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# Generalized weak $\varepsilon$-subdifferential and applications

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**Abstract** A concept of subdifferential of a vector-valued mapping is introduced, called generalized weak $\varepsilon$-subdifferential.We establish existence theorems and investigate their main properties, and provide illustrative examples to clarify the construction. This construction extends and unifies several existing notions of approximate subgradients in vector optimization, including the Pareto weak subdifferential. We establish some formulas of the generalized weak $\varepsilon$-subdifferential for the sum and the difference of two vector-valued mappings. A relationship between the generalized weak $\varepsilon$-subdifferential and a directional derivative is presented. We discuss the positive homogeneity of the generalized weak $\varepsilon$-subdifferential. As application of the calculus rules, we establish necessary and sufficient optimality conditions for a constrained vector optimization problem with the difference of two vector-valued mappings.

**Keywords** Generalized weak $\varepsilon$-subdifferential, Calculus rule, Optimality condition, Vector optimization problem

**AMS 2010 subject classifications** 49J52, 90C29, 58C20

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## 1. Introduction

In the past few decades, there has been notable theoretical progress in the development of the approximate vector subdifferential for extended vector-valued mappings [1, 2, 3, 4, 5, 6, 7, 8, 9]. It is one of the crucial concepts in vector optimization, since it characterizes the approximate efficient optimal solutions and admits a rich calculus. Important contributions date back to the 1980s, when Kutateladze [9] introduced the framework of $\varepsilon$-convex programming and Loridan [10] proposed the notion of $\varepsilon$-solutions in vector minimization problems. El Maghri [4, 5] and Gutiérrez, Huerga, López, Novo and their collaborators [1, 2, 3, 11, 12] developed various versions of $\varepsilon$-subdifferentials and investigated their calculus rules.

Thirteen years ago, the authors Li and Guo [8] introduced a concept of approximate subdifferential of a vector mapping. This concept, which was defined via a norm, called generalized strong $\varepsilon$-subdifferential. This notion of the subdifferential is global and weaker than the strong subdifferential and adapted to nonconvex mappings. The authors also investigated its properties, provided some characterizations, and established certain calculus rules.

Motivated by [8], we define and study the concept of Pareto approximate subdifferential, called generalized weak $\varepsilon$-subdifferential. Our motivation arises from the fact that the existing generalized strong $\varepsilon$-subdifferential (Li and Guo [8]) is often too restrictive and may even be empty in many practical situations, which limits its applicability in vector optimization problems. As emphasized by El Maghri and Laghdir [4], the weak subdifferential provides a more flexible and stable framework than the strong one, since it remains nonempty under weaker regularity assumptions and allows the establishment of necessary and sufficient optimality conditions in vector optimization problems.

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The present study falls within the broad framework of approximate subdifferential concepts in vector optimization. Indeed, the rise of nonconvex and nonsmooth problems has led to the development of generalized subdifferentials, such as the Clarke subdifferential [13] and the limiting subdifferential of Mordukhovich [14], which offer greater robustness and stability than the classical strong subdifferential. The recent work of Van Ackooij et al. [15] illustrates the relevance of the $\varepsilon$-subdifferential concept in the study of weakly convex functions. In the vector setting, O. Benslimane et al. [16] have recently established sufficient optimality conditions for an $\varepsilon$-weak Pareto minimal point by introducing the $\varepsilon$-pseudo-Diff-Max property. Likewise, A. Ed-dahdah et al. [17] derived a calculus formula for the subdifferential of the difference of two vector-valued functions, which allows one to obtain necessary and sufficient optimality conditions for constrained optimization problems. In this context, our contribution aims to establish some new formulas for the generalized weak $\varepsilon$-subdifferential of the sum and the difference of two vector mappings and give sufficient and necessary optimality conditions for weak efficient minimizers of vector optimization problems.

The organization of the paper is as follows: Section 2 provides a review of relevant concepts and previous findings that are utilized in the current study. In section 3, we will be interested in proving the existence theorem of the generalized weak $\varepsilon$-subgradient. We also discuss the relationship with the notions of weak $\varepsilon$-subgradient [4, 5, 6] and generalized strong $\varepsilon$-subgradient [8]. In section 4, we investigate various properties of the generalized weak $\varepsilon$-subdifferential and examine the relationship between the directional derivative of a vector-valued mapping and the generalized weak $\varepsilon$-subdifferential. So, it presents the calculus rules of the generalized weak $\varepsilon$-subdifferential for the difference and the sum of two vector-valued mappings. In section 5, we study the global necessary and sufficient optimality conditions of a constrained vector optimization problem (VOP) for the difference of two vector-valued mappings. Finally, the paper ends with a conclusion and future work.

## 2. Preliminaries

We present in this section, some preliminaries needed in the sequel. In the entirety of this paper, $\left(X_1, \|.\|_{X_1}\right)$ and $\left(X_2, \|.\|_{X_2}\right)$ are two normed vector spaces. The space $X_2$ is ordered by the following relations: For $u_2,\ w_2 \in X_2$,

$$\begin{aligned} u_2 \quad &\leq_{C_2} \quad w_2 \iff w_2 - u_2 \in C_2 \\ u_2 \quad &<_{C_2} \quad w_2 \iff w_2 - u_2 \in \mathrm{int} C_2, \end{aligned}$$

where $\emptyset \neq C_2 \subset X_2$ is a convex cone such that $\mathrm{int} C_2 \neq \emptyset$ (topological interior). We adjoin to $X_2$ two abstract elements $+\infty_2$ and $-\infty_2$ such that $-(+\infty_2) = -\infty_2$, $(+\infty_2) - (+\infty_2) = +\infty_2$ and

$$\begin{cases} u_2 - \infty_2 \leq_{C_2} w_2,\ \forall u_2, w_2 \in X_2 \\ u_2 \leq_{C_2} w_2 + \infty_2 = +\infty_2,\ \forall u_2, w_2 \in X_2 \cup \{+\infty_2\} \\ \alpha.(+\infty_2) = +\infty_2,\ \forall \alpha \geq 0 \end{cases}$$

We use the notations $X_1^*$ and $X_2^*$ to represent the topological dual spaces of $X_1$ and $X_2$, respectively. The duality pairing in $X_1$ (resp. in $X_2$) is denoted by $\langle u_1^*, u_1 \rangle$ ($u_1^* \in X_1^*, u_1 \in X_1$) (resp. $\langle u_2^*, u_2 \rangle$, $u_2^* \in X_2^*$ and $u_2 \in X_2$). Let $L(X_1, X_2)$ be the set of all continuous linear operators from $X_1$ to $X_2$. We denote by $C_2^*$ for the positive polar cone of $C_2$ where

$$C_2^* := \{u_2^* \in X_2^* : \langle u_2^*, u \rangle \geq 0,\ \forall u \in C_2\}.$$

We say that a mapping $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ is $C_2$-convex if for every $b \in [0,1]$ and $u_1, w_1 \in X_1$,

$$g(bu_1 + (1-b)w_1) \leq_{C_2} bg(u_1) + (1-b)g(w_1).$$

$g$ is said to be $C_2$-concave if $-g$ is $C_2$-convex. We denote the effective domain of $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ by

$$\mathrm{dom} g := \{u \in X_1 :\ g(u) \in X_2\}.$$

If $\mathrm{dom} g \neq \emptyset$, we say that $g$ is proper.

For every $\bar{v} \in X_1$, we define the function $\varphi_{\bar{v}}(u) := \|u - \bar{v}\|_{X_1},\ \forall u \in X_1$.

*Definition 2.1*
([8], [22]) Let $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a mapping and $\bar{v} \in \mathrm{dom} g$.

1. The strong subdifferential of $g$ at $\bar{v}$ is the set
$$\partial^s g(\bar{v}) := \{R \in L(X_1, X_2),\ g(\bar{v}) - R(\bar{v}) \leq_{C_2} g(u) - R(u),\ \forall u \in X_1\}.$$
2. The set
$$\partial^w g(\bar{v}) := \{R \in L(X_1, X_2), \nexists u \in X_1, g(u) - R(u) <_{C_2} g(\bar{v}) - R(\bar{v})\}.$$
is the weak subdifferential of $g$ at $\bar{v}$.
3. For every $\varepsilon \in C_2$, the generalized strong $\varepsilon$-subdifferential of $g$ at $\bar{v}$ is the set
$$\partial^s_\varepsilon g(\bar{v}) := \{R \in L(X_1, X_2) : g(\bar{v}) - R(\bar{v}) - \varphi_{\bar{v}}(u)\varepsilon \leq_{C_2} g(u) - R(u),\ \forall u \in X_1\}.$$
If $\bar{v} \notin \mathrm{dom} g$, we set $\partial^s g(\bar{v}) = \partial^w g(\bar{v}) = \partial^s_\varepsilon g(\bar{v}) := \emptyset$.

