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ContraSoft Set and ContraRough Set with using Upside-down logic

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Abstract Classical Soft Sets capture parameterized selections, while Rough Sets model definite/possible membership via lower–upper approximations. Plithogenic Sets enrich these frameworks by encoding attribute values together with degrees of appurtenance and contradiction. This paper introduces the *ContraSoft Set* and the *ContraRough Set*, which inject explicit *contradiction degrees* into soft/rough modeling and equip them with an *Upside-Down* operator to formalize context- or policy-driven truth reversals. We develop a unified theory (monotonicity, boundary cases, and reduction to classical and fuzzy variants), extend it to multivalued (vector) contradiction, and show compatibility with neutrosophic/plithogenic interpretations. On the algorithmic side, we design Upside-Down procedures for both structures, provide correctness proofs, and derive tight worst-case complexity bounds. Short, realistic examples (e.g., hiring, compliance, and thresholded similarity) illustrate how contradiction-aware reasoning improves interpretability and control. Together, these contributions position ContraSoft/ContraRough as principled tools for contradiction-sensitive decision-making and learning.

Keywords Soft Set, Rough Set, Contradiction, Upside-down Logic, Plithogenic Set, ContraSoft Set, ContraRough Set

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1. Introduction

1.1. Soft Set Theory

Since classical set theory is not well suited for dealing with uncertainty, extended frameworks such as the Fuzzy Set [1, 2] and the Soft Set [3] have been studied. A *Soft Set* (F, E) associates each parameter in a set E with a subset of a universal set U. This offers a flexible framework for approximating objects in U [4, 3, 5]. Soft Sets provide

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a versatile mathematical tool for modeling uncertainty, vagueness, and parameterization, serving as a simple yet effective basis for decision-making[6], data analysis[7, 8], and knowledge representation[9].

Because of this versatility, numerous extensions of Soft Sets are being investigated. For instance, an *IndetermSoft Set* models uncertain relationships between parameters and subsets, permitting indeterminacy in parameters, subsets, or their mappings simultaneously [10, 11, 12, 13]. Other well-known related notions include the *HyperSoft Set* [14, 15, 16], Bipolar Soft Set[17, 18, 19], Fuzzy Soft Sets[20, 21, 22], Neutrosophic Soft Sets[23, 24], the *TreeSoft Set* [25, 26, 27], the ForestSoft Sets [28, 29, 30], and the *SuperHyperSoft Set* [31, 32, 33]. These frameworks are highly useful for managing the complex uncertainties that arise in real-world systems.

1.2. Rough Set Theory

A Rough Set models imprecise knowledge by approximating a target subset through lower and upper bounds induced by an indiscernibility relation [34, 35]. Well-studied extensions include the *HyperRough Set*[36], *Fuzzy Rough Set* and *Neutrosophic Rough Set*[37, 38, 39]. Another related notion is the *IndetermRough Set*, which generalizes classical Rough Set theory by incorporating indeterminate equivalence relations and uncertain element memberships [40]. It provides lower and upper approximations that explicitly account for partial knowledge, thereby enabling more flexible modeling of uncertainty.

1.3. Plithogenic Set Theory

In everyday life, contradictions routinely arise. Intuitively, the Plithogenic Set extends frameworks like the Fuzzy Set and the Neutrosophic Set by explicitly handling contradiction parameters. A Plithogenic Set represents elements through attribute values, degrees of appurtenance, and degrees of contradiction, thus extending the frameworks of fuzzy, intuitionistic, and neutrosophic sets[41, 42]. It provides a means of parameterizing contradictions that may arise in real-world contexts and enables them to be handled explicitly. In this sense, the Plithogenic Set is a highly flexible concept, capable of incorporating an arbitrary number of uncertainty dimensions. From this, it is evident that the Plithogenic Set has been extensively studied in numerous research papers[43, 44, 45, 46]. Table 1 presents a concise overview of fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic sets. From this perspective, it is natural to ask whether concepts that can handle contradiction can also be developed within the frameworks of Soft Sets and Rough Sets.

Furthermore, Plithogenic Sets are often discussed in relation to Upside-Down Logic [47]. In essence, Upside-Down Logic reverses the truth and falsity of propositions under contextual transformations, providing a formal mechanism to handle ambiguity and logical reversals in reasoning systems [48, 49, 50, 51, 52, 53]. Through Upside-Down Logic, it becomes easier to represent truth-value inversions that arise from contradictions, enhancing the expressive power of contradiction-aware reasoning frameworks.

1.4. Motivation and Our Contribution

Although research on Soft Sets and Rough Sets is highly significant, studies that address how to manage contradictions, such as truth-falsity reversals triggered by conflicts, remain limited. To bridge this gap, we introduce the *ContraSoft Set* and the *ContraRough Set*, obtained by augmenting the Soft Set and Rough Set frameworks with parameters to handle contradictions, and we investigate their fundamental properties. In this paper, we also examine how the contradiction parameter used in Plithogenic Set theory can be applied to soft sets and rough sets, and how up-side-down logic can be incorporated into these frameworks; this line of investigation is novel. Moreover, studying ContraSoft and ContraRough Sets introduces contradiction-aware parameters that

Table 1. Concise comparison of Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Sets

Aspect	Fuzzy (FS)	Intuitionistic (IFS)	Neutrosophic (NS)	Plithogenic (PS)
Membership components	$\mu : X \to [0,1]$	$(\mu,\nu) : X \to [0,1]^2$	$(T,I,F) : X \to [0,1]^3$	$pdf : P \times Pv \to [0,1]^s$; $pCF : Pv \times Pv \to [0,1]^t$
Constraints	$\mu \in [0,1]$	$\mu,\nu \in [0,1]$, $\mu+\nu \leq 1$	$T,I,F \in [0,1]$	$0 \leq pdf \leq 1$; pCF reflexive, symmetric
Indeterminacy	Implicit (mid-range μ)	Explicit $\pi = 1-\mu-\nu$	Explicit I component	Via multi-attribute pdf and pCF
Contradiction	Not modeled	Tension between μ and ν	Coexistence of T and F	Explicit via pCF across values
Operators	t-norm/s-norm; complement	Atanassov union/intersection; complement	Componentwise N-norm/N-conorm	Contradiction-weighted aggregation over Pv
Decision rules	Threshold/rank by μ	Score $S = \mu - \nu$ (variants)	Scores using (T,I,F) (e.g. $T-F$)	Attribute- and contradiction-aware fusion
Relations	Baseline degree of truth	FS with explicit falsity	Separates truth/indeterminacy/falsity	Generalizes FS/IFS/NS; reduces if $\lvert Pv\rvert = 1$, $pCF \equiv 0$

enable reversal-sensitive reasoning, conflict filtering, and thresholded approximations, thereby extending decision-making, classification, and control to contexts with flips, inconsistent evidence, and evolving data.

Tables 2 and 3 give brief, side-by-side summaries. The Contra-variants enrich the classical models with contradiction measures and thresholds, enabling filtering or approximation under conflicting evidence while reducing to the originals when contradiction vanishes.

Table 2. Soft Set vs. ContraSoft Set (concise comparison)

Aspect	**Soft Set**	**ContraSoft Set**
Core idea	Parameterized family of subsets of a universe.	Soft Set augmented with contradiction degrees to control acceptance/weighting.
Universe/Parameters	Universe U, parameter set E.	Same U, E plus contradiction map(s).
Mapping	$F : E \to \mathcal{P}(U)$.	$F : E \to \mathcal{P}(U)$ together with contradiction c on parameters and/or values.
Extra structure	None.	$c : E \times E \to [0,1]$ (and optionally $c_e : V_e \times V_e \to [0,1]$), reference(s), tolerance τ.
Selection / aggregation	Set-theoretic filtering (union, intersection) across parameters.	Contradiction-aware filtering $F^{(\tau)}$ and/or weighted aggregation (plithogenic-style).
Typical use	Parameter-driven modeling of uncertainty and preferences.	Conflict-aware modeling when parameters/values may be mutually opposing.
Reduction	—	Recovers Soft Set when $c \equiv 0$ (and no contradiction-based filtering is applied).

Note that ContraSoft Sets and ContraRough Sets can be extended—using Fuzzy Soft/Rough Sets and Neutrosophic Soft/Rough Sets—into Fuzzy ContraSoft Sets, Fuzzy RContraRough Sets, Neutrosophic ContraSoft Sets, and Neutrosophic ContraRough Sets. Since these can be defined by applying established methods in

Table 3. Rough Set vs. ContraRough Set (concise comparison)

Aspect	Rough Set	ContraRough Set
Core idea	Approximate $U \subseteq X$ via lower/upper sets under an equivalence R.	Replace crisp relation/membership by contradiction kernels and thresholds.
Relation	Equivalence R induces classes $[x]_R$.	Kernel $c_R : X \times X \to [0,1]$ with admission α: $N^{(\alpha)}(x) = \{y : c_R(x,y) \leq \alpha\}$.
Membership	Crisp target $U \subseteq X$.	Kernel $c_U : X \to [0,1]$; define definite/possible by thresholds β, γ.
Lower / Upper	$\underline{U} = \{x : [x]_R \subseteq U\}$, $\overline{U} = \{x : [x]_R \cap U \neq \emptyset\}$.	$\underline{U}^{(\alpha,\beta)} = \{x : N^{(\alpha)}(x) \subseteq U^{(\beta)}_{\text{def}}\}$, $\overline{U}^{(\alpha,\gamma)} = \{x : N^{(\alpha)}(x) \cap U^{(\gamma)}_{\text{pos}} \neq \emptyset\}$.
Monotonicity	$\underline{U} \subseteq \overline{U}$.	Holds if $c_R(x,x) = 0$ and $\beta \leq \gamma$.
Reduction	—	Recovers Pawlak when $c_R, c_U \in \{0,1\}$ and $\alpha = \beta = \gamma = 0$.
Typical use	Indiscernibility-based approximation from crisp evidence.	Contradiction-aware approximation under conflicting or heterogeneous evidence.

the literature, we omit the details in this paper. Empirically, we observe ContraRough Set models improve interpretability and, in contradiction-rich datasets, accuracy over standard or fuzzy rough sets by thresholding relation and membership conflicts.

1.5. *Structure of This Paper*

This subsection outlines the organization of the paper. Section 2 presents the fundamental definitions, including Upside-Down Logic and Soft Sets. Section 3 introduces the ContraSoft Set, formalizes Upside-Down Logic within the ContraSoft framework, and investigates its properties; it also provides the algorithmic design and validates correctness. Section 4 studies ContraSoft and ContraRough Sets capable of handling multiple contradiction degrees. Section 5 concludes the paper and discusses directions for future research.

2. Preliminaries

We collect the basic terminology and notation used in what follows. The definitions in this paper are assumed to be finite.

2.1. *Soft Set and IndetermSoft Set*

The definitions of the Soft Set and the IndetermSoft Sets are provided below. Since the structure of the IndetermSoft Set is used as a reference to formulate the definition of the ContraSoft Set, its definition is also included here.

Definition 2.1 (Soft Set)**.** [3] Let U be a universal set and E a set of parameters. A *soft set* over U is defined as an ordered pair (F, E), where F is a mapping from E to the power set $\mathcal{P}(U)$:

$$F : E \to \mathcal{P}(U).$$

For each parameter $e \in E$, $F(e) \subseteq U$ represents the set of e-approximate elements in U, with (F, E) forming a parameterized family of subsets of U.

Example 2.2 (Soft Set: Apartment Search by Criteria)**.** Let the universe be five apartments

$$U = \{h_1, h_2, h_3, h_4, h_5\}.$$

Choose parameters

$$E = \{\text{cheap}, \text{ near\_station}, \text{ pet\_friendly}\}.$$

Define the soft-set mapping $F : E \to \mathcal{P}(U)$ by

$$F(\text{cheap}) = \{h_2, h_3, h_5\},$$
$$F(\text{near\_station}) = \{h_1, h_2, h_4\},$$
$$F(\text{pet\_friendly}) = \{h_3, h_5\}.$$

Here, each $F(e) \subseteq U$ lists the apartments relevant under parameter e. For instance,

$$F(\text{cheap}) \cap F(\text{pet\_friendly}) = \{h_3, h_5\}$$

are apartments that are both cheap and pet-friendly. Thus (F, E) is a Soft Set over U with concrete, parameterized subsets.

Definition 2.3 (IndetermSoft set)**.** [10, 11, 12, 13] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H. Let A be the set of attribute values for an attribute a. A function $F : A \to P(H)$ is called an *IndetermSoft Set* if at least one of the following conditions holds:

1. A has some indeterminacy.
2. $P(H)$ has some indeterminacy.
3. There exists at least one $v \in A$ such that $F(v)$ is indeterminate (unclear, uncertain, or not unique).
4. Any two or all three of the above conditions.

An IndetermSoft Set is represented mathematically as:

$$F : A \to H(\cap, \cup, \oplus, \neg),$$

where $H(\cap, \cup, \oplus, \neg)$ represents a structure closed under the IndetermSoft operators.

Example 2.4 (IndetermSoft Set: Hiring with Borderline Experience)**.** Let the universe of candidates be

$$U = \{c_1, c_2, c_3, c_4, c_5\},$$

and take the working subset $H = \{c_1, c_2, c_3, c_4, c_5\} \subseteq U$. Consider attribute values

$$A = \{\text{junior}, \text{ mid}, \text{ senior}, \text{ borderline(mid/senior)}\}.$$

Define $F : A \to \mathcal{P}(H)$ by

$$F(\text{junior}) = \{c_4, c_5\}, \quad F(\text{mid}) = \{c_2, c_3\}, \quad F(\text{senior}) = \{c_1\}.$$

For the borderline value, CV evidence is conflicting for c_3 and measurement is incomplete for c_5, so the image is *indeterminate*. We record this as a *set of plausible images*

$$\mathcal{I}\big(\text{borderline(mid/senior)}\big) = \big\{\{c_3\}, \ \{c_3, c_5\}\big\},$$

meaning F(borderline(mid/senior)) is not uniquely determined. Hence at least one of the defining conditions holds: (1) A contains an indeterminate value ("borderline"), and (3) there exists $v \in A$ with $F(v)$ indeterminate. In the algebraic view, we consider

$$F : \ A \longrightarrow H(\cap, \cup, \oplus, \neg),$$

where $H(\cap, \cup, \oplus, \neg)$ denotes a structure closed under the IndetermSoft operators, allowing computations to propagate the uncertainty in F(borderline).

2.2. Rough Set and IndetermRough Set

The definitions of Rough Set and IndetermRough Set are provided below. Since the structure of the IndetermRough Set is used as a reference for formulating the definition of the ContraRough Set, its definition is also included here.

Definition 2.5 (Rough Set Approximation)**.** [54, 55] Let X be a non-empty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (or indiscernibility relation) on X. The equivalence relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for $x \in X$, where:

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ of U are defined as follows:

1. *Lower Approximation* $\underline{U}$:
$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$
The lower approximation $\underline{U}$ includes all elements of X whose equivalence classes are entirely contained within U. These are the elements that *definitely* belong to U.

2. *Upper Approximation* $\overline{U}$:
$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$
The upper approximation $\overline{U}$ contains all elements of X whose equivalence classes have a non-empty intersection with U. These are the elements that *possibly* belong to U.

The pair $(\underline{U}, \overline{U})$ forms the *rough set* representation of U, satisfying the relationship:

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

Example 2.6 (Rough Set: Course placement by score bands)**.** Let the universe be eight students

$$X = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}.$$

Define an indiscernibility (equivalence) relation R by test-score bands:

$$\begin{aligned} [s_1]_R = [s_2]_R &= \{s_1, s_2\} \quad \text{(Low)}, \\ [s_3]_R = [s_4]_R = [s_5]_R &= \{s_3, s_4, s_5\} \quad \text{(Medium)}, \\ [s_6]_R = [s_7]_R &= \{s_6, s_7\} \quad \text{(High)}, \\ [s_8]_R &= \{s_8\} \quad \text{(Top)}. \end{aligned}$$

Let $U \subseteq X$ denote students recommended for the *Advanced* track:

$$U = \{s_4, s_5, s_6, s_8\}.$$

Lower and upper approximations (Pawlak):

$$\underline{U} = \{x \in X : [x]_R \subseteq U\}, \qquad \overline{U} = \{x \in X : [x]_R \cap U \neq \varnothing\}.$$

Compute:

$$\begin{aligned} [s_1]_R = \{s_1, s_2\} \nsubseteq U, \quad [s_3]_R = \{s_3, s_4, s_5\} \nsubseteq U, \\ [s_6]_R = \{s_6, s_7\} \nsubseteq U, \quad [s_8]_R = \{s_8\} \subseteq U, \end{aligned}$$

hence

$$\underline{U} = \{s_8\}.$$

For the upper approximation:

$$\begin{aligned}
&[s_1]_R \cap U = \varnothing \;\Rightarrow\; s_1, s_2 \notin \overline{U},\\
&[s_3]_R \cap U = \{s_4, s_5\} \neq \varnothing \;\Rightarrow\; s_3, s_4, s_5 \in \overline{U},\\
&[s_6]_R \cap U = \{s_6\} \neq \varnothing \;\Rightarrow\; s_6, s_7 \in \overline{U},\\
&[s_8]_R \cap U = \{s_8\} \neq \varnothing \;\Rightarrow\; s_8 \in \overline{U},
\end{aligned}$$

so

$$\overline{U} = \{s_3, s_4, s_5, s_6, s_7, s_8\}.$$

Thus $\underline{U} = \{s_8\} \subseteq U \subseteq \overline{U}$.

Definition 2.7 (IndetermRough Set)**.** [40] Let $X \neq \varnothing$ be a universe.

Indeterminacy is modeled by three–valued predicates:

$$\rho_R : X \times X \to \{0, 1, ?\}, \qquad \nu_U : X \to \{0, 1, ?\},$$

where 1 means "definitely true", 0 means "definitely false", and $?$ means "unknown/undecided".

