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A Minimizing Sequence Proof of the Banach Fixed Point Theorem

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Abstract In this manuscript, we present a novel proof of the Banach contraction principle based on the concept of minimizing sequences. By examining the distances between each point and its image under the mapping, we construct a sequence that converges to a unique fixed point and explore extensions to both b-metric and incomplete metric spaces. A central technical contribution is the derivation of an optimal inequality linking the contraction coefficients with the b-metric constant, which enables the extension of the method to generalized settings. Illustrative examples demonstrate how the proposed framework ensures the existence of fixed points in incomplete spaces when the set of point-to-image distances reaches its minimum.

Keywords Fixed point theorems, Contraction mappings, Complete metric spaces, b-metric spaces, Direct methods, Distance minimization, Minimizing sequences, Picard iteration, Infimum, Incomplete metric spaces

AMS 2010 subject classifications 47H10, 54H25

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1. Introduction

The goal of this work is to introduce a new minimization-based framework for proving fixed-point theorems and to demonstrate its utility in classical, b-metric, and incomplete settings.. By introducing a set $\mathcal{A}$ defined by distances between points and their images under the contraction, we demonstrate that its infimum is attained and corresponds to the fixed point. This approach not only simplifies the existence-uniqueness argument but also avoids the computational and analytical challenges associated with Picard iterations. We extend our results further to both b-metric and incomplete metric spaces. In b-metric spaces, our approach guarantees fixed-point existence under the relaxed condition $c < 1$, based on the lemma we introduced, in contrast to the classical requirement $c < 1/s$.

The minimizing sequence framework introduced in this work conceptually parallels classical boundary-condition approaches such as those of [1] for nonexpansive mappings. While the result of Kirk's rely on conditions on the boundary of convex sets to ensure the existence of fixed point, our method employs a metric functional $\phi(x) = d(x, T(x))$, to establish new approach to prove fixed point result through minimization.

Suzuki [2] established key inequalities for b-metric spaces through sequential analysis, however, we obtain comparable convergence results via an alternative approach based on the properties of the distance-minimizing set $\mathcal{A} = \{d(x, Tx) : x \in X\}$.

However, the study of fixed point theory for contraction mappings in complete metric spaces is a fundamental topic in mathematical analysis. The *Banach Fixed-Point Theorem* [3] (or *Contraction Mapping Theorem*, CMT) guarantees the existence and uniqueness of a fixed point for any contraction mapping on a complete metric space. The classical proof relies on constructing the *Picard sequence* and demonstrating its convergence, a method that has inspired extensive research on extensions and alternatives. Significant developments include

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geometric approaches [4], set-valued generalizations [5], and recent efforts to unify terminology for contraction-type mappings [6]. Further advances include generalized criteria relaxing Lipschitz conditions [7]. A discussion on b-metric spaces and related results in metric and G-metric spaces can be found in [8]. Contributions on fixed point results involving new distance structures and quasi contractions in neutrosophic fuzzy metric spaces can be found in [10, 11]. Recent work by Taleb et al. [9] developed fixed point results in extended (ϕ, ψ)-metric spaces.

2. Main Result

Before we proceed with proving the existence of a fixed point, we recall the well-known Banach Fixed-Point Theorem, which states that if a mapping $T : X \to X$ is a contraction on a complete metric space (X, d), then T has a unique fixed point in X. This theorem is powerful in establishing the existence and uniqueness of fixed points under the contraction condition. In this context, we will use the properties of contraction mappings, the infimum of a specific set, and the completeness of X to demonstrate that such a fixed point exists and is unique.

To provide context, we first briefly review the traditional Picard iteration method. Given a contraction mapping T, the Picard sequence is defined as

$$x_{n+1} = T(x_n),$$

starting from an initial point x_0. In this case the sequence will converge to a unique fixed point x^*. While this method is mostly effective, in some cases it can be computationally intensive, especially when the contraction constant is close to 1.