Let us note that
$$\partial^s_\varepsilon g\,(\bar{v}) = \partial^s\left(\varphi_{\bar{v}}(.)\varepsilon + g\right)(\bar{v}).$$
If $X_2 = \mathbb{R}$ and $C_2 = \mathbb{R}_+$, the strong and weak subdifferential are just the subdifferential of convex analysis :
$$\partial g(\bar{v}) = \{u_1^* \in X_1^*, g(\bar{v}) - \langle u_1^*, \bar{v}\rangle \leq g(u) - \langle u_1^*, u\rangle\,,\ \forall u \in X_1\}.$$
For $\bar{v} \in X_1$ and $r > 0$, the open ball of radius $r$ and center $\bar{v}$ is defined by
$$B(\bar{v}, r) = \{u \in X_1, \varphi_{\bar{v}}(u) < r\}.$$

*Definition 2.2*
([8]) Let $\varepsilon \in C_2$. A mapping $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ is said to be generalized lower locally $\varepsilon$-Lipschitz at $\bar{v}$ if
$$\exists r > 0, g(\bar{v}) - \varphi_{\bar{v}}(u)\varepsilon \leq_{C_2} g(u),\ \forall u \in B(\bar{v}, r). \tag{1}$$
If (1) holds for every $u \in X_1$, we say that $g$ is generalized lower $\varepsilon$-Lipschitz at $\bar{v}$.

## 3. Generalized weak $\varepsilon$-subdifferential

Motivated by the paper [8], we introduce the concept of generalized weak $\varepsilon$-subdifferential and we discuss some properties and calculus rules.

*Definition 3.1*
Let $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a mapping, $\varepsilon \in C_2$ and $\bar{v} \in \mathrm{dom} g$. The set
$$\partial^w_\varepsilon g(\bar{v}) := \{R \in L(X_1, X_2) : \nexists u \in X_1,\ g(u) - R(u) <_{C_2} g(\bar{v}) - R(\bar{v}) - \varphi_{\bar{v}}(u)\varepsilon\},$$
is called the generalized weak $\varepsilon$-subdifferential of $g$ at $\bar{v}$. Every $R \in \partial^w_\varepsilon g(\bar{v})$ is called a generalized weak $\varepsilon$-subgradient of $g$ at $\bar{v}$.
If $\bar{v} \notin \mathrm{dom} g$, we set $\partial^w_\varepsilon g(\bar{v}) := \emptyset$.

Let us note that
$$\partial^s_\varepsilon g\,(\bar{v}) = \partial^s\left(\varphi_{\bar{v}}(.)\varepsilon + g\right)(\bar{v}) \subseteq \partial^w\left(\varphi_{\bar{v}}(.)\varepsilon + g\right)(\bar{v}) = \partial^w_\varepsilon g\,(\bar{v}),$$
and when $\varepsilon = 0$, $\partial^w_\varepsilon g\,(\bar{v}) = \partial^w g\,(\bar{v})$.

*Example 3.1*
Let

$$\begin{array}{rccc} g: & \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ & u & \longmapsto & (0,-|u|) \end{array} \tag{2}$$

The space $\mathbb{R}^2$ is equipped with the convex cone $\mathbb{R}^2_+$. For $\varepsilon = \left(1, \frac{1}{2}\right)$, $R = (0,0)$ and $\bar{v} = 0$. Since $-|u| \le 0$ and $-|u| \le -\frac{1}{2}|u|$, then

$$\nexists u \in \mathbb{R},\ g(u) - R(u) <_{\mathbb{R}^2_+} g(\bar{v}) - R(\bar{v}) - |u - \bar{v}|\,\varepsilon,$$

i.e. $R \in \partial^w_\varepsilon g(\bar{v})$.

*Remark 3.1*
Let us note that if $X_1 = \mathbb{R}^m$, $X_2 = \mathbb{R}^n$, and $(\varepsilon + \text{int}(C_2)) \cap (-C_2) = \emptyset$ then, the generalized weak $\varepsilon$-subdifferential of $g$ becomes the weak $(C,h)$-subdifferential of f at $\bar{v}$, where $C = \varepsilon + \text{int}(C_2)$ and $h(.,\bar{v}) = \varphi_{\bar{v}}$. defined in [12].

Now, by introducing the notion of generalized weakly lower locally $\varepsilon$-Lipschitz mapping, we present existence theorems of generalized weak $\varepsilon$-subgradient.

*Definition 3.2*
Let $\varepsilon \in C_2$. A mapping $g: X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ is called generalized weakly lower locally $\varepsilon$-Lipschitz at $\bar{v}$ if

$$\exists r > 0, \nexists u \in B(\bar{v}, r), g(u) - g(\bar{v}) <_{C_2} -\varphi_{\bar{v}}(u)\varepsilon. \tag{3}$$

If (3) does not hold for every $u \in X_1$, then $g$ is said to be generalized weakly lower $\varepsilon$-Lipschitz at $\bar{v}$.

*Proposition 3.1*
Let $\varepsilon \in C_2$. If a mapping $g: X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ is generalized lower locally $\varepsilon$-Lipschitz at $\bar{v}$ then it is generalized weakly lower locally $\varepsilon$-Lipschitz at $\bar{v}$.

*Proof*
As $g$ is generalized lower locally $\varepsilon$-Lipschitz at $\bar{v}$, then there exists $r > 0$ :

$$-\varphi_{\bar{v}}(u)\varepsilon \le_{C_2} g(u) - g(\bar{v}),\ \forall u \in B(\bar{v}, r). \tag{4}$$

We proceed by contradiction. Suppose that $g$ is not generalized weakly lower locally $\varepsilon$-Lipschitz at $\bar{v}$, then for $B(\bar{v}, r)$, there exists some $u_0 \in B(\bar{v}, r)$ such that

$$g(u_0) - g(\bar{v}) <_{C_2} -\varphi_{\bar{v}}(u_0)\varepsilon$$

i.e.

$$g(u_0) - g(\bar{v}) + \varphi_{\bar{v}}(u_0)\varepsilon \in -\text{int}(C_2). \tag{5}$$

For $u = u_0$ in (4), we have

$$-\varphi_{\bar{v}}(u_0)\varepsilon - g(u_0) + g(\bar{v}) \in -C_2. \tag{6}$$

From (5) and (6), we deduce $0 \in -\text{int}(C_2) - C_2 \subset -\text{int}(C_2)$. This is a contradiction. □

*Remark 3.2*
The reverse of Proposition 3.1 is false. In fact, we consider the space $\mathbb{R}^2$ equipped with convex cone $\mathbb{R}^2_+$ and the following mapping

$$\begin{array}{rccc} g: & \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ & u & \longmapsto & (0,-|u|) \end{array} \tag{7}$$

The mapping $g$ is generalized weakly lower locally $\varepsilon$-Lipschitz at $\bar{v} = 0$, where $\varepsilon = (1, \frac{1}{2})$.

Let us prove that $g$ is not generalized lower locally $\varepsilon$-Lipschitz at $\bar{v}$. Let $r > 0$ and $u_0 \in B(\bar{v}, r) \setminus \{0\}$. We have

$$g(u_0) - g(\bar{v}) + |u_0 - \bar{v}|\varepsilon = \left(|u_0|, -\frac{1}{2}|u_0|\right) \notin \mathbb{R}^2_+,$$

then $g$ is not generalized lower locally $\varepsilon$-Lipschitz at $\bar{v}$.

*Proposition 3.2*
Let $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a mapping and $\varepsilon \in C_2$. If $g$ is generalized weakly lower $\varepsilon$-Lipschitz at $\bar{v}$, then $g$ is generalized weakly $\varepsilon$-subdifferentiable at $\bar{v}$.

*Proof*
As $g$ is generalized weakly lower $\varepsilon$-Lipschitz at $\bar{v}$, i.e.

$$\nexists u \in X_1,\ g(u) - g(\bar{v}) <_{C_2} -\varphi_{\bar{v}}(u)\varepsilon,$$

then

$$\nexists u \in X_1,\ g(u) - 0 <_{C_2} g(\bar{v}) - 0 - \varphi_{\bar{v}}(u)\varepsilon$$

i.e. $0 \in \partial_{\varepsilon}^{w} g(\bar{v})$, then $g$ is generalized weakly $\varepsilon$-subdifferentiable at $\bar{v}$. □

*Remark 3.3*
Let us note that the class of mappings $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ satisfying $0 \in \partial_{\varepsilon}^{w} g(\bar{v})$ are generalized weakly lower $\varepsilon$-Lipschitz at $\bar{v}$.