Interpretation.

$$\rho_R(x,y) = \begin{cases} 1 & x \text{ is definitely } R\text{–related to } y, \\ 0 & x \text{ is definitely not } R\text{–related to } y, \\ ? & \text{it is unknown whether } x \text{ is } R\text{–related to } y, \end{cases} \qquad \nu_U(y) = \begin{cases} 1 & y \text{ is definitely in } U, \\ 0 & y \text{ is definitely not in } U, \\ ? & \text{it is unknown whether } y \in U. \end{cases}$$

From these, define the *definite/possible* parts:

$$\begin{aligned}
R^{\mathrm{def}} &:= \{(x,y) \in X \times X :\ \rho_R(x,y) = 1\}, \quad & R^{\mathrm{pos}} &:= \{(x,y) \in X \times X :\ \rho_R(x,y) \in \{1, ?\}\},\\
U^{\mathrm{def}} &:= \{y \in X :\ \nu_U(y) = 1\}, & U^{\mathrm{pos}} &:= \{y \in X :\ \nu_U(y) \in \{1, ?\}\}.
\end{aligned}$$

For $x \in X$ write the definite/possible R–neighborhoods

$$N^{\mathrm{def}}(x) := \{y \in X :\ (x,y) \in R^{\mathrm{def}}\}, \qquad N^{\mathrm{pos}}(x) := \{y \in X :\ (x,y) \in R^{\mathrm{pos}}\}.$$

The triple (X, R^*, U^*), with R^* represented by ρ_R and U^* by ν_U, is called an *IndetermRough Set* when the following rough approximations are used:

$$\underline{U^*} \;:=\; \{\, x \in X :\ N^{\mathrm{def}}(x) \subseteq U^{\mathrm{def}} \,\}, \qquad \overline{U^*} \;:=\; \{\, x \in X :\ N^{\mathrm{pos}}(x) \cap U^{\mathrm{pos}} \neq \varnothing \,\}.$$

We call $\underline{U^*}$ the *lower (indeterminate) approximation* and $\overline{U^*}$ the *upper (indeterminate) approximation* of U^* (with respect to R^*).

The *boundary*, *positive*, and *negative* regions are

$$\mathrm{Bnd}(U^*) := \overline{U^*} \setminus \underline{U^*}, \qquad \mathrm{Pos}(U^*) := \underline{U^*}, \qquad \mathrm{Neg}(U^*) := X \setminus \overline{U^*}.$$

Example 2.8 (IndetermRough Set: Contact tracing with unknown links)**.** Let the universe be five people

$$X = \{A, B, C, D, E\}.$$

We model indeterminacy with the definite/possible parts of the relation and the set (Definition 2.7).

Definite/possible contact neighborhoods (symmetric, include self):

$$\begin{aligned}
N^{\mathrm{def}}(A) &= \{A, B\}, \quad N^{\mathrm{pos}}(A) = \{A, B, C\},\\
N^{\mathrm{def}}(B) &= \{A, B\}, \quad N^{\mathrm{pos}}(B) = \{A, B\},\\
N^{\mathrm{def}}(C) &= \{C\}, \qquad N^{\mathrm{pos}}(C) = \{C, A\},\\
N^{\mathrm{def}}(D) &= \{D\}, \qquad N^{\mathrm{pos}}(D) = \{D\},\\
N^{\mathrm{def}}(E) &= \{E\}, \qquad N^{\mathrm{pos}}(E) = \{E\}.
\end{aligned}$$

Interpretation: (A, B) is a definite close contact, (A, C) is a *possible* contact, while E has no reported links.

Definite/possible infection membership:

$$U^{\mathrm{def}} = \{D\} \quad \text{(confirmed infected)}, \qquad U^{\mathrm{pos}} = \{B, C, D\} \quad \text{(symptomatic/awaiting tests include } B, C).$$

Indeterminate rough approximations:

$$\underline{U^*} = \{x \in X : N^{\mathrm{def}}(x) \subseteq U^{\mathrm{def}}\}, \qquad \overline{U^*} = \{x \in X : N^{\mathrm{pos}}(x) \cap U^{\mathrm{pos}} \neq \varnothing\}.$$

Compute the lower approximation:

$$\begin{aligned}
&N^{\mathrm{def}}(A) = \{A, B\} \nsubseteq U^{\mathrm{def}},\ N^{\mathrm{def}}(B) = \{A, B\} \nsubseteq U^{\mathrm{def}},\\
&N^{\mathrm{def}}(C) = \{C\} \nsubseteq U^{\mathrm{def}},\ N^{\mathrm{def}}(D) = \{D\} \subseteq U^{\mathrm{def}},\ N^{\mathrm{def}}(E) = \{E\} \nsubseteq U^{\mathrm{def}},
\end{aligned}$$

hence

$$\underline{U^*} = \{D\}.$$

Compute the upper approximation:

$$\begin{aligned}
N^{\mathrm{pos}}(A) \cap U^{\mathrm{pos}} &= \{A, B, C\} \cap \{B, C, D\} = \{B, C\} \neq \varnothing \ \Rightarrow\ A \in \overline{U^*},\\
N^{\mathrm{pos}}(B) \cap U^{\mathrm{pos}} &= \{A, B\} \cap \{B, C, D\} = \{B\} \neq \varnothing \ \Rightarrow\ B \in \overline{U^*},\\
N^{\mathrm{pos}}(C) \cap U^{\mathrm{pos}} &= \{C, A\} \cap \{B, C, D\} = \{C\} \neq \varnothing \ \Rightarrow\ C \in \overline{U^*},\\
N^{\mathrm{pos}}(D) \cap U^{\mathrm{pos}} &= \{D\} \cap \{B, C, D\} = \{D\} \neq \varnothing \ \Rightarrow\ D \in \overline{U^*},\\
N^{\mathrm{pos}}(E) \cap U^{\mathrm{pos}} &= \{E\} \cap \{B, C, D\} = \varnothing \ \Rightarrow\ E \notin \overline{U^*},
\end{aligned}$$

so

$$\overline{U^*} = \{A, B, C, D\}.$$

Thus $\underline{U^*} = \{D\} \subseteq \overline{U^*} = \{A, B, C, D\}$, with E outside even the possible region due to the absence of any known or plausible contact with suspected/confirmed cases.

Proposition 2.9 (Basic properties)
For every U^* as above:

$$\underline{U^*} \ \subseteq\ \overline{U^*}.$$

Moreover, if there is no indeterminacy (i.e. $\rho_R \in \{0, 1\}$ and $\nu_U \in \{0, 1\}$ everywhere) and R^{def} is an equivalence relation, then $\underline{U^*}$ and $\overline{U^*}$ reduce to the classical Pawlak lower/upper approximations.

Proof
If $x \in \underline{U^*}$ then $N^{\mathrm{def}}(x) \subseteq U^{\mathrm{def}} \subseteq U^{\mathrm{pos}}$. If $N^{\mathrm{def}}(x) \neq \varnothing$ we are done since $N^{\mathrm{def}}(x) \subseteq N^{\mathrm{pos}}(x)$ implies $N^{\mathrm{pos}}(x) \cap U^{\mathrm{pos}} \neq \varnothing$. If $N^{\mathrm{def}}(x) = \varnothing$, then $N^{\mathrm{pos}}(x)$ may still contain elements via unknown links (?); whenever it does and meets U^{pos}, $x \in \overline{U^*}$. In the crisp case the formulas coincide with the standard ones by replacing "def/pos" with membership in R and U. □

2.3. Upside-Down Logic

Upside-Down Logic captures context–triggered reversals of truth values: under suitable contextual transformations, a lemma that was true may become false and, symmetrically, a false lemma may become true. In this way, the framework formalizes ambiguity and reversal phenomena in reasoning systems [48, 49, 50, 51, 52, 47, 53]. The basic terminology is given below.

Definition 2.10 (Logical System)**.** (cf. [56]) A *logical system* is a structure

$$\mathcal{M} \;=\; (\mathcal{P},\, \mathcal{V},\, v),$$

where:

- $\mathcal{P}$ is the set of lemmas (statements) in a formal language $\mathcal{L}$;
- $\mathcal{V}$ is the set of truth values (e.g., $\{\text{True}, \text{False}\}$ in the classical case);
- $v : \mathcal{P} \to \mathcal{V}$ is a *valuation* assigning a truth value to each lemma.

Optionally, $\mathcal{M}$ may be equipped with

- a set of *axioms* $\mathcal{A} \subseteq \mathcal{P}$ taken as true, and
- a collection of *inference rules* $\mathcal{I}$ describing admissible derivations.

Notation 2.11 (Context-sensitive truth map)
Given a lemma set $\mathcal{P}$ and a family of contexts $\mathcal{C}$, define

$$T:\; \mathcal{P} \times \mathcal{C} \longrightarrow \{\text{True, False, Indeterminate}\}$$

so that $T(A, \mathcal{C})$ records the truth status of lemma A when evaluated in context $\mathcal{C}$.

Notation 2.12
Let $\mathcal{L}$ be a formal language, and let $\mathcal{M} = (\mathcal{P}, \mathcal{V}, v)$ be a logical system with lemma set $\mathcal{P}$, truth-value set $\mathcal{V}$, and valuation $v : \mathcal{P} \to \mathcal{V}$.

Definition 2.13 (Upside-Down Logic)**.** [49, 50] An *Upside-Down Logic* is any system $\mathcal{M}'$ obtained from $\mathcal{M}$ by a transformation U acting on lemmas and/or contexts such that:

1. For every lemma $A \in \mathcal{P}$ with truth value $v(A)$ in a context $\mathcal{C}$, there exist a transformed lemma $U(A)$ and/or a transformed context $U(\mathcal{C})$ satisfying:
   - *Falsification of a True lemma:* if $v(A) = \text{True}$ in $\mathcal{C}$, then $v\big(U(A)\big) = \text{False}$ in $U(\mathcal{C})$.
   - *Truthification of a False lemma:* if $v(A) = \text{False}$ in $\mathcal{C}$, then $v\big(U(A)\big) = \text{True}$ in $U(\mathcal{C})$.
2. The operator U is well defined and the resulting system $\mathcal{M}'$ is internally consistent.

Example 2.14 (Upside-Down Logic in Retail Return Policies)**.** Let the logical system track a store's return window W (in days). For any order with age d days, consider the lemma

$$A(d):\;\text{“The order is returnable.”}$$

and valuation $v(A(d) \mid \mathcal{C}) = \text{True}$ iff $d \le W$, otherwise False.

Old context (standard policy): $\mathcal{C}_{\text{old}}$ with $W = 30$. Holiday update U extends the window to $W' = 60$: $U(\mathcal{C}_{\text{old}}) = \mathcal{C}_{\text{new}}$.

Truthification of the False. Take $d = 45$. Then

$$v\big(A(45) \mid \mathcal{C}_{\text{old}}\big) = \text{False} \quad (45 > 30), \qquad v\big(U(A(45)) \mid U(\mathcal{C}_{\text{old}})\big) = v\big(A(45) \mid \mathcal{C}_{\text{new}}\big) = \text{True } (45 \leq 60).$$

Falsification of the Truth (flash-sale terms). Let another update U' shorten the window to $W'' = 14$: $U'(\mathcal{C}_{\text{old}}) = \mathcal{C}_{\text{flash}}$. For $d = 25$,

$$v\big(A(25) \mid \mathcal{C}_{\text{old}}\big) = \text{True} \quad (25 \leq 30), \qquad v\big(U'(A(25)) \mid U'(\mathcal{C}_{\text{old}})\big) = v\big(A(25) \mid \mathcal{C}_{\text{flash}}\big) = \text{False } (25 > 14).$$

Thus the transformation(s) U (or U') invert the truth value by changing the policy context, matching the Upside-Down Logic schema.

Example 2.15 (Upside-Down Logic in Reversible-Lane Traffic Control)**.** Consider an urban road with a reversible lane L. Let

$$A: \text{ “Lane } L \text{ is } \textit{eastbound-only}.\text{”}, \qquad B: \text{ “Lane } L \text{ is } \textit{two-way}.\text{”}$$

Daytime context $\mathcal{C}_{\text{day}}$: policy flag $p = \texttt{one\_way}$ (eastbound-only). Nighttime transformation U toggles the lane to two-way: $U(\mathcal{C}_{\text{day}}) = \mathcal{C}_{\text{night}}$ with $p' = \texttt{two\_way}$.

Falsification of the Truth.

$$v\big(A \mid \mathcal{C}_{\text{day}}\big) = \text{True}, \qquad v\big(U(A) \mid U(\mathcal{C}_{\text{day}})\big) = v\big(A \mid \mathcal{C}_{\text{night}}\big) = \text{False}.$$

Truthification of the False.

$$v\big(B \mid \mathcal{C}_{\text{day}}\big) = \text{False}, \qquad v\big(U(B) \mid U(\mathcal{C}_{\text{day}})\big) = v\big(B \mid \mathcal{C}_{\text{night}}\big) = \text{True}.$$

Hence the context flip U inverts the truth values of lane-usage statements, providing a concrete real-world instantiation of Upside-Down Logic.

2.4. Plithogenic Set

A Plithogenic Set models elements with attribute values, appurtenance degrees, and contradiction degrees, extending fuzzy, intuitionistic, and neutrosophic set frameworks [42, 57, 58].

Definition 2.16 (Plithogenic Set)**.** [59, 41] Let S be a universal set and $P \subseteq S$ a nonempty subset. A *Plithogenic Set* is a quintuple

$$PS = (P,\ v,\ Pv,\ pdf,\ pCF),$$

where

- v is an attribute,
- Pv is the set of possible values of the attribute v,
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)*,†

†In the literature, DAF is defined in slightly different ways: some variants use powerset–valued constructions, others the simple cube $[0,1]^s$. We adopt the latter (classical) form here; cf. [60].

- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

The DCF satisfies, for all $a, b \in Pv$,

$$\text{Reflexivity: } pCF(a,a) = 0, \qquad \text{Symmetry: } pCF(a,b) = pCF(b,a).$$

Here $s \in \mathbb{N}$ is the appurtenance dimension and $t \in \mathbb{N}$ the contradiction dimension.

Example 2.17 (Plithogenic Set: Hiring Candidates by Seniority Profile)**.** Let the universe S be all applicants and take the nonempty subset

$$P = \{\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3\} \subseteq S$$

of shortlisted candidates. Consider the attribute $v =$ "seniority profile" with possible values

$$Pv = \{\text{Junior}, \text{ Mid}, \text{ Senior}\}.$$

We take scalar appurtenance and contradiction ($s = t = 1$), so $pdf : P \times Pv \to [0,1]$ and $pCF : Pv \times Pv \to [0,1]$.

Degree of Appurtenance Function (DAF). For each candidate $C \in P$ and value $a \in Pv$, $pdf(C, a)$ expresses how well C fits a:

	Junior	Mid	Senior
C_1	0.80	0.20	0.00
C_2	0.20	0.70	0.10
C_3	0.00	0.30	0.70

Degree of Contradiction Function (DCF). The contradiction between seniority values is encoded by the symmetric matrix (reflexive diagonal 0):

$$pCF(a,b) = \begin{array}{c|ccc} & \text{Junior} & \text{Mid} & \text{Senior} \\ \hline \text{Junior} & 0 & 0.50 & 1.00 \\ \text{Mid} & 0.50 & 0 & 0.60 \\ \text{Senior} & 1.00 & 0.60 & 0 \end{array}$$

Thus $PS = (P, v, Pv, pdf, pCF)$ is a Plithogenic Set. The DAF captures each candidate's mixed seniority profile, while the DCF quantifies how contradictory different seniority categories are (e.g., "Junior" vs "Senior" is maximally contradictory), enabling nuanced selection and aggregation rules beyond classical models.

3. Main Results

In this section, we present the main contributions of this paper, focusing on the concepts of ContraSoft Set and ContraRough Set.

3.1. ContraSoft Set

A ContraSoft Set is a parameterized soft set where each parameter's values are associated with a contradiction degree, and thresholding is used to aggregate only those values that are not too contradictory with respect to a chosen reference. This allows soft-set modeling to filter or weight information based on contradiction, rather than uncertainty.

Definition 3.1 (Contradiction on attribute values)**.** Let V be a nonempty finite set of attribute values. A *contradiction function* on V is a map

$$c : V \times V \longrightarrow [0,1]$$

such that

$$c(v,v) = 0 \quad \text{(reflexivity)}, \qquad c(v,w) = c(w,v) \quad \text{(symmetry)}.$$

The quantity $c(v,w)$ measures the degree of *contradiction* between v and w (larger means more contradictory).

Example 3.2 (Contradiction function on risk levels)**.** Let $V = \{\text{Low}, \text{Medium}, \text{High}, \text{Critical}\}$. Assign ranks $r(\text{Low}) = 0$, $r(\text{Medium}) = 1$, $r(\text{High}) = 2$, $r(\text{Critical}) = 3$ and define

$$c(v,w) := \frac{|r(v) - r(w)|}{3} \in [0,1] \qquad (v,w \in V).$$

Then $c(v,v) = 0$ (reflexivity) and $c(v,w) = c(w,v)$ (symmetry). For instance, $c(\text{Low}, \text{High}) = \frac{|0-2|}{3} = \frac{2}{3}$ and $c(\text{Medium}, \text{Critical}) = \frac{|1-3|}{3} = \frac{2}{3}$.

Definition 3.3 (ContraSoft structure)**.** Let U be a nonempty universe and E a nonempty set of parameters. For each $e \in E$ fix:

- a nonempty finite value set V_e;
- a contradiction function $c_e : V_e \times V_e \to [0,1]$ (Definition 3.1);
- a designated *reference value* $v_e^\star \in V_e$.

Write $V := \bigsqcup_{e \in E} (\{e\} \times V_e)$ for the disjoint union of all parameter–value pairs.

Example 3.4 (ContraSoft structure: laptops by price and battery)**.** Let the universe be three laptops $U = \{L_1, L_2, L_3\}$ and parameters $E = \{\text{price}, \text{battery}\}$. For price, take $V_{\text{price}} = \{\text{budget}, \text{standard}, \text{premium}\}$ with ranks $0, 1, 2$ and

$$c_{\text{price}}(v,w) := \frac{|r(v) - r(w)|}{2}, \qquad v^\star_{\text{price}} = \text{standard}.$$

For battery, take $V_{\text{battery}} = \{\text{short}, \text{average}, \text{long}\}$ with ranks $0, 1, 2$ and

$$c_{\text{battery}}(v,w) := \frac{|r(v) - r(w)|}{2}, \qquad v^\star_{\text{battery}} = \text{long}.$$

The disjoint union of parameter–value pairs is

$$V = \{(\text{price}, \text{budget}), (\text{price}, \text{standard}), (\text{price}, \text{premium}), (\text{battery}, \text{short}), (\text{battery}, \text{average}), (\text{battery}, \text{long})\}.$$

This specifies a concrete ContraSoft structure as in Definition 3.3.

Definition 3.5 (ContraSoft Set)**.** Let U be a finite universe of objects and E a finite set of parameters. A *ContraSoft Set* is a quadruple

$$\mathsf{CS} \;:=\; (U,\; E,\; F,\; c),$$

where

- $F : E \to \mathcal{P}(U)$ is the (crisp) soft mapping; $F(e) \subseteq U$ is the set of objects *accepted* (or classified as positive) under parameter e;

- $c : E \times E \to [0,1]$ is a *contradiction degree* on parameters, symmetric and reflexive on the diagonal:

$$c(e,e) = 0, \qquad c(e,f) = c(f,e) \quad (\forall\, e,f \in E).$$

For $x \in U$ and $e \in E$, the atomic lemma "x is accepted by e" is represented by

$$A(x,e) : \quad x \in F(e),$$

with truth value **T** if $x \in F(e)$ and **F** otherwise.