In contrast, our proposed method focuses on minimizing the set

$$\mathcal{A} = \{d(x, T(x)) : x \in X\}.$$

We aim by constructing a sequence $\{x_n\}$ so that $d(x_n, T(x_n))$ converges to the infimum of A, we prove the existence of fixed point. This approach is advantageous when the infimum can be computed analytically or approximated efficiently.

Theorem 2.1 (CMT)
Let (X, d) be a complete metric space, and let $T : X \to X$ be a contraction mapping. That is, there exists a constant $0 \leq c < 1$ such that

$$d(T(x), T(y)) \leq c\, d(x, y), \quad \forall x, y \in X.$$

Then, T has a unique fixed point $x^* \in X$.

Proof
Define the set

$$\mathcal{A} = \{d(x, T(x)) : x \in X\} \subseteq [0, \infty).$$

Since the metric d is always non-negative, it follows that $\inf \mathcal{A} \geq 0$.
Let $\alpha = \inf \mathcal{A}$. By the definition of infimum, there exists a sequence $\{x_n\} \subseteq X$ such that:

$$d(x_n, T(x_n)) \to \alpha \quad \text{as } n \to \infty.$$

Claim: $\alpha = 0$.
Using the contraction inequality, we obtain

$$d(T(x_n), T(T(x_n))) \leq c d(x_n, T(x_n)).$$

Taking limits on both sides, we get

$$\lim_{n\to\infty} d(T(x_n), T(T(x_n))) \leq c \lim_{n\to\infty} d(x_n, T(x_n)) = c\alpha.$$

Since $T(x_n) \in X$, we also have

$$d(T(x_n), T(T(x_n))) \in \mathcal{A}.$$

Thus, taking the limit on both sides

$$\alpha \leq \lim_{n\to\infty} \{d(T(x_n), T(T(x_n)))\} \leq c\alpha.$$

Since $0 \leq c < 1$, the only possibility is $\alpha = 0$. Hence,

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0. \tag{1}$$

Claim: (x_n) is Cauchy.
Applying the triangle inequality

$$d(x_n, x_m) \leq d(x_n, T(x_n)) + d(T(x_n), T(x_m)) + d(T(x_m), x_m).$$

Using the contraction condition and rearranging, gives

$$(1-c)d(x_n, x_m) \leq d(x_n, T(x_n)) + d(x_m, T(x_m)).$$

Taking the limit as $n, m \to \infty$, and using Equation 1, we obtain

$$d(x_n, x_m) \to 0.$$

Thus, (x_n) is a Cauchy sequence. Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$.
Taking the limit as $n \to \infty$, we get

$$d(x^*, T(x^*)) = \lim_{n\to\infty} d(x_n, T(x_n)) = \alpha = 0.$$

Thus, $x^* = T(x^*)$, so x^* is a fixed point.

The uniqueness follows directly from the contraction condition. □

Having established the classical case, we now demonstrate the versatility of our approach by extending it to incomplete metric spaces.

2.1. *Fixed Points in Incomplete Metric Spaces*

The classical Banach Fixed-Point Theorem fundamentally relies on the completeness of the metric space to ensure that a Cauchy sequence converges. We show in this section that a fixed point exists in incomplete spaces if the function $\phi(x) = d(x, T(x))$ reaches its minimum value. This approach doesn't require iterative methods.

Theorem 2.2
Let (X, d) be a metric space (not necessarily complete) and $T : X \to X$ a contraction mapping with constant $c \in [0, 1)$. If the function $\phi(x) = d(x, T(x))$ attains its minimum at a point $x^* \in X$, then x^* is the unique fixed point of T.

Proof
Since $\phi(x^*)$ is the global minimum, we have

$$d(x^*, T(x^*)) \leq d(x, T(x)) \quad \forall x \in X.$$

Using the contraction property:

$$d(T(x^*), T(T(x^*))) \leq cd(x^*, T(x^*)).$$

But since x^* minimizes ϕ, we must have

$$d(x^*, T(x^*)) \leq d(T(x^*), T(T(x^*))) \leq cd(x^*, T(x^*))).$$

Since $0 \leq c < 1$, this implies $d(x^*, T(x^*)) = 0$. Uniqueness follows from the contraction condition. □

This theorem provides a powerful and simple criterion for fixed point existence that is independent of the completeness of the space. The problem now is reduced to finding conditions on $\phi(x)$ where it attains its minimum.

