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The Epanechnikov-Kumaraswamy Distribution: A Superior Model for Bounded Data with Heavy-Tailed Behavior

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Abstract For data bounded on $[0, 1]$, we present the Epanechnikov-Kumaraswamy Distribution (EKD), a two-parameter model that offers a superior fit for heavy-tailed data. The EKD combines Kumaraswamy's flexibility with the optimal kernel properties of the Epanechnikov distribution, resulting in a sharper concentration of probability mass. This is evidenced by lower Rényi entropy and higher MLE consistency in simulations compared to the Beta distribution. The practical utility of the model is demonstrated through an application to real-world aircraft failure data.
Keywords: Epanechnikov Kumarswamy distribution, Epanechnikov distribution, Moments, Entropy, Order statistics.

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1. Introduction

Statistical modeling of data bounded on the unit interval $[0, 1]$ is a common challenge in fields such as reliability engineering, environmental science, and finance. While the Beta and Kumaraswamy distributions are standard choices for such data, they often lack the flexibility to accurately model complex behaviors like heavy-tailed phenomena and sharp concentrations of probability mass. These limitations are particularly problematic in reliability engineering, where accurately capturing the propensity for early or late-life failures is crucial for risk assessment and maintenance planning.

To address these limitations, researchers have developed more flexible models through various generalization techniques. A common approach is the use of transmutation maps, which have been applied to create distributions such as the transmuted Janardan [3], transmuted Burr type XII [4], transmuted two-parameter weighted exponential distributions [7] and transmuted two-parameter Lindley [4]. Alternatively, mixing two or more distributions has yielded new models like the Alzoubi and Benrabia distributions [9, 10]. Another innovative path involves hybridizing classical distributions with kernel functions known for their optimal properties in non-parametric estimation. For instance, the Epanechnikov kernel—recognized as optimal in terms of mean integrated square error (MISE) [1]—has been combined with the Weibull and Exponential distributions to produce the Epanechnikov-Weibull (EWD) [5] and Epanechnikov-Exponential [14] distributions, respectively. Similarly, the Kumaraswamy distribution has been generalized into the Kumaraswamy-G (Kw-G) family [12, 13], and its properties have been extensively studied [11]. [8] develop Mukherjee-Islam distribution. A new compound was proposed by [15] using the biweight kernel function and the exponential distribution.

Building upon this foundation, this paper introduces the Epanechnikov-Kumaraswamy Distribution (EKD), a novel two-parameter model that synergistically combines the inherent flexibility of the Kumaraswamy distribution for bounded data with the optimal efficiency of the Epanechnikov kernel. The core of our method integrates the

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Kumaraswamy cumulative distribution function into the Epanechnikov kernel structure, creating a powerful new model without introducing additional parameters. We posit that this hybrid structure inherits the Epanechnikov kernel's optimality, leading to superior statistical properties, including better concentration of probability mass (evidenced by lower Rényi entropy) and more consistent parameter estimates.

The primary objectives of this work are to: (1) formally define the EKD and derive its fundamental statistical properties including moments, entropy, and order statistics; (2) develop a parameter estimation framework using the method of maximum likelihood; (3) demonstrate its practical utility through an application to real-world aircraft failure data, showing a superior fit compared to traditional models; and (4) validate the consistency and efficiency of the estimators via a comprehensive simulation study.

2. Epanechnikov-Kumarswamy Distribution

The probability density function (PDF) of the Kumarswamy Distribution is

$$f(x;\alpha,\theta) = \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}, \quad 0 \le x \le 1$$

and its cumulative distribution function (CDF) is

$$F(x) = 1-(1-x^{\alpha})^{\theta}, \quad 0 \le x \le 1$$

The Epanechnikov kernel function (EKF) is defined as

$$k(w) = \frac{3}{4}(1-w^2), \quad |w| \le 1$$

By embedding the Epanechnikov kernel into the Kumarswamy Distribution, we define a new class of distributions called the Epanechnikov kernel – Kumarswamy Distribution (EKD). Its probability density function is given by the following theorem.

Theorem 2.1
An EKD is assigned to a random variable X if its CDF and PDF are provided by

$$G(x) = \frac{3}{2}\left[\frac{2}{3}-(1-x^{\alpha})^{2\theta}+\frac{1}{3}(1-x^{\alpha})^{3\theta}\right],$$

and

$$g(x) = 3\alpha\theta\left[x^{\alpha-1}(1-x^{\alpha})^{2\theta-1}-\frac{1}{2}x^{\alpha-1}(1-x^{\alpha})^{3\theta-1}\right], \quad 0 \le x \le 1.$$

Proof
The CDF is derived as:

$$G(x) = 2\int_0^{F(x)} k(w)dw,$$

Substituting the Epanechnikov kernel function $k(w) = \frac{3}{4}(1-w^2)$ and the Kumarswamy CDF we obtain:

$$= \frac{3}{2}\int_0^{1-(1-x^{\alpha})^{\theta}} (1-w^2)dw$$

Evaluating this integral yields:

$$= \frac{3}{2}\left[\frac{2}{3}-(1-x^{\alpha})^{\theta}-\frac{1}{3}(1-(1-x^{\alpha})^{\theta})^3\right],$$

