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Bipolar Valued Vague Subfields of a Field by Mapping

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Abstract The fuzzy concept dives from the crisp, so a niche obtained in the world. A substantial achievement was gotten in fuzzy from many research people. In this way, we are trying to work in the extension of fuzzy with help of function. Here bipolar valued vague subfield of a field is introduced and defined by the previous work; a crucial part of bipolar valued vague subfield of a field is explored that homomorphism and translations are used in this field and these two are vital role in algebra. In this paper, statements and their proof are given as detail.

Keywords Fuzzy set, vague set, bipolar valued vague set, bipolar valued vague subfield, pseudo bipolar valued vague coset, translation.

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Introduction

The crisp set was generalized into many valued logic that was fuzzy set, it was introduced by L.A.Zadeh in 1965.Succeeding years, fuzzy sets grew in different ways. The following are extensions of fuzzy sets: vague set, intuitionistic fuzzy set, bipolar valued fuzzy set, etc. Rosenfeld [15], bipolar valued fuzzy subset by Zhang [19], vague group by Biswas [5], bipolar vague set by Cicily Flora and Arockiarani [6], bipolar valued fuzzy subgroup by Anitha et al. [2], bipolar valued vague subfield by Bala Bavithra et al. [3], and works by Deepa et al. [7, 8, 9] introduced bipolar valued vague subrings of a ring and their properties. In a similar way, other works [1, 4, 12, 13, 14, 16, 17, 18] were useful to write this paper. It is necessary to introduce this paper in the current growth of human life. Upto 16th century we are working in two valued logic, after the introduction of fuzzy set, the growth of people life has been being grown with respect to the many valued logic. We are giving some examples, theorems and characterization of BVVSF in this paper.

1. Preliminaries

Definition 1.1. [18] A map $R : G \to [0, 1]$ is called a fuzzy subset of G.

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Definition 1.2. [10] The ordered structure $\mho = \{(z, [\mho_T(z), 1 - \mho_F(z)]) : z \in W\}$ is called a vague set of W, where $\mho_T : W \to [0, 1]$ is a truth membership map and $\mho_F : W \to [0, 1]$ is a false membership map, such that $\mho_T(z) + \mho_F(z) \leq 1$, for all z in W.

Definition 1.3. [10] The interval $[\mho_T(z), 1 - \mho_F(z)]$ is called the vague value of z in $\mho$ and it is denoted by $\mho(z)$, i.e., $\mho(z) = [\mho_T(z), 1 - \mho_F(z)]$.

Example 1.4. Let $\mho = \{< z, [0.04, 0.07] >, < v, [0.02, 0.06] >, < n, [0.03, 0.08] >\}$ is a vague set of $R = \{z, v, n\}$.

Definition 1.5. [19] The ordered structure $T = \{(z, T^+(z), T^-(z)) : z \in W\}$ is called a bipolar valued fuzzy subset of W, where $T^+ : W \to [0, 1]$ is a positive membership map and $T^- : W \to [-1, 0]$ is a negative membership map.

Definition 1.6. [6] The ordered structure $\mho = \{(z, [\mho^+_T(z), 1 - \mho^+_F(z)], [-1 - \mho^-_F(z), \mho^-_T(z)]) : z \in W\}$ is called a bipolar valued vague subset of W, where $\mho^+_T : W \to [0, 1], \mho^+_F : W \to [0, 1], \mho^-_T : W \to [-1, 0]$, and $\mho^-_F : W \to [-1, 0]$ are mapping such that $\mho^+_T(z) + \mho^+_F(z) \leq 1$, $-1 \leq \mho^-_T(z) + \mho^-_F(z)$, for all z in W. Bipolar valued vague subset $\mho$ is denoted as $\mho = \{(z, [\mho^+(z), \mho^-(z)]) : z \in W\}$, where $\mho^+(z) = [\mho^+_T(z), 1 - \mho^+_F(z)]$ and $\mho^-(z) = [-1 - \mho^-_F(z), \mho^-_T(z)]$. It is denoted as B_{VV}S.