Remark 2.3. The question of when the function

$$\phi(x) = d(x, T(x))$$

actually attains its minimum is important. Finding general conditions that guarantee this minimum is attained, such as various forms of compactness, demands a separate investigation.

Since this direction opens broader possibilities—especially in extending the minimization framework beyond standard completeness assumptions—we postpone its full treatment to a forthcoming work devoted specifically to the attainability and stability of infimum-based fixed point principles.

Example 2.4. This example shows a contraction on an incomplete function space where the fixed point exists in the space, and the minimizing sequence approach successfully finds it.

Let

$$X = C^1([0,1], \mathbb{R}),$$

the space of continuously differentiable functions on $[0,1]$, equipped with the metric

$$d(f,g) = \|f-g\|_\infty = \sup_{x\in[0,1]} |f(x)-g(x)|.$$

This space is not complete (a sequence of differentiable functions can converge uniformly to a non-differentiable function).

Define the operator $T : X \to X$ by

$$(Tf)(x) = \frac{1}{2}\int_0^x f(t)\,dt.$$

We check that T is a contraction. For $f, g \in X$,

$$|(Tf)(x) - (Tg)(x)| = \frac{1}{2}\left|\int_0^x (f(t)-g(t))\,dt\right| \le \frac{1}{2}\int_0^x |f(t)-g(t)|\,dt \le \tfrac{1}{2}\|f-g\|_\infty.$$

Thus,

$$d(Tf, Tg) \le \tfrac{1}{2} d(f,g).$$

The set of distances is

$$\mathcal{A} = \{d(f, Tf) : f \in X\} = \left\{\sup_{x\in[0,1]} \left|f(x) - \tfrac{1}{2}\int_0^x f(t)\,dt\right|\right\}.$$

The fixed point equation is

$$f(x) = \tfrac{1}{2}\int_0^x f(t)\,dt.$$

Differentiating gives

$$f'(x) = \tfrac{1}{2} f(x), \qquad f(0) = 0.$$

The unique solution is $f(x) \equiv 0$, the zero function.

Now, consider

$$f_n(x) = \tfrac{1}{n}\sin(nx).$$

Then

$$(Tf_n)(x) = \frac{1}{2}\int_0^x \tfrac{1}{n}\sin(nt)\,dt = \frac{1}{2n^2}\big(1-\cos(nx)\big).$$

Hence

$$d(f_n, Tf_n) = \sup_{x\in[0,1]} \left|\tfrac{1}{n}\sin(nx) - \tfrac{1}{2n^2}(1-\cos(nx))\right| \le \tfrac{1}{n} + \tfrac{1}{2n^2} \to 0.$$

Thus (f_n) is a minimizing sequence with

$$d(f_n, Tf_n) \to 0.$$

Moreover, $f_n \to 0$ uniformly on $[0,1]$. Since $0 \in X$, the infimum of A is 0, and it is attained at $f^*(x) \equiv 0$.

Remark 2.5. The Picard iteration starting from a nonzero function involves repeated integrals of trigonometric terms, which can be computationally messy. In contrast, the minimizing sequence approach directly shows convergence to the fixed point, even though the space $X = C^1([0,1], \mathbb{R})$ is not complete.

Example 2.6. Let X be the set of all 2×2 real matrices with Frobenius norm less than 1, i.e.,

$$X = \{A \in M_2(\mathbb{R}) : \ \|A\|_F < 1\}.$$

Define the metric d on X as the Frobenius norm distance:

$$d(A,B) = \|A-B\|_F = \sqrt{\sum_{i,j}(A_{ij}-B_{ij})^2}.$$

This space is not complete because it excludes singular matrices (matrices with $\det(A) = 0$), which are limits of Cauchy sequences in X. For example, the sequence

$$A_n = \begin{pmatrix} 1 & 0 \\ 0 & 1-\frac{1}{n} \end{pmatrix}$$

converges to

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is not in X.