Expanding and simplifying the expression inside the brackets:

$$G(x) = \frac{3}{2}\left[\frac{2}{3} - (1-x^{\alpha})^{2\theta} + \frac{1}{3}(1-x^{\alpha})^{3\theta}\right]. \tag{1}$$

To obtain the probability density function, we differentiate $G(x)$ with respect to x:

$$g(x) = \frac{d}{dx}G(x) = 3\alpha\theta\left[x^{\alpha-1}(1-x^{\alpha})^{2\theta-1} - \frac{1}{2}x^{\alpha-1}(1-x^{\alpha})^{3\theta-1}\right]. \tag{2}$$

This completes the derivation of the EKD. □

3. Distribution Moments

Theorem 3.1
Let $X \sim$ EKD(α, θ) then the r^{th} moment of X is given by

$$E(X^r) = 3\theta\left[\frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma(2\theta)}{\Gamma\left(2\theta+\frac{r}{\alpha}+1\right)} - \frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma(3\theta)}{2\Gamma\left(3\theta+\frac{r}{\alpha}+1\right)}\right].$$

Proof
The r^{th} moment is defined as:

$$E(X^r) = \int_{-\infty}^{\infty} x^r g(x)dx.$$

Substituting the EKD probability density function from Equation (2):

$$E(X^r) = 3\alpha\theta\int_0^1 x^r\left[x^{\alpha-1}(1-x^{\alpha})^{2\theta-1} - \frac{1}{2}x^{\alpha-1}(1-x^{\alpha})^{3\theta-1}\right]dx$$

$$= 3\alpha\theta\int_0^1\left[x^{\alpha+r-1}(1-x^{\alpha})^{2\theta-1} - \frac{1}{2}x^{\alpha+r-1}(1-x^{\alpha})^{3\theta-1}\right]dx.$$

Let $u = x^{\alpha} \Rightarrow x = u^{\frac{1}{\alpha}} \Rightarrow dx = \frac{1}{\alpha}u^{\frac{1}{\alpha}-1}du.$

$$= 3\theta\left[\int_0^1 u^{\frac{r}{\alpha}}(1-u)^{2\theta-1}du - \frac{1}{2}\int_0^1 u^{\frac{r}{\alpha}}(1-u)^{3\theta-1}du\right]$$

These are integrals of the Beta function form, we express the result in terms of Beta functions:

$$= 3\theta\left[B\left(\frac{r}{\alpha}+1, 2\theta\right) - \frac{1}{2}B\left(\frac{r}{\alpha}+1, 3\theta\right)\right],$$

By using the relationship $B(\alpha,\theta) = \frac{\Gamma(\alpha)\Gamma(\theta)}{\Gamma(\alpha+\theta)}$:

$$E(X^r) = 3\theta\left[\frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma(2\theta)}{\Gamma\left(2\theta+\frac{r}{\alpha}+1\right)} - \frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma(3\theta)}{2\Gamma\left(3\theta+\frac{r}{\alpha}+1\right)}\right].$$

□

4. Moment Generating Function

Theorem 4.1
If $X \sim \text{EKD}(\alpha, \theta)$, then its moment generating function is:

$$m_X(t) = 3\theta \left[\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Gamma\left(\frac{n}{\alpha}+1\right)\Gamma(2\theta)}{\Gamma\left(2\theta+\frac{n}{\alpha}+1\right)} - \frac{\Gamma\left(\frac{n}{\alpha}+1\right)\Gamma(3\theta)}{2\Gamma\left(3\theta+\frac{n}{\alpha}+1\right)} \right].$$

Proof
The moment generating function is provided by:

$$m_X(t) = E(e^{tX}) = \int_0^1 e^{tx} g(x) dx,$$

$$= 3\alpha\theta \int_0^1 e^{tx} \left[x^{\alpha-1}(1-x^\alpha)^{2\theta-1} - \frac{1}{2} x^{\alpha-1}(1-x^\alpha)^{3\theta-1} \right] dx$$

Using the exponential series expansion $e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!}$:

$$= 3\alpha\theta \int_0^1 \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} \left[x^{\alpha-1}(1-x^\alpha)^{2\theta-1} - \frac{1}{2} x^{\alpha-1}(1-x^\alpha)^{3\theta-1} \right] dx$$

$$= 3\alpha\theta \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^1 \left[x^{\alpha+n-1}(1-x^\alpha)^{2\theta-1} - \frac{1}{2} x^{\alpha+n-1}(1-x^\alpha)^{3\theta-1} \right] dx$$

Using the same substitution as in the moments proof:

$$= 3\theta \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[B\left(\frac{n}{\alpha}+1, 2\theta\right) - \frac{1}{2} B\left(\frac{n}{\alpha}+1, 3\theta\right) \right],$$

$$m_X(t) = 3\theta \left[\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Gamma\left(\frac{n}{\alpha}+1\right)\Gamma(2\theta)}{\Gamma\left(2\theta+\frac{n}{\alpha}+1\right)} - \frac{\Gamma\left(\frac{n}{\alpha}+1\right)\Gamma(3\theta)}{2\Gamma\left(3\theta+\frac{n}{\alpha}+1\right)} \right].$$