Example 1.7. Let $\mho = \{< z, [0.05, 0.07], [-0.05, -0.02] >, < v, [0.04, 0.08], [-0.06, -0.03] >, < n, [0.14, 0.19], [-0.25, -0.22] >\}$ is a bipolar value vague set of $R = \{z, v, n\}$.

Definition 1.8. [6] Let $\mho = (\mho^+, \mho^-)$ and $G = (G^+, G^-)$ be two bipolar valued vague sets on W. Then

1. $\mho \subset G$ if and only if $\mho^+(z) \leq G^+(z)$ and $\mho^-(z) \geq G^-(z)$, for all $z \in W$.
2. $\mho \cap G = \{ \langle z, \text{rmin}(\mho^+(z), G^+(z)), \text{rmax}(\mho^-(z), G^-(z)) \rangle / z \in W \}$.

Definition 1.9. [3] A $B_{VV}S$ $C =< C^+, C^- >$ of a field K is said to ba a bipolar valued vague subfield C of $K(B_{VV}SF)$ if C has,

1. $C^+(y - w) \geq \text{rmin}\{C^+(y), C^+(w)\}$,
2. $C^+(yw) \geq \text{rmin}\{C^+(y), C^+(w)\}$,
3. $C^-(y - w) \leq \text{rmax}\{C^-(y), C^-(w)\}$,
4. $C^-(yw) \leq \text{rmax}\{C^-(y), C^-(w)\}$ for all $y, w \in K$,
5. $C^+(y^{-1}) \geq C^+(y)$ $\forall$ $y \neq 0 \in K$,
6. $C^-(y^{-1}) \leq C^-(y)$ $\forall$ $y \neq 0 \in K$,

where 0 is an first operation identity element of K, $\text{rmin}\{[r, s], [t, u]\} = [min\{r, t\}, min\{s, u\}]$ and $\text{rmax}\{[r, s], [t, u]\} = [max\{r, t\}, max\{s, u\}]$.

Example 1.10. $C^+(y)$=[0.36, 0.38] for $y \in < 2 >$,$C^-(y)$=[-0.39, -0.36] for $y \in < 2 >$, $C^+(y)$=[0.35, 0.37], $C^-(y)$=[-0.38, -0.35] for $y \in R- < 2 >$ is a $B_{VV}SF$ of the field R.

Definition 1.11. [6] Let $A =< A^+, A^- >$ and $W =< W^+, W^- >$ be B_{VV}Ss of the sets V_1 and V_2 respectively. The product of A and W, denoted by $A \times W$, is defined as $A \times W = \{< (\chi, \zeta), (A \times W)^+(\chi, \zeta), (A \times W)^-(\chi, \zeta) > / \ \forall \ (\chi, \zeta) \in V_1 \times V_2\}$, where $(A \times W)^+(\chi, \zeta) = \text{rmin}\{A^+(\chi), W^+(\zeta)\}$ and $(A \times W)^-(\chi, \zeta) = \text{rmax}\{A^-(\chi), W^-(\zeta)\}$.

Definition 1.12. [8] Let X and Y be any two nonempty sets and $f : X \to Y$ be a mapping.If $A =< V^+_A, V^-_A >$ be a bipolar valued vague set in X,then the image of A under f,denoted by $f(A) = B =< V^+_B, V^-_B >$ (say), is $B_{VV}S$ in Y defined by $V^+_B(y) = \underset{x \in f^{-1}(y)}{\text{rsup}} V^+_A(x)$,$V^-_B(y) = \underset{x \in f^{-1}(y)}{\text{rinf}} V^-_A(x)$ if $f^{-1}(y) \neq \phi$ for all $y \in Y$ and $V^+_B(y) = [0, 0]$, $V^-_B(y) = [-1, -1]$ otherwise. If $B =< V^+_B, V^-_B >$ is a $B_{VV}S$ in Y, then the *preimage* of B under f, denoted by $f^{-1}(B) = A =< V^+_A, V^-_A >$ (say),is the $B_{VV}S$ in X defined by $V^+_A(x) = V^+_B(f(x))$,$V^-_A(x) = V^-_B(f(x))$ for all $x \in X$.