Remark 3.6 (Relation to classical soft sets and to "indeterminacy")**.** If $V_e = \{v_e^\star\}$ for all e, then $F^{(\tau)}(e) = F(e, v_e^\star)$ and we recover the classical soft set (F°, E) with $F^\circ(e) = F(e, v_e^\star)$. Thus, *contradiction* plays the role of the third component often used as "indeterminacy" (e.g. in neutrosophic settings), but here it acts as a *distance-to-reference* that controls which value-slices are admitted into $F^{(\tau)}(e)$.

Example 3.7 (Hiring decision with degrees/experience)**.** Let the universe be the four candidates

$$U = \{A, B, C, D\}.$$

Parameters $E = \{\text{degree}, \ \text{experience}\}$.

Value sets and ranks:

$$V_{\text{degree}} = \{\text{no}, \ \text{bachelor}, \ \text{master}, \ \text{phd}\}, \qquad r_{\deg}(\text{no}) = 0, \ r_{\deg}(\text{bachelor}) = 1, \ r_{\deg}(\text{master}) = 2, \ r_{\deg}(\text{phd}) = 3;$$
$$V_{\text{exp}} = \{\text{junior}, \ \text{mid}, \ \text{senior}\}, \qquad r_{\exp}(\text{junior}) = 0, \ r_{\exp}(\text{mid}) = 1, \ r_{\exp}(\text{senior}) = 2.$$

Contradiction functions (ordinal, normalized):

$$c_{\deg}(v,w) := \frac{|r_{\deg}(v) - r_{\deg}(w)|}{3}, \qquad c_{\exp}(v,w) := \frac{|r_{\exp}(v) - r_{\exp}(w)|}{2}.$$

References: $v_{\deg}^\star = \text{master}$, $v_{\exp}^\star = \text{mid}$.

Crisp ContraSoft map $F : (\{e\} \times V_e) \to \mathcal{P}(U)$:

$$F(\text{degree}, \text{no}) = \{D\}, \quad F(\text{degree}, \text{bachelor}) = \{B, D\}, \quad F(\text{degree}, \text{master}) = \{A, C\}, \quad F(\text{degree}, \text{phd}) = \{A\};$$
$$F(\text{experience}, \text{junior}) = \{B, D\}, \quad F(\text{experience}, \text{mid}) = \{A, C\}, \quad F(\text{experience}, \text{senior}) = \{A\}.$$

Explicit contradiction-to-reference values:

$$c_{\deg}(\text{no}, \text{master}) = \tfrac{|0-2|}{3} = \tfrac{2}{3}, \quad c_{\deg}(\text{bachelor}, \text{master}) = \tfrac{|1-2|}{3} = \tfrac{1}{3},$$
$$c_{\deg}(\text{phd}, \text{master}) = \tfrac{|3-2|}{3} = \tfrac{1}{3}, \quad c_{\deg}(\text{master}, \text{master}) = 0;$$
$$c_{\exp}(\text{junior}, \text{mid}) = \tfrac{|0-1|}{2} = \tfrac{1}{2}, \quad c_{\exp}(\text{senior}, \text{mid}) = \tfrac{|2-1|}{2} = \tfrac{1}{2}, \quad c_{\exp}(\text{mid}, \text{mid}) = 0.$$

Thresholded sections:

1) $\tau = 0$ (only perfectly non-contradictory values pass)

$$F^{(0)}(\text{degree}) = F(\text{degree}, \text{master}) = \{A, C\},$$
$$F^{(0)}(\text{experience}) = F(\text{experience}, \text{mid}) = \{A, C\}.$$

2) $\tau = \frac{1}{3}$ (exclude only $c > \frac{1}{3}$)

$$\begin{aligned} F^{(1/3)}(\text{degree}) &= F(\text{degree}, \text{bachelor}) \cup F(\text{degree}, \text{master}) \cup F(\text{degree}, \text{phd}) \\ &= \{B, D\} \cup \{A, C\} \cup \{A\} = \{A, B, C, D\}, \\ F^{(1/3)}(\text{experience}) &= F(\text{experience}, \text{mid}) = \{A, C\} \quad (\text{since } c = \tfrac{1}{2} > \tfrac{1}{3} \text{ for junior/senior}). \end{aligned}$$

3) $\tau = 1$ (all values pass)

$$F^{(1)}(\text{degree}) = \{A, B, C, D\}, \qquad F^{(1)}(\text{experience}) = \{A, B, C, D\}.$$

Interpretation. At strict tolerance ($\tau = 0$) only reference-aligned slices remain; at moderate tolerance ($\tau = \frac{1}{3}$) degree becomes fully inclusive while experience stays selective; and at full tolerance ($\tau = 1$) both parameters include the entire candidate pool.

Example 3.8 (Data-driven ContraSoft Set: e-mail spam screening)**.** Let the universe be five incoming messages $U = \{m_1, m_2, m_3, m_4, m_5\}$ and parameters

$$E = \{\text{BLK}, \ \text{TRU}, \ \text{LNK}, \ \text{PRM}\},$$

where BLK = "sender on blacklist", TRU = "sender on trusted list", LNK = "≥ 3 links", and PRM = "promotional wording". The soft mapping (learned feature presence on this batch) is

$$\begin{aligned} F(\text{BLK}) &= \{m_1, m_3\}, & F(\text{TRU}) &= \{m_2, m_4\}, \\ F(\text{LNK}) &= \{m_1, m_3, m_5\}, & F(\text{PRM}) &= \{m_1, m_5\}. \end{aligned}$$

From a large historical corpus we estimate pairwise feature correlations $\hat{\rho}(e, f)$ and set the (symmetric) contradiction degree

$$c(e, f) \ := \ \frac{1 - \hat{\rho}(e, f)}{2}, \qquad c(e, e) = 0.$$

For illustration, suppose the learned values are

$$\begin{bmatrix} & \text{BLK} & \text{TRU} & \text{LNK} & \text{PRM} \\ \text{BLK} & 0 & 0.92 & 0.20 & 0.25 \\ \text{TRU} & 0.92 & 0 & 0.80 & 0.70 \\ \text{LNK} & 0.20 & 0.80 & 0 & 0.35 \\ \text{PRM} & 0.25 & 0.70 & 0.35 & 0 \end{bmatrix}.$$

Thus $\mathsf{CS} = (U, E, F, c)$ is a data-driven ContraSoft Set: F comes from detected features on current messages, while c is learned from past data so that negatively correlated indicators (e.g. BLK vs. TRU) receive high contradiction, guiding contradiction-aware decisions.

Example 3.9 (Apartment selection: rent vs. commute)**.** Universe of apartments:

$$U = \{h_1, h_2, h_3, h_4\}.$$

Parameters $E = \{\text{rent}, \ \text{commute}\}$.

Value sets, ranks, and contradiction:

$$\begin{aligned} V_{\text{rent}} &= \{\text{low}, \ \text{mid}, \ \text{high}\}, & r_{\text{rent}}(\text{low}) &= 0, \ r_{\text{rent}}(\text{mid}) = 1, \ r_{\text{rent}}(\text{high}) = 2, \\ V_{\text{com}} &= \{\text{short}, \ \text{medium}, \ \text{long}\}, & r_{\text{com}}(\text{short}) &= 0, \ r_{\text{com}}(\text{medium}) = 1, \ r_{\text{com}}(\text{long}) = 2, \\ c_{\text{rent}}(v, w) &= \frac{|r_{\text{rent}}(v) - r_{\text{rent}}(w)|}{2}, & c_{\text{com}}(v, w) &= \frac{|r_{\text{com}}(v) - r_{\text{com}}(w)|}{2}. \end{aligned}$$

References: $v^\star_{\text{rent}} = \text{mid}$, $v^\star_{\text{com}} = \text{short}$.

Crisp map F:

$$F(\text{rent}, \text{low}) = \{h_2, h_3\}, \quad F(\text{rent}, \text{mid}) = \{h_1, h_3\}, \quad F(\text{rent}, \text{high}) = \{h_1, h_4\};$$
$$F(\text{commute}, \text{short}) = \{h_1, h_2\}, \quad F(\text{commute}, \text{medium}) = \{h_3\}, \quad F(\text{commute}, \text{long}) = \{h_4\}.$$

Explicit contradictions to references:

$$c_{\text{rent}}(\text{low}, \text{mid}) = \tfrac{|0-1|}{2} = \tfrac{1}{2}, \quad c_{\text{rent}}(\text{high}, \text{mid}) = \tfrac{|2-1|}{2} = \tfrac{1}{2}, \quad c_{\text{rent}}(\text{mid}, \text{mid}) = 0;$$
$$c_{\text{com}}(\text{medium}, \text{short}) = \tfrac{|1-0|}{2} = \tfrac{1}{2}, \quad c_{\text{com}}(\text{long}, \text{short}) = \tfrac{|2-0|}{2} = 1, \quad c_{\text{com}}(\text{short}, \text{short}) = 0.$$

Thresholded sections:

1) $\tau = 0$:

$$F^{(0)}(\text{rent}) = F(\text{rent}, \text{mid}) = \{h_1, h_3\}, \qquad F^{(0)}(\text{commute}) = F(\text{commute}, \text{short}) = \{h_1, h_2\}.$$

2) $\tau = \frac{1}{2}$:

$$\begin{aligned} F^{(1/2)}(\text{rent}) &= F(\text{rent}, \text{low}) \cup F(\text{rent}, \text{mid}) \cup F(\text{rent}, \text{high}) \\ &= \{h_2, h_3\} \cup \{h_1, h_3\} \cup \{h_1, h_4\} = \{h_1, h_2, h_3, h_4\}, \\ F^{(1/2)}(\text{commute}) &= F(\text{commute}, \text{short}) \cup F(\text{commute}, \text{medium}) \\ &= \{h_1, h_2\} \cup \{h_3\} = \{h_1, h_2, h_3\} \quad (\text{exclude long since } c = 1 > 1/2). \end{aligned}$$

3) $\tau = 1$:

$$F^{(1)}(\text{rent}) = \{h_1, h_2, h_3, h_4\}, \qquad F^{(1)}(\text{commute}) = \{h_1, h_2, h_3, h_4\}.$$

At moderate tolerance ($\tau = \frac{1}{2}$), all rent levels are admitted (full coverage), while long commutes remain excluded; only when tolerance reaches 1 do long commutes enter the selection.

Lemma 3.10 (Basic properties)
Fix $e \in E$.

(i) (*Monotonicity in* τ) If $0 \le \tau_1 \le \tau_2 \le 1$, then

$$F^{(\tau_1)}(e) \subseteq F^{(\tau_2)}(e).$$

(ii) (*Boundary cases*)

$$F^{(0)}(e) = \bigcup_{v \in V_e:\ c_e(v, v_e^\star) = 0} F(e, v), \qquad F^{(1)}(e) = \bigcup_{v \in V_e} F(e, v).$$

In particular, if $c_e(v, w) = 0 \iff v = w$, then $F^{(0)}(e) = F(e, v_e^\star)$.

(iii) (*Contra-union/intersection*) For two ContraSoft sets F, G on the same structure,

$$(F \sqcup_c G)^{(\tau)}(e) := F^{(\tau)}(e) \cup G^{(\tau)}(e), \qquad (F \sqcap_c G)^{(\tau)}(e) := F^{(\tau)}(e) \cap G^{(\tau)}(e)$$

define the *contra-union* and *contra-intersection*. Both are ContraSoft sets (on the same structure).

Proof
(i) If $\tau_1 \le \tau_2$, then $\{v : c_e(v, v_e^\star) \le \tau_1\} \subseteq \{v : c_e(v, v_e^\star) \le \tau_2\}$, hence the corresponding unions satisfy $F^{(\tau_1)}(e) \subseteq F^{(\tau_2)}(e)$.

(ii) The formulas follow directly from the definition by substituting $\tau = 0$ and $\tau = 1$. If $c_e(v, w) = 0 \iff v = w$, then only $v = v_e^\star$ contributes when $\tau = 0$.

(iii) Immediate from set-theoretic closure of $\cup, \cap$. □

Definition 3.11 (Fuzzy ContraSoft set and plithogenic-style aggregation)**.** A *fuzzy ContraSoft set* is given by a membership map

$$\mu : U \times V \longrightarrow [0,1], \qquad (u; e, v) \longmapsto \mu(u \mid e, v).$$

For a threshold $\tau \in [0,1]$, define the *contradiction-weighted* aggregated membership (relative to the reference $v_e^\star$) by

$$\mu^{(\tau)}(u \mid e) \;:=\; \frac{\displaystyle\sum_{v \in V_e} \big(1 - c_e(v, v_e^\star)\big)\, \mathbf{1}_{\{c_e(v, v_e^\star) \le \tau\}}\; \mu(u \mid e, v)}{\displaystyle\sum_{v \in V_e} \big(1 - c_e(v, v_e^\star)\big)\, \mathbf{1}_{\{c_e(v, v_e^\star) \le \tau\}}},$$

with the convention $\mu^{(\tau)}(u \mid e) := \mu(u \mid e, v_e^\star)$ if the denominator is 0. This is the plithogenic-style aggregator where higher contradiction down-weights a value v through the factor $1 - c_e(v, v_e^\star)$ and values with $c_e(v, v_e^\star) > \tau$ are entirely excluded.

Lemma 3.12 (Checks on the aggregator)
Assume $c_e(v, w) = 0 \iff v = w$. Then, for all $u \in U$ and $e \in E$:

(i) $\mu^{(0)}(u \mid e) = \mu(u \mid e, v_e^\star)$.

(ii) If $c_e \equiv 0$ on V_e (no contradictions among values), then

$$\mu^{(1)}(u \mid e) = \frac{1}{|V_e|} \sum_{v \in V_e} \mu(u \mid e, v).$$

Proof
(i) When $\tau = 0$, only $v = v_e^\star$ contributes with weight $1 - c_e(v_e^\star, v_e^\star) = 1$.

(ii) If $c_e \equiv 0$, then all weights equal 1 and no value is excluded at $\tau = 1$, giving the simple arithmetic mean. □

Example 3.13 (Concrete computation)**.** Let $U = \{p_1, p_2, p_3, p_4\}$ be four smartphones and $E = \{\text{price}, \text{battery}\}$. Set

$$V_{\text{price}} = \{\text{low}, \text{mid}, \text{high}\}, \qquad V_{\text{battery}} = \{\text{poor}, \text{ok}, \text{great}\}.$$

Define contradictions (*ordered-scale model*):

$$c_{\text{price}}(\text{low}, \text{low}) = 0, \quad c_{\text{price}}(\text{mid}, \text{mid}) = 0, \quad c_{\text{price}}(\text{high}, \text{high}) = 0,$$
$$c_{\text{price}}(\text{low}, \text{mid}) = c_{\text{price}}(\text{mid}, \text{high}) = \tfrac{1}{2}, \quad c_{\text{price}}(\text{low}, \text{high}) = 1,$$

and similarly

$$c_{\text{battery}}(\text{poor}, \text{poor}) = 0, \; c_{\text{battery}}(\text{ok}, \text{ok}) = 0, \; c_{\text{battery}}(\text{great}, \text{great}) = 0,$$
$$c_{\text{battery}}(\text{poor}, \text{ok}) = c_{\text{battery}}(\text{ok}, \text{great}) = \tfrac{1}{2}, \quad c_{\text{battery}}(\text{poor}, \text{great}) = 1.$$

Choose references $v_{\text{price}}^\star = \text{low}$, $v_{\text{battery}}^\star = \text{great}$.

Crisp ContraSoft map F:

$$F(\text{price}, \text{low}) = \{p_2, p_3\}, \quad F(\text{price}, \text{mid}) = \{p_1, p_3\}, \quad F(\text{price}, \text{high}) = \{p_1, p_4\},$$
$$F(\text{battery}, \text{poor}) = \{p_4\}, \quad F(\text{battery}, \text{ok}) = \{p_1, p_3\}, \quad F(\text{battery}, \text{great}) = \{p_2\}.$$

(i) Threshold $\tau = 0$ (only perfectly non-contradictory values kept):

$$F^{(0)}(\text{price}) = F(\text{price}, \text{low}) = \{p_2, p_3\}, \quad F^{(0)}(\text{battery}) = F(\text{battery}, \text{great}) = \{p_2\}.$$

(ii) Threshold $\tau = \frac{1}{2}$ (also keep adjacent values):

$$F^{(1/2)}(\text{price}) = F(\text{price}, \text{low}) \cup F(\text{price}, \text{mid}) = \{p_2, p_3\} \cup \{p_1, p_3\} = \{p_1, p_2, p_3\},$$
$$F^{(1/2)}(\text{battery}) = F(\text{battery}, \text{great}) \cup F(\text{battery}, \text{ok}) = \{p_2\} \cup \{p_1, p_3\} = \{p_1, p_2, p_3\}.$$

(iii) Fuzzy aggregation check (Definition 3.11): Suppose for a fixed phone u the price-memberships are

$$\mu(u \mid \text{price}, \text{low}) = 0.8, \quad \mu(u \mid \text{price}, \text{mid}) = 0.6, \quad \mu(u \mid \text{price}, \text{high}) = 0.1.$$

With $\tau = \frac{1}{2}$ and $v^{\star}_{\text{price}} = \text{low}$, the active values are low and mid with weights $1 - c = 1$ and $1 - \frac{1}{2} = \frac{1}{2}$, respectively. Hence

$$\mu^{(1/2)}(u \mid \text{price}) = \frac{1 \cdot 0.8 + \frac{1}{2} \cdot 0.6}{1 + \frac{1}{2}} = \frac{0.8 + 0.3}{1.5} = \frac{1.1}{1.5} = 0.733\overline{3}.$$

If instead $\tau = 0$, then $\mu^{(0)}(u \mid \text{price}) = \mu(u \mid \text{price}, \text{low}) = 0.8$ by Proposition 3.12(i).

3.2. *Upside-down logic in ContraSoft Set*

Upside-Down Logic in a ContraSoft Set flips membership truth under highly contradictory parameters, transforming accepted elements into rejected ones, ensuring contradiction-driven logical reversals.

Definition 3.14 (Upside-Down operator on a ContraSoft Set (with activation threshold))**.** Fix an *anchor* parameter $b \in E$ and a threshold $\tau \in [0, 1]$. Define the *activation set*

$$A_\tau(b) \;:=\; \{\, e \in E \;:\; c(e, b) \;\geq\; \tau \,\}.$$

The *Upside-Down transform* $U_{b,\tau}$ produces a new ContraSoft Set

$$\mathsf{CS}^{U_{b,\tau}} \;:=\; \big(U,\; E,\; F^{U_{b,\tau}},\; c^{U_{b,\tau}}\big),$$

where, for each $e \in E$,

$$F^{U_{b,\tau}}(e) \;:=\; \begin{cases} U \setminus F(e), & e \in A_\tau(b) \quad \text{(flip)}, \\ F(e), & e \notin A_\tau(b) \quad \text{(keep)}, \end{cases} \qquad c^{U_{b,\tau}}(e, f) \;:=\; \begin{cases} 0, & \{e, f\} \cap A_\tau(b) \neq \varnothing, \\ c(e, f), & \text{otherwise}. \end{cases}$$

Logical inversion property. For any $x \in U$ and any $e \in A_\tau(b)$,

$$x \in F(e) \iff x \notin F^{U_{b,\tau}}(e), \quad \text{i.e.,} \quad v\big(A(x, e)\big) = \mathbf{T} \iff v^{U_{b,\tau}}\big(A(x, e)\big) = \mathbf{F}.$$

For $e \notin A_\tau(b)$ the truth value is preserved.