*Theorem 3.1*
Let $\varepsilon \in C_2$ and $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a $C_2$-convex mapping. If $g$ is generalized weakly lower locally $\varepsilon$-Lipschitz at $\bar{v}$, then $g$ is generalized weakly $\varepsilon$-subdifferentiable at $\bar{v}$.

*Proof*
Let us prove that $g$ is generalized weakly lower $\varepsilon$-Lipschitz at $\bar{v}$. Suppose the contrary, i.e. there exists $u_0 \in X_1$ such that

$$g(u_0) - g(\bar{v}) <_{C_2} -\varphi_{\bar{v}}(u_0)\varepsilon$$

i.e.

$$g(u_0) - g(\bar{v}) + \varphi_{\bar{v}}(u_0)\varepsilon \in -\operatorname{int}(C_2). \tag{8}$$

Let $r > 0$ and $b \in \,]0, 1]$, then there exists $a = \bar{v} + b\,(u_0 - \bar{v}) \in B(\bar{v}, r)$. Since $g$ is $C_2$-convex, we get

$$g(a) = g(bu_0 + (1-b)\bar{v}) \leq_{C_2} bg(u_0) + (1-b)g(\bar{v}),$$

i.e.

$$g(a) - bg(u_0) - (1-b)g(\bar{v}) \in -C_2. \tag{9}$$

Since $b.\operatorname{int}(C_2) \subset \operatorname{int}(C_2)$, it follows from (8) that

$$bg(u_0) - bg(\bar{v}) + \|b(u_0 - \bar{v})\|_{X_1} \varepsilon \in -\operatorname{int}(C_2). \tag{10}$$

Adding (9) and (10) and by taking into account that $-C_2 - \operatorname{int}(C_2) \subseteq -\operatorname{int}(C_2)$, we get

$$g(a) - g(\bar{v}) + \varphi_{\bar{v}}(a)\varepsilon \in -\operatorname{int}(C_2)$$

This contradicts the fact that $g$ is generalized weakly lower locally $\varepsilon$-Lipschitz at $\bar{v}$. Hence according to Proposition 3.2, $g$ is generalized weakly $\varepsilon$-subdifferentiable at $\bar{v}$. □

*Proposition 3.3*
Let $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a mapping. We have

$$\partial^{w} g(\bar{v}) \subset \partial_{\varepsilon}^{w} g(\bar{v}),\ \forall \varepsilon \in C_2.$$

*Proof*
Let $R \in \partial^w g(\bar{v})$, and suppose that for some $\varepsilon_0 \in C_2$, $R \notin \partial^w_{\varepsilon_0} g(\bar{v})$. Hence, there exists some $u_0 \in X$ such that

$$g(u_0) - R(u_0) <_{C_2} g(\bar{v}) - R(\bar{v}) - \varphi_{\bar{v}}(u_0)\varepsilon_0,$$

i.e.

$$g(u_0) - R(u_0) - g(\bar{v}) + R(\bar{v}) + \varphi_{\bar{v}}(u_0)\varepsilon_0 \in -\text{int}(C_2). \tag{11}$$

As $-\varphi_{\bar{v}}(u_0)\varepsilon_0 \in -C_2$ and $-C_2 - \text{int}(C_2) \subseteq -\text{int}(C_2)$, it follows from (11) that

$$g(u_0) - R(u_0) - g(\bar{v}) + R(\bar{v}) \in -\text{int}(C_2),$$

which yields that $R \notin \partial^w g(\bar{v})$. This is a contradiction. □

*Remark 3.4*
From Proposition 3.3, we have $\partial^w g(\bar{v}) \subset \partial^w_\varepsilon g(\bar{v})$ for any $\varepsilon \in C_2$, but the reverse inclusion does not hold. In fact, let us consider the space $\mathbb{R}^2$ equipped with the convex cone $\mathbb{R}^2_+$ and the following mapping

$$\begin{array}{rccc} g: & \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ & u & \longmapsto & (1, u). \end{array}$$

Let $\varepsilon = (1, 2)$, $\bar{v} = 0$ and $R = (-1, -1)$. Then

$$\begin{cases} \nexists u \in \mathbb{R}, g(u) - R(u) <_{\mathbb{R}^2_+} g(\bar{v}) - R(\bar{v}) - |u - \bar{v}|\, \varepsilon \\ g(u) - R(u) <_{\mathbb{R}^2_+} g(\bar{v}) - R(\bar{v}), \ \forall u < 0 \end{cases}$$

which yields that $R \in \partial^w_\varepsilon g(\bar{v})$ and $R \notin \partial^w g(\bar{v})$.

## 4. Properties and calculus rules

This section presents some properties of the generalized weak $\varepsilon$-subdifferential. Additionally, it presents the calculus rules for the generalized weak $\varepsilon$-subdifferential of the difference and sum of two vector-valued mappings.

*Proposition 4.1*
Let $\bar{v} \in X_1$ and $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a mapping. We have

1. $\bigcap\limits_{\varepsilon \in C_2 \setminus \{0\}} \partial^w_\varepsilon g(\bar{v}) = \partial^w g(\bar{v})$.
2. $\bigcap\limits_{\eta \in C_2 \setminus \{0\}} \partial^w_{\varepsilon+\eta} g(\bar{v}) = \partial^w_\varepsilon g(\bar{v}), \ \forall \varepsilon \in C_2$.

*Proof*
1. If $\bar{v} \notin \text{dom}g$, then the equality is evident. Let $\bar{v} \in \text{dom}g$ and $R \in \bigcap\limits_{\varepsilon \in C_2 \setminus \{0\}} \partial^w_\varepsilon g(\bar{v})$, we have

$$R \in \partial^w_{\frac{\varepsilon}{n}} g(\bar{v}), \ \forall n \in \mathbb{N} \setminus \{0\}, \ \forall \varepsilon \in C_2 \setminus \{0\},$$

that is

$$g(u) - R(u) - g(\bar{v}) + R(\bar{v}) + \varphi_{\bar{v}}(u)\frac{\varepsilon}{n} \in (-\text{int}C_2)^c, \ \forall u \in X_1, \ \forall n \geq 1, \ \forall \varepsilon \in C_2 \setminus \{0\}.$$

where $(-\text{int}(C_2))^c$ stands for the complement of $(-\text{int}(C_2))$. Since $(-\text{int}C_2)^c$ is closed, hence when $n \longrightarrow +\infty$, we have

$$g(u) - R(u) - g(\bar{v}) + R(\bar{v}) \in (-\text{int}C_2)^c, \forall u \in X_1,$$

i.e $R \in \partial^w g\left(\bar{v}\right)$. The reverse inclusion follows from Proposition 3.3.
2. We have, for any $\varepsilon \in C_2$

$$
\begin{aligned}
\bigcap_{\eta \in C_2\setminus\{0\}} \partial^w_{\varepsilon+\eta} g(\bar{v}) &= \bigcap_{\eta \in C_2\setminus\{0\}} \partial^w \left(g + \varphi_{\bar{v}}(.)\varepsilon + \varphi_{\bar{v}}(.)\eta\right)(\bar{v}) \\
&= \bigcap_{\eta \in C_2\setminus\{0\}} \partial^w_{\eta} \left(g + \varphi_{\bar{v}}(.)\varepsilon\right)(\bar{v}) \\
&= \partial^w \left(g + \varphi_{\bar{v}}(.)\varepsilon\right)(\bar{v}) \\
&= \partial^w_{\varepsilon} g(\bar{v}).
\end{aligned}
$$

□

We say that $R_n \longrightarrow R$ in $L(X_1, X_2)$ in the sense of the topology of pointwise convergence, if for any $u \in X_1,\ \|R_n\left(u\right) - R\left(u\right)\|_{X_2} \longrightarrow 0.$

*Proposition 4.2*
For any $\varepsilon \in C_2$, the set $\partial^w_\varepsilon g(\bar{v})$ is closed in $L(X_1, X_2)$.