Definition 1.13. [7] Let $C =< V_C^+, V_C^- >$ be a $B_{VV}SF$ of a field F_1 and $s \in F_1$. Then the pseudo bipolar valued vague coset $(sC)^p =< (sV_C^+)^{V_p^+}, (sV_C^-)^{V_p^-} >$ is defined by $(sV_C^+)^{V_p^+}(a) = V_p^+(s)V_C^+(a)$ and $(sV_C^-)^{V_p^-}(a) = -V_p^-(s)V_C^-(a)$ for every $a \in F_1$ and $p \in P$, where P is a collection of $B_{VV}SSs$ of F_1.

Definition 1.14. [9] Let $A =< V_A^+, V_A^- >$ be a $B_{VV}S$ of a set X. Then the height $H(A) =< H(V_A^+), H(V_A^-) >$ is defined as $H(V_A^+) = \text{rsup}V_A^+(x)$ and $H(V_A^-) = \text{rinf}V_A^-(x)$ for all $x \in X$.

Definition 1.15. [9] Let $A =< V_A^+, V_A^- >$ be a $B_{VV}S$ of a set X. Then A is called bipolar valued normal vague subset of X if $H(V_A^+) = [1]$ and $H(V_A^-) = [-1]$.

Definition 1.16. [9] Let $A =< V_A^+, V_A^- >$ be a $\mathbf{B}_{VV}S$ of X. Then $^\circ A =<^\circ V_A^+, ^\circ V_A^- >$ is defined as $^\circ V_A^+(x) = V_A^+(x)H(V_A^+)$ and $^\circ V_A^-(x) = -V_A^-(x)H(V_A^-)$ for all x in X.

Definition 1.17. [9] Let $A =< V_A^+, V_A^- >$ be a $\mathbf{B}_{VV}S$ of X. Then $^\Delta A =<^\Delta V_A^+, ^\Delta V_A^- >$ is defined as $^\Delta V_A^+(x) = V_A^+(x)/H(V_A^+)$ and $^\Delta V_A^-(x) = -V_A^-(x)/H(V_A^-)$ for all x in X.

2. Bipolar Valued Vague Subfield of a Field

Theorem 2.1
If $N =< N^+, N^- >$ and $V =< V^+, V^- >$ are $B_{VV}SFs$ of fields F_1 and F_2, respectively, then $N \times V$ is a B_{VV}SF of the field of $F_1 \times F_2$.

Proof
Let $\rho, d \in F_1$ and $\zeta, \xi \in F_2$. Then $(\rho, \zeta), (d, \xi) \in F_1 \times F_2$. Then

$$\begin{aligned}
(N \times V)^+[(\rho, \zeta) - (d, \xi)] &= (N \times V)^+[(\rho - d, \zeta - \xi)] \\
&= \text{rmin}\{N^+(\rho - d), V^+(\zeta - \xi)\} \\
&\geq \text{rmin}\{\text{rmin}\{N^+(\rho), N^+(d)\}, \text{rmin}\{V^+(\zeta), V^+(\xi)\}\} \\
&= \text{rmin}\{\text{rmin}\{N^+(\rho), V^+(\zeta)\}, \text{rmin}\{N^+(d), V^+(\xi)\}\} \\
&= \text{rmin}\{(N \times V)^+(\rho, \zeta), (N \times V)^+(d, \xi)\}
\end{aligned}$$