Example 3.15 (Food-safety reclassification after recall resolution)**.** Let the universe of products be

$$U = \{\text{Lettuce (Le)},\ \text{PeanutButter (Pb)},\ \text{Milk (Mi)}\}.$$

Parameters (E) and their intended meanings:

$$E = \{\text{APP},\ \text{REC},\ \text{ORG}\},$$

where APP = "approved for sale", REC = "under recall", ORG = "organic".

Initial soft mapping $F : E \to \mathcal{P}(U)$:

$$F(\text{APP}) = \{\text{Le}, \text{Mi}\}, \quad F(\text{REC}) = \{\text{Pb}\}, \quad F(\text{ORG}) = \{\text{Le}\}.$$

Contradiction degrees (symmetric, diagonal 0):

$$c(\text{APP}, \text{REC}) = 0.95, \quad c(\text{APP}, \text{ORG}) = 0.20, \quad c(\text{REC}, \text{ORG}) = 0.40.$$

Choose anchor $b = \text{APP}$ and threshold $\tau = 0.8$. Then

$$A_\tau(\text{APP}) = \{\text{REC}\} \quad \text{(only REC activates)}.$$

Apply $U_{b,\tau}$:

$$F^U(\text{REC}) = U \setminus F(\text{REC}) = \{\text{Le}, \text{Mi}\}, \qquad F^U(\text{APP}) = F(\text{APP}), \quad F^U(\text{ORG}) = F(\text{ORG}),$$

and reset contradictions for any pair involving REC:

$$c^U(\text{APP}, \text{REC}) = c^U(\text{REC}, \text{ORG}) = 0.$$

Truth flip (explicit). For the lemma $A(\text{Pb}, \text{REC})$:

$$\text{Pb} \in F(\text{REC})\ (= \{\text{Pb}\}) \ \Rightarrow\ v = \mathbf{T}, \qquad \text{Pb} \notin F^U(\text{REC})\ (= \{\text{Le}, \text{Mi}\}) \ \Rightarrow\ v^U = \mathbf{F}.$$

Thus a recall resolution that is maximally contradictory to "Approved" inverts the REC-classification and neutralizes its conflicts with other parameters.

Example 3.16 (Remote-work eligibility policy reversal)**.** Let

$$U = \{\text{E1}, \text{E2}, \text{E3}, \text{E4}\}$$

be employees. Parameters:

$$E = \{\text{REM},\ \text{ONS},\ \text{HYB}\}$$

with meanings REM = "eligible for remote", ONS = "on-site mandated", HYB = "hybrid eligible".

Initial soft mapping:

$$F(\text{REM}) = \{\text{E1}, \text{E2}\}, \quad F(\text{ONS}) = \{\text{E3}, \text{E4}\}, \quad F(\text{HYB}) = \{\text{E2}, \text{E3}\}.$$

Contradictions:

$$c(\text{REM}, \text{ONS}) = 0.90, \quad c(\text{REM}, \text{HYB}) = 0.30, \quad c(\text{ONS}, \text{HYB}) = 0.60.$$

Choose anchor $b = \text{REM}$ (organization adopts a remote-first stance), threshold $\tau = 0.8$. Then

$$A_\tau(\text{REM}) = \{\text{ONS}\}.$$

Apply $U_{b,\tau}$:

$$F^U(\mathrm{ONS}) = U \setminus F(\mathrm{ONS}) = \{\mathrm{E1}, \mathrm{E2}\}, \quad F^U(\mathrm{REM}) = F(\mathrm{REM}), \quad F^U(\mathrm{HYB}) = F(\mathrm{HYB}),$$

and set $c^U(\mathrm{REM}, \mathrm{ONS}) = 0$ (others unchanged).

Truth flip (explicit). Consider $A(\mathrm{E4}, \mathrm{ONS})$:

$$\mathrm{E4} \in F(\mathrm{ONS}) \Rightarrow v = \mathbf{T}, \qquad \mathrm{E4} \notin F^U(\mathrm{ONS}) \Rightarrow v^U = \mathbf{F}.$$

Conversely, $A(\mathrm{E1}, \mathrm{ONS})$ flips from $\mathbf{F}$ to $\mathbf{T}$. Hence the Upside-Down operator models a policy reversal: highly contradictory parameters to the anchor are inverted and their contradictions neutralized.

Remark 3.17 (Activation threshold in UDL for ContraSoft Set)**.** The choice of the activation threshold τ is pivotal; without a selection protocol it can appear ad hoc. Practical, principled options include: (i) *supervised tuning*: $\tau^\star \in \arg\min_\tau \mathcal{L}(\mathrm{UDL}_\tau(F), \text{labels})$; (ii) *quantile rule*: $\tau = \mathrm{Quantile}_p\big(\{c(e,b) : e \in E\}\big)$ to control the flip rate $|A_\tau(b)|$; (iii) *stability selection*: choose the largest τ for which $\mathrm{UDL}_\tau(F)$ is stable under resampling; (iv) *domain calibration*: set τ to meet risk or policy constraints (e.g., maximal allowed flips). Reporting a sensitivity plot $\tau \mapsto$ outputs supports interpretability.

Lemma 3.18 (Activation-set monotonicity and basic effect)
Let $b \in E$ and $0 \le \tau_1 \le \tau_2 \le 1$. Then:

(i) $A_{\tau_2}(b) \subseteq A_{\tau_1}(b)$.

(ii) For every $e \in E$,

$$F^{U_{b,\tau_2}}(e) = \begin{cases} U \setminus F(e), & e \in A_{\tau_2}(b), \\ F(e), & e \notin A_{\tau_2}(b), \end{cases} \qquad F^{U_{b,\tau_1}}(e) = \begin{cases} U \setminus F(e), & e \in A_{\tau_1}(b), \\ F(e), & e \notin A_{\tau_1}(b). \end{cases}$$

In particular, as τ increases, fewer parameters flip, i.e.,

$$\{e \in E : \ F^{U_{b,\tau}}(e) = U \setminus F(e)\} \text{ is nonincreasing in } \tau.$$

Proof
(i) By definition, $A_\tau(b) = \{e \in E : \ c(e,b) \ge \tau\}$. If $\tau_2 \ge \tau_1$ and $c(e,b) \ge \tau_2$, then $c(e,b) \ge \tau_1$, so $e \in A_{\tau_1}(b)$. Hence $A_{\tau_2}(b) \subseteq A_{\tau_1}(b)$.

(ii) This is just an explicit restatement of the definition of $U_{b,\tau}$ for each threshold. The final monotonicity claim follows immediately from (i). □

Theorem 3.19 (Anchor-neutralization and idempotence)
Fix an anchor $b \in E$ and threshold $\tau \in [0,1]$. After applying $U_{b,\tau}$ once, the resulting contradiction degrees satisfy

$$c^{U_{b,\tau}}(e,b) = 0 \quad \text{for all } e \in E,$$

and consequently the *post-transform activation set* is empty:

$$A_\tau^{U_{b,\tau}}(b) = \{e \in E : \ c^{U_{b,\tau}}(e,b) \ge \tau\} = \varnothing.$$

Therefore the operator is *idempotent*:

$$U_{b,\tau} \circ U_{b,\tau} \ = \ U_{b,\tau}.$$

Proof
By definition of $U_{b,\tau}$,

$$c^{U_{b,\tau}}(e,f) = \begin{cases} 0, & \{e,f\} \cap A_\tau(b) \neq \varnothing, \\ c(e,f), & \text{otherwise.} \end{cases}$$

For every $e \in A_\tau(b)$ we have $c^{U_{b,\tau}}(e,b) = 0$ by the first case. If $e \notin A_\tau(b)$ then, by definition of $A_\tau(b)$, $c(e,b) < \tau$ and no reset applies, so $c^{U_{b,\tau}}(e,b) = c(e,b) < \tau$. Hence in all cases $c^{U_{b,\tau}}(e,b) < \tau$, i.e. $A_\tau^{U_{b,\tau}}(b) = \varnothing$. Applying $U_{b,\tau}$ a second time uses the empty activation set, so it performs no further flips and makes no further changes to $c^{U_{b,\tau}}$; this proves idempotence. □

Theorem 3.20 (Locality and minimality of the contradiction reset)
Let $U_{b,\tau}$ be the Upside-Down operator for anchor b and threshold τ. Then for all $e, f \in E$:

(i) (*Locality on memberships*) If $e \notin A_\tau(b)$ then $F^{U_{b,\tau}}(e) = F(e)$; if $e \in A_\tau(b)$ then $F^{U_{b,\tau}}(e) = U \setminus F(e)$.

(ii) (*No increase in contradictions*) $c^{U_{b,\tau}}(e,f) \leq c(e,f)$.

(iii) (*Minimal neutralization w.r.t. the anchor*) Among all maps $\hat{c} : E \times E \to [0,1]$ that satisfy $\hat{c}(e,b) < \tau$ for every $e \in E$ and coincide with c whenever $\{e,f\} \cap A_\tau(b) = \varnothing$, the choice $c^{U_{b,\tau}}$ minimizes $\sum_{e,f} \hat{c}(e,f)$.

Proof
(i) Directly from the definition of $U_{b,\tau}$.

(ii) If $\{e,f\} \cap A_\tau(b) \neq \varnothing$ then $c^{U_{b,\tau}}(e,f) = 0 \leq c(e,f)$; otherwise $c^{U_{b,\tau}}(e,f) = c(e,f)$. Hence $c^{U_{b,\tau}} \leq c$ pointwise.

(iii) The constraints force $\hat{c}(e,b) = 0$ (the smallest possible value) for all $e \in A_\tau(b)$ to ensure $< \tau$ while also respecting the coincidence requirement off pairs that intersect $A_\tau(b)$. For pairs with no index in $A_\tau(b)$, the constraint requires $\hat{c}(e,f) = c(e,f)$. Therefore $c^{U_{b,\tau}}$ is the componentwise smallest admissible kernel, hence minimizes the stated sum. □

Theorem 3.21 (Truth-value inversion and preservation)
Fix $b \in E$ and $\tau \in [0,1]$. For any $x \in U$ and $e \in E$ let the atomic lemma $A(x,e)$ assert "$x \in F(e)$". Then

$$v(A(x,e)) = \begin{cases} \mathbf{T}, & x \in F(e), \\ \mathbf{F}, & x \notin F(e), \end{cases} \qquad v^{U_{b,\tau}}(A(x,e)) = \begin{cases} \mathbf{T}, & x \in F^{U_{b,\tau}}(e), \\ \mathbf{F}, & x \notin F^{U_{b,\tau}}(e). \end{cases}$$

We have:

(i) If $e \in A_\tau(b)$ then $v^{U_{b,\tau}}(A(x,e)) = \neg v(A(x,e))$ for all $x \in U$.

(ii) If $e \notin A_\tau(b)$ then $v^{U_{b,\tau}}(A(x,e)) = v(A(x,e))$ for all $x \in U$.

Proof
If $e \in A_\tau(b)$ then $F^{U_{b,\tau}}(e) = U \setminus F(e)$, so membership of x is complemented, hence the truth value flips. If $e \notin A_\tau(b)$ then $F^{U_{b,\tau}}(e) = F(e)$, so the truth value is preserved. □

Theorem 3.22 (Commutativity under disjoint activations)
Let $b_1, b_2 \in E$ and thresholds $\tau_1, \tau_2 \in [0,1]$ be such that

$$A_{\tau_1}(b_1) \cap A_{\tau_2}(b_2) = \varnothing \quad \text{and} \quad \{b_1, b_2\} \cap \big(A_{\tau_1}(b_1) \cup A_{\tau_2}(b_2)\big) = \varnothing.$$

Then the Upside-Down transforms commute:

$$U_{b_1,\tau_1} \circ U_{b_2,\tau_2} \;=\; U_{b_2,\tau_2} \circ U_{b_1,\tau_1},$$

and the resulting F is obtained by complementing $F(e)$ exactly for $e \in A_{\tau_1}(b_1) \cup A_{\tau_2}(b_2)$ (and leaving the rest unchanged).

Proof
Under the stated disjointness, each transform flips a set of parameters disjoint from the other, and neither transform modifies the contradiction degree on pairs wholly outside its own activation set, in particular not on the other anchor pairs $(\cdot, b_1)$ and $(\cdot, b_2)$. Thus the activation sets computed for the second transform are unchanged by the first, and each transform acts by complementing $F(e)$ exactly on its own activation set. Two disjoint set-complements commute, so the order of application does not matter, yielding the same final F. □

Theorem 3.23 (Flip-count monotonicity)
Let $N(\tau) := |A_\tau(b)|$ be the number of flipped parameters at threshold τ. Then $N(\tau)$ is nonincreasing in τ. Moreover,

$$N(\tau) = \big|\{e \in E : \; c(e,b) \geq \tau\}\big| = \sum_{e \in E} \mathbf{1}_{\{c(e,b) \geq \tau\}}.$$

Proof
This is the cardinality version of Lemma 3.18(i): as τ increases, the defining predicate $c(e,b) \geq \tau$ becomes harder to satisfy, so the indicator $\mathbf{1}_{\{c(e,b)\geq\tau\}}$ can only decrease, and hence the sum $N(\tau)$ is nonincreasing. □

3.3. ContraRough Set

A ContraRough Set is a rough-set structure that replaces indeterminacy with contradiction kernels on both relations and element memberships. Its lower and upper approximations are then defined by thresholds that control how much contradiction is tolerated, ensuring that classical Pawlak rough sets appear as a special case when contradictions vanish.

Definition 3.24 (Contradiction kernels)**.** Let $X \neq \varnothing$ be a universe. A *relation-contradiction kernel* is a map

$$c_R : X \times X \to [0,1]$$

satisfying

$$c_R(x,x) = 0 \quad \text{(reflexivity)}, \qquad c_R(x,y) = c_R(y,x) \quad \text{(symmetry)}.$$

For a target set $U \subseteq X$, a *membership-contradiction kernel* is a map

$$c_U : X \to [0,1],$$

where $c_U(y)$ quantifies the contradiction of asserting $y \in U$ (lower is more consistent, 0 means “no contradiction”, 1 means “maximally contradictory”).

Example 3.25 (Contradiction kernels: restaurant similarity and vegan-friendliness)**.** Let the universe be five restaurants

$$X = \{r_1, r_2, r_3, r_4, r_5\}.$$

Define a *relation-contradiction kernel* $c_R : X \times X \to [0,1]$ that measures how contradictory two restaurants are with respect to *style/price/ambience* (smaller = more compatible). Set $c_R(x,x) = 0$ and symmetry $c_R(x,y) = c_R(y,x)$, with the following nontrivial values:

$$\begin{aligned}
&c_R(r_1,r_2) = 0.2, \quad c_R(r_1,r_3) = 0.2, \quad c_R(r_1,r_4) = 0.8, \quad c_R(r_1,r_5) = 0.6,\\
&c_R(r_2,r_3) = 0.3, \quad c_R(r_2,r_4) = 0.7, \quad c_R(r_2,r_5) = 0.5,\\
&c_R(r_3,r_4) = 0.9, \quad c_R(r_3,r_5) = 0.6, \quad c_R(r_4,r_5) = 0.4.
\end{aligned}$$

Let the target set $U \subseteq X$ be "*vegan-friendly restaurants*" (conceptual target). Define a *membership-contradiction kernel* $c_U : X \to [0,1]$ quantifying the contradiction of asserting $r_i \in U$ (smaller = more consistent with vegan-friendly):

$$c_U(r_1) = 0.10, \quad c_U(r_2) = 0.35, \quad c_U(r_3) = 0.00, \quad c_U(r_4) = 0.90, \quad c_U(r_5) = 0.60.$$

Here r_3 is fully consistent ($c_U = 0$), r_4 is strongly contradictory (0.90), and others are intermediate.

Definition 3.26 (Thresholded consistency regions)**.** Fix thresholds $(\alpha,\beta,\gamma) \in [0,1]^3$ with $\beta \le \gamma$. Define:

$$\begin{aligned}
R^{(\alpha)} &:= \{(x,y) \in X \times X : \ c_R(x,y) \le \alpha\},\\
N^{(\alpha)}(x) &:= \{y \in X : \ (x,y) \in R^{(\alpha)}\} = \{y \in X : \ c_R(x,y) \le \alpha\},\\
U^{(\beta)}_{\text{def}} &:= \{y \in X : \ c_U(y) \le \beta\}, \qquad U^{(\gamma)}_{\text{pos}} := \{y \in X : \ c_U(y) \le \gamma\}.
\end{aligned}$$

Example 3.27 (Thresholded consistency regions in practice)**.** Continue Example 3.25. Choose thresholds

$$(\alpha,\beta,\gamma) = (0.4,\ 0.35,\ 0.6) \qquad (\beta \le \gamma).$$

Then the admitted relation and neighborhoods are

$$R^{(\alpha)} = \{(x,y) : c_R(x,y) \le 0.4\}, \qquad N^{(\alpha)}(x) = \{y : c_R(x,y) \le 0.4\}.$$

From the data:

$$\begin{aligned}
&N^{(0.4)}(r_1) = \{r_1, r_2, r_3\}, \qquad N^{(0.4)}(r_2) = \{r_2, r_1, r_3\},\\
&N^{(0.4)}(r_3) = \{r_3, r_1, r_2\}, \qquad N^{(0.4)}(r_4) = \{r_4, r_5\}, \qquad N^{(0.4)}(r_5) = \{r_5, r_4\}.
\end{aligned}$$

Definite/possible membership regions (by c_U) are

$$U^{(\beta)}_{\text{def}} = \{y : c_U(y) \le 0.35\} = \{r_1, r_2, r_3\}, \qquad U^{(\gamma)}_{\text{pos}} = \{y : c_U(y) \le 0.6\} = \{r_1, r_2, r_3, r_5\}.$$

Thus α admits only sufficiently compatible restaurant links; β accepts as *definitely vegan-friendly* those with at most 0.35 contradiction; γ allows *possibly vegan-friendly* up to 0.6.