*Proof*
Let $R_n \in \partial^w_\varepsilon g(\bar{v})$ and $R_n \longrightarrow R$ in the sense of the topology of pointwise convergence as $n \longrightarrow +\infty$. By $R_n \in \partial^w_\varepsilon g(\bar{v})$, we have

$$g(u) - R_n(u) - g(\bar{v}) + R_n(\bar{v}) + \varphi_{\bar{v}}(u)\varepsilon \in \left(-\text{int}(C_2)\right)^c,\ \forall u \in X_1,\ \forall n \in \mathbb{N},$$

Since $\left(-\text{int}(C_2)\right)^c$ is closed, and we get as $n \longrightarrow +\infty$

$$g(u) - R(u) - g(\bar{v}) + R(\bar{v}) + \varphi_{\bar{v}}(u)\varepsilon \in \left(-\text{int}(C_2)\right)^c,\ \forall u \in X_1$$

i.e. $R \in \partial^w_\varepsilon g(\bar{v})$. □

For a $C_2$-convex function $g$, the generalized weak $\varepsilon$-subdifferential is described by the directional derivative.

*Proposition 4.3*
Let $\varepsilon \in C_2$, $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a $C_2$-convex mapping and $\bar{v} \in \mathrm{dom} g$. Suppose that $g'\left(\bar{v}, .\right)$, the directional derivative of $g$ at $\bar{v}$, exists i.e. for any $h \in X_1$,

$$g'\left(\bar{v}, h\right) := \lim_{\alpha \to 0^+} \frac{g\left(\bar{v} + \alpha h\right) - g\left(\bar{v}\right)}{\alpha}$$

exists in $X_2$, in sense of norm convergence, and the cone $C_2$ is closed, then

$$\partial^w_\varepsilon g(\bar{v}) = \left\{R \in L(X_1, X_2) : \nexists h \in X,\ g'(\bar{v}, h) <_{C_2} R\left(h\right) - \|h\|_{X_1} \varepsilon\right\}.$$

*Proof*
This proposition follows directly from Proposition 3.2 in [22], taking into account that $\partial^w_\varepsilon g(\bar{v}) = \partial^w g_\varepsilon(u)$, where $g_\varepsilon := g + \varphi_{\bar{v}}(.)\varepsilon$, and that $g'_\varepsilon\left(\bar{v}, h\right) = g'\left(\bar{v}, h\right) + \|h\|_{X_1} \varepsilon$. □

*Remark 4.1*
The following example shows that $\partial^w_\varepsilon g\left(\bar{v}\right)$ is not generally a convex subset in $L\left(X_1, X_2\right)$. Let

$$
\begin{aligned}
g : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\
u &\longmapsto (1, u)
\end{aligned}
$$

The space $\mathbb{R}^2$ is equipped with the convex cone $\mathbb{R}^2_+$. We have

$$R \in \partial^w_\varepsilon g(\bar{v}) \Longleftrightarrow \nexists u \in \mathbb{R}, g(u) - g(\bar{v}) + |u - \bar{v}|\, \varepsilon <_{\mathbb{R}^2_+} R(u - \bar{v}).$$

Let $\varepsilon = \left(0, \frac{1}{2}\right)$, $\bar{v} = 0$, $R_1 = (1, 0)$ and $R_2 = (0, 4)$. then

$$\begin{cases} \nexists u \in \mathbb{R}, \left(0, u + \frac{1}{2}|u|\right) <_{\mathbb{R}^2_+} (u, 0) \\ \nexists u \in \mathbb{R}, \left(0, u + \frac{1}{2}|u|\right) <_{\mathbb{R}^2_+} (0, 4u). \end{cases}$$

which means that $R_1 \in \partial^w_\varepsilon g(\bar{v})$ and $R_2 \in \partial^w_\varepsilon g(\bar{v})$. But, for $u = 1$, we have

$$g(1) - g(0) + |1 - 0|\, \varepsilon <_{\mathbb{R}^2_+} \frac{1}{2} R_1 (1 - 0) + \frac{1}{2} R_2 (1 - 0).$$

i.e. $\frac{1}{2} R_1 + \frac{1}{2} R_2 \notin \partial^w_\varepsilon g(\bar{v})$ and hence $\partial^w_\varepsilon g(\bar{v})$ is not a convex subset in $L(X_1, X_2)$.

*Proposition 4.4*
Let $\varepsilon \in C_2$, $\bar{v} \in X_1$ and $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a mapping. Then

$$\alpha \partial^w_\varepsilon g(\bar{v}) = \partial^w_{\alpha\varepsilon} (\alpha g)(\bar{v}), \ \forall \alpha > 0.$$

*Proof*
If $\bar{v} \notin \text{dom} g$, the equality is obvious. Let $\bar{v} \in \text{dom} g$ and $\alpha > 0$, we have

$$R \in \alpha \partial^w_\varepsilon g(\bar{v}) \Longleftrightarrow \frac{1}{\alpha} R \in \partial^w_\varepsilon g(\bar{v})$$

which is equivalent to

$$\frac{1}{\alpha} R(u) - g(u) - \frac{1}{\alpha} R(\bar{v}) + g(\bar{v}) - \varphi_{\bar{v}}(u)\varepsilon \in (\text{int}(C_2))^c, \ \forall u \in X_1$$

i.e.

$$R(u) - \alpha g(u) - R(\bar{v}) + \alpha g(\bar{v}) - \varphi_{\bar{v}}(u)\alpha\varepsilon \in \alpha\, (\text{int}(C_2))^c, \ \forall u \in X_1.$$

As $\alpha. (\text{int}(C_2))^c = (\text{int} C_2)^c$, for any $\alpha > 0$, then

$$R(u) - \alpha g(u) - R(\bar{v}) + \alpha g(\bar{v}) - \varphi_{\bar{v}}(u)\alpha\varepsilon \in (\text{int}(C_2))^c, \ \forall u \in X_1.$$

i.e.

$$R \in \alpha \partial^w_\varepsilon g(\bar{v}) \Longleftrightarrow R \in \partial^w_{\alpha\varepsilon} (\alpha g)(\bar{v}).$$

□

We recall the notion of star-difference, which will be used to derive calculus rules for the generalized weak $\varepsilon$-subdifferential of a difference.

*Definition 4.1*
([8]) The star difference of two subsets $E$ and $F$ of $X_1$ is given by

$$E \overset{*}{-} F = \{u \in X_1 : u + F \subset E\}.$$

We adopt the convention that $E \overset{*}{-} F := \emptyset$ if $E = \emptyset, F \neq \emptyset$ and $E \overset{*}{-} F := X_1$ if $F = \emptyset$. Such an operation has been used in several works among which are [18, 19, 20, 21, 17]. Clearly, for $F$ nonempty, one has $E \overset{*}{-} F + F \subset E$ and $E \overset{*}{-} F \subset E - F$.

The following concept is an extension of the notion of approximately pseudo-dissipativity, which was introduced by Penot in [18].

*Definition 4.2*
([23]) Let $K : X_1 \rightrightarrows L(X_1, X_2)$ be a set-valued mapping. $K$ is said to be approximately pseudo-dissipative (APD) at $\bar{v} \in X_1$ if for any $\varepsilon \in \text{int}C_2$,

$$\exists r > 0, \forall u \in B\left(\bar{v}, r\right), \exists R \in K\left(u\right), \exists \widetilde{R} \in K\left(\bar{v}\right) : (R - \widetilde{R})\left(u - \bar{v}\right) \leq_{C_2} \varphi_{\bar{v}}(u)\varepsilon.$$

*Example 4.1*
Let $X_1 = \mathbb{R}$, $X_2 = \mathbb{R}^2$, $C_2 = \mathbb{R}^2_+$ and $f(u) = (u^2, |u|)$. We have

$$\partial^s f(u) = \begin{cases} \{0\} \times [-1, 1], \; u = 0 \\ \{(2u, 1)\}, \; u > 0 \\ \{(2u, -1)\}, \; u < 0. \end{cases}$$

Then $\partial^s f$ is APD at $\bar{v} = 0$ (see [23]).

Now, we introduce the notion of pseudo-dissipativity as below and we will discuss its relationship with the approximately pseudo-dissipativity.

*Definition 4.3*
A set valued mapping $K : X_1 \rightrightarrows L\left(X_1, X_2\right)$ is called pseudo-dissipative (PD) at $\bar{v} \in X_1$ if

$$\exists r > 0, \forall u \in B\left(\bar{v}, r\right), \exists R \in K\left(u\right), \exists \widetilde{R} \in K\left(\bar{v}\right) : \left(R - \widetilde{R}\right)\left(u - \bar{v}\right) \leq_{C_2} 0.$$

Let us note that if $f$ is generalized lower $\varepsilon$-Lipschitz at $\bar{v}$ and generalized $\varepsilon$-subdifferentiable on a neighborhood of $\bar{v}$, then we can verify that the generalized strong $\varepsilon$-subdifferential of $f$ at $\bar{v}$, $\partial^s_\varepsilon f(\bar{v})$, is PD at $\bar{v}$.

