form $\min(u_1, \ldots, u_n, v_1, \ldots, v_n) \in [0, 1]$. Thus the union defining $\tilde{\mu}_Y(y)$ is nonempty and clearly a subset of $[0, 1]$. □

Theorem 3.7 (Generalization of Fuzzy Weighted Average)
If each $\tilde{A}_i$ and $\tilde{W}_i$ is a degenerate hyperfuzzy set given by singletons $\tilde{\mu}_{A_i}(x) = \{\mu_{A_i}(x)\}$ and $\tilde{\mu}_{W_i}(w) = \{\mu_{W_i}(w)\}$ for fuzzy numbers A_i, W_i, then $\tilde{Y}$ reduces to the classical fuzzy weighted average Y with membership function μ_Y as in Definition 3.1.

Proof
Under the singleton assumption, every union in the definition of $\tilde{\mu}_Y(y)$ contains exactly one value $\min\{\mu_{A_1}(x_1), \dots, \mu_{A_n}(x_n), \mu_{W_1}(w_1), \dots, \mu_{W_n}(w_n)\}$. Taking the supremum over these coincides with the extension-principle formula for $\mu_Y(y)$, so $\tilde{\mu}_Y(y) = \{\mu_Y(y)\}$. □

3.3. SuperHyperFuzzy weighted average

An (m,n)-SuperHyperfuzzy Weighted Average generalizes hyperfuzzy weighting, aggregating hierarchical multi-level memberships of inputs and weights, yielding multi-tiered set-valued means.

Definition 3.8
[(m,n)-SuperHyperfuzzy Weighted Average] Let U be the real line $\mathbb{R}$. For $i = 1, \dots, n$ let

$$\tilde{\mu}_{A_i} \colon \mathcal{P}_m^*(U) \longrightarrow \mathcal{P}\big(\mathcal{P}_n([0,1])\big) \setminus \{\emptyset\}, \quad \tilde{\mu}_{W_i} \colon \mathcal{P}_m^*(\mathbb{R}_{\geq 0}) \longrightarrow \mathcal{P}\big(\mathcal{P}_n([0,1])\big) \setminus \{\emptyset\}$$

be two (m,n)-superhyperfuzzy numbers, one on the values A_i, the other on the weights W_i [11, 41]. We define their (m,n)-*superhyperfuzzy weighted average* to be the map

$$\tilde{\mu}_Y \colon \mathcal{P}_m^*(U) \longrightarrow \mathcal{P}\big(\mathcal{P}_n([0,1])\big) \setminus \{\emptyset\}$$

given, for each nonempty m-level set $B \subseteq U$, by

$$\tilde{\mu}_Y(B) = \bigcup_{\substack{A_i \in \mathcal{P}_m^*(U),\ W_i \in \mathcal{P}_m^*(\mathbb{R}_{\geq 0}), \\ \exists\, x_i \in A_i,\ w_i \in W_i \colon B = \left\{ \frac{\sum w_i x_i}{\sum w_i} \right\}}} \Big\{ \min\big(u_1, \dots, u_n, v_1, \dots, v_n\big) \ \Big|\ u_i \in \tilde{\mu}_{A_i}(A_i),\ v_i \in \tilde{\mu}_{W_i}(W_i) \Big\}.$$

Here the condition $B = \{\sum w_i x_i / \sum w_i\}$ means B is the singleton containing the usual weighted average of one choice of $x_i \in A_i$ with weights $w_i \in W_i$.

Example 3.9 (Smartphone Overall Score)
Let $U = \{3, 4, 5\}$ be the possible star-ratings for a smartphone feature. Take $m = 1$, so $\mathcal{P}_1^*(U)$ consists of all nonempty subsets of U. We define two $(1,2)$-superhyperfuzzy numbers on $\mathcal{P}_1^*(U)$: one for the raw feature scores $\tilde{A}$, and one for their importance weights $\tilde{W}$.

