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On Fixed Points of Tirado-Type Contractions in k-Neutrosophic Metric Spaces

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Abstract This paper introduces the concept of a generalized k-Neutrosophic Metric Spaces (abbreviated k-NMS) and its intrinsic properties. Tirado-type k-neutrosophic contraction mappings are a new class of contraction mappings that apply the concepts of classical contraction to this new framework. The generalized naturalness of k-neutrosophic metric spaces is illustrated by an example, and a fixed-point theorem is established under the G-completeness condition. By connecting and expanding fixed-point theories in broader contexts, these discoveries close this gap.

Keywords Fuzzy Metric Spaces; k-Neutrosophic Metric Spaces; k-Neutrosophic Tirado-type contractive mapping; Fixed Point.

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1. Introduction

Atanassov introduced the concept of intuitionistic fuzzy sets, which expanded the framework of fuzzy sets by incorporating the degree of hesitation alongside membership and non-membership [1]. This significant advancement has inspired further research in generalized metric spaces and fixed-point theory.

George explored fuzzy metric spaces and established foundational results that became integral to studies in this domain [2]. Kramosil developed the concept of fuzzy metrics, linking them with statistical metric spaces to provide a broader understanding of their theoretical underpinnings [9]. Poovaragavan extended this work to multidimensional common fixed-point theorems and contraction in V-fuzzy metric spaces, providing new avenues for research [5, 6, 10, 11].

Gopal investigated k-NMS, presenting the first contraction principle in such spaces, which has broadened the application of fixed-point theorems [3]. Johnsy contributed to generalized NMS, particularly focusing on fixed-point results for $(\psi - \phi)$-contractions [7].

Huang explored fuzzy f-contractions and established fixed-point theorems in fuzzy metric spaces, highlighting their practical applications in mathematics and engineering [4]. Park introduced intuitionistic fuzzy metric spaces,

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providing an extension to classical metric spaces and enabling more robust modeling of uncertainty [12]. Patel et al. [13] explored k-fuzzy metric spaces and their applications, providing valuable insights into the theoretical and practical aspects of these spaces in nonlinear analysis.

Sowndrarajan expanded fixed-point theory to NMS, demonstrating the potential of contraction theorems in this context [14]. Simsek further advanced the study of NMS by proving fixed-point results specific to these spaces [15]. Kirisci developed a comprehensive theory of NMS, focusing on their unique properties and applications [8].

Smarandache introduced neutrosophy, a generalized theory for dealing with uncertainty, which has been applied extensively in various mathematical frameworks [16]. Vasuki contributed significantly to the study of Cauchy sequences and fixed-point theorems in fuzzy metric spaces, establishing results that have found broad applicability [17].

Wardowski explored fuzzy contractive mappings, providing new insights into fixed points in fuzzy metric spaces and their practical applications [18]. Finally, Zadeh's seminal work on fuzzy sets laid the foundation for all subsequent developments in this domain [19]. In this present paper, the authors investigate Tirado-type contractions and their fixed-point results within the framework of k-NMS. The study extends existing fixed-point theorems by introducing these contractions, which are useful in handling imprecision and uncertainty inherent in neutrosophic spaces. The findings offer new insights into applying these contraction mappings in various real-world problems where ambiguity and partial truth are present.

2. Preliminaries

Definition 2.1. [2] A transformation $\odot$ defined on $[0,1]^2$ to $[0,1]$ is referred to as a continuous triangular norm (CTN) if it meets the subsequent assertions:
(i) **Commutativity and Associativity:** $\odot(e,f) = \odot(f,e)$ and $\odot(\odot(e,f),g) = \odot(e,\odot(f,g))$ for all $e,f,g \in [0,1]$,
(ii) **Continuity:** $\odot$ is a continuous function;
(iii) **Neutral Element:** $1 \odot e = e$ for all $e \in [0,1]$,
(iv) **Monotonicity:** $e \odot f \leq g \odot h$ whenever $e \leq g$ and $f \leq h$ for all $e,f,g,h \in [0,1]$.