*Example 4.2*
Let $F = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ where each $f_i$ is a convex function. For each $x \in \mathbb{R}^n$, define

$$K(u) := -\,\partial^s F(u) = \big\{ -(\xi_1, \ldots, \xi_m) : \; \xi_i \in \partial^s f_i(u) \big\}.$$

Then $K$ is pseudo-dissipative.

Indeed, for any $u, \bar{v} \in \mathbb{R}^n$, take $R = -\xi \in K(u)$ and $\widetilde{R} = -\tilde{\xi} \in K(\bar{u})$, with $\xi_i \in \partial^s f_i(u)$ and $\tilde{\xi}_i \in \partial^s f_i(\bar{u})$. Since each $\partial^s f_i$ is monotone, we have

$$\langle \xi_i - \tilde{\xi}_i, \; u - \bar{v} \rangle \geq 0 \quad \text{for all } i = 1, ..., n,$$

which implies

$$(R - \widetilde{R})(u - \bar{v}) = \big( -\langle \xi_1 - \tilde{\xi}_1, \; u - \bar{v} \rangle, \ldots, -\langle \xi_m - \tilde{\xi}_m, \; u - \bar{v} \rangle \big) \in -\mathbb{R}^m_+ = -C_2.$$

Therefore, $K$ is pseudo-dissipative.

*Proposition 4.5*
If a set valued mapping $K : X_1 \rightrightarrows L\left(X_1, X_2\right)$ is PD at $\bar{v} \in X_1$, then $K$ is APD at $\bar{v}$.

*Proof*
Let $\varepsilon \in \text{int}(C_2)$. Since $K$ is PD at $\bar{v}$ , then

$$\exists r > 0, \forall u \in B(\bar{v}, r), \exists R \in K\left(u\right), \exists \widetilde{R} \in K\left(\bar{v}\right) : -\left(R - \widetilde{R}\right)\left(u - \bar{v}\right) \in C_2. \tag{12}$$

As

$$\varphi_{\bar{v}}(u)\varepsilon \in C_2. \tag{13}$$

By adding (12) and (13), we have

$$\exists r > 0, \forall u \in B\left(\bar{v}, r\right), \exists R \in K\left(u\right), \exists \widetilde{R} \in K\left(\bar{v}\right) : \varphi_{\bar{v}}(u)\varepsilon - \left(R - \widetilde{R}\right)\left(u - \bar{v}\right) \in C_2,$$

i.e.

$$\exists r > 0, \forall u \in B(\bar{v}, r), \exists R \in K(u), \exists \widetilde{R} \in K(\bar{v}) : \left(R - \widetilde{R}\right)(u - \bar{v}) \leq_{C_2} \varphi_{\bar{v}}(u)\varepsilon,$$

then $K$ is APD at $\bar{v}$. □

*Remark 4.2*
The reverse of Proposition 4.5 is false. In fact, the mapping $\partial^s g$ in Example 4.1 is APD at $\bar{v} = 0$, however is not PD at $\bar{v}$. Indeed, let $r > 0$. For $u = \frac{-r}{2} \in B(\bar{v}, r)$ and for any $R \in \partial^s g(u)$ and any $\widetilde{R} \in \partial^s g(\bar{v})$ we have

$$R = (2u, -1) \text{ and } \widetilde{R} = (0, a), \text{ for some } a \in [-1, 1],$$

then

$$\left(R - \widetilde{R}\right)(u - \bar{v}) = \left(2x^2, x - au\right) \notin -\mathbb{R}^2_+$$

hence $\partial^s g$ is not PD at $\bar{v} = 0$.

*Lemma 4.1*
Let $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a mapping and $\bar{v} \in \mathrm{dom} g$. If $g$ is $C_2$-concave and $\partial^s g$ is PD at $\bar{v}$, then for all $u \in X_1$, there exists a mapping $R_u \in \partial^s g(\bar{v})$ such that

$$g(u) - R_u(u) - g(\bar{v}) + R_u(\bar{v}) \in -C_2.$$

*Proof*
Since $\partial^s g$ is PD at $\bar{v}$, then there exists some $r > 0$ such that, for any $z \in B(\bar{v}, r)$, there exist $A_z \in \partial^s g(z)$ and $B_z \in \partial^s g(\bar{v})$ satisfying

$$(A_z - B_z)(z - \bar{v}) \in -C_2. \tag{14}$$

Let $u \in X_1$, there exist some $y(u) \in B(\bar{v}, r)$ and $b \in ]0, 1]$ satisfying

$$y(u) = \bar{v} + b(u - \bar{v}).$$

Therefore, from (14), there exist $L_{y(u)} \in \partial^s g(y(u))$ and $S_{y(u)} \in \partial^s g(\bar{v})$ satisfying

$$\left(L_{y(u)} - S_{y(u)}\right)(y(u) - \bar{v}) \in -C_2 \Longleftrightarrow b\left(L_{y(u)} - S_{y(u)}\right)(u - \bar{v}) \in -C_2.$$

Since $b > 0$ and $C_2$ is a cone, we get

$$\left(L_{y(u)} - S_{y(u)}\right)(u - \bar{v}) \in -C_2. \tag{15}$$

Since $L_{y(u)} \in \partial^s g(y(u))$, i.e

$$g(y(u)) - g(\bar{v}) - L_{y(u)}(y(u) - \bar{v}) \in -C_2,$$

and $y(u) = bu + (1 - b)\bar{v}$, we have

$$g(bu + (1 - b)\bar{v}) - g(\bar{v}) - bL_{y(u)}(u - \bar{v}) \in -C_2. \tag{16}$$

Since $g$ is $C_2$-concave, we get

$$bg(u) - g(bu + (1 - b)\bar{v}) + (1 - b)g(\bar{v}) \in -C_2. \tag{17}$$

From (16) and (17) we have

$$bg(u) - bL_{y(u)}(u - \bar{v}) - bg(\bar{v}) \in -C_2,$$

i.e

$$g(u) - L_{y(u)}(u - \bar{v}) - g(\bar{v}) \in -C_2. \tag{18}$$

By adding (15) and (18), we get

$$g(u) - S_{y(u)}(u - \bar{v}) - g(\bar{v}) \in -C_2.$$

By setting $S_{y(u)} := R_u$, we obtain

$$g(u) - R_x(u) - g(\bar{v}) + R_u(\bar{v}) \in -C_2.$$

□

*Theorem 4.1*
Let $\varepsilon \in C_2$, $g,\ h : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be two mappings and $\bar{v} \in \mathrm{dom}h \cap \mathrm{dom}g$. Then

$$\partial^w_\varepsilon (h-g)(\bar{v}) \subseteq \partial^w_\varepsilon h(\bar{v}) \overset{*}{-} \partial^s g(\bar{v}),$$

with equality if $g$ is $C_2$-concave and $\partial^s g$ is PD at $\bar{v}$.

*Proof*
Let $A \in \partial^w_\varepsilon (h-g)(\bar{v})$ i.e.

$$\nexists u \in X_1, (h-g)(u) - A(u) - (h-g)(\bar{v}) + A(\bar{v}) + \varphi_{\bar{v}}(u)\varepsilon \in -\mathrm{int}(C_2).$$

We prove that

$$A + R \in \partial^w_\varepsilon h(\bar{v}),\ \forall R \in \partial^s g(\bar{v}).$$

Suppose, on the contrary, that there exists $T \in \partial^s g(\bar{v})$ such that $A + T \notin \partial^w_\varepsilon h(\bar{v})$, then there exists $u_0 \in X_1$ such that

$$h(u_0) - A(u_0 - \bar{v}) - h(\bar{v}) - T(u_0 - \bar{v}) + \varphi_{\bar{v}}(u_0)\varepsilon \in -\mathrm{int}(C_2). \tag{19}$$

As $T \in \partial^s g(\bar{v})$, i.e

$$-g(u_0) + T(u_0 - \bar{v}) + g(\bar{v}) \in -C_2, \tag{20}$$

hence by adding (19) and (20), and using the fact that $-C_2 - \mathrm{int}C_2 \subset -\mathrm{int}C_2$, we obtain

$$(h-g)(u_0) - A(u_0) - (h-g)(\bar{v}) + A(\bar{v}) + \varphi_{\bar{v}}(u_0)\varepsilon \in -\mathrm{int}(C_2),$$

which contradicts the fact that $A \in \partial^w_\varepsilon (h-g)(\bar{v})$. Now, let us show the reverse inclusion under, the conditions that $g$ is $C_2$-concave and $\partial^s g$ is PD at $\bar{v}$. Let $L \in \partial^w_\varepsilon h(\bar{v}) \overset{*}{-} \partial^s g(\bar{v})$. Suppose, on the contrary that $L \notin \partial^w_\varepsilon (h-g)(\bar{v})$, then, there exists $u_0 \in X_1$ such that

$$(h-g)(u_0) - L(u_0) - (h-g)(\bar{v}) + L(\bar{v}) + \varphi_{\bar{v}}(u_0)\varepsilon \in -\mathrm{int}(C_2). \tag{21}$$