for all $(\rho, \zeta), (d, \xi) \in F_1 \times F_2$. And

$$\begin{aligned}
(N \times V)^+[(\rho, \zeta)(d, \xi)^{-1}] &= (N \times V)^+[(\rho d^{-1}, \zeta\xi^{-1})] \\
&= \text{rmin}\{N^+(\rho d^{-1}), V^+(\zeta\xi^{-1})\} \\
&\geq \text{rmin}\{\text{rmin}\{N^+(\rho), N^+(d)\}, \text{rmin}\{V^+(\zeta), V^+(\xi)\}\} \\
&= \text{rmin}\{\text{rmin}\{N^+(\rho), V^+(\zeta)\}, \text{rmin}\{N^+(d), V^+(\xi)\}\} \\
&= \text{rmin}\{(N \times V)^+(\rho, \zeta), (N \times V)^+(d, \xi)\}
\end{aligned}$$

for all $(\rho, \zeta), (d, \xi) \in F_1 \times F_2$. Also

$$\begin{aligned}
(N \times V)^-[(\rho, \zeta) - (d, \xi)] &= (N \times V)^-[(\rho - d, \zeta - \xi)] \\
&= \text{rmax}\{N^-(\rho - d), V^-(\zeta - \xi)\} \\
&\leq \text{rmax}\{\text{rmax}\{N^-(\rho), N^-(d)\}, \text{rmax}\{V^-(\zeta), V^-(\xi)\}\} \\
&= \text{rmax}\{\text{rmax}\{N^-(\rho), V^-(\zeta)\}, \text{rmax}\{N^-(d), V^-(\xi)\}\} \\
&= \text{rmax}\{(N \times V)^-(\rho, \zeta), (N \times V)^-(d, \xi)\}
\end{aligned}$$

for all $(\rho,\zeta),(d,\xi)\in F_1\times F_2$. And

$$\begin{aligned}(N\times V)^-[(\rho,\zeta)(d,\xi)^{-1}] &= (N\times V)^-[(\rho d^{-1},\zeta\xi^{-1})]\\ &= \text{rmax}\{N^-(\rho d^{-1}),V^-(\zeta\xi^{-1})\}\\ &\leq \text{rmax}\{\text{rmax}\{N^-(\rho),N^-(d)\},\text{rmax}\{V^-(\zeta),V^-(\xi)\}\}\\ &= \text{rmax}\{r\ max\{N^-(\rho),V^-(\zeta)\},\text{rmax}\{N^-(d),V^-(\xi)\}\}\\ &= \text{rmax}\{(N\times V)^-(\rho,\zeta),(N\times V)^-(d,\xi)\}\end{aligned}$$

for all $(\rho,\zeta),(d,\xi)\in F_1\times F_2$. Hence $N\times V$ is a B_{VV}SFs of $F_1\times F_2$. □

Theorem 2.2
Let $C=(V_C^+,V_C^-)$ be a $B_{VV}SF$ of a field F_1. Then the pseudo bipolar valued vague coset $(sC)^p$ is a $B_{VV}SF$ of F_1, for every $s\in F_1$ and $p\in P$, where P is a collection of $B_{VV}SSs$ of F_1.

Proof
Let $a,b\in F_1$,

$$\begin{aligned}(V_{sC}^+)^{V_p^+}(a-b) &= V_p^+(s)V_C^+(a-b)\\ &\geq V_p^+(s)\ \text{rmin}\{V_C^+(a),V_C^+(b)\}\\ &= \text{rmin}\{V_p^+(s)V_C^+(a),V_p^+(s)V_C^+(b)\}\\ &= \text{rmin}\{(V_{sC}^+)^{V_p^+}(a),(V_{sC}^+)^{V_p^+}(b)\}\end{aligned}$$

for all $a,b\in F_1$. And

$$\begin{aligned}(V_{sC}^+)^{V_p^+}(ab^{-1}) &= V_p^+(s)V_C^+(ab^{-1})\\ &\geq V_p^+(s)\ \text{rmin}\{V_C^+(a),V_C^+(b)\}\\ &= \text{rmin}\{V_p^+(s)V_C^+(a),V_p^+(s)V_C^+(b)\}\\ &= \text{rmin}\{(V_{sC}^+)^{V_p^+}(a),(V_{sC}^+)^{V_p^+}(b)\}\end{aligned}$$