Definition 2.2. [12] A transformation $\oplus$ defined on $[0,1] \times [0,1]$ to $[0,1]$ is referred to as a continuous triangular conorm (CTCN) if it meets the subsequent assertions:
(i) **Commutativity and Associativity:** $\oplus(e,f) = \oplus(f,e)$ and $\oplus(\oplus(e,f),g) = \oplus(e,\oplus(f,g))$ for all $e,f,g \in [0,1]$,
(ii) **Continuity:** $\oplus$ is a continuous function,
(iii) **Neutral Element:** $0 \oplus e = e$ for all $e \in [0,1]$,
(iv) **Monotonicity:** $e \oplus f \geq g \oplus h$ whenever $e \geq g$ and $f \geq h$, for all $e,f,g,h \in [0,1]$.

Examples of t-norms and t-conorms include:
(i) $e \odot f = e \cdot f, \quad e \oplus f = \min\{e,f\}$,
(ii) $e \odot f = \min\{e,f\}, \quad e \oplus f = \max\{e,f\}$,
(iii) $e \odot f = \max\{e+f-1, 0\}, \quad e \oplus f = \min\{e+f, 1\}$.

Definition 2.3. Let U be a universe of discourse. A neutrosophic set A in U is defined as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in U\},$$

where

- $T_A(x)$ denotes the degree of truth-membership of the element x in A,
- $I_A(x)$ denotes the degree of indeterminacy-membership of the element x in A,
- $F_A(x)$ denotes the degree of falsity-membership of the element x in A.

Here,

$$T_A(x),\ I_A(x),\ F_A(x) \subseteq [0,1],$$

and they are independent, i.e.,

$$0 \leq \inf T_A(x) + \inf I_A(x) + \inf F_A(x) \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3.$$

Definition 2.4. An ordered triple $(A, B_y, \odot)$ is a FMS if A is nonempty. $\odot$ is a CTN. B_y is a FS on $A^2 \times (0, \infty)$ fulfilling the following claims, for all $\varphi, \iota, k \in A$ and $\lambda, s > 0$
(i) $B_y(\varphi, \iota, \lambda) > 0$,
(ii) $B_y(\varphi, \iota, \lambda) = 1$ if and only if $\varphi = \iota$,
(iii) $B_y(\varphi, \iota, \lambda) = B_y(\iota, \varphi, \lambda)$,
(iv) $B_y(\varphi, \iota, \lambda) \odot B_y(\iota, k, s) \leq B_y(\varphi, k, \lambda + s)$,
(v) $B_y(\varphi, \iota, .) : (0, +\infty) \to [0, 1]$ is continuous.

Definition 2.5. A IFMS is a 5-tuple $(A, B_y, D_y, \odot, \oplus)$ where A is a non-empty set. $\odot$ and $\oplus$ are CTN and CTCN respectively. B_y, D_y be FS, on $A^2 \times (0, +\infty)$ fulfills the following assertions, for all $\varphi, \iota \in A$ and $\lambda, s > 0$,
(i) $B_y(\varphi, \iota, \lambda) + D_y(\varphi, \iota, \lambda) \leq 1$,
(ii) $B_y(\varphi, \iota, \lambda) > 0$,
(iii) $B_y(\varphi, \iota, \lambda) = 1 \Leftrightarrow \varphi = \iota$,
(iv) $B_y(\varphi, \iota, \lambda) = B_y(\iota, \varphi, \lambda)$,
(v) $B_y(\varphi, k, \lambda + s) \leq B_y(\varphi, \iota, \lambda) \odot B_y(\iota, k, s)$,
(vi) $B_y(\varphi, \iota, .) : (0, \infty) \to [0, 1]$ is continuous mapping and $\lim\limits_{\lambda \to \infty} B_y(\varphi, \iota, \lambda) = 1$ for all $\lambda > 0$,
(vii) $D_y(\varphi, \iota, \lambda) > 0$,
(viii) $D_y(\varphi, \iota, \lambda) = 0 \Leftrightarrow \varphi = \iota$,
(ix) $D_y(\varphi, \iota, \lambda) = D_y(\iota, \varphi, \lambda)$,
(x) $D_y(\varphi, k, \lambda + s) \leq D_y(\varphi, \iota, \lambda) \oplus D_y(\iota, k, s)$,
(xi) $D_y(\varphi, \iota, .)$:The mapping from $(0, \infty) \to [0, 1]$ is continuous.and $\lim\limits_{\lambda \to \infty} D_y(\varphi, \iota, r) = 0$ for all $r > 0$.
Then $(A, B_y, D_y, \odot, \oplus)$ is an IFMS.

