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Even and Odd Antimagic Total Labeling on Total Graph of Paths and Cycles

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Abstract Consider a graph $G = (V, E)$ with n nodes and m links. In this paper, we introduce two types of labeling, namely odd-vertex(edge) antimagic total labeling and even-vertex(edge) antimagic total labeling. A vertex(edge) antimagic total labeling is called odd vertex(edge) antimagic total labeling if $\varphi(V(G)) = \{1, 3, ..., 2n-1\}$ and it is even vertex(edge) antimagic total labeling if $\varphi(V(G)) = \{2, 4, ..., 2n\}$. Here we discuss even and odd vertex antimagic total labeling and edge antimagic total labeling on total graph of paths and cycles.

Keywords VATL, EATL, Odd vertex(edge) antimagic total labeling, Even vertex(edge) antimagic total labeling, Totally Antimagic Total Labeling

AMS 2010 subject classifications 05C78

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1. Introduction

In this study, we focus on simple and undirected graphs. The vertex set and edge set of G are identified by $V = V(G)$ and $E = E(G)$ respectively. Let $|V(G)| = n$ and $|E(G)| = m$.

Essentially, the labeling φ of a graph G, is a bijection from $V \cup E$ to $\{1, 2, ..., n+m\}$. The vertex weight Λ_v of a vertex $x \in V$ is $\Lambda_v(x) = \varphi(x) + \sum \varphi(xy)$, where the sum varies over all vertices y incident to x and the edge weight Λ_e of an edge $xy \in E$ is $\Lambda_e(xy) = \varphi(x) + \varphi(xy) + \varphi(y)$ where $x, y \in V$. The labeling φ is a vertex antimagic total labeling(VATL) if $\Lambda_v(x)$ for all $x \in V$ are distinct, whereas it is an edge antimagic total labeling(EATL) if $\Lambda_e(xy)$ for all $xy \in E$ are distinct.

Further, if a total labeling is both vertex antimagic and edge antimagic then it is called totally antimagic total(TAT) labeling. A graph which admits TAT labeling is called a TAT graph. In this paper, we introduce new labelings such as odd-VATL, even-VATL, odd-EATL and even-EATL. A VATL is called odd-VATL if $\varphi(V(G)) = \{1, 3, ..., 2n-1\}$ and it is even-VATL if $\varphi(V(G)) = \{2, 4, ..., 2n\}$. Similarly, an EATL is called odd-EATL if $\varphi(V(G)) = \{1, 3, ..., 2n-1\}$ and it is even-EATL if $\varphi(V(G)) = \{2, 4, ..., 2n\}$. In addition a graph is an odd(even)-vertex TAT graph, if it has both odd(even)-VATL and odd(even)-EATL.

The concept of VMTL was initiated by MacDougall et al. [5], while Baca et al. [1] presented the idea of VATL and $(h, r)-$ VATL. In addition, the concept of EMTL and EATL was initiated by Kotzig et al. [4], while Simanjuntak et al. [11] presented the idea of $(h, r)-$ EATL. In [12] Stewart discussed the super-magic properties of complete graphs and complete bipartite graphs. Baca et al. [2] investigated the totally antimagic total labeling property for some class of graphs. Nagaraj et al. in [8], introduced odd VMTL of some graphs and in [9] & [6] they extended the hypothesis to odd (even) $(h, r)-$ VATL and $(h, r)-$ EATL by labeling the vertices with odd

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(even) numbers. J. Gallian's dynamic inspection of graph labeling [3] elaborates on the works done so far in graph labeling.

The total graph $T(G)$ of a graph G is a graph such that the vertex set of $T(G)$ corresponds to the vertices and edges of G and two vertices are adjacent in $T(G)$ iff their corresponding elements are either adjacent or incident in G. Let the number of vertices and edges of the total graph be denoted as s and t respectively. In this paper, we discuss the existence of odd(even)-VATL and odd(even)-EATL on total graph of paths and cycles.

2. Total Graph of Paths P_n

Let $\{v_1, v_2, ..., v_n\}$ and $\{e_i = v_iv_{i+1} : 1 \leq i \leq n-1\}$ be the vertex and edge sets of a path P_n respectively. Then we have $V(T[P_n]) = \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n-1\}$ and $E(T[P_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where $E_1 = \{v_iv_{i+1} : 1 \leq i \leq n-1\}, E_2 = \{v_ie_i : 1 \leq i \leq n-1\}, E_3 = \{v_ie_{i-1} : 2 \leq i \leq n\}$ and $E_4 = \{e_ie_{i+1} : 1 \leq i \leq n-2\}$. Clearly $s = |V(T[P_n])| = 2n-1$ and $t = |E(T[P_n])| = 4n-5$.