for all $a,b\in F_1$. Also

$$\begin{aligned}(V_{sC}^-)^{V_p^-}(a-b) &= V_p^-(s)V_C^-(a-b)\\ &\leq V_p^+(s)\ \text{rmax}\{V_C^-(a),V_C^-(b)\}\\ &= \text{rmax}\{V_p^-(s)V_C^-(a),V_p^-(s)V_C^-(b)\}\\ &= \text{rmax}\{(V_{sC}^-)^{V_p^-}(a),(V_{sC}^-)^{V_p^-}(b)\}\end{aligned}$$

for all $a,b\in F_1$.And

$$\begin{aligned}(V_{sC}^-)^{V_p^-}(ab^{-1}) &= V_p^-(s)V_C^-(ab^{-1})\\ &\leq V_p^-(s)\ \text{rmax}\{V_C^-(a),V_C^-(b)\}\\ &= \text{rmax}\{V_p^-(s)V_C^-(a),V_p^-(s)V_C^-(b)\}\\ &= \text{rmax}\{(V_{sC}^-)^{V_p^-}(a),(V_{sC}^-)^{V_p^-}(b)\}\end{aligned}$$

for all $a,b\in F_1$. Hence $(sC)^\rho$ is a $B_{VV}SF$ of F_1. □

Theorem 2.3
Let C be a $B_{VV}SS$ of a field F_1. Then C is a $B_{VV}SF$ of F_1 if and only if each (V_C^+,V_C^-) is a B_V-fuzzy subfield of F_1.

Proof
Let $a, b \in F_1$. Suppose C is a $B_{VV}SF$ of F_1,

$$\begin{aligned} V_C^+(a-b) &\geq \text{rmin}\{V_C^+(a), V_C^+(b)\} \\ V_C^+(ab^{-1}) &\geq \text{rmin}\{V_C^+(a), V_C^+(b)\} \\ V_C^-(a-b) &\leq \text{rmax}\{V_C^-(a), V_C^-(b)\} \\ V_C^-(ab^{-1}) &\leq \text{rmax}\{V_C^-(a), V_C^-(b)\} \end{aligned}$$

Hence each (V_C^+, V_C^-) is a B_V-fuzzy subfield of F_1.

Conversely, assume that each (V_C^+, V_C^-) is a B_V-fuzzy subfield of F_1. As per the definition of $B_{VV}SF$ of F_1, C is a $B_{VV}SF$ of F_1. □

Theorem 2.4
If $K = \langle V_K^+, V_K^- \rangle$ is a $B_{VV}SF$ of a field F_1, then $^\circ K = \langle ^\circ V_K^+, ^\circ V_K{}^- \rangle$ is a $B_{VV}SF$ of F_1 .

Proof
For any $u, v \in F_1$,

$$\begin{aligned} ^\circ V_k{}^+(u-v) &= V_k{}^+(u-v)H(V_k{}^+) \\ &\geq \text{rmin}\{V_k{}^+(u), V_k{}^+(v))\}H(V_k{}^+)\} \\ &= \text{rmin}\{V_k{}^+(u)H(V_k{}^+), V_k{}^+(v))H(V_k{}^+)\} \\ &= \text{rmin}\{^\circ V_k{}^+(u), ^\circ V_k{}^+(v)\} \end{aligned}$$

$\forall\ u, v \in F_1$.

And

$$\begin{aligned} ^\circ V_k{}^+(uv^{-1}) &= V_k{}^+(uv^{-1})H(V_k{}^+) \\ &\geq \text{rmin}\{V_k{}^+(u), V_k{}^+(v))\}H(V_k{}^+) \\ &= \text{rmin}\{V_k{}^+(u)H(V_k{}^+), V_k{}^+(v))H(V_k{}^+)\} \\ &= \text{rmin}\{^\circ V_k{}^+(u), ^\circ V_k{}^+(v)\} \end{aligned}$$