Definition 2.6. [14] A NMS is a 6-tuple $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$, where: A is a non-empty set. $\odot$ and $\oplus$ are CTN and CTCN, respectively. $\mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}$ are neutrosophic sets defined on $\mathrm{A} \times \mathrm{A} \times (0, +\infty)$ mapping to $[0, 1]$. These components meets the subsequent assertions for all $\varphi, \iota, \kappa \in \mathrm{A}$ and $\lambda, s > 0$:
(i) $\mathrm{B_Y}(\varphi, \iota, \lambda) + \mathrm{D_Y}(\varphi, \iota, \lambda) + \mathrm{E_Y}(\varphi, \iota, \lambda) \leq 3$,
(ii) $\mathrm{B_Y}(\varphi, \iota, \lambda) > 0$,
(iii) $\mathrm{B_Y}(\varphi, \iota, \lambda) = 1 \Leftrightarrow \varphi = \iota$,
(iv) $\mathrm{B_Y}(\varphi, \iota, \lambda) = \mathrm{B_Y}(\iota, \varphi, \lambda)$,
(v) $\mathrm{B_Y}(\varphi, \kappa, \lambda + s) \leq \mathrm{B_Y}(\varphi, \iota, \lambda) \odot \mathrm{B_Y}(\iota, \kappa, s)$,
(vi) $\mathrm{B_Y}(\varphi, \iota, \cdot) : (0, \infty) \to [0, 1]$ is a continuous mapping,
(vii) $\mathrm{D_Y}(\varphi, \iota, \lambda) > 0$,
(viii) $\mathrm{D_Y}(\varphi, \iota, \lambda) = 1 \Leftrightarrow \varphi = \iota$,
(ix) $\mathrm{D_Y}(\varphi, \iota, \lambda) = \mathrm{D_Y}(\iota, \varphi, \lambda)$,
(x) $\mathrm{D_Y}(\varphi, \kappa, \lambda + s) \geq \mathrm{D_Y}(\varphi, \iota, \lambda) \oplus \mathrm{D_Y}(\iota, \kappa, s)$,
(xi) $\mathrm{D_Y}(\varphi, \iota, \cdot) : (0, \infty) \to [0, 1]$ is a continuous mapping,
(xii) $\mathrm{E_Y}(\varphi, \iota, \lambda) > 0$,
(xiii) $\mathrm{E_Y}(\varphi, \iota, \lambda) = 1 \Leftrightarrow \varphi = \iota$,
(xiv) $\mathrm{E_Y}(\varphi, \iota, \lambda) = \mathrm{E_Y}(\iota, \varphi, \lambda)$,
(xv) $\mathrm{E_Y}(\varphi, \kappa, \lambda + s) \geq \mathrm{E_Y}(\varphi, \iota, \lambda) \oplus \mathrm{E_Y}(\iota, \kappa, s)$,
(xvi) $\mathrm{E_Y}(\varphi, \iota, \cdot) : (0, \infty) \to [0, 1]$ is a continuous mapping.

Definition 2.7. A space $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{D_Y}, \odot, \oplus)$ is referred to as a natural NMS if, and only if, it meets the subsequent assertions: $\lim\limits_{\lambda \to +\infty} \mathrm{B_Y}(\varphi, \iota, \lambda) = 1$, $\lim\limits_{\lambda \to +\infty} \mathrm{D_Y}(\varphi, \iota, \lambda) = 0$, $\lim\limits_{\lambda \to +\infty} \mathrm{E_Y}(\varphi, \iota, \lambda) = 0$, for all $\varphi, \iota \in \mathrm{A}$.

Example 2.8. Let $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{D_Y}, \odot, \oplus)$ be a NMS. Define $\mathrm{B_Y}, \mathrm{D_Y}$ and $\mathrm{E_Y}$ on $\mathrm{A}^2 \times (0, +\infty)$ by

$$\mathrm{B_Y}(\varphi, \iota, \varpi(\varphi, \iota)) = \frac{1}{1 + \varpi(\varphi, \iota)^2}, \quad \mathrm{D_Y}(\varphi, \iota, \varpi(\varphi, \iota)) = \frac{\varpi(\varphi, \iota)}{1 + \varpi(\varphi, \iota)}, \quad \mathrm{E_Y}(\varphi, \iota, \varpi(\varphi, \iota)) = \frac{1}{1 + \varpi(\varphi, \iota)},$$

for all $\varphi, \iota > 0$. Then $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{D_Y}, \odot, \oplus)$ is a natural NMS.