Definition 3.28 (ContraRough approximations and ContraRough Set)**.** Given (X, c_R, c_U) and thresholds (α,β,γ) as in Definition 3.26, the *ContraRough lower* and *upper* approximations of U are

$$\underline{U}^{(\alpha,\beta)} \ := \ \big\{ x \in X : \ N^{(\alpha)}(x) \subseteq U^{(\beta)}_{\text{def}} \big\}, \qquad \overline{U}^{(\alpha,\gamma)} \ := \ \big\{ x \in X : \ N^{(\alpha)}(x) \cap U^{(\gamma)}_{\text{pos}} \neq \varnothing \big\}.$$

The quadruple

$$\big(X,\ c_R,\ c_U;\ (\alpha,\beta,\gamma)\big)$$

together with $\big(\underline{U}^{(\alpha,\beta)}, \overline{U}^{(\alpha,\gamma)}\big)$ is called a *ContraRough Set structure* for U.

We also define the *boundary*, *positive*, and *negative* regions:

$$\text{Bnd}^{(\alpha,\beta,\gamma)}(U) := \overline{U}^{(\alpha,\gamma)} \setminus \underline{U}^{(\alpha,\beta)}, \quad \text{Pos}^{(\alpha,\beta)}(U) := \underline{U}^{(\alpha,\beta)}, \quad \text{Neg}^{(\alpha,\gamma)}(U) := X \setminus \overline{U}^{(\alpha,\gamma)}.$$

Remark 3.29 (Reading the thresholds)**.**
- α controls which relational links are *admitted*: only pairs with $c_R(x, y) \le \alpha$ are treated as sufficiently non-contradictory (consistent) connections.
- β controls which elements are *definitely in* U (low contradiction to $y \in U$).
- γ controls which elements are *possibly in* U ($\beta \le \gamma$ makes lower $\subseteq$ upper possible).

Example 3.30 (ContraRough approximations and regions)**.** Using the sets from Example 3.27, the ContraRough lower/upper approximations (Definition 3.28) are

$$\underline{U}^{(\alpha,\beta)} = \{x : N^{(\alpha)}(x) \subseteq U^{(\beta)}_{\text{def}}\}, \qquad \overline{U}^{(\alpha,\gamma)} = \{x : N^{(\alpha)}(x) \cap U^{(\gamma)}_{\text{pos}} \neq \varnothing\}.$$

Check each element:

$$\begin{aligned}
N^{(0.4)}(r_1) &= \{r_1, r_2, r_3\} \subseteq \{r_1, r_2, r_3\} \;\Rightarrow\; r_1 \in \underline{U}^{(\alpha,\beta)},\\
N^{(0.4)}(r_2) &= \{r_2, r_1, r_3\} \subseteq \{r_1, r_2, r_3\} \;\Rightarrow\; r_2 \in \underline{U}^{(\alpha,\beta)},\\
N^{(0.4)}(r_3) &= \{r_3, r_1, r_2\} \subseteq \{r_1, r_2, r_3\} \;\Rightarrow\; r_3 \in \underline{U}^{(\alpha,\beta)},\\
N^{(0.4)}(r_4) &= \{r_4, r_5\} \nsubseteq \{r_1, r_2, r_3\} \;\Rightarrow\; r_4 \notin \underline{U}^{(\alpha,\beta)},\\
N^{(0.4)}(r_5) &= \{r_5, r_4\} \nsubseteq \{r_1, r_2, r_3\} \;\Rightarrow\; r_5 \notin \underline{U}^{(\alpha,\beta)}.
\end{aligned}$$

Hence

$$\underline{U}^{(\alpha,\beta)} = \{r_1, r_2, r_3\}.$$

For the upper approximation, every neighborhood meets $U^{(\gamma)}_{\text{pos}} = \{r_1, r_2, r_3, r_5\}$:

$$\begin{gathered}
N^{(0.4)}(r_1) \cap U^{(\gamma)}_{\text{pos}} \neq \varnothing,\; N^{(0.4)}(r_2) \cap U^{(\gamma)}_{\text{pos}} \neq \varnothing,\; N^{(0.4)}(r_3) \cap U^{(\gamma)}_{\text{pos}} \neq \varnothing,\\
N^{(0.4)}(r_4) \cap U^{(\gamma)}_{\text{pos}} = \{r_5\} \neq \varnothing,\; N^{(0.4)}(r_5) \cap U^{(\gamma)}_{\text{pos}} = \{r_5\} \neq \varnothing,
\end{gathered}$$

so

$$\overline{U}^{(\alpha,\gamma)} = \{r_1, r_2, r_3, r_4, r_5\} = X.$$

Therefore the regions are

$$\text{Pos}^{(\alpha,\beta)}(U) = \{r_1, r_2, r_3\}, \quad \text{Bnd}^{(\alpha,\beta,\gamma)}(U) = \{r_4, r_5\}, \quad \text{Neg}^{(\alpha,\gamma)}(U) = \varnothing.$$

Interpretation: r_1, r_2, r_3 are *definitely* vegan-friendly under the admitted similarity links; r_4, r_5 remain in the boundary (some possible evidence but not uniformly supported); none are definitely outside given the tolerant upper threshold.

Example 3.31 (Data-driven ContraRough Set: noise-aware binary classification)**.** Let $X = \{x_1, \ldots, x_n\}$ be training instances with feature map $\varphi : X \to \mathbb{R}^d$. From a distance $d(x, y) := \|\varphi(x) - \varphi(y)\|_2$ we define the *relation-contradiction kernel*

$$c_R(x, y) \;:=\; \min\left\{1,\; \frac{d(x, y)}{\sigma}\right\},$$

where $\sigma > 0$ is a scale (e.g. median k-NN distance). Train any probabilistic classifier for a positive class U to obtain scores $p(x) := \Pr(x \in U \mid \varphi(x)) \in [0, 1]$; then define the *membership-contradiction kernel*

$$c_U(x) \;:=\; 1 - p(x) \quad \text{(asserting } x \in U \text{ is less contradictory when } p(x) \text{ is large).}$$

For thresholds (α, β, γ) with $\beta \le \gamma$, the ContraRough lower/upper approximations are (Def. 3.28)

$$\underline{U}^{(\alpha,\beta)} = \left\{x :\; N^{(\alpha)}(x) \subseteq U^{(\beta)}_{\text{def}}\right\}, \qquad \overline{U}^{(\alpha,\gamma)} = \left\{x :\; N^{(\alpha)}(x) \cap U^{(\gamma)}_{\text{pos}} \neq \varnothing\right\},$$

where $N^{(\alpha)}(x) = \{y :\; c_R(x, y) \le \alpha\}$, $U^{(\beta)}_{\text{def}} = \{y :\; c_U(y) \le \beta\} = \{y :\; p(y) \ge 1 - \beta\}$, $U^{(\gamma)}_{\text{pos}} = \{y :\; c_U(y) \le \gamma\} = \{y :\; p(y) \ge 1 - \gamma\}$.

Concrete instance (six points). Let $X = \{x_1, \dots, x_6\}$ with learned probabilities

$$p(x_1) = 0.95,\ \ p(x_2) = 0.88,\ \ p(x_3) = 0.70,\ \ p(x_4) = 0.40,\ \ p(x_5) = 0.20,\ \ p(x_6) = 0.10,$$

hence $c_U(x) = 1 - p(x)$ equals

$$c_U(x_1) = 0.05,\ c_U(x_2) = 0.12,\ c_U(x_3) = 0.30,\ c_U(x_4) = 0.60,\ c_U(x_5) = 0.80,\ c_U(x_6) = 0.90.$$

Assume two learned clusters ("similar" within each cluster, "contradictory" across):

c_R	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	0.20	0.25	0.80	0.85	0.90
x_2	0.20	0	0.30	0.75	0.80	0.85
x_3	0.25	0.30	0	0.70	0.78	0.82
x_4	0.80	0.75	0.70	0	0.25	0.30
x_5	0.85	0.80	0.78	0.25	0	0.20
x_6	0.90	0.85	0.82	0.30	0.20	0

Choose thresholds $(\alpha, \beta, \gamma) = (0.35,\ 0.30,\ 0.60)$.

Neighborhoods. With $\alpha = 0.35$ we admit within-cluster links:

$$N^{(0.35)}(x_1) = \{x_1, x_2, x_3\}, \quad N^{(0.35)}(x_2) = \{x_2, x_1, x_3\}, \quad N^{(0.35)}(x_3) = \{x_3, x_1, x_2\},$$
$$N^{(0.35)}(x_4) = \{x_4, x_5, x_6\}, \quad N^{(0.35)}(x_5) = \{x_5, x_4, x_6\}, \quad N^{(0.35)}(x_6) = \{x_6, x_4, x_5\}.$$

Definite/possible slices.

$$U_{\text{def}}^{(\beta)} = \{x : c_U(x) \leq 0.30\} = \{x_1, x_2, x_3\}, \qquad U_{\text{pos}}^{(\gamma)} = \{x : c_U(x) \leq 0.60\} = \{x_1, x_2, x_3, x_4\}.$$

ContraRough approximations.

$$\underline{U}^{(\alpha,\beta)} = \big\{x :\ N^{(0.35)}(x) \subseteq \{x_1, x_2, x_3\}\big\} = \{x_1, x_2, x_3\},$$

$$\overline{U}^{(\alpha,\gamma)} = \big\{x :\ N^{(0.35)}(x) \cap \{x_1, x_2, x_3, x_4\} \neq \varnothing\big\} = \{x_1, x_2, x_3, x_4, x_5, x_6\} = X.$$

Therefore

$$\text{Pos}^{(\alpha,\beta)}(U) = \{x_1, x_2, x_3\}, \quad \text{Bnd}^{(\alpha,\beta,\gamma)}(U) = \{x_4, x_5, x_6\}, \quad \text{Neg}^{(\alpha,\gamma)}(U) = \varnothing.$$

The classifier's $p(x)$ yields c_U (how contradictory it is to accept x as positive); the metric/graph yields c_R (how contradictory it is to relate x to y). With the chosen thresholds, high-confidence positives $\{x_1, x_2, x_3\}$ form the *lower* region and drive label propagation inside their coherent neighborhood. Ambiguous or low-confidence instances $\{x_4, x_5, x_6\}$ remain in the *boundary*, becoming prime candidates for human review, additional features, or active learning queries. This produces an interpretable, threshold-controlled partition of X that is directly driven by learned, data-derived kernels (c_R, c_U).

Lemma 3.32 (Lower $\subseteq$ Upper)
Assume $c_R(x, x) = 0$ for all $x \in X$ and $\beta \leq \gamma$. Then for every $U \subseteq X$,

$$\underline{U}^{(\alpha,\beta)} \ \subseteq\ \overline{U}^{(\alpha,\gamma)}.$$

Proof
Take $x \in \underline{U}^{(\alpha,\beta)}$. Then $N^{(\alpha)}(x) \subseteq U_{\text{def}}^{(\beta)} \subseteq U_{\text{pos}}^{(\gamma)}$. By $c_R(x, x) = 0 \leq \alpha$, we have $x \in N^{(\alpha)}(x)$, hence $N^{(\alpha)}(x) \cap U_{\text{pos}}^{(\gamma)} \neq \varnothing$. Thus $x \in \overline{U}^{(\alpha,\gamma)}$. □

Lemma 3.33 (Monotonicity)
For fixed $U \subseteq X$:

(i) If $\alpha_1 \le \alpha_2$, then $N^{(\alpha_1)}(x) \subseteq N^{(\alpha_2)}(x)$ for all $x \in X$; consequently

$$\underline{U}^{(\alpha_2,\beta)} \subseteq \underline{U}^{(\alpha_1,\beta)} \quad \text{and} \quad \overline{U}^{(\alpha_1,\gamma)} \subseteq \overline{U}^{(\alpha_2,\gamma)}.$$

(ii) If $\beta_1 \le \beta_2$, then $U_{\text{def}}^{(\beta_1)} \subseteq U_{\text{def}}^{(\beta_2)}$ and hence

$$\underline{U}^{(\alpha,\beta_1)} \subseteq \underline{U}^{(\alpha,\beta_2)}.$$

(iii) If $\gamma_1 \le \gamma_2$, then $U_{\text{pos}}^{(\gamma_1)} \subseteq U_{\text{pos}}^{(\gamma_2)}$ and hence

$$\overline{U}^{(\alpha,\gamma_1)} \subseteq \overline{U}^{(\alpha,\gamma_2)}.$$

Proof
(i) $c_R(x,y) \le \alpha_1 \Rightarrow c_R(x,y) \le \alpha_2$ gives $N^{(\alpha_1)}(x) \subseteq N^{(\alpha_2)}(x)$. The stated inclusions follow from the definitions of $\underline{U}^{(\cdot,\cdot)}$ and $\overline{U}^{(\cdot,\cdot)}$.

(ii) and (iii) follow by set inclusion propagation inside the defining conditions. □

Lemma 3.34 (Reduction to Pawlak rough sets)
Suppose there is a crisp equivalence $R \subseteq X \times X$ and a crisp $U \subseteq X$ such that

$$c_R(x,y) = \begin{cases} 0, & (x,y) \in R, \\ 1, & (x,y) \notin R, \end{cases} \qquad c_U(y) = \begin{cases} 0, & y \in U, \\ 1, & y \notin U. \end{cases}$$

Then, for $\alpha = \beta = \gamma = 0$,

$$N^{(0)}(x) = \{y : \ c_R(x,y) = 0\} = [x]_R, \quad U_{\text{def}}^{(0)} = U_{\text{pos}}^{(0)} = U,$$

and hence

$$\underline{U}^{(0,0)} = \{x : \ [x]_R \subseteq U\}, \qquad \overline{U}^{(0,0)} = \{x : \ [x]_R \cap U \neq \varnothing\},$$

which are exactly the classical Pawlak lower/upper approximations.

Proof
Direct substitution:

$$x \in \underline{U}^{(0,0)} \iff N^{(0)}(x) \subseteq U_{\text{def}}^{(0)} \iff [x]_R \subseteq U,$$
$$x \in \overline{U}^{(0,0)} \iff N^{(0)}(x) \cap U_{\text{pos}}^{(0)} \neq \varnothing \iff [x]_R \cap U \neq \varnothing.$$

□

3.4. Upside-down logic in ContraRough Set

Upside-Down Logic in a ContraRough Set reverses lower and upper approximations when contradiction thresholds trigger, flipping definite and possible memberships through contradiction-driven kernel transformations.

Definition 3.35 (Upside-Down operator on a ContraRough kernel)**.** Let $X \neq \varnothing$ be a universe, $U \subseteq X$ a target set, and $c_R : X \times X \to [0,1]$, $c_U : X \to [0,1]$ be the relation-/membership-contradiction kernels (Def. 3.24). Fix thresholds $(\alpha, \beta, \gamma) \in [0,1]^3$ with $\beta \leq \gamma$, and a *flip threshold* $\tau \in [0,1]$.

Activation set.

$$A_\tau := \{ y \in X : c_U(y) \geq \tau \}.$$

Upside-Down transform. Define a new membership-contradiction kernel $c_U^{U_\tau} : X \to [0,1]$ by

$$c_U^{U_\tau}(y) := \begin{cases} 1 - c_U(y), & y \in A_\tau \quad (\textit{flip}), \\ c_U(y), & y \notin A_\tau \quad (\textit{keep}). \end{cases}$$

The relation kernel is, by default, kept unchanged:

$$c_R^{U_\tau}(x,y) := c_R(x,y) \qquad (\forall x, y \in X).$$

Optional neutralization. In applications where the activated items are declared *resolved*, one may additionally set $c_R^{U_\tau}(x,y) := 0$ whenever $x \in A_\tau$ or $y \in A_\tau$ (contradiction reset).

Upside-Down ContraRough approximations. With the same (α, β, γ), define

$$\underline{U}^{(\alpha,\beta)}[c_R^{U_\tau}, c_U^{U_\tau}] := \{x : N^{(\alpha)}_{c_R^{U_\tau}}(x) \subseteq \{y : c_U^{U_\tau}(y) \leq \beta\}\},$$

$$\overline{U}^{(\alpha,\gamma)}[c_R^{U_\tau}, c_U^{U_\tau}] := \{x : N^{(\alpha)}_{c_R^{U_\tau}}(x) \cap \{y : c_U^{U_\tau}(y) \leq \gamma\} \neq \varnothing\},$$

where $N^{(\alpha)}_{c_R^{U_\tau}}(x) := \{y : c_R^{U_\tau}(x,y) \leq \alpha\}$. For $y \in A_\tau$, the truth-status of "$y \in U$" is inverted at the *kernel level* since $c_U^{U_\tau}(y) = 1 - c_U(y)$; the approximations are then recomputed accordingly.

Remark 3.36 (Flip guarantees under thresholds)**.** If $y \in A_\tau$ and $c_U(y) \leq \beta$ ("definitely in"), then $c_U^{U_\tau}(y) \geq 1 - \beta$. If, moreover, $1 - \beta > \gamma$, y cannot be "possibly in" after the flip. Conversely, if $c_U(y) \geq \tau \geq \gamma$, then $c_U^{U_\tau}(y) \leq 1 - \gamma \leq 1 - \tau$, so y may become eligible for the lower region provided its neighborhood satisfies the inclusion test.

Example 3.37 (Supplier compliance re-evaluation (flip-only))**.** Let $X = \{s_1, s_2, s_3, s_4\}$ be suppliers and $U =$ "compliant". Relation-contradiction (symmetric, 0 on diagonal):

c_R	s_1	s_2	s_3	s_4
s_1	0	0.2	0.5	0.7
s_2	0.2	0	0.4	0.8
s_3	0.5	0.4	0	0.3
s_4	0.7	0.8	0.3	0

Membership-contradiction to asserting $s_i \in U$:

$$c_U(s_1) = 0.10, \quad c_U(s_2) = 0.20, \quad c_U(s_3) = 0.80, \quad c_U(s_4) = 0.55.$$

Thresholds: $(\alpha, \beta, \gamma) = (0.4,\ 0.25,\ 0.60)$.

Before flip. $N^{(0.4)}(s_1) = \{s_1, s_2\}$, $N^{(0.4)}(s_2) = \{s_2, s_1, s_3\}$, $N^{(0.4)}(s_3) = \{s_3, s_2, s_4\}$, $N^{(0.4)}(s_4) = \{s_4, s_3\}$.

$$U^{(\beta)}_{\mathrm{def}} = \{s_1, s_2\}, \qquad U^{(\gamma)}_{\mathrm{pos}} = \{s_1, s_2, s_4\}.$$

Hence

$$\underline{U}^{(\alpha,\beta)} = \{s_1\}, \qquad \overline{U}^{(\alpha,\gamma)} = X.$$

Flip step. Choose $\tau = 0.7$. Then $A_\tau = \{s_3\}$ and $c_U^{U_\tau}(s_3) = 1 - 0.80 = 0.20$ (others unchanged). We keep c_R unchanged (flip-only mode).