$\forall\ u, v \in F_1$. Also

$$\begin{aligned} ^\circ V_k{}^-(u-v) &= -V_k{}^-(u-v)H(V_k{}^-) \\ &\leq -\text{rmax}\{V_k{}^-(u), V_k{}^-(v))\}H(V_k{}^-) \\ &= \text{rmax}\{-V_k{}^-(u)H(V_k{}^-), -V_k{}^-(v))H(V_k{}^-)\} \\ &= \text{rmax}\{^\circ V_k{}^-(u), ^\circ V_k{}^-(v)\} \end{aligned}$$

$\forall\ u, v \in F_1$. And

$$\begin{aligned} ^\circ V_k{}^-(uv^{-1}) &= -V_k{}^-(uv^{-1})H(V_k{}^-) \\ &\leq -\text{rmax}\{V_k{}^-(u), V_k{}^-(v))\}H(V_k{}^-) \\ &= \text{rmax}\{-V_k{}^-(u)H(V_k{}^-), V_k{}^-(v))H(V_k{}^-)\} \\ &= \text{rmax}\{^\circ V_k{}^-(u), ^\circ V_k{}^-(v)\} \end{aligned}$$

$\forall\ u, v \in F_1$. Hence $^\circ K$ is a $\mathbf{B}_{VV}SF$ of F_1. □

Theorem 2.5
If $K = \langle V_K^+, V_K^- \rangle$ is a $\mathbf{B}_{VV}SF$ of a field F_1, then:
(i) if $H(V_K{}^+) < [1]$, then $^\circ V_K^+ < V_K^+$;
(ii) if $H(V_K{}^-) > [-1]$, then $^\circ V_K^- > V_K^-$;
(iii) if $H(V_K{}^+) < [1]$ and $H(V_K{}^-) > [-1]$, then $^\circ K < K$.

Proof
(i), (ii) and (iii) are trivial. □

Theorem 2.6
If $K = \langle V_K^+, V_K^- \rangle$ is a $\mathbf{B}_{VV}SF$ of a field F_1, then ${}^{\Delta}K = \langle {}^{\Delta}V_K^+, {}^{\Delta}V_K^- \rangle$ is a $\mathbf{B}_{VV}SF$ of F_1.

Proof
For any $u, v \in F_1$, Then

$$\begin{aligned} {}^{\Delta}V_K{}^+(u-v) &= V_K{}^+(u-v)/H(V_K{}^+) \\ &\geq \text{rmin}\{V_K{}^+(u), V_K{}^+(v)\}/H(V_K{}^+) \\ &= \text{rmin}\{V_K{}^+(u)/H(V_K{}^+), V_K{}^+(v)/H(V_K{}^+)\} \\ &= \text{rmin}\{{}^{\Delta}V_K{}^+(u), {}^{\Delta}V_K{}^+(v)\} \end{aligned}$$

$\forall\, u, v \in F_1$.
And

$$\begin{aligned} {}^{\Delta}V_K{}^+(uv^{-1}) &= V_K{}^+(uv^{-1})/H(V_K{}^+) \\ &\geq \text{rmin}\{V_K{}^+(u), V_K{}^+(v)\}/H(V_K{}^+) \\ &= \text{rmin}\{V_K{}^+(u)/H(V_K{}^+), V_K{}^+(v))/H(V_K{}^+)\} \\ &= \text{rmin}\{{}^{\Delta}V_K{}^+(u), {}^{\Delta}V_K{}^+(v)\} \end{aligned}$$

$\forall\, u, v \in F_1$. Also

$$\begin{aligned} {}^{\Delta}V_K{}^-(u-v) &= -V_K{}^-(u-v)/H(V_K{}^-) \\ &\leq -\text{rmax}\{V_K{}^-(u), V_K{}^-(v)\}/H(V_K{}^-) \\ &= \text{rmax}\{-V_K{}^-(u)/H(V_K{}^-), -V_K{}^-(v))/H(V_K{}^-)\} \\ &= \text{rmax}\{{}^{\Delta}V_K{}^-(u), {}^{\Delta}V_K{}^-(v)\} \end{aligned}$$