By Lemma 4.1, there exists a mapping $R_{u_0} \in \partial^s g(\bar{v})$ satisfying

$$g(u_0) - R_{u_0}(u_0) - g(\bar{v}) + R_{u_0}(\bar{v}) \in -C_2, \tag{22}$$

Adding (21) and (22), we get

$$h(u_0) - (L + R_{u_0})(u_0) - h(\bar{v}) + (L + R_{u_0})(\bar{v}) + \varphi_{\bar{v}}(u_0)\varepsilon \in -\mathrm{int}(C_2),$$

which means that $L + R_{u_0} \notin \partial^w_\varepsilon h(\bar{v})$. The fact that $L \in \partial^w_\varepsilon h(\bar{v}) \overset{*}{-} \partial^s g(\bar{v})$ and $R_{u_0} \in \partial^s g(\bar{v})$, it follows that $L + R_{u_0} \in \partial^w_\varepsilon h(\bar{v})$. This is a contradiction. Then $L \in \partial^w_\varepsilon (h-g)(\bar{v})$ and thus we obtain the equality $\partial^w_\varepsilon (h-g)(\bar{v}) = \partial^w_\varepsilon h(\bar{v}) \overset{*}{-} \partial^s g(\bar{v})$. □

*Corollary 4.1*
Let $h,\ g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be two mappings, $\bar{v} \in \mathrm{dom}h \cap \mathrm{dom}g$. If for some $\eta \in C_2$, $g + \varphi_{\bar{v}}(.)\eta$ is $C_2$-concave and $\partial^s_\eta g$ is PD at $\bar{v}$, then for all $\varepsilon \in C_2$, we have

$$\partial^w_\varepsilon (h-g)(\bar{v}) = \partial^w_{\varepsilon+\eta} h(\bar{v}) \overset{*}{-} \partial^s_\eta g(\bar{v}).$$

*Proof*
Note that

$$h - g = (h + \varphi_{\bar{v}}(.)\eta) - (g + \varphi_{\bar{v}}(.)\eta).$$

As $g + \varphi_{\bar{v}}(.)\eta$ is $C_2$-concave, $\partial^s_\eta g(\bar{v}) = \partial^s (g + \varphi_{\bar{v}}(.)\eta)$ is PD at $\bar{v}$, then the desired result is obtained by applying Theorem 4.1, and by replacing $h$ and $g$ respectively by $h + \varphi_{\bar{v}}(.)\eta$ and $g + \varphi_{\bar{v}}(.)\eta$. □

In what follows, we establish the calculus rules of the generalized weak $\varepsilon$-subdifferential for the sum. For this aim, we will need the concept of weak-regular subdifferentiability of a vector-valued mapping.

*Definition 4.4*
([22]) A mapping $h : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ is said to be weak-regular subdifferentiable at $\bar{v} \in \mathrm{dom}h$ if

$$\partial \left(u_2^* \circ h\right)(\bar{v}) = u_2^* \circ \partial^s h\left(\bar{v}\right), \; \forall u_2^* \in C_2^* \backslash \{0\}.$$

*Theorem 4.2*
Let $g, \; h : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be two $C_2$-convex vector-valued mappings and $\bar{v} \in \mathrm{dom}h \cap \mathrm{dom}g$. If $g$ is weak-regular subdifferentiable at $\bar{v}$ and continuous at a point $v_0 \in \mathrm{dom}h \cap \mathrm{dom}g$, then for every $\varepsilon \in C_2$, we have

$$\partial_\varepsilon^w \left(h+g\right)(\bar{v}) = \partial_\varepsilon^w h\left(\bar{v}\right) + \partial^s g\left(\bar{v}\right).$$

*Proof*
Since $h$ is $C_2$-convex then, the mapping $h_\varepsilon := h + \varphi_{\bar{v}}(.)\varepsilon$ is $C_2$-convex, and since $g$ is continuous at $v_0 \in \mathrm{dom}h \cap \mathrm{dom}g$ then,the qualification conditions of Moreau–Rockafellar holds. As $g$ is $C_2$-convex and weak-regular subdifferentiable at $\bar{v}$, then the hypotheses of Theorem 4.1 in [22] is satisfied, it follows that

$$\partial_\varepsilon^w (h+g)(\bar{v}) = \partial^w \left(h_\varepsilon + g\right)(\bar{v}) = \partial^w h_\varepsilon(\bar{v}) + \partial^s g(\bar{v}) = \partial_\varepsilon^w h(\bar{v}) + \partial^s g(\bar{v}).$$

□

*Corollary 4.2*
Let $h, \; g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be two $C_2$-convex mappings with $g$ is continuous at a point $v_0 \in \mathrm{dom}h \cap \mathrm{dom}g$. If for some $\eta \in C_2$, the mapping $g_\eta := g + \varphi_{\bar{v}}(.)\eta$ is weak-regular subdifferentiable at $\bar{v}$, then

$$\partial_{\varepsilon+\eta}^w \left(h+g\right)(\bar{v}) = \partial_\varepsilon^w h\left(\bar{v}\right) + \partial_\eta^s g\left(\bar{v}\right).$$

*Proof*
Just apply Theorem 4.2 for $h$ and $g_\eta$. □

*Remark 4.3*
The computation of the generalized weak $\varepsilon$-subdifferential may be challenging in vector-valued optimization. Scalarization techniques, where the vector problem is reduced to a family of scalar optimization problems, provide a natural way to approximate weak $\varepsilon$-subgradients by applying scalar subdifferential calculus to the resulting scalarized functions. Developing efficient numerical algorithms based on these ideas is an interesting direction for future research, and we plan to investigate this approach for computing or approximating the generalized weak $\varepsilon$-subdifferential.

## 5. Application to VOPs

Let the following VOP:

$$(P) \left\{ \begin{array}{l} \min(h(u) - g(u)) \\ u \in C \end{array} \right.$$

where $g, \; h : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ are two vector-valued mappings and $\emptyset \neq C \subset X_1$. The concept $\varepsilon$-quasi Pareto solution was given by Loridan [10] for multiobjective optimization problems with the Pareto order and extended and generalized later by Gutiérrez, López and Novo [11] and Huerga et al. [12]. This type of solution is usually known in the literature by quasi efficient solution.

*Definition 5.1*
([10]) Let $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a proper mapping, $\emptyset \neq C \subset X_1$, $\bar{v} \in C \cap \mathrm{dom}g$ and $\varepsilon \in C_2$. We say that $\bar{v}$

is an $\varepsilon$-quasi Pareto solution of $g$ on $C$ if

$$\nexists u \in C,\ g(u) <_{C_2} g(\bar{v}) - \varphi_{\bar{v}}(u)\varepsilon.$$

The set of all $\varepsilon$-quasi Pareto solutions of $g$ on $C$ will be denoted by $S^w_\varepsilon(g, C)$.

Let us note that if $\varepsilon = 0$, $\varepsilon$-quasi Pareto minimizer becomes weak minimizer i.e.

$$\nexists u \in C,\ g(u) <_{C_2} g(\bar{v}).$$

*Proposition 5.1*
Let $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a proper mapping, $\emptyset \neq C \subset X_1,\ \bar{v} \in C \cap \mathrm{dom} g$ and $\varepsilon \in C_2$. If $\bar{v}$ is a weak minimizer of $g$ on $C$, then $\bar{v}$ is an $\varepsilon$-quasi Pareto solution of $g$ on $C$.