Define $T : X \to X$ by:

$$T(A) = \frac{A+I}{2},$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix. This mapping averages A with the identity matrix. We will verify that T is a contraction mapping.

For any $A, B \in X$,

$$d(T(A), T(B)) = \left\| \frac{A+I}{2} - \frac{B+I}{2} \right\|_F = \frac{1}{2}\|A-B\|_F = \frac{1}{2}d(A,B).$$

Thus, T is a contraction mapping with $k = \frac{1}{2}$.

The set $\mathcal{A}$ is defined as:

$$\mathcal{A} = \{d(A, T(A)) : A \in X\}.$$

For $T(A) = \frac{A+I}{2}$, we have:

$$d(A, T(A)) = \left\| A - \frac{A+I}{2} \right\|_F = \left\| \frac{A-I}{2} \right\|_F = \frac{1}{2}\|A-I\|_F.$$

Thus:

$$\mathcal{A} = \left\{ \frac{1}{2}\|A-I\|_F : A \in X \right\}.$$

The minimum of this set is attained when $A = I$, since

$$d(I, T(I)) = \left\| I - \frac{I + I}{2} \right\|_F = \|I - I\|_F = 0.$$

The fixed point of T is the identity matrix I, since:

$$T(I) = \frac{I + I}{2} = I.$$

3. Sharp Bound Lemma and Fixed Point in b-Metric Spaces

A b-metric space is a generalization of a metric space where the triangle inequality is relaxed (see [12, 13]). Specifically, a b-metric on a set X is a function $d : X \times X \to [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

1. $q(x, y) = 0$ if and only if $x = y$.
2. $q(x, y) = q(y, x)$.
3. There exists a constant $s \geq 1$ such that $q(x, z) \leq s(q(x, y) + q(y, z))$.

Convergence, Cauchyness, and completeness in b-metric spaces are defined as follows:

Definition 3.1. [13] Let (X, q) be a b-metric space with parameter $s \geq 1$ and let (x_n) be a sequence in X. Then we say that

1. (x_n) is converges to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$.
2. (x_n) is Cauchy if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$.
3. (X, q) is complete if every Cauchy sequence converges in X.

The following technical lemma provides a crucial inequality for our main results, ensuring that we maintain the standard bound $c < 1$ rather than the weaker condition $c < \frac{1}{s}$ that would naturally arise from the b-metric structure. This sharper control of the contraction parameter is essential in our approach. The existence of such natural number l reflects a deeper compatibility between the contraction coefficient c and the b-metric parameter s.

In extending our minimizing sequence approach to b-metric spaces, we have a fundamental geometric challenge which is the relaxed triangle inequality parameter $s \geq 1$ that gives distortions which accumulate when applying the contraction mapping multiple times. The purpose of this section is to establish a precise relationship between the contraction coefficient c and the b-metric parameter s to control these distortions.

The main vision is that while a single application of the contraction T reduces distances by factor c the triangle inequality may amplify distances by factor s when chaining inequalities. To ensure overall convergence we need to find an iteration count l such that the cumulative contraction c^l dominates the cumulative triangle inequality amplification that appears in our estimates.

Lemma 3.2
For any fixed $c, s \in \mathbb{R}$ with $0 \leq c < 1$ and $s \geq 1$, there exists a natural number $l \in \mathbb{N}$ such that $c < s^{-2/l}$.

Proof
We consider three cases based on the values of c and s.
Case 1: $c = 0$.
Since $e^x > 0\ \forall x \in \mathbb{R}$. Then, for any $l \in \mathbb{N}$, $s^{-2/l} = e^{-\frac{2}{l}\ln s} > 0 = c$.
Case 2: $s = 1$.
The inequality reduces to $c < 1$, which holds by assumption. Thus, any $l \in \mathbb{N}$ satisfies the condition.