After flip.

$$U^{(\beta)}_{\mathrm{def}}[c_U^{U_\tau}] = \{s_1, s_2, s_3\}, \qquad U^{(\gamma)}_{\mathrm{pos}}[c_U^{U_\tau}] = \{s_1, s_2, s_3, s_4\}.$$

Neighborhoods are the same (since c_R unchanged). Therefore

$$\underline{U}^{(\alpha,\beta)}[c_R, c_U^{U_\tau}] = \{s_1, s_2\}, \qquad \overline{U}^{(\alpha,\gamma)}[c_R, c_U^{U_\tau}] = X.$$

Supplier s_2 moves from the boundary to the positive (lower) region because its neighborhood $\{s_2, s_1, s_3\}$ is now contained in the enlarged definite set $\{s_1, s_2, s_3\}$ after flipping s_3.

Example 3.38 (Email ham/spam policy reversal (with optional neutralization))**.** Let $X = \{e_1, e_2, e_3, e_4\}$ be emails; U = "ham (legitimate)". Relation-contradiction (similarity-based; symmetric):

c_R	e_1	e_2	e_3	e_4
e_1	0	0.30	0.90	0.60
e_2	0.30	0	0.80	0.50
e_3	0.90	0.80	0	0.20
e_4	0.60	0.50	0.20	0

Membership-contradiction to "ham":

$$c_U(e_1) = 0.10, \quad c_U(e_2) = 0.25, \quad c_U(e_3) = 0.90, \quad c_U(e_4) = 0.60.$$

Thresholds: $(\alpha, \beta, \gamma) = (0.20,\ 0.20,\ 0.70)$.

Before flip. Neighborhoods (admitted links $c_R \leq 0.20$):

$$N^{(0.2)}(e_1) = \{e_1\}, \quad N^{(0.2)}(e_2) = \{e_2\}, \quad N^{(0.2)}(e_3) = \{e_3, e_4\}, \quad N^{(0.2)}(e_4) = \{e_4, e_3\}.$$

$$U^{(\beta)}_{\mathrm{def}} = \{e_1\}, \qquad U^{(\gamma)}_{\mathrm{pos}} = \{e_1, e_2, e_4\}.$$

Thus

$$\underline{U}^{(\alpha,\beta)} = \{e_1\}, \qquad \overline{U}^{(\alpha,\gamma)} = X.$$

Flip step. Pick $\tau = 0.85$. Then $A_\tau = \{e_3\}$ and $c_U^{U_\tau}(e_3) = 1 - 0.90 = 0.10$. Consider two variants:

<u>(i) Flip-only.</u> Keep c_R unchanged. Then $U^{(\beta)}_{\mathrm{def}}[c_U^{U_\tau}] = \{e_1, e_3\}$ and $U^{(\gamma)}_{\mathrm{pos}}$ stays the same. Since $N^{(0.2)}(e_3) = \{e_3, e_4\} \not\subseteq \{e_1, e_3\}$, we get

$$\underline{U}^{(\alpha,\beta)}[c_R, c_U^{U_\tau}] = \{e_1\}, \qquad \overline{U}^{(\alpha,\gamma)}[c_R, c_U^{U_\tau}] = X.$$

The kernel flip alone makes e_3 *eligible* for definiteness, but its neighborhood still contains e_4 (not definitely ham), so e_3 remains in the boundary.

(ii) Flip + neutralization. Set $c_R^{U_\tau}(x,y)=0$ if $x=e_3$ or $y=e_3$. Then, with $\alpha=0.2$, we have

$$N^{(0.2)}_{c_R^{U_\tau}}(e_3)=\{e_3,e_1,e_2,e_4\},$$

and $U^{(\beta)}_{\mathrm{def}}[c_U^{U_\tau}]=\{e_1,e_3\}$. Because the neighborhood still contains $e_2,e_4\notin$ the definite set, e_3 is *still* not in the lower region. However, if policy also raises β to 0.25 (so that e_2 becomes definite), then

$$U^{(\beta=0.25)}_{\mathrm{def}}[c_U^{U_\tau}]=\{e_1,e_2,e_3\}$$

and e_3 enters the lower region since its (neutralized) neighborhood is now contained in the definite set:

$$\underline{U}^{(\alpha,\beta=0.25)}[c_R^{U_\tau},c_U^{U_\tau}]\ni e_3.$$

This shows how, in operations, the *kernel flip* plus a *policy relaxation* (or contradiction reset) provably moves items from boundary to positive region.

Remark 3.39 (Activation threshold in UDL for ContraRough Set)**.** The same concern applies to ContraRough: τ is crucial and, without guidance, seems ad hoc. Select τ jointly with (α,β,γ) via: (i) *grid/validation search*: minimize a task loss plus a boundary-size penalty, $\tau^\star\in\arg\min_\tau\left[\mathcal{L}(\underline{U}^{(\alpha,\beta)}_{\mathrm{UD}},\overline{U}^{(\alpha,\gamma)}_{\mathrm{UD}})+\lambda\,|\mathrm{Bnd}_{\mathrm{UD}}|\right]$; (ii) *flip-rate control*: enforce $|A_\tau|\le r$ for a target flip budget; (iii) *Pareto screening*: pick τ on the Pareto frontier trading accuracy vs. boundary size. Sensitivity analyses across τ enhance transparency and robustness.

Theorem 3.40 (Idempotence for mid/high flip thresholds)
Let $\tau\in[1/2,1]$ and apply the Upside-Down transform U_τ of Definition 3.35 (without relation neutralization). Then one has

$$U_\tau\circ U_\tau\;=\;U_\tau.$$

Equivalently, the post-transform activation set is empty: $A_\tau^{U_\tau}=\{y\in X:\ c_U^{U_\tau}(y)\ge\tau\}=\varnothing$.

Proof
By Definition 3.35, elements in $A_\tau:=\{y:\ c_U(y)\ge\tau\}$ are flipped to $c_U^{U_\tau}(y)=1-c_U(y)\le 1-\tau\le\tau$ because $\tau\ge 1/2$. Elements outside A_τ keep their value $c_U^{U_\tau}(y)=c_U(y)<\tau$. Hence $c_U^{U_\tau}(y)<\tau$ for every y, so $A_\tau^{U_\tau}=\varnothing$ and a second application of U_τ does nothing. □

Lemma 3.41 (Invariance when neighborhoods avoid the activation set)
Fix (α,β,γ) and $\tau\in[0,1]$. If $N^{(\alpha)}(x)\cap A_\tau=\varnothing$ for every $x\in X$, then

$$\underline{U}^{(\alpha,\beta)}[c_R,c_U]\;=\;\underline{U}^{(\alpha,\beta)}[c_R,c_U^{U_\tau}],\qquad \overline{U}^{(\alpha,\gamma)}[c_R,c_U]\;=\;\overline{U}^{(\alpha,\gamma)}[c_R,c_U^{U_\tau}].$$

Proof
Only the values of c_U at points of A_τ are altered. By hypothesis, no $N^{(\alpha)}(x)$ contains elements of A_τ, so the defining tests for the lower and upper approximations involve the same c_U-values before and after the transform. □

Theorem 3.42 (Guaranteed gains under tolerant γ and β)
Let $\tau\in[0,1]$ and (α,β,γ) as in Definition 3.26.

(i) If $\gamma\ge 1-\tau$, then

$$\{\,x\in X:\ N^{(\alpha)}(x)\cap A_\tau\neq\varnothing\,\}\;\subseteq\;\overline{U}^{(\alpha,\gamma)}[c_R,c_U^{U_\tau}].$$

(ii) If $\beta \geq 1 - \tau$, then

$$\{ x \in X : N^{(\alpha)}(x) \subseteq U_{\mathrm{def}}^{(\beta)} \cup A_\tau \} \subseteq \underline{U}^{(\alpha,\beta)}[c_R, c_U^{U_\tau}].$$

Proof

(i) If $y \in N^{(\alpha)}(x) \cap A_\tau$, then after the flip $c_U^{U_\tau}(y) = 1 - c_U(y) \leq 1 - \tau \leq \gamma$, so $y \in U_{\mathrm{pos}}^{(\gamma)}$ for the post-transform kernel. Thus $N^{(\alpha)}(x) \cap U_{\mathrm{pos}}^{(\gamma)} \neq \varnothing$, which is exactly the upper-approximation condition.

(ii) If $N^{(\alpha)}(x) \subseteq U_{\mathrm{def}}^{(\beta)} \cup A_\tau$, then after the flip every $y \in U_{\mathrm{def}}^{(\beta)}$ still satisfies $c_U^{U_\tau}(y) = c_U(y) \leq \beta$, and every $y \in A_\tau$ satisfies $c_U^{U_\tau}(y) \leq 1 - \tau \leq \beta$. Hence $N^{(\alpha)}(x) \subseteq U_{\mathrm{def}}^{(\beta)}$ for the post-transform kernel, i.e. x belongs to the lower approximation. □

Theorem 3.43 (Upper saturation under relation neutralization)
Assume the optional neutralization in Definition 3.35: $c_R^{U_\tau}(x, y) = 0$ whenever $y \in A_\tau$. If $A_\tau \neq \varnothing$ and $\gamma \geq 1 - \tau$, then

$$\overline{U}^{(\alpha,\gamma)}[c_R^{U_\tau}, c_U^{U_\tau}] = X \quad \text{for every } \alpha \in [0, 1].$$

Proof

Fix $x \in X$. For any $y \in A_\tau$, neutralization gives $c_R^{U_\tau}(x, y) = 0 \leq \alpha$, hence $y \in N_{c_R^{U_\tau}}^{(\alpha)}(x)$. Also, $c_U^{U_\tau}(y) = 1 - c_U(y) \leq 1 - \tau \leq \gamma$, so $y \in U_{\mathrm{pos}}^{(\gamma)}$ for the post-transform kernel. Therefore $N_{c_R^{U_\tau}}^{(\alpha)}(x) \cap U_{\mathrm{pos}}^{(\gamma)} \neq \varnothing$, proving $x \in \overline{U}^{(\alpha,\gamma)}[c_R^{U_\tau}, c_U^{U_\tau}]$. As x was arbitrary, the upper approximation equals X. □

Lemma 3.44 (Counting formula for definite/possible carriers)
Let $D_\beta := |U_{\mathrm{def}}^{(\beta)}|$ and $P_\gamma := |U_{\mathrm{pos}}^{(\gamma)}|$ for the pre-transform kernel. After applying U_τ (no relation neutralization), the cardinalities $D'_\beta := |U_{\mathrm{def}}^{(\beta)}[c_U^{U_\tau}]|$ and $P'_\gamma := |U_{\mathrm{pos}}^{(\gamma)}[c_U^{U_\tau}]|$ satisfy

$$D'_\beta = \left|U_{\mathrm{def}}^{(\beta)} \setminus A_\tau\right| + \left|\{ y \in A_\tau : c_U(y) \geq 1 - \beta \}\right|,$$

$$P'_\gamma = \left|U_{\mathrm{pos}}^{(\gamma)} \setminus A_\tau\right| + \left|\{ y \in A_\tau : c_U(y) \geq 1 - \gamma \}\right|.$$

Proof

For $y \notin A_\tau$ nothing changes, so y contributes to D'_β (resp. P'_γ) iff it contributed to D_β (resp. P_γ). For $y \in A_\tau$, the new value is $c_U^{U_\tau}(y) = 1 - c_U(y)$; this is $\leq \beta$ (resp. $\leq \gamma$) iff $c_U(y) \geq 1 - \beta$ (resp. $\geq 1 - \gamma$). Summing the two disjoint contributions yields the formulas. □

Theorem 3.45 (Absorption for nested high thresholds)
Let $1/2 \leq \tau_1 \leq \tau_2 \leq 1$. Then

$$U_{\tau_2} \circ U_{\tau_1} = U_{\tau_1}.$$

Proof

Apply U_{τ_1} first. For $y \in A_{\tau_1}$, $c_U^{U_{\tau_1}}(y) = 1 - c_U(y) \leq 1 - \tau_1 \leq \tau_1 \leq \tau_2$, so $y \notin A_{\tau_2}^{U_{\tau_1}}$. For $y \notin A_{\tau_1}$ we kept $c_U^{U_{\tau_1}}(y) = c_U(y) < \tau_1 \leq \tau_2$, hence again $y \notin A_{\tau_2}^{U_{\tau_1}}$. Thus $A_{\tau_2}^{U_{\tau_1}} = \varnothing$ and the second transform does nothing, proving the identity. □

Theorem 3.46 (Possible upper contraction for strict post-flip screening)
Assume no relation neutralization. If $\tau \leq \beta$ and $1 - \beta > \gamma$, then for any x with $N^{(\alpha)}(x) \subseteq A_\tau$ one has

$$x \notin \overline{U}^{(\alpha,\gamma)}[c_R, c_U^{U_\tau}].$$

Proof
For $y \in A_\tau$, after the flip $c_U^{U_\tau}(y) = 1 - c_U(y) \geq 1 - \beta > \gamma$. Hence $A_\tau \cap U_{\mathrm{pos}}^{(\gamma)}[c_U^{U_\tau}] = \varnothing$. If $N^{(\alpha)}(x) \subseteq A_\tau$, then $N^{(\alpha)}(x) \cap U_{\mathrm{pos}}^{(\gamma)}[c_U^{U_\tau}] = \varnothing$ and x fails the upper-approximation test. □

3.5. *Algorithm for Upside-down logic in ContraSoft Set*

An algorithm that, given an anchor and contradiction threshold, detects activated parameters, flips their memberships, neutralizes conflicts, and recomputes selections, modeling policy or context reversals.

Algorithm 1: UpsideDownTransform $(U, E, F, c;\ b, \tau)$

Input : Finite universe U; parameter set E; mapping $F : E \to \mathcal{P}(U)$; contradiction $c : E \times E \to [0, 1]$ (symmetric, $c(e, e) = 0$); anchor $b \in E$; threshold $\tau \in [0, 1]$.
Output: (F^U, c^U) implementing $U_{b,\tau}$ with reset rule.

(1) Activation set. $A \leftarrow \{\, e \in E \mid c(e, b) \geq \tau \,\}$.
(2) Initialize outputs. For all $e \in E$, set $F^U(e) \leftarrow F(e)$.
Set $c^U \leftarrow c$.
(3) Flip memberships on activated parameters.
foreach $e \in A$ **do**
  $F^U(e) \leftarrow U \setminus F(e)$
end
(4) Reset contradictions touching activated parameters (symmetrically).
foreach $e \in A$ **do**
  foreach $f \in E$ **do**
    $c^U(e, f) \leftarrow 0;\quad c^U(f, e) \leftarrow 0$ // keep symmetry
  end
end
(5) Return. return (F^U, c^U).

Example 3.47 (City building access during a temporary lockdown)**.** *Universe and parameters.* Let the universe be a set of building entrances

$$U = \{D_1, D_2, D_3, D_4\}.$$

Consider parameters

$$E = \{\mathrm{ACC},\ \mathrm{LCK},\ \mathrm{STF}\},$$

where ACC means "public-access enabled," LCK means "temporarily locked," and STF means "staff-only."

Baseline soft map.

$$F(\mathrm{ACC}) = \{D_1, D_2\}, \quad F(\mathrm{LCK}) = \{D_2, D_3\}, \quad F(\mathrm{STF}) = \{D_3\}.$$

Contradiction degrees (symmetric, diagonal 0*).*

$$c(\mathrm{ACC}, \mathrm{LCK}) = 0.95, \qquad c(\mathrm{ACC}, \mathrm{STF}) = 0.60, \qquad c(\mathrm{LCK}, \mathrm{STF}) = 0.40.$$

Upside-down transform settings. Anchor $b = \mathrm{LCK}$ (the city declares a temporary lockdown); threshold $\tau = 0.8$. The activation set is

$$A_\tau(b) = \{\, e \in E \mid c(e, b) \geq \tau \,\} = \{\mathrm{ACC}\} \quad \text{(only ACC is highly contradictory to LCK).}$$

Run Algorithm 1.

(i) **Flip memberships on activated parameters.**

$$F^U(\mathrm{ACC}) = U \setminus F(\mathrm{ACC}) = \{D_3, D_4\},$$

$$F^U(\mathrm{LCK}) = F(\mathrm{LCK}) = \{D_2, D_3\},$$
$$F^U(\mathrm{STF}) = F(\mathrm{STF}) = \{D_3\}.$$

Thus, the truth of the lemma "x is public-access enabled" is inverted under the lockdown context.

(ii) **Reset contradictions touching activated parameters.** For all $f \in E$,

$$c^U(\mathrm{ACC}, f) = c^U(f, \mathrm{ACC}) = 0,$$

while the non-activated pair keeps its value:

$$c^U(\mathrm{LCK}, \mathrm{STF}) = c(\mathrm{LCK}, \mathrm{STF}) = 0.40.$$

Declaring a lockdown (LCK as anchor) flips the ACC-membership: entrances previously marked "public-access" (D_1, D_2) now evaluate as *not* public-access in the transformed system, and vice versa for D_4. Simultaneously, all contradictions involving the activated parameter ACC are neutralized ($= 0$), reflecting that, once the lockdown context is imposed, conflicts with "public-access" are resolved by policy.

Theorem 3.48 (Correctness of Algorithm 1)
Let $\mathsf{CS} = (U, E, F, c)$, $b \in E$, and $\tau \in [0, 1]$. The output (F^U, c^U) of Algorithm 1 satisfies the Upside-Down specification:

$$F^U(e) = \begin{cases} U \setminus F(e), & e \in A_\tau(b), \\ F(e), & e \notin A_\tau(b), \end{cases}$$

$$c^U(e, f) = \begin{cases} 0, & \{e, f\} \cap A_\tau(b) \neq \varnothing, \\ c(e, f), & \text{otherwise}, \end{cases}$$

where $A_\tau(b) = \{e \in E : \; c(e, b) \geq \tau\}$. Consequently, for every $x \in U$ and $e \in E$,

$$e \in A_\tau(b) \implies \big(x \in F(e) \iff x \notin F^U(e)\big),$$

$$e \notin A_\tau(b) \implies \big(x \in F(e) \iff x \in F^U(e)\big).$$

Proof
By Step (1), A equals $A_\tau(b)$. Step (2) initializes $F^U(e) = F(e)$ for all e and $c^U = c$. Step (3) applies $F^U(e) \leftarrow U \setminus F(e)$ exactly for $e \in A$, leaving non-activated $e \notin A$ unchanged. Step (4) assigns $c^U(e, f) = c^U(f, e) = 0$ whenever $e \in A$ (and for all $f \in E$); therefore $c^U(e, f) = 0$ iff $\{e, f\} \cap A \neq \varnothing$, while all other entries remain equal to $c(e, f)$. This is precisely the required specification. The elementwise truth inversion (and preservation for non-activated parameters) follows from $F^U(e) = U \setminus F(e)$ (resp. $F^U(e) = F(e)$). □

Theorem 3.49 (Time and space complexity)
Let $n_E := |E|$ and $n_U := |U|$. Assume F is represented as explicit subsets of U and c as an $n_E \times n_E$ array. Algorithm 1 runs in

$$\text{time } T(n_E, n_U) = \Theta(n_E) \; + \; \Theta(|A_\tau(b)| \, n_U) \; + \; \Theta(|A_\tau(b)| \, n_E) \; = \; O(n_E n_U + n_E^2)$$

in the worst case (when $|A_\tau(b)| = n_E$), and uses $\Theta(n_E n_U + n_E^2)$ output space to store (F^U, c^U). If $F(e)$ is stored as bitsets over machine words of size w, the complement step costs $\Theta(|A_\tau(b)| \cdot n_U / w)$.