*Proof*
Assume that $\bar{v}$ is a weak minimizer of $g$ on $C$ and $\bar{v}$ is not an $\varepsilon$-quasi Pareto solution of $g$ on $C$. Then, there exists some $u_0 \in C$ such that

$$g(u_0) + \varphi_{\bar{v}}(u_0)\varepsilon - g(\bar{v}) \in -\mathrm{int}(C_2)$$

i.e.

$$g(u_0) - g(\bar{v}) \in -\mathrm{int}(C_2) - \varphi_{\bar{v}}(u_0)\varepsilon.$$

As $-\varphi_{\bar{v}}(u_0)\varepsilon \in -C_2$ and $-\mathrm{int}(C_2) - C_2 \subset -\mathrm{int}(C_2)$, it follows that

$$g(u_0) - g(\bar{v}) \in -\mathrm{int}(C_2)$$

which yields that $\bar{v}$ is not a weak minimizer of $g$ on $C$, contradiction. □

The following example proves that the $\varepsilon$-quasi Pareto minimizer of $g$ on $C$ is weaker than the weak minimizer. Let

$$\begin{array}{rcl} g : \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ u & \longmapsto & (u, u) \end{array}$$

where the space $\mathbb{R}^2$ is equipped with its natural order $\mathbb{R}^2_+$. For $\bar{v} = 0$ and $\varepsilon = (1, 0)$, we have

$$\nexists u \in \mathbb{R}, g\left(u\right) <_{\mathbb{R}^2_+} g\left(0\right) - \varepsilon \left|u - 0\right|$$

is fulfilled, then $0$ is an $\varepsilon$-quasi Pareto solution for $g$ on $\mathbb{R}$, and since

$$g\left(u\right) <_{\mathbb{R}^2_+} g\left(0\right),\ \forall u < 0$$

it follows that $0$ is not a weak efficient minimizer of $g$ on $\mathbb{R}$.

Now, we give a sufficient and necessary optimality conditions for $\bar{v}$ to be an $\varepsilon$-quasi Pareto solution for a vector-valued mapping.

*Proposition 5.2*
Let $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a proper mapping, $\bar{v} \in \mathrm{dom} g$ and $\varepsilon \in C_2$. Then $\bar{v}$ is an $\varepsilon$-quasi Pareto solution of $g$ on $X_1$ if and only if $0 \in \partial^w_\varepsilon g\left(\bar{v}\right)$.

*Proof*
We have $\bar{v}$ is an $\varepsilon$-quasi Pareto solution of $g$ on $X_1$ if and only if

$$\nexists u \in X_1,\ g(u) <_{C_2} g(\bar{v}) - \varphi_{\bar{v}}(u)\varepsilon$$

i.e.

$$\nexists u \in X_1,\ g(u) - 0 <_{C_2} g(\bar{v}) - 0 - \varphi_{\bar{v}}(u)\varepsilon,$$

which means that $0 \in \partial^w_\varepsilon g\left(\bar{v}\right)$. □

The vector indicator mapping for a nonempty subset $C \subseteq X$, $\delta_C^v : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ is defined by

$$\delta_C^v(u) := \begin{cases} 0 & \text{if } u \in C \\ +\infty_2 & \text{else.} \end{cases}$$

and the vector normal cone at $\bar{v} \in C$, in a vector sense, is the set

$$N_s^v(C,\bar{v}) := \{R \in L(X_1,X_2) : R(u-\bar{v})\} \leq_{C_2} 0,\ \forall u \in C\}.$$

One can easily verify that $\partial^s \delta_C^v(\bar{v}) = N_s^v(C,\bar{v})$. The properness and the $C_2$-convexity of $\delta_C^v$ follows immediately since $C$ is nonempty and convex.

*Lemma 5.1*
Let $g : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be a mapping and $\emptyset \neq C \subset X_1$. We have

$$S_\varepsilon^w(g,C) = S_\varepsilon^w(g+\delta_C^v, X_1).$$

*Proof*
Let $\bar{v} \in S_\varepsilon^w(g+\delta_C^v, X_1)$, then $\bar{v} \in C \cap \mathrm{dom} g$. Suppose that $\bar{v} \notin S_\varepsilon^w(g,C)$, hence

$$\exists u_0 \in C,\ g(u_0) <_{C_2} g(\bar{v}) - \varphi_{\bar{v}}(u_0)\varepsilon,$$

which yields

$$\exists u_0 \in C,\ g(u_0) + \delta_C^v(u_0) <_{C_2} g(\bar{v}) + \delta_C^v(\bar{v}) - \varphi_{\bar{v}}(u_0)\varepsilon,$$

contradicting $\bar{v} \in S_\varepsilon^w(g+\delta_C^v, X_1)$. Conversely, let $\bar{v} \in S_\varepsilon^w(g,C)$, then $\bar{v} \in \mathrm{dom} g \cap C$. If we suppose $\bar{v} \notin S_\varepsilon^w(g+\delta_C^v, X_1)$, then

$$\exists u_0 \in X_1,\ g(u_0) + \delta_C^v(u_0) <_{C_2} g(\bar{v}) + \delta_C^v(\bar{v}) - \varphi_{\bar{v}}(u_0)\varepsilon,$$

which implies $u_0 \in C \cap \mathrm{dom} g$ and therefore

$$\exists u_0 \in C,\ g(u_0) <_{C_2} g(\bar{v}) - \varphi_{\bar{v}}(u_0)\varepsilon,$$

contradicting $\bar{v} \in S_\varepsilon^w(g,C)$. □

In order to derive optimality conditions of VOP $(P)$, we recall that the vector indicator mapping $\delta_C^v$ is weak-regular subdifferentiable (see [22]).

*Theorem 5.1*
Let $g,\ h : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ be two given mappings, $\emptyset \neq C \subset X_1$, $\bar{v} \in \mathrm{dom} h \cap \mathrm{dom} g \cap C$ and $\varepsilon \in C_2$. If $h$ is $C_2$-convex and continuous at a point $v_0 \in C \cap \mathrm{dom} h$, $g$ is $C_2$-concave and $\partial^s g$ is PD at $\bar{v}$, then $\bar{v}$ is an $\varepsilon$-quasi Pareto solution of $(P)$ if and only if

$$\partial^s g(\bar{v}) \subseteq \partial_\varepsilon^w h(\bar{v}) + N_s^v(C,\bar{v}).$$

*Proof*
By virtue of Lemma 5.1, we have

$$\min_{u \in C}(h(u) - g(u)) = \min_{u \in X_1}(h(u) + \delta_C^v(u) - g(u)).$$

From Proposition 5.2, we have $\bar{v}$ is an $\varepsilon$-quasi Pareto solution of $(P)$ if and only if

$$0 \in \partial_\varepsilon^w (h + \delta_C^v - g)(\bar{v}). \tag{23}$$

According to Theorem 4.1, we have

$$\partial_\varepsilon^w (h + \delta_C^v - g)(\bar{v}) = \partial_\varepsilon^w (h + \delta_C^v)(\bar{v}) \stackrel{*}{-} \partial^s g(\bar{v}).$$

Therefore (23) becomes equivalent to

$$0 \in \partial_\varepsilon^w (h + \delta_C^v)(\bar{v}) \overset{*}{-} \partial^s g(\bar{v})$$

i.e.

$$\partial^s g(\bar{v}) \subseteq \partial_\varepsilon^w (h + \delta_C^v)(\bar{v}).$$

Note that $\delta_C^v$ is $C_2$-convex, proper and weak-regular subdifferentiable at $\bar{v}$ and since $h$ is continuous at $v_0 \in C \cap \mathrm{dom} h = \mathrm{dom}\delta_C^v \cap \mathrm{dom} h$, it follows from Theorem 4.2 that

$$\begin{aligned} \partial_\varepsilon^w (h + \delta_C^v)(\bar{v}) &= \partial_\varepsilon^w h(\bar{v}) + \partial^s \delta_C^v(\bar{v}) \\ &= \partial_\varepsilon^w h(\bar{v}) + N_s^v(C, \bar{v}), \end{aligned}$$

which yields

$$\partial^s g(\bar{v}) \subseteq \partial_\varepsilon^w h(\bar{v}) + N_s^v(C, \bar{v}).$$

Thus $\bar{v}$ is an $\varepsilon$-quasi Pareto solution of problem $(P)$ if and only if $\partial^s g(\bar{v}) \subseteq \partial_\varepsilon^w h(\bar{v}) + N_s^v(C, \bar{v})$. □

*Example 5.1*
Let $X_1 = \mathbb{R}$, $X_2 = \mathbb{R}^2$ with $C_2 = \mathbb{R}^2_+$ and $C = [-1, 2]$. Define

$$h(u) = (|u| + 1, 0), \qquad g(u) = (-u, 0), \forall u \in \mathbb{R}.$$

Then $h$ is $C_2$-convex and continuous, $g$ is $C_2$-concave. Take $\bar{v} = 0 \in C$ and $\varepsilon \in \mathbb{R}^2_+$. We check that the vector normal cone at 0 is $N_s^v(C, 0) = \{(0, 0)\}$, $\partial^s g(0) = \{(-1, 0)\}$ and $(-1, 0) \in \partial_\varepsilon^w h(0)$. Hence

$$\partial^s g(0) \subset \partial_\varepsilon^w h(0) + N_s^v(C, 0),$$

Since $\partial^s g(0)$ is PD at 0, then the inclusion in Theorem 5.1 holds. Thus $\bar{v} = 0$ is an $\varepsilon$-quasi Pareto solution of the problem $(P)$.