$\forall\, u, v \in F_1$. And

$$\begin{aligned} {}^{\Delta}V_K{}^-(uv^{-1}) &= -V_K{}^-(uv^{-1})/H(V_K{}^-) \\ &\leq -\text{rmax}\{V_K{}^-(u), V_K{}^-(v)\}/H(V_K{}^-) \\ &= \text{rmax}\{-V_K{}^-(u)/H(V_K{}^-), -V_K{}^-(v))/H(V_K{}^-)\} \\ &= \text{rmax}\{{}^{\Delta}V_K{}^-(u), {}^{\Delta}V_K{}^-(v)\} \end{aligned}$$

$\forall\, u, v \in F_1$. Hence ${}^{\Delta}K$ is a $\mathbf{B}_{VV}SF$ of F_1. □

Theorem 2.7
If $K = \langle V_K^+, V_K^- \rangle$ is a $\mathbf{B}_{VV}SF$ of a field F_1, then:

(i) If $H(V_K^+) \quad < [1]$, then ${}^{\Delta}V_K{}^+ \quad > V_K^+$;
(ii) If $H(V_K^-) > [-1]$, then ${}^{\Delta}V_K{}^+ < V_K^-$;
(iii) If $H(V_K^+) \quad < [1]$ and $H(V_K^-) \quad > [-1]$, then ${}^{\Delta}K \quad > K$;
(iv) If $H(V_K^+) < [1]$ and $H(V_K^-) > [-1]$, then ${}^{\Delta}K$ is a normal $\mathbf{B}_{VV}SF$ of F_1.

Proof
(i), (ii), (iii), and (iv) are trivial. □

Theorem 2.8
If $K = \langle V_K^+, V_K^- \rangle$ is a normal $\mathbf{B}_{VV}SF$ of a field F_1, then:

(i) $^{\circ}K = K$,
(ii) $^{\triangle}K = K$.

Proof
It can be easily proved. □

3. Homomorphism on Bipolar Valued Vague Subfield of a Field

Theorem 3.1
The homomorphic image of a $\mathbf{B}_{VV}SF$ of a field F_1 is a $\mathbf{B}_{VV}SF$ of a field F_2.

Proof
Let $f : F_1 \to F_2$ be a homomorphism. Let $V = f(A) = \langle V_V^+, V_V^- \rangle$, where $A = \langle V_A^+, V_A^- \rangle$ is a $\mathbf{B}_{VV}SF$ of F_1. We have to prove that V is a $\mathbf{B}_{VV}SF$ of F_2.
Now for $f(x), f(y) \in F_2$,

$$V_V^+(f(x)-f(y)) = V_V^+(f(x-y)) \geq V_A^+(x-y) \geq \text{rmin}\{V_A^+(x), V_A^+(y)\} = \text{rmin}\{V_V^+(f(x)), V_V^+(f(y))\},$$

which implies that

$$V_V^+(f(x)-f(y)) \geq \text{rmin}\{V_V^+(f(x)), V_V^+(f(y))\}.$$

Also,

$$V_V^+(f(x)f(y)^{-1}) = V_V^+(f(xy^{-1})) \geq V_A^+(xy^{-1}) \geq \text{rmin}\{V_A^+(x), V_A^+(y)\} = \text{rmin}\{V_V^+(f(x)), V_V^+(f(y))\},$$

which implies that

$$V_V^+(f(x)f(y)^{-1}) \geq \text{rmin}\{V_V^+(f(x)), V_V^+(f(y))\}.$$

Similarly,

$$V_V^-(f(x)-f(y)) = V_V^-(f(x-y)) \leq V_A^-(x-y) \leq \text{rmax}\{V_A^-(x), V_A^-(y)\} = \text{rmax}\{V_V^-(f(x)), V_V^-(f(y))\},$$