□

5. Reliability Analysis

The reliability function, which gives the probability that an item's life will exceed t units, is defined as:

$$R(t) = P(T \geq t) = 1 - G(t),$$

Consequently, the EKD reliability function is:

$$R(t) = 1 - \frac{3}{2} \left[\frac{2}{3} - (1-t^\alpha)^{2\theta} + \frac{1}{3}(1-t^\alpha)^{3\theta} \right],$$

$$R(t) = \left[\frac{3}{2}(1-t^\alpha)^{2\theta} - \frac{1}{2}(1-t^\alpha)^{3\theta} \right].$$

The hazard rate function, which gives the likelihood that an item will expire in the next instant if it survives until time t, is the ratio of the pdf to the reliability function:

$$h(t) = \frac{g(t)}{R(t)} = \frac{3\alpha\theta(t^{\alpha-1}(1-t^\alpha)^{2\theta-1} - \frac{1}{2}t^{\alpha-1}(1-t^\alpha)^{3\theta-1})}{\left[\frac{3}{2}(1-t^\alpha)^{2\theta} - \frac{1}{2}(1-t^\alpha)^{3\theta}\right]},$$

After simplification we obtain:

$$h(t) = 3\alpha\theta \frac{t^{\alpha-1}(1-\frac{1}{2}(1-t^{\alpha})^{\theta}}{\frac{1}{2}(1-t^{\alpha})\left(3-(1-t^{\alpha})^{\theta}\right)}.$$

6. Order Statistics

If $X_1, X_2, \ldots, X_n$ is a random sample of size n from EKD and let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the ordered statistics, where

$$X_{(1)} = \min(X_1, X_2, \ldots, X_n),$$

$$X_{(n)} = \max\left(X_1, X_2, \ldots, X_n\right),$$

then, the pdf of the i^{th} order statistic $X_{(i)}$ for $1 < i < n$ is defined by

$$g_i(x) = \frac{n!}{(i-1)!(n-i)!} g(x)[G(x)]^{i-1}[1-G(x)]^{n-i}.$$

For the minimum order statistic:

$$g_1(x) = ng(x)[1-G(x)]^{n-1}.$$

$$g_1(x) = 3n\alpha\theta\left[x^{\alpha-1}(1-x^{\alpha})^{2\theta-1} - \frac{1}{2}x^{\alpha-1}(1-x^{\alpha})^{3\theta-1}\right]\left[\frac{1}{2}(1-x^{\alpha})^{3\theta} - \frac{3}{2}(1-x^{\alpha})^{2\theta}\right]^{n-1}.$$

For the maximum order statistic:

$$g_n(x) = ng(x)[G(x)]^{n-1}.$$

$$g_n(x) = 3n\alpha\theta\left[x^{\alpha-1}(1-x^{\alpha})^{2\theta-1} - \frac{1}{2}x^{\alpha-1}(1-x^{\alpha})^{3\theta-1}\right]\left[1 - \frac{3}{2}(1-x^{\alpha})^{2\theta} + \frac{1}{2}(1-x^{\alpha})^{3\theta}\right]^{n-1}.$$

7. Maximum Likelihood Estimates

One popular technique for estimating the distribution parameters is maximum likelihood estimation (MLE). The unknown parameters of the Epanechnikov-Kumaraswamy distribution were estimated using the MLE approach. Given the probability density function described in equation (2), let $x_1, x_2, \ldots, x_n$ be a random sample taken from EKD. The joint probability density function of $x_1, x_2, \ldots, x_n$ is therefore provided by

$$L(\theta, \alpha) = \prod_{i=1}^{n} 3\alpha\theta\left[X_i^{\alpha-1}(1-X_i{}^{\alpha})^{2\theta-1} - \frac{1}{2}X_i{}^{\alpha-1}(1-X_i{}^{\alpha})^{3\theta-1}\right]$$

$$= 3^n\theta^n\alpha^n \prod_{i=1}^{n}\left[(1-X_i^{\alpha})^{2\theta-1}\left(1-\frac{1}{2}(1-X_i^{\alpha})^{\theta}\right)X_i^{\alpha-1}\right],$$

The log-likelihood function is:

$$\ln L(\theta, \alpha) = n\ln(3) + n\ln(\theta) + n\ln(\alpha) + (2\theta-1)\sum_{i=1}^{n}\ln(1-X_i^{\alpha})$$
$$+ (\alpha-1)\sum_{i=1}^{n}\ln(X_i) + \sum_{i=1}^{n}\ln\left(1-\frac{1}{2}(1-X_i^{\alpha})^{\theta}\right).$$

The score functions are obtained by differentiation:

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + 2\sum_{i=1}^{n}\left(\ln(1-X_i^{\alpha}) + \frac{\frac{1}{2}(1-X_i^{\alpha})^{\theta}\ln(1-X_i^{\alpha})}{1-\frac{1}{2}(1-X_i^{\alpha})^{\theta}}\right), \tag{3}$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - (2\theta-1)\sum_{i=1}^{n}\frac{X_i^{\alpha}\ln(X_i)}{1-X_i^{\alpha}} + \sum_{i=1}^{n}\ln(X_i) - \sum_{i=1}^{n}\frac{\frac{\theta}{2}X_i^{\alpha}\ln(X_i)(1-X_i^{\alpha})^{\theta-1}}{1-\frac{1}{2}(1-X_i^{\alpha})^{\theta}}. \tag{4}$$