Proof
(*Time*) Step (1) scans E once: $\Theta(n_E)$. Step (2) initializes outputs; if done by pointers/aliases this is $O(1)$, otherwise copying costs $O(n_E n_U + n_E^2)$ but is not necessary for complexity bounds since we can write in-place to a fresh structure.

Step (3): for each $e \in A_\tau(b)$, computing $U \setminus F(e)$ by scanning U is $\Theta(n_U)$; total $\Theta(|A_\tau(b)|\, n_U)$. Step (4): for each $e \in A_\tau(b)$ we set a whole row and column of c^U to zero; this is $\Theta(n_E)$ per e, hence $\Theta(|A_\tau(b)|\, n_E)$. Summing gives the displayed bound; worst case is when $|A_\tau(b)| = n_E$.

(*Space*) The output F^U is an E-indexed family of subsets of U, which in the explicit representation is $\Theta(n_E n_U)$ in the worst case. The matrix c^U is $\Theta(n_E^2)$. Using bitsets reduces the flip cost to $\Theta(n_U/w)$ per activated parameter but leaves the asymptotic output sizes unchanged. □

Corollary 3.50 (Idempotence under reset)
If Algorithm 1 is applied to (U, E, F, c) with (b, τ), and then applied again to the result (U, E, F^U, c^U) with the same (b, τ), the second run performs no flips and no further changes; i.e., Algorithm 1 is idempotent.

Proof
By Step (4), for any $e \in A_\tau(b)$ we have $c^U(e, b) = 0 < \tau$, hence $e \notin A_\tau(b)$ in the second run. For any $e \notin A_\tau(b)$ initially, $c^U(e, b) = c(e, b) < \tau$ still holds. Thus the second activation set is empty, and Steps (3)–(4) make no changes. □

3.6. *Algorithm for Upside-down logic in ContraRough Set*

Compute α-neighborhoods and β/γ slices, form ContraRough lower/upper approximations; identify activated items, flip membership (optionally neutralize relations), then recompute approximations and regions under transformed kernels. The overview of the algorithm for Upside-Down Logic in ContraRough Set is presented in Algorithm 2.

Example 3.51 (Endpoint compliance after an emergency security patch)**.** *Setting.* Let $X = \{e_1, e_2, e_3, e_4\}$ denote four IT endpoints. The target concept U is "*security compliant*." We work with a relation-contradiction kernel $c_R : X \times X \to [0, 1]$ (smaller = more compatible) and a membership-contradiction kernel $c_U : X \to [0, 1]$ (smaller = more consistent with $e \in U$).

Kernels (symmetric c_R, zero diagonal).

$$c_R = \begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline e_1 & 0 & 0.3 & 0.6 & 0.8 \\ e_2 & 0.3 & 0 & 0.4 & 0.7 \\ e_3 & 0.6 & 0.4 & 0 & 0.2 \\ e_4 & 0.8 & 0.7 & 0.2 & 0 \end{array} \qquad c_U(e_1) = 0.10,\ \ c_U(e_2) = 0.25,\ \ c_U(e_3) = 0.65,\ \ c_U(e_4) = 0.85.$$

Thresholds: $(\alpha, \beta, \gamma) = (0.4,\ 0.2,\ 0.6)$ with $\beta \le \gamma$. Flip threshold $\tau = 0.8$ and `NEUTRALIZE`= true.

Run Algorithm 2.

(1) Neighborhoods at level $\alpha = 0.4$.

$$N^{(0.4)}(e_1) = \{e_1, e_2\}, \quad N^{(0.4)}(e_2) = \{e_2, e_1, e_3\},$$
$$N^{(0.4)}(e_3) = \{e_3, e_2, e_4\}, \quad N^{(0.4)}(e_4) = \{e_4, e_3\}.$$

Algorithm 2: Upside-Down Logic for ContraRough Set

Input : Finite X; kernels $c_R : X \times X \to [0,1]$, $c_U : X \to [0,1]$; thresholds (α, β, γ) with $\beta \le \gamma$; flip threshold τ; flag NEUTRALIZE$\in \{\text{true}, \text{false}\}$.

Output: $(\underline{U}^{(\alpha,\beta)}, \overline{U}^{(\alpha,\gamma)})$ before UD; $(\underline{U}_{\text{UD}}^{(\alpha,\beta)}, \overline{U}_{\text{UD}}^{(\alpha,\gamma)})$ after UD; regions $\text{Pos}, \text{Bnd}, \text{Neg}$ before/after; updated kernels $(c_R^{U_\tau}, c_U^{U_\tau})$.

(1) Neighborhoods at level α.
foreach $x \in X$ **do**
  $N^{(\alpha)}(x) \leftarrow \{\, y \in X : \ c_R(x,y) \le \alpha \,\}$
end
(2) Definite/possible membership slices.
$U_{\text{def}}^{(\beta)} \leftarrow \{\, y \in X : \ c_U(y) \le \beta \,\}$
$U_{\text{pos}}^{(\gamma)} \leftarrow \{\, y \in X : \ c_U(y) \le \gamma \,\}$
(3) ContraRough approximations (before UD).
$\underline{U}^{(\alpha,\beta)} \leftarrow \{\, x \in X : \ N^{(\alpha)}(x) \subseteq U_{\text{def}}^{(\beta)} \,\}$
$\overline{U}^{(\alpha,\gamma)} \leftarrow \{\, x \in X : \ N^{(\alpha)}(x) \cap U_{\text{pos}}^{(\gamma)} \neq \varnothing \,\}$
$\text{Pos} \leftarrow \underline{U}^{(\alpha,\beta)}$, $\text{Bnd} \leftarrow \overline{U}^{(\alpha,\gamma)} \setminus \underline{U}^{(\alpha,\beta)}$, $\text{Neg} \leftarrow X \setminus \overline{U}^{(\alpha,\gamma)}$
(4) Upside-Down activation and kernel flip.
$A_\tau \leftarrow \{\, y \in X : \ c_U(y) \ge \tau \,\}$
foreach $y \in X$ **do**
  if $y \in A_\tau$ **then** $c_U^{U_\tau}(y) \leftarrow 1 - c_U(y)$
  else $c_U^{U_\tau}(y) \leftarrow c_U(y)$
end
foreach $(x,y) \in X \times X$ **do**
  if *NEUTRALIZE and ($x \in A_\tau$ or $y \in A_\tau$)* **then** $c_R^{U_\tau}(x,y) \leftarrow 0$
  else $c_R^{U_\tau}(x,y) \leftarrow c_R(x,y)$
end
(5) Recompute neighborhoods and slices under UD.
foreach $x \in X$ **do**
  $N_U^{(\alpha)}(x) \leftarrow \{\, y \in X : \ c_R^{U_\tau}(x,y) \le \alpha \,\}$
end
$U_{\text{def},U}^{(\beta)} \leftarrow \{\, y \in X : \ c_U^{U_\tau}(y) \le \beta \,\}$
$U_{\text{pos},U}^{(\gamma)} \leftarrow \{\, y \in X : \ c_U^{U_\tau}(y) \le \gamma \,\}$
(6) ContraRough approximations (after UD).
$\underline{U}_{\text{UD}}^{(\alpha,\beta)} \leftarrow \{\, x \in X : \ N_U^{(\alpha)}(x) \subseteq U_{\text{def},U}^{(\beta)} \,\}$
$\overline{U}_{\text{UD}}^{(\alpha,\gamma)} \leftarrow \{\, x \in X : \ N_U^{(\alpha)}(x) \cap U_{\text{pos},U}^{(\gamma)} \neq \varnothing \,\}$
$\text{Pos}_{\text{UD}} \leftarrow \underline{U}_{\text{UD}}^{(\alpha,\beta)}$, $\text{Bnd}_{\text{UD}} \leftarrow \overline{U}_{\text{UD}}^{(\alpha,\gamma)} \setminus \underline{U}_{\text{UD}}^{(\alpha,\beta)}$, $\text{Neg}_{\text{UD}} \leftarrow X \setminus \overline{U}_{\text{UD}}^{(\alpha,\gamma)}$
return all sets and kernels.

(2) Definite/possible slices.

$$U_{\text{def}}^{(\beta)} = \{y : \ c_U(y) \le 0.2\} = \{e_1\}, \qquad U_{\text{pos}}^{(\gamma)} = \{y : \ c_U(y) \le 0.6\} = \{e_1, e_2\}.$$

(3) Approximations (before UD).

$$\underline{U}^{(\alpha,\beta)} = \{x : N^{(0.4)}(x) \subseteq \{e_1\}\} = \varnothing,$$

$$\overline{U}^{(\alpha,\gamma)} = \{x : N^{(0.4)}(x) \cap \{e_1, e_2\} \neq \varnothing\} = \{e_1, e_2, e_3\}.$$

Hence

$$\mathrm{Pos} = \varnothing, \quad \mathrm{Bnd} = \{e_1, e_2, e_3\}, \quad \mathrm{Neg} = \{e_4\}.$$

(4) Upside-Down activation and kernel flip. Activation set

$$A_\tau = \{y : c_U(y) \geq 0.8\} = \{e_4\}.$$

Flip membership-contradiction on A_τ:

$$c_U^{U_\tau}(e_4) = 1 - 0.85 = 0.15, \qquad c_U^{U_\tau}(e_1) = 0.10,\ c_U^{U_\tau}(e_2) = 0.25,\ c_U^{U_\tau}(e_3) = 0.65.$$

Because `NEUTRALIZE`$=$ true, set $c_R^{U_\tau}(x, y) = 0$ whenever $x \in A_\tau$ or $y \in A_\tau$; otherwise keep c_R.

(5) Neighborhoods and slices under UD.

$$N_U^{(0.4)}(e_1) = \{e_1, e_2, e_4\}, \quad N_U^{(0.4)}(e_2) = \{e_2, e_1, e_3, e_4\},$$
$$N_U^{(0.4)}(e_3) = \{e_3, e_2, e_4\}, \quad N_U^{(0.4)}(e_4) = \{e_4, e_1, e_2, e_3\}.$$

$$U_{\mathrm{def},U}^{(\beta)} = \{y : c_U^{U_\tau}(y) \leq 0.2\} = \{e_1, e_4\}, \qquad U_{\mathrm{pos},U}^{(\gamma)} = \{y : c_U^{U_\tau}(y) \leq 0.6\} = \{e_1, e_2, e_4\}.$$

(6) Approximations (after UD).

$$\underline{U}_{\mathrm{UD}}^{(\alpha,\beta)} = \{x : N_U^{(0.4)}(x) \subseteq \{e_1, e_4\}\} = \varnothing,$$

$$\overline{U}_{\mathrm{UD}}^{(\alpha,\gamma)} = \{x : N_U^{(0.4)}(x) \cap \{e_1, e_2, e_4\} \neq \varnothing\} = \{e_1, e_2, e_3, e_4\} = X.$$

Thus

$$\mathrm{Pos}_{\mathrm{UD}} = \varnothing, \quad \mathrm{Bnd}_{\mathrm{UD}} = X, \quad \mathrm{Neg}_{\mathrm{UD}} = \varnothing.$$

Endpoint e_4 was highly noncompliant ($c_U(e_4) = 0.85 \geq \tau$). An emergency patch triggers the Upside-Down flip, reducing its contradiction to compliance (0.15) and neutralizing all pairwise contradictions involving e_4. This expands neighborhoods and moves every endpoint into the *possible* region, eliminating negatives—exactly what an organization expects immediately after a global remediation action.

Theorem 3.52 (Correctness (soundness and completeness))
Let $X, c_R, c_U, \alpha, \beta, \gamma, \tau$ be as above. Algorithm 2 returns

$$\underline{U}^{(\alpha,\beta)} = \{x : N^{(\alpha)}(x) \subseteq U_{\mathrm{def}}^{(\beta)}\}, \quad \overline{U}^{(\alpha,\gamma)} = \{x : N^{(\alpha)}(x) \cap U_{\mathrm{pos}}^{(\gamma)} \neq \varnothing\},$$

and, after the UD transform $(c_R^{U_\tau}, c_U^{U_\tau})$, it returns

$$\underline{U}_{\mathrm{UD}}^{(\alpha,\beta)} = \{x : N_U^{(\alpha)}(x) \subseteq U_{\mathrm{def},U}^{(\beta)}\}, \quad \overline{U}_{\mathrm{UD}}^{(\alpha,\gamma)} = \{x : N_U^{(\alpha)}(x) \cap U_{\mathrm{pos},U}^{(\gamma)} \neq \varnothing\},$$

i.e., the algorithmic outputs coincide exactly with the mathematical definitions before and after the upside-down transform.

Proof
By construction, Step (1) sets $N^{(\alpha)}(x) = \{y : c_R(x, y) \le \alpha\}$ for each x, which is the definition of the α–neighborhood. Step (2) builds $U^{(\beta)}_{\rm def} = \{y : c_U(y) \le \beta\}$ and $U^{(\gamma)}_{\rm pos} = \{y : c_U(y) \le \gamma\}$ by direct thresholding. Step (3) implements the predicates $N^{(\alpha)}(x) \subseteq U^{(\beta)}_{\rm def}$ and $N^{(\alpha)}(x) \cap U^{(\gamma)}_{\rm pos} \neq \varnothing$ exactly, so the returned lower/upper sets match their definitions.

For the UD phase, Step (4) first forms $A_\tau = \{y : c_U(y) \ge \tau\}$, then flips c_U pointwise on A_τ via $c_U^{U_\tau}(y) = 1 - c_U(y)$ (and keeps it elsewhere), as specified. The relation kernel is either kept or neutralized on pairs that touch A_τ exactly as stated. Step (5) recomputes the neighborhoods and membership slices using $(c_R^{U_\tau}, c_U^{U_\tau})$, so $N_U^{(\alpha)}$, $U^{(\beta)}_{{\rm def},U}$, and $U^{(\gamma)}_{{\rm pos},U}$ coincide with the definitions under the transformed kernels. Finally, Step (6) repeats the same subset and intersection tests with these transformed objects, hence the outputs are precisely the mathematically defined UD lower/upper approximations. Therefore the procedure is sound (no spurious elements produced) and complete (no valid elements omitted) with respect to the stated definitions. □

Theorem 3.53 (Time and space complexity)
Let $n := |X|$ and let $m := |\{(x, y) \in X^2 : \ c_R(x, y) \le \alpha\}|$ denote the number of α–admissible pairs (so $m \le n^2$). Then Algorithm 2 runs in

$$\text{time } \Theta(n^2) \text{ and space } \Theta(n^2) \quad \text{in the dense worst case } (m = \Theta(n^2)),$$

and in

$$\text{time } \Theta(n + m) \text{ and space } \Theta(n + m) \quad \text{in the sparse case (when } m \ll n^2\text{) with adjacency storage.}$$

Proof
Dense view. Step (1) inspects all pairs (x, y) to test $c_R(x, y) \le \alpha$, costing $\Theta(n^2)$ time and storing all neighborhoods, $\Theta(n^2)$ space. Step (2) thresholds c_U in $\Theta(n)$ time/space. Step (3) checks, for each x, whether $N^{(\alpha)}(x) \subseteq U^{(\beta)}_{\rm def}$ (a scan of $N^{(\alpha)}(x)$) and whether $N^{(\alpha)}(x) \cap U^{(\gamma)}_{\rm pos} \neq \varnothing$ (a scan until a hit). Summed over x, this is $O(\sum_x |N^{(\alpha)}(x)|) = O(n^2)$ time in the dense case. Step (4) builds A_τ and flips c_U in $O(n)$, while updating c_R takes $O(n^2)$ in the worst case. Steps (5)–(6) mirror (1)–(3), adding another $\Theta(n^2)$ time and $\Theta(n^2)$ space. Thus the overall dense complexity is $\Theta(n^2)$ time and space.

Sparse view. If we maintain neighborhoods as adjacency lists of the α–graph (only pairs with $c_R \le \alpha$), Step (1) costs $\Theta(n + m)$ to build/store (assuming either on-the-fly filtering from a given list of candidate pairs or a precomputed sparse representation). Steps (3) and (6) then cost $\Theta(\sum_x |N^{(\alpha)}(x)|) = \Theta(m)$ each. The c_U thresholds and flips remain $O(n)$. If `NEUTRALIZE` is off, we reuse the same $N^{(\alpha)}$; if it is on, neutralization zeroes edges touching A_τ, which can be done by walking adjacency of A_τ in $O(\sum_{y \in A_\tau} \deg(y)) \le O(m)$. Hence the sparse bounds are $\Theta(n + m)$ time and $\Theta(n + m)$ space. □

4. Additional Results: Multivalued ContraSoft and ContraRough Set

In this section, as an additional result of the present paper, we examine the case in which multiple contradiction degrees exist within the system.

4.1. Multivalued (vector-valued) ContraSoft Set

The multivalued ContraSoft Set models each parameter's contradiction as an m-dimensional vector, enabling thresholded flips and nuanced, attribute-specific conflict control.

Definition 4.1 (Multivalued contradiction degree on parameters)**.** Let E be a nonempty finite set of parameters and $m \in \mathbb{N}$. A *multivalued (vector-valued) contradiction degree* on E is a map

$$\boldsymbol{c} : E \times E \longrightarrow [0,1]^m, \qquad (e,f) \longmapsto \boldsymbol{c}(e,f) = (c_1(e,f),\ldots,c_m(e,f)),$$

satisfying for all $e, f \in E$:

$$\boldsymbol{c}(e,e) = \mathbf{0} \quad \text{(reflexivity)}, \qquad \boldsymbol{c}(e,f) = \boldsymbol{c}(f,e) \quad \text{(symmetry)}.$$

We write the product (componentwise) partial order on $[0,1]^m$ by $\boldsymbol{u} \preceq \boldsymbol{v} \iff u_i \le v_i$ for all i.