Case 3: $s > 1$ and $0 < c < 1$.
We manipulate the inequality as follows:

$$c < s^{-2/l} \iff \ln(1/c) > \frac{2}{l}\ln s \iff l > \frac{2\ln s}{\ln(1/c)}.$$

Since $\ln(1/c) > 0$ (as $c < 1$) and $\ln s \geq 0$ (as $s \geq 1$), the right-hand side is finite. Hence, the Archimedean property of $\mathbb{R}$ ensures the existence of such $l \in \mathbb{N}$ which satisfying this inequality.
Explicitly, we may take

$$l = \left\lfloor \frac{2\ln s}{\ln(1/c)} \right\rfloor + 1,$$

where $\lfloor\cdot\rfloor$ denotes the floor function. □

Example 3.3. Consider $s = 2$ and $c = 0.8$. Then

$$\frac{2\ln s}{\ln(1/c)} = \frac{2\ln 2}{\ln(1.25)} \approx \frac{1.386}{0.223} \approx 6.21.$$

Thus we can take $l = 7$, and indeed $s^{-2/l} = 2^{-2/7} \approx 0.82 > 0.8 = c$.

Theorem 3.4
[Fixed Point in b-Metric Spaces]
Let (X, q) be a complete b-metric space with constant $s \geq 1$, and let $T : X \to X$ be a contraction mapping. That is, there exists a constant $0 \leq c < 1$ such that

$$q(T(x), T(y)) \leq c\, q(x, y), \quad \forall x, y \in X.$$

Then, T has a unique fixed point $x^* \in X$.

Proof
Define the set $\mathcal{A}$ as follows

$$\mathcal{A} = \{q(x, T(x)) : x \in X\} \subseteq [0, \infty).$$

Since q is non-negative, $\inf \mathcal{A} \geq 0$.
Let $\alpha = \inf \mathcal{A}$. By the definition of infimum, there exists a sequence $\{x_n\} \subseteq X$ such that

$$q(x_n, T(x_n)) \to \alpha \quad \text{as } n \to \infty.$$

Claim: $\alpha = 0$.
Using the contraction property:

$$q(T(x_n), T(T(x_n))) \leq cq(x_n, T(x_n)).$$

Taking limits as $n \to \infty$:

$$\limsup_{n\to\infty} q(T(x_n), T(T(x_n))) \leq c \lim_{n\to\infty} q(x_n, T(x_n)) = c\alpha.$$

Since $T(x_n) \in X$, $q(T(x_n),\ \text{then}\ T(T(x_n))) \in \mathcal{A}$, so $\alpha \leq q(T(x_n), T(T(x_n)))$, and hence

$$\alpha \leq \liminf_{n\to\infty} q(T(x_n), T(T(x_n))).$$

Thus,

$$\alpha \leq \liminf_{n\to\infty} q(T(x_n), T(T(x_n))) \leq \limsup_{n\to\infty} q(T(x_n), T(T(x_n))) \leq c\alpha.$$

Hence $\alpha \leq c\alpha$. Since $0 \leq c < 1$, then $\alpha = 0$.

Hence, we have

$$\lim_{n\to\infty} q(x_n, T(x_n)) = 0. \tag{2}$$

According to Lemma 3.2, there exists a fixed natural number $l \in \mathbb{N}$ such that $c < s^{-2/l}$.
Now, we prove by induction on k for $(1 \le k \le l)$, that

$$\lim_{n\to\infty} q(x_n, T^k(x_n)) = 0. \tag{3}$$

For $k = 1$, it is true from Equation 2 that

$$\lim_{n\to\infty} q(x_n, T^1(x_n)) = 0.$$

Now, assume that it is true for natural number k, where $1 < k < l$. i.e.,

$$\lim_{n\to\infty} q(x_n, T^k(x_n)) = 0.$$

We now observe that for each $j \in \mathbb{N}$,

$$q(T^j(x_n), T^{j+1}(x_n)) \le c^j q(x_n, T(x_n)). \tag{4}$$

This follows by induction on j. For $j = 0$, the inequality is trivial. Assume it holds for some $j \ge 0$. Then using the contraction property of T, we have

$$q(T^{j+1}(x_n), T^{j+2}(x_n)) \le c\, q(T^j(x_n), T^{j+1}(x_n)) \le c \cdot c^j q(x_n, T(x_n)) = c^{j+1} q(x_n, T(x_n)).$$

Thus, (4) holds for all $j \in \mathbb{N}$.