Theorem 2.1
For every path $P_n, n \geq 3$, the graph $T[P_n]$ is an odd-TAT graph.

Proof
Consider the graph $T[P_n]$ with vertex set $V(T[P_n]) = \{v_1, v_2, ..., v_n\} \cup \{e_1, e_2, ..., e_{n-1}\}$ and edge set $E(T[P_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where $E_1 = \{v_iv_{i+1} : 1 \leq i \leq n-1\}, E_2 = \{v_ie_i : 1 \leq i \leq n-1\}, E_3 = \{v_ie_{i-1} : 2 \leq i \leq n\}, E_4 = \{e_ie_{i+1} : 1 \leq i \leq n-2\}$.

Let us define φ as follows:

$$\begin{aligned}
\varphi(v_i) &= 4i-3, \text{ for } 1 \leq i \leq n \\
\varphi(e_i) &= 4i-1, \text{ for } 1 \leq i \leq n-1 \\
\varphi(v_iv_{i+1}) &= 6i-2, \text{ for } 1 \leq i \leq n-1 \\
\varphi(v_ie_i) &= 6i-4, \text{ for } 1 \leq i \leq n-1 \\
\varphi(v_ie_{i-1}) &= 6(i-1), \text{ for } 2 \leq i \leq n \\
\varphi(e_ie_{i+1}) &= 6n-2i-5, \text{ for } 1 \leq i \leq n-2
\end{aligned}$$

It is straight forward from the above labeling that the vertex labels form the set $\{1, 3, ..., 2s-1\}$ and the edge labels form the set $\{1, 2, ..., s+t\} \backslash \{1, 3, ..., 2s-1\}$. To complete the proof, we have to prove that $\Lambda_v(x)$ for all $x \in V(T[P_n])$ are distinct and $\Lambda_e(xy)$ for all $xy \in E(T[P_n])$ are distinct.

The vertex weights are as follows:

$$\begin{aligned}
\Lambda_v(v_1) &= \varphi(v_1) + \varphi(v_1v_2) + \varphi(v_1e_1) = 7 \\
\Lambda_v(v_n) &= \varphi(v_n) + \varphi(v_{n-1}v_n) + \varphi(v_ne_{n-1}) = 16n-17 \\
\Lambda_v(e_1) &= \varphi(e_1) + \varphi(e_1e_2) + \varphi(v_1e_1) + \varphi(v_2e_1) = 6n+4 \\
\Lambda_v(e_{n-1}) &= \varphi(e_{n-1}) + \varphi(e_{n-2}e_{n-1}) + \varphi(v_ne_{n-1}) + \varphi(v_{n-1}e_{n-1}) = 20n-22
\end{aligned}$$

For $2 \leq i \leq n-1$

$$\Lambda_v(v_i) = \varphi(v_i) + \varphi(v_{i-1}v_i) + \varphi(v_iv_{i+1}) + \varphi(v_ie_i) + \varphi(v_ie_{i-1}) = 28i-23$$

For $2 \leq i \leq n-2$

$$\Lambda_v(e_i) = \varphi(e_i) + \varphi(v_ie_i) + \varphi(v_{i+1}e_i) + \varphi(e_{i-1}e_i) + \varphi(e_ie_{i+1}) = 12n+12i-13$$

One can easily verify that the vertex weights are distinct and hence we can say that $T[P_n]$ has an odd-VATL.