George and Veeramani presented the idea of k-NMS, where $k \in \{1, 2, 3, \ldots\}$, Fuzzy metric spaces were expanded upon and generalized in 1994.. In this paradigm, the degree of nearness between two places is specified by k- parameters. The basic concept behind the k-neutrosophic metric is shown here.

Definition 2.9. A definite set A is specified, $\odot$ and $\oplus$ are CTN and CTCN, respectively, k is a positive integer, and $\mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}$ are the neutrosophic sets on $\mathrm{A}^2 \times (0, +\infty)^k$ known as k-Neutrosophic metric for every $\varphi, \iota, \kappa \in \mathrm{A}$ and $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$ meets the subsequent assertions:
(i) $\mathrm{B_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) + \mathrm{D_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) + \mathrm{E_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) \leq 3$,
(ii) $\mathrm{B_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) > 0$,
(iii) $\mathrm{B_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) = 1$ if and only if $\varphi = \iota$,
(iv) $\mathrm{B_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) = \mathrm{B_Y}(\iota, \varphi, \lambda_1, \lambda_2, \ldots, \lambda_k)$,
(v) For any $\mathrm{q} \in \{1, 2, 3, \ldots, k\}$, we have:

$$\begin{aligned}\mathrm{B_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{q}-1}, t, \lambda_{\mathrm{q}+1}, \ldots, \lambda_k) \odot \mathrm{B_Y}(\iota, \kappa, \lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{q}-1}, s, r_{\mathrm{q}+1}, \ldots, \lambda_k) \\ \leq \mathrm{B_Y}(\varphi, \kappa, \lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{q}-1}, t+s, \lambda_{\mathrm{q}+1}, \ldots, \lambda_k);\end{aligned}$$

(vi) $\mathrm{B_Y}(\varphi, \iota, \cdot) : (0, +\infty)^k \to [0, 1]$ is a continuous mapping,
(vii) $\mathrm{D_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) > 0$,
(viii) $\mathrm{D_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) = 1$ if and only if $\varphi = \iota$,
(ix) $\mathrm{D_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) = \mathrm{D_Y}(\iota, \varphi, \lambda_1, \lambda_2, \ldots, \lambda_k)$,
(x) For any $\mathrm{q} \in \{1, 2, 3, \ldots, k\}$, we have:

$$\begin{aligned}\mathrm{D_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{q}-1}, t, \lambda_{\mathrm{q}+1}, \ldots, \lambda_k) \oplus \mathrm{D_Y}(\iota, \kappa, \lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{q}-1}, s, \lambda_{\mathrm{q}+1}, \ldots, \lambda_k) \\ \geq \mathrm{D_Y}(\varphi, \kappa, \lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{q}-1}, t+s, \lambda_{\mathrm{q}+1}, \ldots, \lambda_k),\end{aligned}$$

(xi) $\mathrm{D_Y}(\varphi, \iota, \cdot) : (0, +\infty)^k \to [0, 1]$ is a continuous mapping,
(xii) $\mathrm{E_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) > 0$,
(xiii) $\mathrm{E_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) = 1$ if and only if $\varphi = \iota$,
(xiv) $\mathrm{E_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) = \mathrm{E_Y}(\iota, \varphi, \lambda_1, \lambda_2, \ldots, \lambda_k)$,
(xv) For any $\mathrm{q} \in \{1, 2, 3, \ldots, k\}$, we have:

$$\begin{aligned}\mathrm{E_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{q}-1}, t, \lambda \mathrm{q}+1, \ldots, \lambda_k) \oplus \mathrm{E_Y}(\iota, \kappa, \lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{q}-1}, s, \lambda_{\mathrm{q}+1}, \ldots, \lambda_k) \\ \geq \mathrm{E_Y}(\varphi, \kappa, \lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{q}-1}, t+s, \lambda_{\mathrm{q}+1}, \ldots, \lambda_k),\end{aligned}$$

(xvi) $\mathrm{E_Y}(\varphi, \iota, \cdot) : (0, +\infty)^k \to [0, 1]$ is a continuous mapping.
The notation of k-NMS is $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$.