Now, for the inductive step $k \to k+1$, we have from Equation (4) and the b-metric triangle inequality

$$q(x_n, T^{k+1}(x_n)) \le sq(x_n, T^k(x_n)) + sc^k q(x_n, T(x_n)).$$

By taking limsup as $n \to \infty$ and using Equation 2 and Equation 3, we get

$$\limsup_{n\to\infty} q(x_n, T^{k+1}(x_n)) \le 0.$$

Since $q \ge 0$, then $\lim\limits_{n\to\infty} q(x_n, T^{k+1}(x_n)) = 0$.
Claim: (x_n) is Cauchy.

For $n, m \in \mathbb{N}$, we have

$$\begin{aligned} q(x_n, x_m) &\le s(sq(x_n, T^l(x_n)) + sq(T^l(x_n), T^l(x_m))) + sq(T^l(x_m), x_m) \\ &\le s(sq(x_n, T^l(x_n)) + c^l sq(x_n, x_m)) + sq(x_m, T^l(x_m)). \end{aligned}$$

Hence,

$$(1 - s^2 c^l) q(x_n, x_m) \le s^2 q(x_n, T^l(x_n)) + s^2 q(x_m, T^l(x_m)).$$

Since $1 - s^2 c^l > 0$, then taking the limit as $n, m \to \infty$ gives

$$q(x_n, x_m) \to 0.$$

Thus, (x_n) is a Cauchy sequence. Since (X, q) is complete, there exists $x^* \in X$ such that $x_n \to x^*$.
From the triangle inequality,

$$\begin{aligned} q(x^*, T(x^*)) &\le s(sq(x^*, x_n) + sq(x_n, T(x_n))) + sq(T(x_n), T(x^*)) \\ &\le s(sq(x^*, x_n) + sq(x_n, T(x_n))) + scq(x_n, x^*). \end{aligned}$$

Taking the limit as $n \to \infty$, gives

$$q(x^*, T(x^*)) = 0.$$

Thus, $x^* = T(x^*)$, so x^* is a fixed point.

The uniqueness follows directly from the contraction condition.

□

Remark 3.5. While Theorem 3.4 generalizes Theorem 2.1 when $s = 1$, we present both to highlight (i) the conceptual simplicity of our method in classical settings, and (ii) its adaptability to more complex spaces.

4. Conclusion

This study develops a unified framework for fixed point theory based on minimizing sequences, yielding new existence and uniqueness results for contraction mappings in complete, b-metric, and incomplete metric spaces. The approach relies on minimizing the set of point-to-image distances, providing a geometric alternative to classical iterative techniques and offering greater analytical flexibility through several key refinements.

The extension to b-metric spaces rests on a crucial inequality that connects the contraction coefficient with the geometric parameter of the space. This relation enables fixed point existence under the improved condition $c < 1$, surpassing the traditional requirement $c < 1/s$, and thereby demonstrating the method's strength in dealing with relaxed triangle inequalities.

For incomplete metric spaces, the results show that a fixed point exists whenever the distance function $\phi(x) = d(x, T(x))$ attains its minimum, even in the absence of completeness. This shifts attention from Cauchy sequence behavior to the intrinsic geometry of the mapping. The question of when $\phi(x)$ attains its minimum opens further opportunities for extending this minimization approach beyond classical completeness assumptions, a subject planned for future investigation.

By combining direct distance minimization with refined control of b-metric distortions and relaxed completeness criteria, this framework advances fixed point theory in both scope and depth. It not only provides alternative proofs of classical theorems but also extends their reach to new settings, offering a flexible foundation for subsequent research in generalized metric spaces and their applications.

Looking forward, potential directions include applications to broader distance structures, computational implementations of the minimizing sequence approach, and extensions to set-valued contractions within generalized metric frameworks.

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