which implies that

$$V_V^-(f(x)-f(y)) \leq \text{rmax}\{V_V^-(f(x)), V_V^-(f(y))\}.$$

And,

$$V_V^-(f(x)f(y)^{-1}) = V_V^-(f(xy^{-1})) \leq V_A^-(xy^{-1}) \leq \text{rmax}\{V_A^-(x), V_A^-(y)\} = \text{rmax}\{V_V^-(f(x)), V_V^-(f(y))\},$$

which implies that

$$V_V^-(f(x)f(y)^{-1}) \leq \text{rmax}\{V_V^-(f(x)), V_V^-(f(y))\}.$$

Hence V is a $\mathbf{B}_{VV}$SF of F_2. □

Theorem 3.2
The homomorphic preimage of a $\mathbf{B}_{VV}SF$ of a field F_2 is a $\mathbf{B}_{VV}SF$ of a field F_1.

Proof
Let $f : F_1 \to F_2$ be a homomorphism. Let $V = f(A) = \langle V_V^+, V_V^- \rangle$ be a $\mathbf{B}_{VV}SF$ of F_2. We have to prove that $A = \langle V_A^+, V_A^- \rangle$ is a $\mathbf{B}_{VV}SF$ of F_1. Let $x, y \in F_1$. Now

$$V_A^+(x-y) = V_V^+(f(x-y)) = V_V^+(f(x)-f(y)) \geq \text{rmin}\{V_V^+(f(x)), V_V^+(f(y))\} = \text{rmin}\{V_A^+(x), V_A^+(y)\},$$

which implies that

$$V_A^+(x-y) \geq \text{rmin}\{V_A^+(x), V_A^+(y)\}.$$

Also,

$$V_A^+(xy^{-1}) = V_V^+(f(xy^{-1})) = V_V^+(f(x)f(y)^{-1}) \geq \text{rmin}\{V_V^+(f(x)), V_V^+(f(y))\} = \text{rmin}\{V_A^+(x), V_A^+(y)\},$$

which implies that

$$V_A^+(xy^{-1}) \geq \text{rmin}\{V_A^+(x), V_A^+(y)\}.$$

Similarly,

$$V_A^-(x-y) = V_V^-(f(x-y)) = V_V^-(f(x)-f(y)) \leq \text{rmax}\{V_V^-(f(x)), V_V^-(f(y))\} = \text{rmax}\{V_A^-(x), V_A^-(y)\},$$

which implies that

$$V_A^-(x-y) \leq \text{rmax}\{V_A^-(x), V_A^-(y)\}.$$

And,

$$V_A^-(xy^{-1}) = V_V^-(f(xy^{-1})) = V_V^-(f(x)f(y)^{-1}) \leq \text{rmax}\{V_V^-(f(x)), V_V^-(f(y))\} = \text{rmax}\{V_A^-(x), V_A^-(y)\},$$

which implies that

$$V_A^-(xy^{-1}) \leq \text{rmax}\{V_A^-(x), V_A^-(y)\}.$$

Hence A is a $\mathbf{B}_{VV}SF$ of F_1. □

Theorem 3.3
The homomorphic image of a product of two $\mathbf{B}_{VV}SFs$ of the fields F_1 and F_2 is a $\mathbf{B}_{VV}SF$ of a field.

Proof
From Theorems 2.1 and 3.1, the proof follows directly. □

Theorem 3.4
The homomorphic preimage of a product of two $\mathbf{B}_{VV}SFs$ of the fields F_1 and F_2 is a $\mathbf{B}_{VV}SF$ of a field.

Proof
From Theorems 2.1 and 3.2, the proof follows directly. □

Conclusion

Using the above theorems, we can find more results. This study can be extended to different types of $\mathbf{B}_{VV}$ algebra, particular, $\mathbf{B}_{VV}$ subspaces and $\mathbf{B}_{VV}$ normed spaces are natural extension of the $\mathbf{B}_{VV}SF$ of a field of this work.Field theory has varies types of application in natural life that the application depends the crisp field but after the introduction of fuzzy set and their extension the crisp concept is extended to uncertainty concepts and their extension, the one of the extension is bipolar valued vague subfield of the field. Using these we can develop the application in real field.

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