The edge weights are as follows:

$$\begin{aligned}
&\text{For } 1 \le i \le n-1,\\
\Lambda_e(v_iv_{i+1}) &= \varphi(v_i) + \varphi(v_iv_{i+1}) + \varphi(v_{i+1}) = 14i - 4\\
&\text{For } 1 \le i \le n-1,\\
\Lambda_e(v_ie_i) &= \varphi(v_i) + \varphi(v_ie_i) + \varphi(e_i) = 14i - 8\\
&\text{For } 2 \le i \le n,\\
\Lambda_e(v_ie_{i-1}) &= \varphi(v_i) + \varphi(v_ie_{i-1}) + \varphi(e_{i-1}) = 14i - 14\\
&\text{For } 1 \le i \le n-2,\\
\Lambda_e(e_ie_{i+1}) &= \varphi(e_i) + \varphi(e_ie_{i+1}) + \varphi(e_{i+1}) = 6n + 6i - 3
\end{aligned}$$

One can easily verify that the edge weights are distinct and hence we can say that $T[P_n]$ has an odd-EATL. Hence we can conclude that $T[P_n]$ is an odd-TAT graph. □

Example 2.2
The following example demonstrates the existence of TAT labeling of the total graph of the path P_6. The vertices and edges in the table are arranged in ascending order of the vertex weighs and edge weights, respectively.

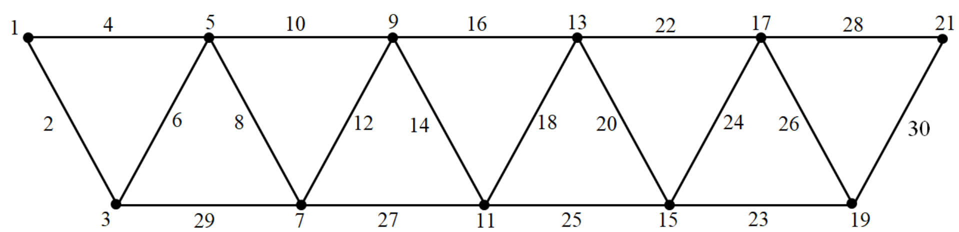

Figure 1. Odd-TAT Labeling of $T[P_6]$

Vertex Label	Vertex Weight	Edge Label	Edge Weight
1	7	2	6
5	33	4	10
3	40	6	14
9	61	8	20
21	79	10	24
7	83	12	28
13	89	14	34
11	95	16	38
19	98	29	39
15	107	18	42
17	117	27	45
		20	48
		25	51
		22	52
		24	56
		23	57
		26	62
		28	66
		30	70

Table 1. Labels and their corresponding weights of vertices and edges of Example 2.2

Theorem 2.3
For every path $P_n, n \geq 3$ the graph $T[P_n]$ is an even-TAT graph.

Proof
Consider the graph $T[P_n]$ with vertex set $V(T[P_n]) = \{v_1, v_2, ..., v_n\} \cup \{e_1, e_2, ..., e_{n-1}\}$ and edge set $E(T[P_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where $E_1 = \{v_i v_{i+1} : 1 \leq i \leq n-1\}, E_2 = \{v_i e_i : 1 \leq i \leq n-1\}, E_3 = \{v_i e_{i-1} : 2 \leq i \leq n\}, E_4 = \{e_i e_{i+1} : 1 \leq i \leq n-2\}$.

Let us define φ as follows:

$$\begin{aligned}
\varphi(v_i) &= 4i-2, \text{ for } 1 \leq i \leq n \\
\varphi(e_i) &= 4i, \text{ for } 1 \leq i \leq n-1 \\
\varphi(v_i v_{i+1}) &= 6i-5, \text{ for } 1 \leq i \leq n-1 \\
\varphi(v_i e_i) &= 6i-1, \text{ for } 1 \leq i \leq n-1 \\
\varphi(v_i e_{i-1}) &= 6i-9, \text{ for } 2 \leq i \leq n \\
\varphi(e_i e_{i+1}) &= 6n-2i-4, \text{ for } 1 \leq i \leq n-2
\end{aligned}$$

It is straight forward from the above labeling that the vertex labels form the set $\{2, 4, ..., 2s\}$ and the edge labels form the set $\{1, 2, ..., s+t\} \backslash \{2, 4, ..., 2s\}$. To complete the proof, we have to prove that $\Lambda_v(x)$ for all $x \in V(T[P_n])$ are distinct and $\Lambda_e(xy)$ for all $xy \in E(T[P_n])$ are distinct.