$$\begin{aligned} \tilde{\mu}_A(\{3\}) &= \big\{\{0.40, 0.50\},\ \{0.45, 0.55\}\big\}, \\ \tilde{\mu}_A(\{4\}) &= \big\{\{0.60, 0.70\},\ \{0.65, 0.75\}\big\}, \\ \tilde{\mu}_A(\{5\}) &= \big\{\{0.80, 0.90\},\ \{0.85, 0.95\}\big\}, \end{aligned}$$

$$\begin{aligned} \tilde{\mu}_W(\{0.3\}) &= \big\{\{0.50, 0.60\},\ \{0.55, 0.65\}\big\}, \\ \tilde{\mu}_W(\{0.4\}) &= \big\{\{0.70, 0.80\},\ \{0.75, 0.85\}\big\}. \end{aligned}$$

By Definition 3.8, their $(1,2)$-superhyperfuzzy weighted average is

$$\tilde{\mu}_Y \colon \mathcal{P}_1^*(U) \longrightarrow \mathcal{P}\big(\mathcal{P}_2([0,1])\big) \setminus \{\emptyset\}.$$

For example, if we select the singleton $\{4\}$ with weight $\{0.4\}$, the crisp average is

$$y = \frac{0.4 \times 4}{0.4} = 4,$$

and the possible membership-value sets arise by taking $\min(u,v)$ for $u \in \{0.60, 0.70\}$ and $v \in \{0.70, 0.80\}$, yielding $\{0.60, 0.70\}$. Hence $\{0.60, 0.70\} \in \tilde{\mu}_Y(\{4\})$.

Example 3.10 (Overall Dining Experience)
Let $U = \{1, 2, 3, 4, 5\}$ denote possible star ratings for a restaurant. We set $m = 1$, $n = 3$, so $\mathcal{P}_1^*(U)$ is all nonempty singletons $\{r\}$, and $\mathcal{P}_3^*([0, 1])$ consists of nonempty collections of collections of subsets of $[0, 1]$.

Define the $(1, 3)$-superhyperfuzzy score $\tilde{\mu}_A$ and weight $\tilde{\mu}_W$ on $\mathcal{P}_1^*(U)$ by

$$\tilde{\mu}_A(\{4\}) = \big\{\{0.60,\, 0.70\},\ \{0.65,\, 0.75\},\ \{0.70,\, 0.80\}\big\},$$

$$\tilde{\mu}_W(\{0.3\}) = \big\{\{0.50,\, 0.60\},\ \{0.55,\, 0.65\},\ \{0.60,\, 0.70\}\big\}.$$

To compute the superhyperfuzzy weighted average for the singleton $\{4\}$ with weight $\{0.3\}$, note the crisp average is

$$y \;=\; \frac{0.3 \times 4}{0.3} \;=\; 4.$$

By Definition 3.8, the membership-value for y is

$$\tilde{\mu}_Y(\{4\}) = \Big\{\{\min(0.60, 0.50)\},\ \{\min(0.65, 0.55)\},\ \{\min(0.70, 0.60)\}\Big\} = \big\{\{0.50\},\ \{0.55\},\ \{0.60\}\big\}.$$

Thus the $(1, 3)$-superhyperfuzzy weighted average yields three nested levels of possible membership degrees $\{0.50\}$, $\{0.55\}$, and $\{0.60\}$, reflecting hierarchical uncertainty in both dining ratings and their importance.

Example 3.11 (Composite PC Performance Score)
We evaluate a PC by three component scores on a 0–100 scale: CPU performance ($\tilde{A}_1$), GPU performance ($\tilde{A}_2$), and RAM throughput ($\tilde{A}_3$). Set $m = 1$, $n = 2$, so $\mathcal{P}_1^*(\mathbb{R})$ consists of singletons. Define two $(1, 2)$-superhyperfuzzy numbers:

$$\tilde{\mu}_{A_1}(\{90\}) = \big\{\{0.70,\, 0.80\},\ \{0.75,\, 0.85\}\big\},$$

$$\tilde{\mu}_{A_2}(\{80\}) = \big\{\{0.60,\, 0.70\},\ \{0.65,\, 0.75\}\big\},$$

$$\tilde{\mu}_{A_3}(\{70\}) = \big\{\{0.50,\, 0.60\},\ \{0.55,\, 0.65\}\big\},$$

$$\tilde{\mu}_{W_1}(\{0.5\}) = \big\{\{0.60,\, 0.70\},\ \{0.65,\, 0.75\}\big\},$$