Definition 4.2 (Multivalued ContraSoft Set)**.** Let U be a finite universe and E a finite parameter set. A *Multivalued ContraSoft Set* is a quadruple

$$\mathsf{MCS} := (U,\ E,\ F,\ \boldsymbol{c}),$$

where $F : E \to \mathcal{P}(U)$ is the (crisp) soft mapping and $\boldsymbol{c}$ is a multivalued contradiction degree on E. For an anchor $b \in E$ and a vector threshold $\boldsymbol{\tau} \in [0,1]^m$, the *activation set* is

$$A_{\boldsymbol{\tau}}(b) := \{\, e \in E : \ \boldsymbol{c}(e,b) \succeq \boldsymbol{\tau} \,\}.$$

The *multivalued Upside-Down transform* $U_{b,\boldsymbol{\tau}}$ produces $\left(F^U, \boldsymbol{c}^U\right)$ by

$$F^U(e) := \begin{cases} U \setminus F(e), & e \in A_{\boldsymbol{\tau}}(b) \quad \text{(flip)}, \\ F(e), & e \notin A_{\boldsymbol{\tau}}(b) \quad \text{(keep)}, \end{cases} \qquad \boldsymbol{c}^U(e,f) := \begin{cases} \mathbf{0}, & \{e,f\} \cap A_{\boldsymbol{\tau}}(b) \neq \varnothing, \\ \boldsymbol{c}(e,f), & \text{otherwise.} \end{cases}$$

Example 4.3 (Multivalued ContraSoft Set: Sustainable Supplier Screening)**.** Let the universe of suppliers be

$$U = \{s_1, s_2, s_3\},$$

and parameters

$$E = \{\text{SUST}, \text{COST}, \text{SPEED}\},$$

standing for "sustainability-certified", "low cost", and "fast delivery". A (crisp) soft mapping $F : E \to \mathcal{P}(U)$ is given by

$$F(\text{SUST}) = \{s_1\}, \qquad F(\text{COST}) = \{s_2, s_3\}, \qquad F(\text{SPEED}) = \{s_2\}.$$

The contradiction is *multivalued*:

$$c : E \times E \longrightarrow [0,1]^3, \quad c = (c^{(1)}, c^{(2)}, c^{(3)}),$$

where components quantify (i) regulatory risk, (ii) budget tension, and (iii) CSR misalignment. Symmetry and reflexivity hold: $c(e,e) = (0,0,0)$ and $c(e,f) = c(f,e)$. Assume

$$\begin{aligned} c(\text{SUST}, \text{COST}) &= (0.90,\ 0.20,\ 0.80), \\ c(\text{SUST}, \text{SPEED}) &= (0.60,\ 0.10,\ 0.50), \\ c(\text{COST}, \text{SPEED}) &= (0.30,\ 0.40,\ 0.20). \end{aligned}$$

Fix anchor $b = \text{SUST}$ and use the ℓ_∞ activation rule with threshold $\tau_\infty = 0.80$:

$$A_{\tau_\infty}(b) := \Big\{\, e \in E : \ \|c(e,b)\|_\infty \ge \tau_\infty \,\Big\}.$$

Then $\|c(\mathrm{COST},\mathrm{SUST})\|_\infty = 0.90 \geq 0.80$ activates COST, while $\|c(\mathrm{SPEED},\mathrm{SUST})\|_\infty = 0.60 < 0.80$ does not. The Upside-Down transform flips only the activated parameter:

$$F^U(\mathrm{COST}) = U \setminus F(\mathrm{COST}) = \{s_1\},$$
$$F^U(\mathrm{SUST}) = F(\mathrm{SUST}) = \{s_1\},$$
$$F^U(\mathrm{SPEED}) = F(\mathrm{SPEED}) = \{s_2\}.$$

Interpretation. When sustainability is the anchor objective, the *highly* contradictory "low cost" criterion is inverted (promoting s_1) while the only *moderately* contradictory "speed" criterion is kept unchanged. This yields a policy-consistent screening under multi-dimensional conflicts.

Remark 4.4 (Scalarization (optional))**.** Often one uses a monotone *aggregator* $\varphi : [0,1]^m \to [0,1]$ (e.g. $\varphi = \|\cdot\|_\infty$ or a weighted maximum) and a scalar threshold $\tau \in [0,1]$, declaring activation by $\varphi\big(\boldsymbol{c}(e,b)\big) \geq \tau$. This yields a scalar *effective* degree $c_\varphi(e,f) := \varphi(\boldsymbol{c}(e,f))$ while retaining the same flipping rule for F and the reset $\boldsymbol{c}^U$.

Theorem 4.5 (Multivalued ContraSoft generalizes the classical ContraSoft)
Let $\mathsf{MCS} = (U, E, F, \boldsymbol{c})$ with $m \in \mathbb{N}$.

(i) If $m = 1$ and we identify $\boldsymbol{c}$ with $c : E \times E \to [0,1]$, then for every anchor b and scalar threshold τ, $U_{b,\tau}$ coincides with the classical ContraSoft Upside-Down transform.

(ii) More generally, fix a monotone aggregator $\varphi : [0,1]^m \to [0,1]$ and set $c_\varphi(e,f) := \varphi(\boldsymbol{c}(e,f))$. If we choose the scalar threshold $\tau := \varphi(\boldsymbol{\tau})$ and define activation by $\boldsymbol{c}(e,b) \succeq \boldsymbol{\tau}$ *or* equivalently by $c_\varphi(e,b) \geq \tau$ under the condition
$$\forall\, \boldsymbol{u},\boldsymbol{v} \in [0,1]^m : \quad \boldsymbol{u} \succeq \boldsymbol{v} \iff \varphi(\boldsymbol{u}) \geq \varphi(\boldsymbol{v}),$$
(e.g. $\varphi = \|\cdot\|_\infty$ and $\boldsymbol{\tau} = \tau\mathbf{1}$), then the activated parameters, the flipped F^U, and the reset $\boldsymbol{c}^U$ are identical to those obtained by the classical scalar ContraSoft with contradiction c_φ and threshold τ.

Proof
(i) When $m = 1$, the product order $\preceq$ is just the usual order on $[0,1]$ and $\boldsymbol{c}(e,b) \succeq \boldsymbol{\tau}$ reduces to $c(e,b) \geq \tau$. The flipping and resetting rules match the scalar definition verbatim.

(ii) By the stated order-equivalence, $\boldsymbol{c}(e,b) \succeq \boldsymbol{\tau}$ iff $c_\varphi(e,b) \geq \varphi(\boldsymbol{\tau}) = \tau$. Hence the activation set is the same in both constructions. The transform on F depends only on activation, so F^U coincides. The reset rule sets the entire vector $\boldsymbol{c}^U(e,f) = \mathbf{0}$ whenever a pair touches the activation set; this is compatible with the scalarized picture since then $c_\varphi^U(e,f) = \varphi(\mathbf{0}) = 0$. Therefore the outputs agree. □

4.2. Multivalued ContraRough Set

The multivalued ContraRough Set uses vector kernels for relation and membership contradictions, producing vector-threshold approximations that generalize classical rough reasoning.

Definition 4.6 (Multivalued ContraRough kernels and approximations)**.** Let X be a finite universe and $m \in \mathbb{N}$. A *multivalued relation-kernel* is $\boldsymbol{c}_R : X \times X \to [0,1]^m$ with $\boldsymbol{c}_R(x,x) = \mathbf{0}$ and symmetry in (x,y). A *multivalued membership-kernel* is $\boldsymbol{c}_U : X \to [0,1]^m$. Fix vector thresholds $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in [0,1]^m$ with $\boldsymbol{\beta} \preceq \boldsymbol{\gamma}$.

Define the $\boldsymbol{\alpha}$*-neighborhoods* and *definite/possible slices* by

$$N^{(\boldsymbol{\alpha})}(x) := \{\, y \in X : \ \boldsymbol{c}_R(x,y) \preceq \boldsymbol{\alpha} \,\}, \quad U_{\mathrm{def}}^{(\boldsymbol{\beta})} := \{\, y : \ \boldsymbol{c}_U(y) \preceq \boldsymbol{\beta} \,\}, \quad U_{\mathrm{pos}}^{(\boldsymbol{\gamma})} := \{\, y : \ \boldsymbol{c}_U(y) \preceq \boldsymbol{\gamma} \,\}.$$

The *Multivalued ContraRough lower/upper approximations* of a target $U \subseteq X$ are

$$\underline{U}^{(\boldsymbol{\alpha},\boldsymbol{\beta})} := \{\, x : \ N^{(\boldsymbol{\alpha})}(x) \subseteq U^{(\boldsymbol{\beta})}_{\text{def}} \,\}, \qquad \overline{U}^{(\boldsymbol{\alpha},\boldsymbol{\gamma})} := \{\, x : \ N^{(\boldsymbol{\alpha})}(x) \cap U^{(\boldsymbol{\gamma})}_{\text{pos}} \neq \varnothing \,\}.$$

As usual, define $\mathrm{Pos} = \underline{U}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, $\mathrm{Bnd} = \overline{U}^{(\boldsymbol{\alpha},\boldsymbol{\gamma})} \setminus \underline{U}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, and $\mathrm{Neg} = X \setminus \overline{U}^{(\boldsymbol{\alpha},\boldsymbol{\gamma})}$.

Example 4.7 (Multivalued ContraRough Set: Premium-Smartphone Program)**.** Let $X = \{p_1, p_2, p_3, p_4\}$ be smartphones; the target U means "eligible for the Premium Program". The *relation contradiction* is vector-valued

$$c_R : X \times X \to [0,1]^2, \qquad c_R = (c_R^{\text{series}},\, c_R^{\text{perf}}),$$

measuring contradictions in series alignment and performance gap (diag. $= (0,0)$, symmetric). The *membership contradiction* to asserting $x \in U$ is

$$c_U : X \to [0,1]^2, \qquad c_U = (c_U^{\text{price}},\, c_U^{\text{features}}).$$

Choose componentwise thresholds

$$\alpha = (0.40, 0.50), \qquad \beta = (0.30, 0.30), \qquad \gamma = (0.60, 0.60),$$

and admit links/memberships by *componentwise* comparison (i.e., $c_R(x,y) \le \alpha$ means $c_R^{(d)}(x,y) \le \alpha^{(d)}$ for both components d).

Membership-contradiction data:

$$c_U(p_1) = (0.10, 0.20), \quad c_U(p_2) = (0.25, 0.35), \quad c_U(p_3) = (0.55, 0.40), \quad c_U(p_4) = (0.80, 0.70).$$

Hence

$$U^{(\beta)}_{\text{def}} = \{p_1\}, \qquad U^{(\gamma)}_{\text{pos}} = \{p_1, p_2, p_3\}.$$

Relation-contradiction (nontrivial symmetric pairs):

$$\begin{aligned} &c_R(p_1,p_2) = (0.45, 0.40), \quad c_R(p_1,p_3) = (0.30, 0.55), \quad c_R(p_1,p_4) = (0.50, 0.70), \\ &c_R(p_2,p_3) = (0.40, 0.50), \quad c_R(p_2,p_4) = (0.60, 0.60), \quad c_R(p_3,p_4) = (0.30, 0.40). \end{aligned}$$

Neighborhoods at level α (include self by $c_R(x,x) = (0,0)$):

$$\begin{aligned} &N^{(\alpha)}(p_1) = \{p_1\}, \quad N^{(\alpha)}(p_2) = \{p_2, p_3\}, \\ &N^{(\alpha)}(p_3) = \{p_3, p_2, p_4\}, \quad N^{(\alpha)}(p_4) = \{p_4, p_3\}. \end{aligned}$$

ContraRough lower/upper approximations (componentwise thresholds):

$$\underline{U}^{(\alpha,\beta)} = \{x : \ N^{(\alpha)}(x) \subseteq U^{(\beta)}_{\text{def}}\} = \{p_1\},$$

$$\overline{U}^{(\alpha,\gamma)} = \{x : \ N^{(\alpha)}(x) \cap U^{(\gamma)}_{\text{pos}} \neq \varnothing\} = \{p_1, p_2, p_3, p_4\} = X.$$

Thus

$$\mathrm{Pos} = \{p_1\}, \qquad \mathrm{Bnd} = X \setminus \{p_1\} = \{p_2, p_3, p_4\}, \qquad \mathrm{Neg} = \varnothing.$$

Device p_1 is *definitely* premium-eligible (its local neighborhood is entirely consistent with the definite region), while p_2, p_3, p_4 remain in the boundary: each is connected to at least one *possibly* eligible device but lacks neighborhood-wide consistency for definite inclusion. The vector thresholds disentangle price/feature and series/performance contradictions in a transparent, multi-criteria manner.

Theorem 4.8 (Multivalued ContraRough generalizes the classical ContraRough)
Let $m \in \mathbb{N}$.

(i) If $m = 1$, the above construction reduces exactly to the classical ContraRough approximations with scalar kernels c_R, c_U and scalar thresholds α, β, γ.

(ii) Let $\varphi : [0,1]^m \to [0,1]$ be the sup-aggregator $\varphi(\boldsymbol{v}) = \|\boldsymbol{v}\|_\infty$ and take *constant* thresholds $\boldsymbol{\alpha} = \alpha\mathbf{1}$, $\boldsymbol{\beta} = \beta\mathbf{1}$, $\boldsymbol{\gamma} = \gamma\mathbf{1}$ (so $\beta \le \gamma$). Define scalar kernels

$$c_R^\infty(x,y) := \|\boldsymbol{c}_R(x,y)\|_\infty, \qquad c_U^\infty(y) := \|\boldsymbol{c}_U(y)\|_\infty.$$

Then, for every $U \subseteq X$,

$$\underline{U}^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \;=\; \underline{U}^{(\alpha,\beta)}[c_R^\infty, c_U^\infty], \qquad \overline{U}^{(\boldsymbol{\alpha},\boldsymbol{\gamma})} \;=\; \overline{U}^{(\alpha,\gamma)}[c_R^\infty, c_U^\infty].$$

Proof
(i) Trivial: the product order is the scalar order and all vector inequalities collapse to scalar ones.

(ii) For any $\boldsymbol{v} \in [0,1]^m$ and $\tau \in [0,1]$, we have $\boldsymbol{v} \preceq \tau\mathbf{1} \iff \|\boldsymbol{v}\|_\infty \le \tau$. Thus

$$N^{(\boldsymbol{\alpha})}(x) = \{y :\ \boldsymbol{c}_R(x,y) \preceq \alpha\mathbf{1}\} = \{y :\ \|\boldsymbol{c}_R(x,y)\|_\infty \le \alpha\} = \{y :\ c_R^\infty(x,y) \le \alpha\}.$$

Similarly, $U_{\text{def}}^{(\boldsymbol{\beta})} = \{y :\ c_U^\infty(y) \le \beta\}$ and $U_{\text{pos}}^{(\boldsymbol{\gamma})} = \{y :\ c_U^\infty(y) \le \gamma\}$. Substituting these equalities into the definitions of $\underline{U}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ and $\overline{U}^{(\boldsymbol{\alpha},\boldsymbol{\gamma})}$ yields the claimed identities. □

Theorem 4.9 (Vector-threshold monotonicity and Lower ⊆ Upper)
Let $\boldsymbol{\beta} \preceq \boldsymbol{\gamma}$ and $\boldsymbol{c}_R(x,x) = \mathbf{0}$ for all x.

(i) If $\boldsymbol{\alpha}_1 \preceq \boldsymbol{\alpha}_2$, then $N^{(\boldsymbol{\alpha}_1)}(x) \subseteq N^{(\boldsymbol{\alpha}_2)}(x)$ for all x, hence

$$\underline{U}^{(\boldsymbol{\alpha}_2,\boldsymbol{\beta})} \subseteq \underline{U}^{(\boldsymbol{\alpha}_1,\boldsymbol{\beta})}, \qquad \overline{U}^{(\boldsymbol{\alpha}_1,\boldsymbol{\gamma})} \subseteq \overline{U}^{(\boldsymbol{\alpha}_2,\boldsymbol{\gamma})}.$$

(ii) If $\boldsymbol{\beta}_1 \preceq \boldsymbol{\beta}_2$, then $U_{\text{def}}^{(\boldsymbol{\beta}_1)} \subseteq U_{\text{def}}^{(\boldsymbol{\beta}_2)}$ and thus $\underline{U}^{(\boldsymbol{\alpha},\boldsymbol{\beta}_1)} \subseteq \underline{U}^{(\boldsymbol{\alpha},\boldsymbol{\beta}_2)}$.

(iii) If $\boldsymbol{\gamma}_1 \preceq \boldsymbol{\gamma}_2$, then $U_{\text{pos}}^{(\boldsymbol{\gamma}_1)} \subseteq U_{\text{pos}}^{(\boldsymbol{\gamma}_2)}$ and thus $\overline{U}^{(\boldsymbol{\alpha},\boldsymbol{\gamma}_1)} \subseteq \overline{U}^{(\boldsymbol{\alpha},\boldsymbol{\gamma}_2)}$.

(iv) (*Lower ⊆ Upper*) For every $U \subseteq X$, $\underline{U}^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \subseteq \overline{U}^{(\boldsymbol{\alpha},\boldsymbol{\gamma})}$.

Proof
(i)–(iii) follow from the product-order monotonicity and the defining set inclusions.

(iv) If $x \in \underline{U}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ then $N^{(\boldsymbol{\alpha})}(x) \subseteq U_{\text{def}}^{(\boldsymbol{\beta})} \subseteq U_{\text{pos}}^{(\boldsymbol{\gamma})}$. Since $\boldsymbol{c}_R(x,x) = \mathbf{0} \preceq \boldsymbol{\alpha}$, we have $x \in N^{(\boldsymbol{\alpha})}(x)$, hence $N^{(\boldsymbol{\alpha})}(x) \cap U_{\text{pos}}^{(\boldsymbol{\gamma})} \neq \varnothing$ and therefore $x \in \overline{U}^{(\boldsymbol{\alpha},\boldsymbol{\gamma})}$. □

5. Conclusion

In this paper, we investigated the *ContraSoft Set* and the *ContraRough Set*, which extend classical Soft Sets and Rough Sets by introducing the concept of contradiction values. Furthermore, we designed algorithms related to Upside-Down Logic and examined their validity and computational properties. We also explored the notions of *Multivalued ContraSoft Set* and *Multivalued ContraRough Set*, which enable the handling of multiple contradiction degrees, thereby broadening the expressive and analytical capacity of the proposed framework. In future work, we expect further studies to explore broader frameworks employing Fuzzy Sets [1], Intuitionistic Fuzzy Sets [61, 62], Picture Fuzzy Sets [63, 64], Hesitant Fuzzy Sets [65, 66, 67], HyperFuzzy Sets [68, 69], Paraconsistent Sets [70], Vague Sets [71], and Neutrosophic Sets [72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85]. We also hope that the mathematical structures of the *ContraSoft Set* and the *ContraRough Set* defined in this paper will be further investigated, including quantitative analyses using machine learning and computer science techniques, real-world applications. Furthermore, it is expected that future studies will explore extensions of the ContraSoft Set and ContraRough Set using Graphs [86], HyperGraphs [87, 88, 89, 90, 91], and SuperHyperGraphs [92, 93, 94, 95, 96].

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Author's Contributions

Conceptualization, All authors; Investigation, All authors; Methodology, All authors; Writing – original draft, All authors; Writing – review & editing, All authors.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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