The vertex weights are as follows:

$$\begin{aligned}
\Lambda_v(v_1) &= \varphi(v_1) + \varphi(v_1 v_2) + \varphi(v_1 e_1) = 8 \\
\Lambda_v(v_n) &= \varphi(v_n) + \varphi(v_{n-1} v_n) + \varphi(v_n e_{n-1}) = 16n - 22 \\
\Lambda_v(e_1) &= \varphi(e_1) + \varphi(e_1 e_2) + \varphi(v_1 e_1) + \varphi(v_2 e_1) = 6n + 6 \\
\Lambda_v(e_{n-1}) &= \varphi(e_{n-1}) + \varphi(e_{n-2} e_{n-1}) + \varphi(v_n e_{n-1}) + \varphi(v_{n-1} e_{n-1}) = 20n - 20
\end{aligned}$$

For $2 \leq i \leq n-1$

$$\Lambda_v(v_i) = \varphi(v_i) + \varphi(v_{i-1} v_i) + \varphi(v_i v_{i+1}) + \varphi(v_i e_i) + \varphi(v_i e_{i-1}) = 28i - 28$$

For $2 \leq i \leq n-2$

$$\Lambda_v(e_i) = \varphi(e_i) + \varphi(v_i e_i) + \varphi(v_{i+1} e_i) + \varphi(e_{i-1} e_i) + \varphi(e_i e_{i+1}) = 12n + 12i - 10$$

One can easily verify that the vertex weights are distinct and hence we can say that $T[P_n]$ has an even-VATL.

The edge weights are as follows:

$$\begin{aligned}
&\text{For } 1 \leq i \leq n-1, \\
\Lambda_e(v_i v_{i+1}) &= \varphi(v_i) + \varphi(v_i v_{i+1}) + \varphi(v_{i+1}) = 14i - 5 \\
&\text{For } 1 \leq i \leq n-1, \\
\Lambda_e(v_i e_i) &= \varphi(v_i) + \varphi(v_i e_i) + \varphi(e_i) = 14i - 3 \\
&\text{For } 2 \leq i \leq n, \\
\Lambda_e(v_i e_{i-1}) &= \varphi(v_i) + \varphi(v_i e_{i-1}) + \varphi(e_{i-1}) = 14i - 15 \\
&\text{For } 1 \leq i \leq n-2, \\
\Lambda_e(e_i e_{i+1}) &= \varphi(e_i) + \varphi(e_i e_{i+1}) + \varphi(e_{i+1}) = 6n + 6i
\end{aligned}$$

One can easily verify that the edge weights are distinct and hence we can say that $T[P_n]$ has an even-EATL. Hence we can conclude that $T[P_n]$ is an even-TAT graph. □

Example 2.4
The following example demonstrates the existence of TAT labeling of the total graph of the path P_4. The vertices and edges in the table are arranged in ascending order of the vertex weighs and edge weights, respectively.

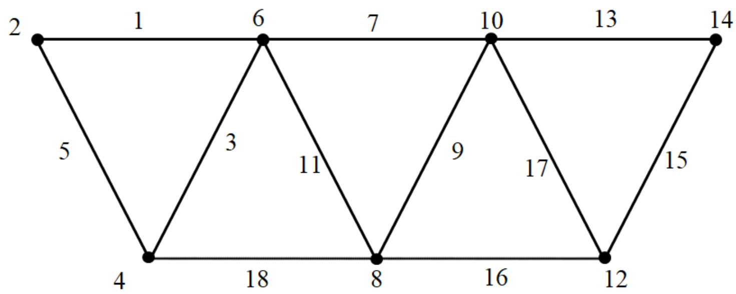

Figure 2. Even-TAT Labeling of $T[P_4]$

Vertex Label	Vertex Weight	Edge Label	Edge Weight
2	8	1	9
6	28	5	11
4	30	3	13
14	42	7	23
10	56	11	25
12	60	9	27
8	62	18	30
		16	36
		13	37
		17	39
		15	41

Table 2. Labels and their corresponding weights of vertices and edges of Example 2.4

3. Total Graph of Cycles C_n

Let $\{v_1, v_2, ..., v_n\}$ and $\{e_i = v_i v_{i+1} : 1 \leq i \leq n\}$(where $i-1$ is n if $i = 1$ and $i+1$ is 1 if $i = n$) be the vertex and edge sets of a cycle C_n, respectively. Then we have $V(T[C_n]) = \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$ and $E(T[C_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where $E_1 = \{v_i v_{i+1} : 1 \leq i \leq n\}, E_2 = \{v_i e_i : 1 \leq i \leq n\}, E_3 = \{v_{i+1} e_i : 1 \leq i \leq n\}$ and $E_4 = \{e_i e_{i+1} : 1 \leq i \leq n\}$. Clearly $s = |V(T[C_n])| = 2n$ and $t = |E(T[P_n])| = 4n$.