$$\tilde{\mu}_{W_2}(\{0.3\}) = \big\{\{0.50,\, 0.60\},\ \{0.55,\, 0.65\}\big\},$$

$$\tilde{\mu}_{W_3}(\{0.2\}) = \big\{\{0.40,\, 0.50\},\ \{0.45,\, 0.55\}\big\}.$$

The crisp weighted average is

$$y \;=\; \frac{0.5 \times 90 + 0.3 \times 80 + 0.2 \times 70}{0.5 + 0.3 + 0.2} = 83.$$

By Definition X, the superhyperweighted membership for $B = \{83\}$ is

$$\tilde{\mu}_Y(B) = \Big\{\{\min(0.70, 0.60, 0.50, 0.60, 0.50, 0.40)\},\ \{\min(0.75, 0.65, 0.55, 0.70, 0.65, 0.45)\}\Big\}$$

$$= \big\{\{0.40\},\ \{0.45\}\big\}.$$

Thus the $(1, 2)$-superhyperfuzzy weighted average yields two nested membership-degree sets $\{0.40\}$ and $\{0.45\}$, capturing hierarchical uncertainty in both component scores and their importance.

In applications, the parameter m reflects the structural depth of the underlying objects (e.g., single elements vs. subsets), while n reflects the number of uncertainty layers in the membership description. We recommend using the smallest values that capture the intended uncertainty: typically $(m, n) = (1, 1)$ for basic hyperfuzzy modeling and $(1, 2)$ when an additional hierarchical level is required. To assess sensitivity, one may compute the results for a small range of (m, n) values and verify that the ranking/aggregation outcomes remain stable.

Theorem 3.12 (Generalization of Hyperfuzzy and Fuzzy Averages)
If each $\tilde{\mu}_{A_i}$ and $\tilde{\mu}_{W_i}$ is a degenerate (m, n)-superhyperfuzzy number concentrated on singletons,

$$\tilde{\mu}_{A_i}(A) = \{\{\mu_{A_i}(x)\}\}, \quad \tilde{\mu}_{W_i}(W) = \{\{\mu_{W_i}(w)\}\},$$

then the above construction reduces first to the hyperfuzzy weighted average and, further, when $n = 0$ to the classical fuzzy weighted average.

Proof
Under the singleton assumption each union in the definition of $\tilde{\mu}_Y(B)$ contains exactly one value $\min\{\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n), \mu_{W_1}(w_1), \ldots, \mu_{W_n}(w_n)\}$. Taking the supremum over these choices reproduces the extension-principle formula for the hyperfuzzy weighted average [20]. If further each hyperfuzzy set collapses to a single real value (i.e. $n = 0$), this coincides with the usual fuzzy weighted average. □

Theorem 3.13 (Well-Definedness)
Let $\tilde{\mu}_{A_i}, \tilde{\mu}_{W_i}$ be (m, n)-superhyperfuzzy numbers as in Definition 3.8. Then the map $\tilde{\mu}_Y \colon \mathcal{P}^*_m(U) \to \mathcal{P}(\mathcal{P}_n([0, 1])) \setminus \{\emptyset\}$ given by the (m, n)-superhyperweighted average is well-defined: for each nonempty $B \subseteq U$, $\tilde{\mu}_Y(B)$ is a nonempty family of n-level subsets of $[0, 1]$.