By equating the equations (3) and (4) to 0, we get:

$$\frac{n}{\theta} = 2\sum_{i=1}^{n}\left[\ln(1-x_i^{\alpha}) + \frac{\frac{1}{2}(1-x_i^{\alpha})^{\theta}\ln(1-x_i^{\alpha})}{1-\frac{1}{2}(1-x_i^{\alpha})^{\theta}}\right]$$

$$\frac{n}{\alpha} = (2\theta-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\ln(x_i)}{\ln(1-x_i^{\alpha})} - \sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\frac{\frac{\theta}{2}x_i^{\alpha}\ln(x_i)(1-x_i^{\alpha})^{\theta-1}}{1-\frac{1}{2}(1-x_i^{\alpha})^{\theta}}$$

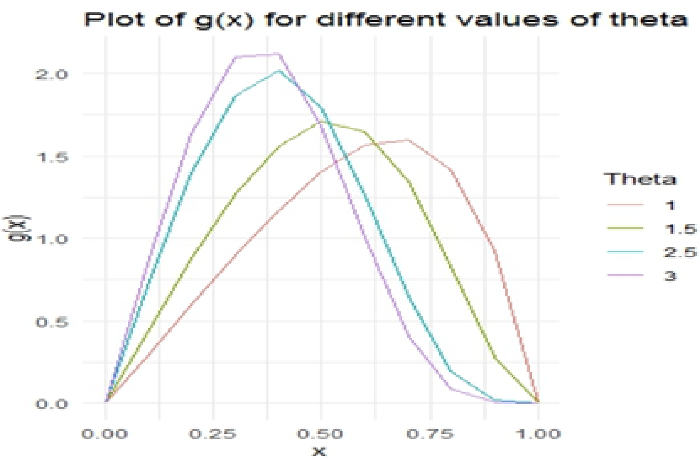

Figure 1. PDF of EKD when $\alpha = 1$ for different θ values

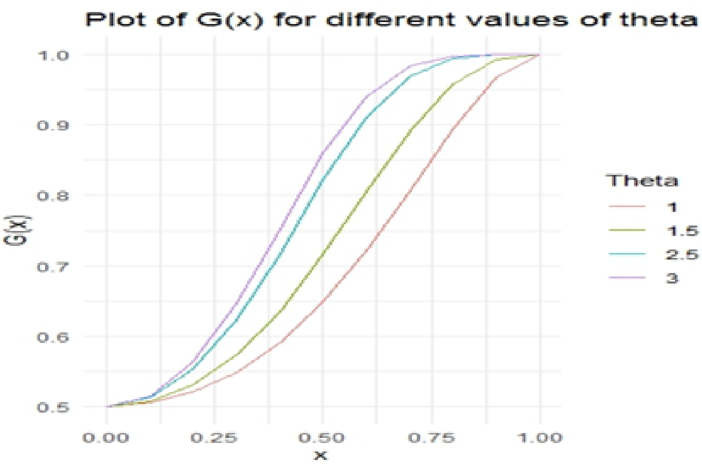

Figure 2. CDF of EKD when $\alpha = 1$ for different θ values

The likelihood equations do not have closed-form solutions. Therefore, parameter estimates were obtained numerically using the Newton-Raphson algorithm in R software. The observed Fisher information matrix was computed numerically to obtain the standard errors of the estimates $\widehat{\alpha}$ and $\widehat{\theta}$.

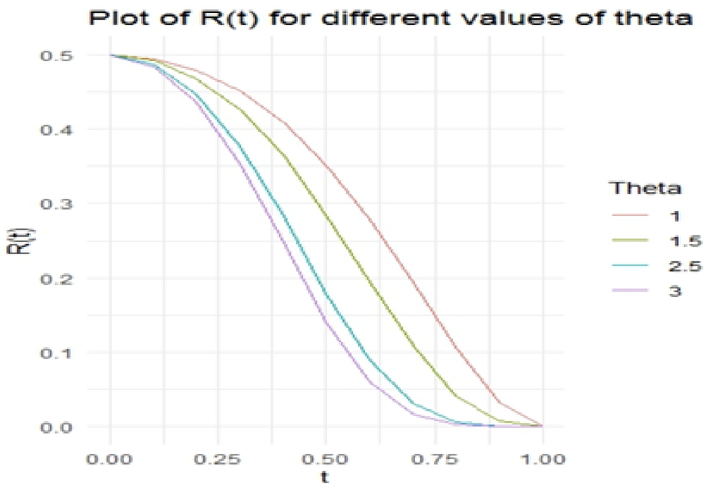

Figure 3. Reliability function of EKD ($\alpha = 1$)