Theorem 3.1
For every cycle $C_n, n \geq 3$, the graph $T[C_n]$ is an odd-TAT graph.

Proof
Consider the graph $T[C_n]$ with vertex set $V(T[C_n]) = \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$ and edge set $E(T[C_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where $E_1 = \{v_i v_{i+1} : 1 \leq i \leq n\}, E_2 = \{v_i e_i : 1 \leq i \leq n\}, E_3 = \{v_{i+1} e_i : 1 \leq i \leq n\}$ and $E_4 = \{e_i e_{i+1} : 1 \leq i \leq n\}$.

Let us define φ as follows:

$$\begin{aligned}
\varphi(v_i) &= 2i-1, \text{ for } 1 \leq i \leq n \\
\varphi(e_i) &= 2n+2i-1, \text{ for } 1 \leq i \leq n \\
\varphi(v_iv_{i+1}) &= 2n-2i+2, \text{ for } 1 \leq i \leq n \\
\varphi(v_ie_i) &= 2n+2i, \text{ for } 1 \leq i \leq n \\
\varphi(v_{i+1}e_i) &= 6n+1-i, \text{ for } 1 \leq i \leq n \\
\varphi(e_ie_{i+1}) &= 5n+1-i, \text{ for } 1 \leq i \leq n
\end{aligned}$$

It is straight forward from the above labeling that the vertex labels form the set $\{1, 3, ..., 2s-1\}$ and the edge labels form the set $\{1, 2, ..., s+t\} \backslash \{1, 3, ..., 2s-1\}$. To complete the proof, we have to prove that $\Lambda_v(x)$ for all $x \in V(T[C_n])$ are distinct and $\Lambda_e(xy)$ for all $xy \in E(T[C_n])$ are distinct.

$$\begin{aligned}
\Lambda_v(v_1) &= \varphi(v_1) + \varphi(v_1v_2) + \varphi(v_1e_1) + \varphi(v_nv_1) + \varphi(v_1e_n) \\
&= 9n+6 \\
\Lambda_v(e_1) &= \varphi(e_1) + \varphi(e_1e_2) + \varphi(v_2e_1) + \varphi(e_ne_1) + \varphi(v_1e_1) \\
&= 19n+4
\end{aligned}$$

For $2 \leq i \leq n$

$$\begin{aligned}
\Lambda_v(v_i) &= \varphi(v_i) + \varphi(v_{i-1}v_i) + \varphi(v_iv_{i+1}) + \varphi(v_ie_i) + \varphi(v_ie_{i-1}) \\
&= 12n-i+7 \\
\Lambda_v(e_i) &= \varphi(e_i) + \varphi(v_ie_i) + \varphi(v_{i+1}e_i) + \varphi(e_{i-1}e_i) + \varphi(e_ie_{i+1}) \\
&= 20n+i+3
\end{aligned}$$

One can easily verify that the vertex weights are distinct and hence we can say that $T[C_n]$ has an odd-VATL.

The edge weights are as follows:

$$\begin{aligned}
\Lambda_e(v_nv_1) &= \varphi(v_n) + \varphi(v_nv_1) + \varphi(v_1) \\
&= 2n+2 \\
\text{For } 1 \leq i \leq n-1, & \\
\Lambda_e(v_iv_{i+1}) &= \varphi(v_i) + \varphi(v_iv_{i+1}) + \varphi(v_{i+1}) \\
&= 2n+2i+2 \\
\text{For } 1 \leq i \leq n, & \\
\Lambda_e(v_ie_i) &= \varphi(v_i) + \varphi(v_ie_i) + \varphi(e_i) \\
&= 4n+6i-2 \\
\Lambda_e(v_{i+1}e_i) &= \varphi(v_{i+1}) + \varphi(v_{i+1}e_i) + \varphi(e_i) \\
&= 8n+3i+1 \\
\Lambda_e(e_ie_{i+1}) &= \varphi(e_i) + \varphi(e_ie_{i+1}) + \varphi(e_{i+1}) \\
&= 9n+3i+1
\end{aligned}$$

One can easily verify that the edge weights are distinct and hence we can say that $T[C_n]$ has an odd-EATL. Hence, we can conclude that $T[C_n]$ is an odd-TAT graph. □

Example 3.2

The following example demonstrates the existence of TAT labeling of the total graph of the path C_5. The vertices and edges in the table are arranged in ascending order of the vertex weighs and edge weights, respectively.