Proof
Fix $B \in \mathcal{P}^*_m(U)$. By construction there exist choices of

$$A_i \in \mathcal{P}^*_m(U), \quad W_i \in \mathcal{P}^*_m(\mathbb{R}_{\geq 0}), \quad x_i \in A_i, \quad w_i \in W_i$$

such that $B = \{\sum_i w_i x_i / \sum_i w_i\}$. Since each $\tilde{\mu}_{A_i}(A_i)$ and $\tilde{\mu}_{W_i}(W_i)$ is by definition nonempty, we can pick

$$u_i \in \tilde{\mu}_{A_i}(A_i), \quad v_i \in \tilde{\mu}_{W_i}(W_i).$$

Then $\min\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ is a well-defined element of $[0, 1]$. Taking the union over all such choices yields a nonempty subset of $[0, 1]$. Hence $\tilde{\mu}_Y(B)$ lies in $\mathcal{P}(\mathcal{P}_n([0, 1])) \setminus \{\emptyset\}$. □

Theorem 3.14 (Idempotency)
If all value inputs coincide, $\tilde{A}_1 = \cdots = \tilde{A}_n$, then for any weights $\{\tilde{W}_i\}$ and any m-level set B, the superhyperweighted average satisfies

$$\tilde{\mu}_Y(B) = \tilde{\mu}_{A_1}(B).$$

Proof
Assume $\tilde{A}_i = \tilde{A}$ for all i. Then every choice of $x_i \in A_i$ with weights w_i yields the same crisp average $\sum w_i x_i / \sum w_i = \bar{x}$. Hence the only relevant m-level set is $B = \{\bar{x}\}$. Moreover, each $\tilde{\mu}_{A_i}(A_i)$ is the same set of n-level subsets. Thus the minima $\min(u_1, \ldots, u_n, v_1, \ldots, v_n)$ range exactly over those obtained by pairing elements of $\tilde{\mu}_A(B)$ with weights; taking the union reproduces $\tilde{\mu}_A(B)$ because $\min(u, \ldots) \leq u$ and every $u \in \tilde{\mu}_A(B)$ occurs when weights saturate to one. Therefore $\tilde{\mu}_Y(B) = \tilde{\mu}_A(B)$. □

It should be noted that compared with hesitant fuzzy and neutrosophic weighted averages, HFWA directly aggregates set-valued membership degrees, while SHFWA further models hierarchical multi-level uncertainty. HFWA is preferred when uncertainty is multi-valued, whereas SHFWA is more suitable for hierarchical uncertainty. The main trade-off is higher computational cost due to more combinations, but improved expressiveness and interpretability.

For HFWA, the computation involves combining membership values from n inputs and n weights, hence the worst-case number of evaluated combinations grows with the product of the sizes of the involved membership sets. For SHFWA, the hierarchical (multi-level) membership structure may increase the search space exponentially with the number of levels. In practice, large-scale instances can be handled using pruning rules (discarding dominated membership values), random sampling/Monte-Carlo selection of representative combinations, or limiting the cardinality of membership sets by clustering. Implementation can be efficiently realized using iterative loops and memoization, and parallelization is possible since combinations are independent.

In HFWA, each membership set represents multiple plausible degrees arising from variability, disagreement, or incomplete information, while in SHFWA the nested structure represents hierarchical uncertainty (uncertainty about uncertainty). For practical use, small hierarchy levels are recommended: $(m, n) = (1, 1)$ for basic multi-valued uncertainty and $(m, n) = (1, 2)$ when a second uncertainty layer is needed. To communicate results to non-experts, the output can be summarized by reporting representative values (e.g., minimum/maximum or median membership) together with a short explanation of the uncertainty levels.

4. Conclusion

In this paper, we extended this concept by defining the Hyperfuzzy Weighted Average and the SuperHyperfuzzy Weighted Average based on Hyperfuzzy and SuperHyperfuzzy Sets. It should be mentioned here that the proposed HFWA/SHFWA operators can support aggregation under multi-level uncertainty in decision support systems, AI/ML evaluation pipelines, and engineering design problems. A limitation is the increased computational cost for large hierarchical memberships, and future work will study efficient implementations, parameter selection, and benchmark-based validation. In future work, we intend to explore further extensions using Vague Sets [51, 52], Intuitionistic Fuzzy Sets [53, 54], Bipolar Fuzzy Sets [55, 56], m-polar Fuzzy Sets [57, 58], and related frameworks. Also, experimental validation on public datasets and comparison with baseline aggregation methods will be considered in future work, as well.

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