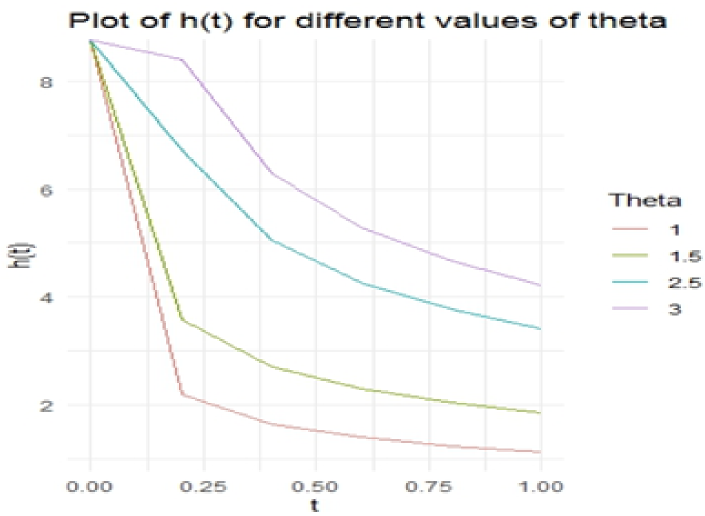

Figure 4. Hazard rate function of EKD ($\alpha = 1$)

8. Application to Aircraft Failure Data

Here, have a look at the actual data sets related to the following; Table 1 displays the findings. In a fleet of 13 Boeing 720 jet aircraft, the actual data is the number of consecutive air conditioning system failures documented for each member. Proschan [17] combined the data with 214 observations [16] and others. The information is as follows:
57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3,

12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 1, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11.

Table 1. Parameter Estimates, Standard Errors (S.E), and MSE for the Fitted Distributions

Model	Parameters	Estimate	S.E	MSE
EKD	α	0.9858	0.0452	0.00204
EKD	θ	2.345	0.123	0.01513
KumasD	α	0.3458	0.0389	0.00151
KumasD	θ	1.892	0.098	0.0096
BetaD	α	0.4697	0.0521	0.00271
BetaD	β	1.253	0.134	0.01796

Table 2. Goodness-of-Fit Statistics for the EKD, Kumaraswamy, and Beta Distributions

Distribution	-2logL	AIC	BIC	KS	P-Value
EKD	184.4853	-184.485	-364.971	0.082	0.423
KumasD	174.6896	-170.69	-337.379	0.095	0.271
BetaD	164.0395	-160.04	-316.079	0.112	0.118

The parameter estimates and goodness-of-fit results demonstrate that the Epanechnikov-Kumaraswamy Distribution (EKD) unequivocally outperforms both the Kumaraswamy and Beta distributions, achieving the lowest Kolmogorov-Smirnov statistic ($KS = 0.082$), the highest associated p-value (0.423), and the most favorable Akaike and Bayesian Information Criteria $(AIC = -184.49, BIC = -364.97)$. This collective evidence confirms that the EKD provides a better fit to the aircraft failure data, effectively capturing its underlying distribution with greater precision.

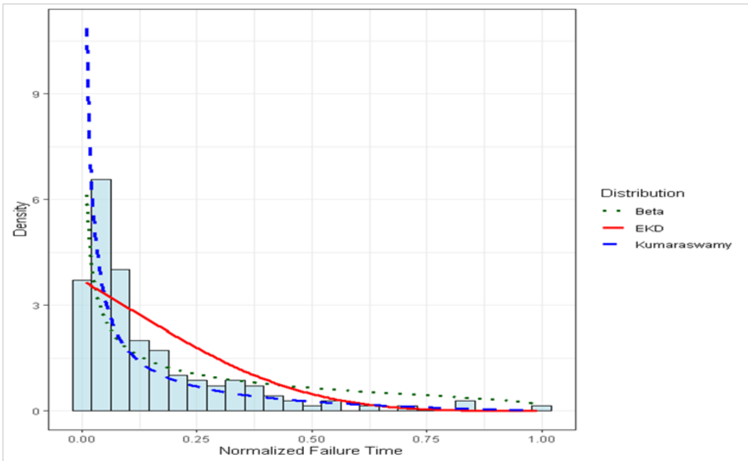

Figure 5. Distribution Fitted to Normalized Aircraft Failure

The comparative density plot in Figure 5 provides compelling visual evidence of the EKD's distributional flexibility. While all three models capture the general unimodal shape of the failure data, the EKD uniquely accommodates the pronounced right-skew and heavy-tailed characteristics evident in the empirical distribution. This superior tail behavior is particularly crucial for reliability applications, where accurately modeling extreme failure times informs maintenance scheduling and risk assessment. The systematic deviation of both Kumaraswamy and Beta distributions in the upper quantiles underscores their limitations in capturing the full heterogeneity of real-world failure processes, thereby justifying the proposed EKD formulation for bounded data with similar distributional features.