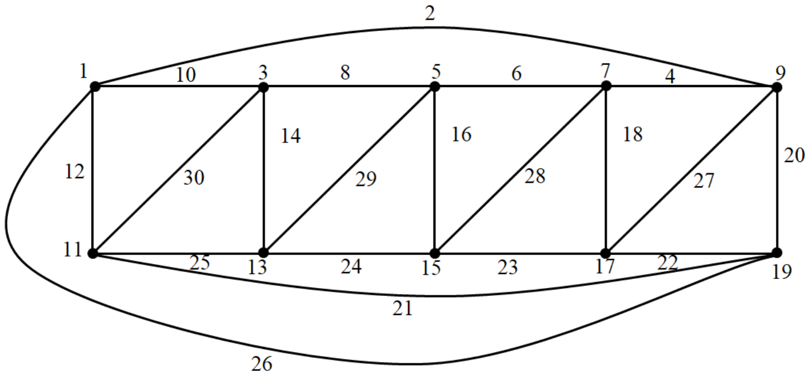

Figure 3. Odd-TAT Labeling of $T[C_5]$

Vertex Label	**Vertex Weight**	**Edge Label**	**Edge Weight**
1	51	2	12
9	62	10	14
7	63	8	16
5	64	6	18
3	65	4	20
11	99	12	24
13	105	14	30
15	106	16	36
17	107	18	42
19	108	30	44
		26	46
		29	47
		20	48
		25	49
		28	50
		21	51
		24	52
		27	53
		23	55
		22	58

Table 3. Labels and their corresponding weights of vertices and edges of Example 3.2

Theorem 3.3
For every cycle $C_n, n \geq 3$, the graph $T[C_n]$ is an even-TAT graph.

Proof
Consider the graph $T[C_n]$ with vertex set $V(T[C_n]) = \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$ and edge set $E(T[C_n]) = E_1 \cup E_2 \cup E_3 \cup E_4$ where $E_1 = \{v_i v_{i+1} : 1 \leq i \leq n\}, E_2 = \{v_i e_i : 1 \leq i \leq n\}, E_3 = \{v_{i+1} e_i : 1 \leq i \leq n\}$ and $E_4 = \{e_i e_{i+1} : 1 \leq i \leq n\}$.

Let us define φ as follows:

$$\begin{aligned}
\varphi(v_i) &= 2i, \text{ for } 1 \leq i \leq n \\
\varphi(e_i) &= 2n + 2i, \text{ for } 1 \leq i \leq n \\
\varphi(v_i v_{i+1}) &= 2i - 1, \text{ for } 1 \leq i \leq n \\
\varphi(v_i e_i) &= 4n + 1 - 2i, \text{ for } 1 \leq i \leq n \\
\varphi(v_{i+1} e_i) &= 6n + 1 - i, \text{ for } 1 \leq i \leq n \\
\varphi(e_i e_{i+1}) &= 5n + 1 - i, \text{ for } 1 \leq i \leq n
\end{aligned}$$

It is straight forward from the above labeling that the vertex labels form the set $\{2, 4, ..., 2s\}$ and the edge labels form the set $\{1, 2, ..., s+t\} \backslash \{2, 4, ..., 2s\}$. To complete the proof, we have to prove that $\Lambda_v(x)$ for all $x \in V(T[C_n])$ are distinct and $\Lambda_e(xy)$ for all $xy \in E(T[C_n])$ are distinct.

$$\begin{aligned}
\Lambda_v(v_1) &= \varphi(v_1) + \varphi(v_1 v_2) + \varphi(v_1 e_1) + \varphi(v_n v_1) + \varphi(v_1 e_n) \\
&= 9n + 10 \\
\Lambda_v(e_1) &= \varphi(e_1) + \varphi(e_1 e_2) + \varphi(v_2 e_1) + \varphi(e_n e_1) + \varphi(v_1 e_1) \\
&= 22n - 2
\end{aligned}$$

For $2 \leq i \leq n$

$$\begin{aligned}
\Lambda_v(v_i) &= \varphi(v_i) + \varphi(v_{i-1} v_i) + \varphi(v_i v_{i+1}) + \varphi(v_i e_i) + \varphi(v_i e_{i-1}) \\
&= 10n + 3i - 1 \\
\Lambda_v(e_i) &= \varphi(e_i) + \varphi(v_i e_i) + \varphi(v_{i+1} e_i) + \varphi(e_{i-1} e_i) + \varphi(e_i e_{i+1}) \\
&= 22n - 3i + 5
\end{aligned}$$

One can easily verify that the vertex weights are distinct and hence we can say that $T[C_n]$ has an even-VATL.