Remark 2.10. A k-NMS reduces to a standard NMS when $k = 1$. To illustrate this, consider the following mathematical formulas for a k-neutrosophic metric in the context of examples. For $k = 1$, the calculations simplify to a neutrosophic metric where the components depend on the distance between elements and a parameter $\lambda_1 > 0$:

$$\mathrm{B_Y}(x_1, x_2, \lambda_1) = e^{-\frac{|\lambda_1|}{\lambda_1}}, \quad \mathrm{D_Y}(x_1, x_2, \lambda_1) = 1 - e^{-\frac{|\lambda_1|}{\lambda_1}}, \quad \mathrm{E_Y}(x_1, x_2, \lambda_1) = \frac{1}{1 + e^{-\frac{|\lambda_1|}{\lambda_1}}}.$$

These formulas represent the truth, falsity, and indeterminacy components of the neutrosophic metric, which depend explicitly on the parameter λ_1. Therefore, the k-NMS $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ transitions into a NMS when $k = 1$.

Example 2.11. The ϖ is a metric on A, $\odot, \oplus$ be CTN and CTCN, $w > 0$, and k be a positive integer. Describe: $\mathrm{B_Y}, \mathrm{D_Y}$ and $\mathrm{E_Y}$ on $\mathrm{A}^2 \times (0, +\infty)^k$ by

$$\mathrm{B_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) = \frac{w\lambda_1\lambda_2\cdots\lambda_k}{w\lambda_1\lambda_2\cdots\lambda_k + \varpi(\varphi,\iota)}$$
$$\mathrm{D_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) = \frac{\varpi(\varphi,\iota)}{w\lambda_1\lambda_2\cdots\lambda_k + \varpi(\varphi,\iota)}$$
$$\mathrm{E_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_k) = \frac{\varpi(\varphi,\iota)}{w\lambda_1\lambda_2\cdots\lambda_k + 2\varpi(\varphi,\iota)}$$

for all $\varphi, \iota \in \mathrm{A}$ and $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$. Then $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ is a k-NMS.

Definition 2.12. A k-NMS $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ known as q-natural k-NMS if there is $\mathrm{q} \in \{1, 2, \ldots, k\}$, corresponding to $\lim_{p_\mathrm{q}\to\infty} \mathrm{B_Y}(\delta, \upsilon, p_1, p_2, \ldots, p_k) = 1$, $\lim_{p_\mathrm{q}\to\infty} \mathrm{D_Y}(\delta, \upsilon, p_1, p_2, \ldots, p_k) = 0$, $\lim_{p_\mathrm{q}\to\infty} \mathrm{E_Y}(\delta, \upsilon, p_1, p_2, \ldots, p_k) = 0$, $\forall\, \delta, \upsilon \in \mathrm{A}$.

To keep things simple, we indicate $\mathrm{B_Y}(\delta, \upsilon, p_1, p_2, \ldots, p_k)$ by $\mathrm{B_Y}(\delta, \upsilon, p_1^k)$.

Example 2.13. Consider a set $\mathrm{A} = \mathbb{R}$ (the set of real numbers) equipped with the usual Euclidean distance ϖ, i.e., for any two points $a, b \in \mathbb{R}$, the distance between them is given by: $\varpi(a, b) = |a - b|$ Define:

$$\mathrm{B_Y}(a, b, p_1, p_2) = \frac{wp_1p_2}{wp_1p_2 + \varpi(a,b)}, \quad \mathrm{D_Y}(a, b, p_1, p_2) = \frac{\varpi(a,b)}{wp_1p_2 + \varpi(a,b)}, \quad \mathrm{E_Y}(a, b, p_1, p_2) = \frac{\varpi(a,b)}{wp_1p_2}.$$

where $w > 0$ is a constant, and $p_1, p_2 > 0$ are positive parameters associated with each of the 2 dimensions. Now, let p_1, p_2 be parameters that we will vary. According to the definition, the space is an q-natural 2-NMS if there exists some $\mathrm{q} \in \{1, 2\}$ such that:
$\lim_{p_\mathrm{q}\to\infty} \mathrm{B_Y}(a, b, p_1, p_2) = 1$, $\lim_{p_\mathrm{q}\to\infty} \mathrm{D_Y}(a, b, p_1, p_2) = 0$, $\lim_{p_\mathrm{q}\to\infty} \mathrm{E_Y}(a, b, p_1, p_2) = 0 \quad \forall\, a, b \in \mathrm{A}$.