*Example 5.2*
Let $X_1 = \mathbb{R}$, $X_2 = \mathbb{R}^2$, $C_2 = \mathbb{R}^2_+$, and $C = [0, 1]$. Define the mappings

$$h(u) = (\,|u| + 1,\ u^2\,), \qquad g(u) = (\,-\max(u, 0),\ -u^2\,), \forall u \in \mathbb{R}.$$

Consider the point $\bar{v} = 0 \in C$ and $\varepsilon \in \mathbb{R}^2_+$.
The mapping $h$ is $C_2$-convex and the mapping $g$ is $C_2$-concave.
At $\bar{v} = 0$, we have

$$\partial^s g(0) = [-1, 0] \times \{0\}, \qquad [-1, 1] \times \{0\} \subset \partial_\varepsilon^w h(0),$$

and

$$N_s^v(C, 0) = ]-\infty, 0] \times ]-\infty, 0].$$

It is easy to verify that $\partial^s g$ is pseudo-dissipative at $\bar{v} = 0$. Indeed, let $u \in B(\bar{v}, 1) = [-1, 1]$. For $u > 0$ one has $R = (-1, -2u) \in \partial^s g(u)$ and for $u < 0$, $R = (0, -2u) \in \partial^s g(u)$; taking $\widetilde{R} = (r_1, 0) \in \partial^s g(0)$ with $r_1 \in [-1, 0]$ yields

$$(R - \widetilde{R})(u - 0) \in -C_2.$$

Moreover,

$$\partial^s g(0) \subset \partial_w^\varepsilon h(0) + N_s^v(C, 0).$$

Therefore, all assumptions of Theorem 5.1 are satisfied, and we conclude that $\bar{v} = 0$ is an $\varepsilon$-quasi Pareto solution of the problem $(P)$.

Now, we consider the following VOP:

$$(P') \begin{cases} \min (h(u) - g(u)) \\ l(u) \in -C_3 \end{cases}$$

where $g,\ h : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ and $l : X_1 \longrightarrow X_3 \cup \{+\infty_3\}$ are vector-valued mappings and $X_3$ is a separated locally convex topological vector space and $C_3$ is a convex cone on $X_3$. In the sequel, $L_+(X_3, X_2)$ stands for the set of positive operators $R \in L(X_3, X_2)$ i.e. $R(C_3) \subset C_2$. The composed vector mapping $R \circ l : X_1 \longrightarrow X_2 \cup \{+\infty_2\}$ is defined by

$$(R \circ l)(u) := \begin{cases} R(l(u)), & \text{if } u \in \text{dom}(l) \\ +\infty_2, & \text{otherwise} \end{cases}$$

*Theorem 5.2*
If $h$ is a $C_2$-convex mapping, $l$ be a $C_3$-convex mapping, $g$ is $C_2$-concave mapping and $\partial^s g$ is PD at $\bar{v}$. if, in addition, $h$ is weak-regular subdifferentiable at $\bar{v} \in \text{dom}h$ and continuous at a point $v_0 \in \text{dom}(l)$ and there exists $R \in L_+(X_3, X_2)$ such that

$$\begin{cases} \partial^s g(\bar{v}) \subseteq \partial^w_\varepsilon (R \circ l)(\bar{v}) + \partial^s h(\bar{v}) \\ (R \circ l)(\bar{v}) = 0 \end{cases}$$

then $\bar{v}$ is an $\varepsilon$-quasi Pareto solution of $(P')$.

*Proof*
Let us note that

$$\partial^s g(\bar{v}) \subset \partial^w_\varepsilon (R \circ l)(\bar{v}) + \partial^s h(\bar{v})$$

is equivalent to

$$0 \in (\partial^w_\varepsilon (R \circ l)(\bar{v}) + \partial^s h(\bar{v})) \overset{*}{-} \partial^s g(\bar{v}). \tag{24}$$

Since $l$ is $C_3$-convex and $R \in L_+(X_3, X_2)$, it is easy to see that $R \circ l$ is $C_2$-convex. As $h$ is $C_2$-convex, weak-regular subdifferentiable at $\bar{v}$ and continuous at $v_0 \in$dom$(l) = \text{dom}(R \circ l)$, it follows from Theorem 4.2 that

$$\partial^w_\varepsilon (R \circ l + h)(\bar{v}) = \partial^w_\varepsilon (R \circ l)(\bar{v}) + \partial^s h(\bar{v})$$

and thus (24) becomes equivalent to

$$0 \in \partial^w_\varepsilon (R \circ l + h)(\bar{v}) \overset{*}{-} \partial^s g(\bar{v}).$$

The vector mappings $h + R \circ l$ and $g$ satisfy together the conditions of Theorem 4.1 and hence we get

$$0 \in \partial^w_\varepsilon (h + R \circ l - g)(\bar{v})$$

that is $\nexists u \in X_1$, such that

$$h(u) + (R \circ l)(u) - g(u) - h(\bar{v}) - (R \circ l)(\bar{v}) + g(\bar{v}) + \varphi_{\bar{v}}(u)\varepsilon \in -\text{int}(C_2).$$

Since $(R \circ l)(\bar{v}) = 0$, then we have

$$\nexists u \in X_1,\ h(u) + (R \circ l)(u) - g(u) - h(\bar{v}) + g(\bar{v}) + \varphi_{\bar{v}}(u)\varepsilon \in -\text{int}(C_2). \tag{25}$$

Now, we only need to prove that $0 \in \partial^w_\varepsilon (h + \delta^v_C - g)(\bar{v})$ where $C := \{u \in X_1 : l(u) \in -C_3\}$, which means according to Proposition 5.2 that $\bar{v}$ is an $\varepsilon$-quasi Pareto solution of problem $(P')$. If we proceed by contradiction, i.e there exists some $u_0$ such that $-l(u_0) \in C_3$ and

$$h(u_0) + \delta^v_C(u_0) - g(u_0) - h(\bar{v}) - \delta^v_C(\bar{v}) + g(\bar{v}) + \varphi_{\bar{v}}(u_0)\varepsilon \in -\text{int}(C_2). \tag{26}$$

As $\bar{v}$, $u_0 \in C$, we have

$$h(u_0) - g(u_0) - h(\bar{v}) + g(\bar{v}) + \varphi_{\bar{v}}(u_0)\varepsilon \in -\text{int}(C_2). \tag{27}$$

Since $-l(u_0) \in C_3$ and $R \in L_+(X_3, X_2)$ , then

$$(R \circ l)(u_0) \in -C_2. \tag{28}$$

Adding (27) and (28), and taking into account that $-C_2 - \text{int}(C_2) \subseteq -\text{int}(C_2)$, we obtain

$$h(u_0) + (R \circ l)(u_0) - g(u_0) - h(\bar{v}) + g(\bar{v}) + \varphi_{\bar{v}}(u_0)\varepsilon \in -\text{int}(C_2),$$

This contradicts (25), and therefore we obtain

$$0 \in \partial_\varepsilon^w (h + \delta_C^v - g)(\bar{v}).$$

□

## 6. Conclusion and Perspectives

In this paper, we introduced and studied a new notion of subdifferential for vector-valued mappings, namely the generalized weak $\varepsilon$-subdifferential. We established several fundamental properties of this concept, including existence results, stability, and closedness. We also derived calculus rules for the sum and difference of vector-valued mappings under suitable regularity and pseudo-dissipativity assumptions. These results extend and complement existing subdifferential concepts in vector optimization, particularly the generalized strong $\varepsilon$-subdifferential and the weak Pareto subdifferential.

As an application, we obtained necessary and sufficient optimality conditions for $\varepsilon$-quasi Pareto solutions of constrained vector optimization problems involving the difference of two vector-valued mappings. These conditions are expressed in terms of generalized weak $\varepsilon$-subdifferentials and vector normal cones, providing a unified and flexible framework for studying approximate solutions in multiobjective optimization.

Future research directions include extending these results by replacing the pseudo-dissipativity condition with the weaker approximate pseudo-dissipativity property, and investigating more general settings, namely extending the definitions and results to Banach or Hilbert spaces. Another promising direction is the development of numerical methods and scalarization-based algorithms for computing or approximating generalized weak $\varepsilon$-subgradients in practical optimization problems.

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