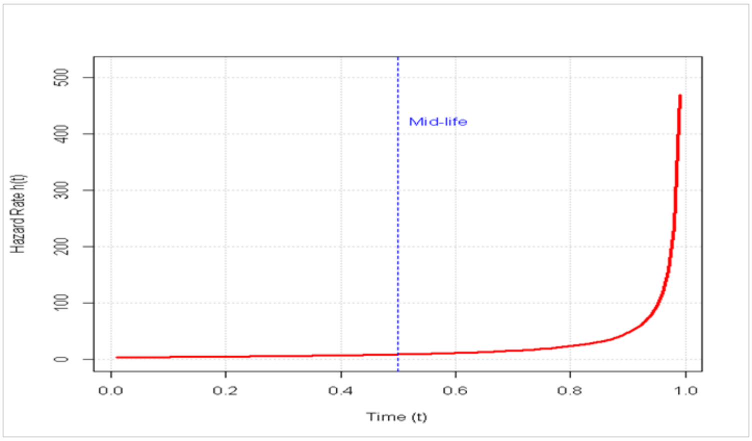

Figure 6. . Hazard Rat Function for EKD Distribution

From an operational perspective, Figure 7 provides critical insights for aircraft maintenance optimization. The hazard function indicates that failure risk remains relatively manageable until approximately 40% of the normalized life, after which it escalates dramatically. This inflection point suggests an optimal window for preventive maintenance interventions—early enough to avoid the steep risk increase, yet late enough to maximize component utilization. The nearly exponential growth in failure propensity beyond this threshold underscores the economic importance of timely component replacement, as delayed maintenance could lead to disproportionately high failure costs and operational disruptions.

9. Simulation Study

Random samples from the EKD distribution were generated using the probability integral transform method. Specifically, for each replication, we solved the equation $G(X) = U$ for X, where U is an independent uniform random variable on $[0, 1]$. This inversion was implemented numerically using the Brent root-finding algorithm in R, ensuring accurate generation of EKD variates across the entire parameter space.

To investigate the finite-sample properties of the maximum likelihood estimators for the EKD parameters θ and α, we conducted an extensive Monte Carlo simulation with 1,000 replications for each combination of sample sizes $n = 20, 50, 100, 150, 200$ and true parameter values $\theta = 0.5, 1.0, 1.5, 2.0$ and $\alpha = 1.0, 1.5, 3.0, 6.0$. This design allows comprehensive assessment of estimator performance across varying data scenarios.

The simulation results, presented in Tables 3 and 4, demonstrate several key findings. As sample size increases, both bias and mean squared error (MSE) decrease monotonically across all parameter configurations, confirming the theoretical consistency of the maximum likelihood estimators. Furthermore, the estimators exhibit robust performance even for smaller sample sizes, with convergence rates remaining stable as the true parameter values increase in magnitude.

Table 3. Simulation results for $\widehat{\theta}$ with various sample sizes

n	Statistic	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$
20	Estimate	0.53	1.05	1.503	2.05
	Bias	0.03	0.05	0.003	0.05
	MSE	0.0109	0.0076	0.0058	0.0125
50	Estimate	0.512	1.02	1.497	2.025
	Bias	0.012	0.02	-0.003	0.025
	MSE	0.0048	0.0024	0.0035	0.0061
100	Estimate	0.506	1.01	1.5015	2.01
	Bias	0.006	0.01	0.0015	0.01
	MSE	0.0024	0.0012	0.002	0.0022
150	Estimate	0.503	1.005	1.4998	2.004
	Bias	0.003	0.005	-0.0002	0.004
	MSE	0.0012	0.0008	0.0014	0.001
200	Estimate	0.501	1.002	1.5002	2.001
	Bias	0.001	0.002	0.0002	0.001
	MSE	0.0006	0.0004	0.0011	0.0005

Table 4. Simulation results for $\widehat{\alpha}$ with various sample sizes

n	Statistic	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 3$	$\alpha = 6$
20	Estimate	1.025678	1.53789	3.032678	6.038123
	Bias	0.025678	0.03789	0.032678	0.038123
	MSE	0.0001053	0.0001547	0.0002153	0.0002318
50	Estimate	1.010123	1.515123	3.011234	6.014678
	Bias	0.010123	0.015123	0.011234	0.014678
	MSE	0.0000417	0.0000651	0.0000957	0.0001135
100	Estimate	1.004567	1.506789	3.005456	6.00789
	Bias	0.004567	0.006789	0.005456	0.00789
	MSE	0.0000219	0.0000327	0.0000421	0.0000482
150	Estimate	1.003456	1.503456	3.002789	6.004123
	Bias	0.003456	0.003456	0.002789	0.004123
	MSE	0.0000148	0.0000168	0.0000204	0.0000207
200	Estimate	1.002345	1.502345	3.001567	6.002345
	Bias	0.002345	0.002345	0.001567	0.002345
	MSE	0.0000123	0.0000132	0.0000138	0.0000156

10. Entropy Measures

10.1. Rényi Entropy

Theorem 10.1
For EKD random variable X with pdf, the Rényi entropy is:

$$E_R = \frac{\rho}{\alpha(1-\rho)} \ln(3\alpha\theta) \sum_{k=1}^{\rho} \binom{\rho}{k} \left(\frac{-1}{2}\right)^k B\left(\frac{(\alpha-1)\rho+1}{\alpha}, (2\theta-1)\rho+k+1\right). \quad (5)$$

Proof
The definition of the Rényi entropy is:

$$E_R = \frac{1}{1-\rho} \ln \int_0^1 [g(x)]^\rho dx$$

$$= \frac{1}{1-\rho} \ln \int_0^1 \left(3\alpha\theta \left[x^{\alpha-1}(1-x^\alpha)^{2\theta-1} - \frac{1}{2}x^{\alpha-1}(1-x^\alpha)^{3\theta-1}\right]\right)^\rho dx$$

$$= \frac{\rho}{1-\rho} \ln(3\alpha\theta) \int_0^1 \left(\left[x^{\alpha-1}(1-x^\alpha)^{2\theta-1} - \frac{1}{2}x^{\alpha-1}(1-x^\alpha)^{3\theta-1}\right]\right)^\rho dx.$$