The edge weights are as follows:

$$\begin{aligned}
\Lambda_e(v_n v_1) &= \varphi(v_n) + \varphi(v_n v_1) + \varphi(v_1) \\
&= 4n + 1 \\
\Lambda_e(v_n e_n) &= \varphi(v_n) + \varphi(v_n e_n) + \varphi(e_n) \\
&= 8n + 1 \\
\Lambda_e(v_1 e_n) &= \varphi(v_1) + \varphi(v_1 e_n) + \varphi(e_n) \\
&= 9n + 3 \\
\Lambda_e(e_n e_1) &= \varphi(e_n) + \varphi(e_n e_1) + \varphi(e_1) \\
&= 10n + 3
\end{aligned}$$

For $1 \leq i \leq n - 1$,

$$\begin{aligned}
\Lambda_e(v_i v_{i+1}) &= \varphi(v_i) + \varphi(v_i v_{i+1}) + \varphi(v_{i+1}) \\
&= 6i + 1 \\
\Lambda_e(v_i e_i) &= \varphi(v_i) + \varphi(v_i e_i) + \varphi(e_i) \\
&= 6n + 2i + 1 \\
\Lambda_e(v_{i+1} e_i) &= \varphi(v_{i+1}) + \varphi(v_{i+1} e_i) + \varphi(e_i) \\
&= 8n + 3i + 3 \\
\Lambda_e(e_i e_{i+1}) &= \varphi(e_i) + \varphi(e_i e_{i+1}) + \varphi(e_{i+1}) \\
&= 9n + 3i + 3
\end{aligned}$$

One can easily verify that the edge weights are distinct and hence we can say that $T[C_n]$ has an even-EATL. Hence, we can conclude that $T[C_n]$ is an even-TAT graph. □

Example 3.4
The following example demonstrates the existence of TAT labeling of the total graph of the path C_4. The vertices and edges in the table are arranged in ascending order of the vertex weighs and edge weights, respectively.

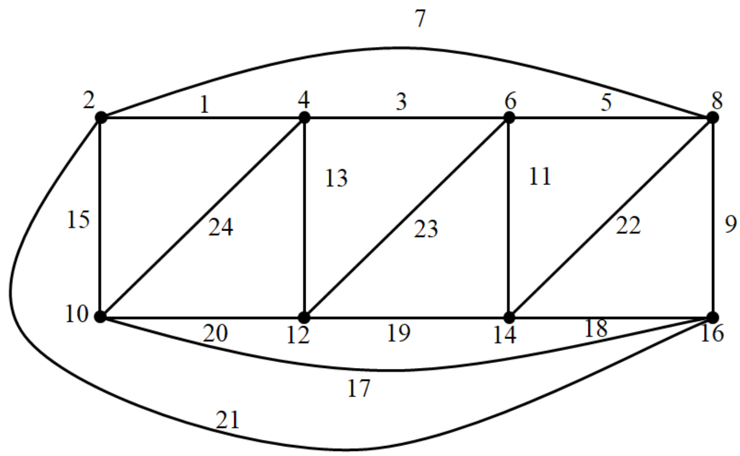

Figure 4. Even-TAT Labeling of $T[C_4]$

Vertex Label	Vertex Weight	Edge Label	Edge Weight
4	45	1	7
2	46	3	13
6	48	7	17
8	51	5	19
16	81	15	27
14	84	13	29
10	86	11	31
12	87	9	33
		24	38
		21	39
		23	41
		20	42
		17	43
		22	44
		19	45
		18	48

Table 4. Labels and their corresponding weights of vertices and edges of Example 3.4

4. Conclusion

In this paper, we discuss the odd and even TAT labeling of total graph of paths and cycles. Moreover, the same idea can be extended for some other graphs like sun graphs, complete graphs, and so on.

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