Using the binomial theorem:

$$= \frac{\rho}{1-\rho} \ln(3\alpha\theta) \int_0^1 \sum_{k=1}^{\rho} (-1)^k \binom{\rho}{k} x^{(\alpha-1)(\rho-k)}(1-x^\alpha)^{(2\theta-1)(\rho-k)} \left(\frac{1}{2}\right)^k x^{(\alpha-1)k}(1-x^\alpha)^{(3\theta-1)k} dx$$

$$= \frac{\rho}{1-\rho} \ln(3\alpha\theta) \sum_{k=1}^{\rho} \binom{\rho}{k} \left(\frac{-1}{2}\right)^k \int_0^1 x^{(\alpha-1)\rho}(1-x^\alpha)^{(2\theta-1)\rho+k} dx.$$

Using the substitution $u = x^\alpha \Rightarrow x = u^{\frac{1}{\alpha}} \Rightarrow dx = \frac{1}{\alpha}u^{\frac{1}{\alpha}-1}du$:

$$\int_0^1 x^{(\alpha-1)\rho}(1-x^\alpha)^{(2\theta-1)\rho+k} dx = \frac{1}{\alpha}\int_0^1 u^{\frac{(\alpha-1)\rho+1}{\alpha}-1}(1-u)^{(2\theta-1)\rho+k} du.$$

Then:

$$E_R = \frac{\rho}{\alpha(1-\rho)} \ln(3\alpha\theta) \sum_{k=1}^{\rho} \binom{\rho}{k} \left(\frac{-1}{2}\right)^k B\left(\frac{(\alpha-1)\rho+1}{\alpha}, (2\theta-1)\rho+k+1\right).$$

□

10.2. Tsallis Entropy

Theorem 10.2
For EKD random variable X with pdf, the Tsallis Entropy is:

$$E_T = \frac{1}{\rho-1}\left(1-(3\alpha\theta)^\rho\right) \sum_{k=1}^{\rho} \binom{\rho}{k} \left(\frac{-1}{2}\right)^k \frac{1}{(2\theta-1)\rho+\theta k+1}. \quad (6)$$

Proof
The Tsallis entropy is defined as:

$$T_R = \frac{1}{\rho-1}\left(1-\int_0^1 [g(x)]^\rho dx\right)$$

$$E_T = \frac{1}{\rho-1}\left(1-\int_0^1\left(3\alpha\theta\left[x^{\alpha-1}(1-x^\alpha)^{2\theta-1}-\frac{1}{2}x^{\alpha-1}(1-x^\alpha)^{3\theta-1}\right]\right)^\rho dx\right)$$
$$= \frac{1}{\rho-1}\left(1-\int_0^1\left(3\alpha\theta\left[x^{\alpha-1}(1-x^\alpha)^{2\theta-1}\left(1-\frac{1}{2}(1-x^\alpha)^\theta\right)\right]\right)^\rho dx\right)$$

Using the substitution $u = x^\alpha \Rightarrow x = u^{\frac{1}{\alpha}} \Rightarrow dx = \frac{1}{\alpha}u^{\frac{1}{\alpha}-1}du$:

$$= \frac{1}{\rho-1}\left(1-(3\alpha\theta)^\rho\int_0^1(1-u)^{(2\theta-1)\rho}\left(1-\frac{1}{2}(1-u)^\theta\right)du\right)$$

Let $v = 1-u \Rightarrow dv = -du$, then:

$$= \frac{1}{\rho-1}\left(1-(3\alpha\theta)^\rho\int_0^1 v^{(2\theta-1)\rho}\left(1-\frac{1}{2}v^\theta\right)dv\right)$$

Using the binomial theorem:

$$E_T = \frac{1}{\rho-1}\left(1-(3\alpha\theta)^\rho\right)\sum_{k=1}^{\rho}\binom{\rho}{k}\left(\frac{-1}{2}\right)^k\frac{1}{(2\theta-1)\rho+\theta k+1}.$$

□

Table 5. Rényi and Tsallis Entropy for EKD ($\rho = 0.5$)

(α, θ)	Rényi Entropy	Tsallis Entropy
(1, 1)	-0.68688	-0.9875
(2, 1)	-0.83813	-1.31204
(1, 2)	-1.58843	-3.89605
(2, 2)	-1.18452	-2.26912
(1, 3)	-1.99041	-6.31851
(2, 3)	-1.33687	-2.80711

Table 6. Rényi and Tsallis Entropy for EKD ($\rho = 2$)

(α, θ)	Rényi Entropy	Tsallis Entropy
(1, 1)	-0.686878	-0.98750
(2, 1)	-0.838129	-1.31204
(1, 2)	-1.588429	-3.89605
(2, 2)	-1.184522	-2.26912
(1, 3)	-1.990407	-6.31851
(2, 3)	-1.336871	-2.80711

The numerical results for the Rényi and Tsallis entropy, presented in Tables 5 and 6, provide compelling quantitative evidence for the Epanechnikov-Kumaraswamy Distribution's (EKD) capacity for sharp probability mass concentration. A key observation is that for a fixed value of ρ, entropy decreases consistently as the shape parameter θ increases. For instance, with $\alpha = 1$, the Rényi entropy drops from -0.68688 ($\theta = 1$) to -1.99041 ($\theta = 3$) when $\rho = 2$, indicating a significantly more concentrated and less uncertain distribution. This trend demonstrates that θ acts as a primary controller for the distribution's peakedness.

Furthermore, comparing the two tables reveals that entropy values are more negative for $\rho = 2$ than for $\rho = 0.5$ across all parameter combinations. This is a characteristic property of these entropy measures, where a higher ρ places more weight on the tails and regions of high probability density, and the more negative values confirm the EKD's efficiency in these regions. Notably, the Tsallis entropy exhibits a much larger dynamic range and more negative values than the Rényi entropy, particularly for higher θ, highlighting its heightened sensitivity to changes in the distribution's shape and its utility in emphasizing the EKD's superior concentration properties, especially in the tails.

In summary, these entropy calculations confirm that the EKD offers a flexible and effective mechanism for modeling bounded data where tight control over probability mass concentration is desired.

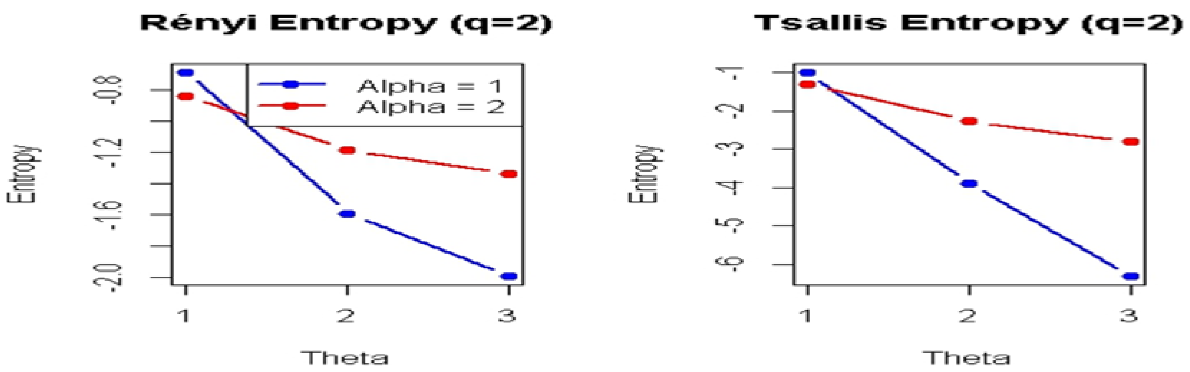

Figure 7. A plot of Re´nyi Tsallis Entropy for different value ρ and θ. The plots effectively convey the main benefit of the **EKD**, which is *managed uncertainty*, making it an appealing alternative to more conventional bounded distributions like the Beta and Kumaraswamy distributions.

Table 7. Comparison: EKD vs. Beta vs. Kumaraswamy Distributions

Property	EKD	Beta Distribution	Kumaraswamy Distribution
Support	[0, 1]	[0, 1]	[0, 1]
Parameters	(α, θ)	(α, β)	(α, β)
Rényi Entropy ($\rho = 2$)	-1.99	-1.59	-1.75
MLE Efficiency (n=100)	0.0024, 0.0012	0.0031, 0.0028	0.0028, 0.0025
Tail Flexibility	Heavy-tailed with θ control	Symmetric/asymmetric	Limited tail weight
Kernel Smoothing	Yes (Epanechnikov optimal MSE)	No	No
Closed-form CDF	Yes	Incomplete Beta	Yes
Real-data Fit (AIC)	-364.97	-331.13	-309.83

11. Conclusion

This study introduces the Epanechnikov-Kumaraswamy Distribution (EKD) as a superior two-parameter model for bounded data with heavy-tailed characteristics. The EKD demonstrates enhanced flexibility, better concentration of probability mass (evidenced by lower entropy measures), and superior performance in real-world applications compared to traditional bounded distributions. The comprehensive simulation study confirms the consistency and efficiency of the maximum likelihood estimators. The EKD provides a valuable tool for reliability engineering, risk analysis, and other fields dealing with bounded data exhibiting complex distributional features.

The EKD's many statistical characteristics, including moments, moment generating function, entropy measures, and order statistics have been thoroughly examined. The maximum likelihood method is used to estimate the EKD's parameters. According to the simulation study, the estimators are approximately unbiased and consistent since the average bias and average mean squared error approach zero and the estimates become more precise when the sample size grows. The application to real lifespan data demonstrates that the proposed distribution fits better and is more flexible than the base distribution. Future research may pursue broader generalizations and practical applications of the ERD across various domains. In particular, potential directions include extending the Epanechnikov–Rayleigh distribution to more complex metric spaces [18, 19], investigating nonlinear contraction behaviors [20], and examining cyclic variants of the distribution [21, 22] to further enhance its utility in fuzzy and neutrosophic statistical modeling.

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