Proposition 2.14. Assume that $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ is a k-NMS, each of $p, p_1, p_2, \ldots, p_k$ is strictly positive. While $p_\mathrm{q} < p$ for a few $\mathrm{q} \in \{1, 2, \ldots, k\}$, then

$$\mathrm{B_Y}(a, b, p_1^k) \leq \mathrm{B_Y}(a, b, p_1, p_2, \ldots, p_{\mathrm{q}-1}, p, p_{\mathrm{q}+1}, \ldots, p_k),$$
$$\mathrm{D_Y}(a, b, p_1^k) \geq \mathrm{D_Y}(a, b, p_1, p_2, \ldots, p_{\mathrm{q}-1}, p, p_{\mathrm{q}+1}, \ldots, p_k),$$
$$\mathrm{E_Y}(a, b, p_1^k) \geq \mathrm{E_Y}(a, b, p_1, p_2, \ldots, p_{\mathrm{q}-1}, p, p_{\mathrm{q}+1}, \ldots, p_k), \quad \forall\, a, b \in \mathrm{A}.$$

Definition 2.15. Assume that $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ is a k-NMS. A sequence $\{a_n\}$ in A is claimed to be convergent to a point a in A iff for each real $\epsilon \in (0, 1)$ there exists n_0 is a natural number as to ensure that $\mathrm{B_Y}\left(a_n, a, p_1^k\right) > 1 - \epsilon$, $\mathrm{D_Y}\left(a_n, a, p_1^k\right) < \epsilon$ and $\mathrm{E_Y}\left(a_n, a, p_1^k\right) < \epsilon$, for each n, $n_0 \leq n$ and $p_1, p_2, \ldots, p_k$ are all non-negative.

Example 2.16. Let $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ be a 2-NMS. Define:

$$\mathrm{B_Y}(\delta, \upsilon, p_1, p_2) = \frac{p_1p_2}{p_1p_2 + |\delta - \upsilon|}, \quad \mathrm{D_Y}(\delta, \upsilon, p_1, p_2) = \frac{|\delta - \upsilon|}{p_1p_2 + |\delta - \upsilon|}, \quad \mathrm{E_Y}(\delta, \upsilon, p_1, p_2) = \frac{|\delta - \upsilon|}{p_1p_2}.$$

Here, $p_1 = 1$, $p_2 = 2$, so $p_1p_2 = 2$. For $a_n = \frac{1}{n}$ and $a = 0$,

$$\mathrm{B_Y}(a_n, 0, p_1, p_2) = \frac{2}{2 + |a_n - 0|} = \frac{2}{2 + 1/n}.$$

As $n \to \infty$, $\mathrm{B_Y}(a_n, 0, p_1, p_2) \to \frac{2}{2} = 1$. For a specific $\epsilon = 0.1$, $1 - \epsilon = 0.9$. We need n_0 such that for $n \geq n_0$, $\mathrm{B_Y}(a_n, 0, p_1, p_2) > 0.9$.

$$\frac{2}{2+1/n} > 0.9 \implies 2 > 0.9(2+1/n) \implies 2 > 1.8 + 0.9/n \implies 0.2 > 0.9/n \implies n > \frac{0.9}{0.2}.$$

Thus, $n_0 \approx 5$. For $n \geq 5$, $\mathrm{B_Y}(a_n, 0, p_1, p_2) > 0.9$.

$$\mathrm{D_Y}(a_n, 0, p_1, p_2) = \frac{|a_n - 0|}{2 + |a_n - 0|} = \frac{1/n}{2+1/n}.$$

As $n \to \infty$, $\mathrm{D_Y}(a_n, 0, p_1, p_2) \to 0$. For $\epsilon = 0.1$, we need n_0 such that for $n \geq n_0$, $\mathrm{D_Y}(a_n, 0, p_1, p_2) < 0.1$.

$$\frac{1/n}{2+1/n} < 0.1 \implies \frac{1/n}{2} < 0.1 \implies \frac{1}{2n} < 0.1 \implies n > \frac{1}{0.2} = 5.$$

Thus, $n_0 \approx 5$. For $n \geq 5$, $\mathrm{D_Y}(a_n, 0, p_1, p_2) < 0.1$. And

$$\mathrm{E_Y}(a_n, 0, p_1, p_2) = \frac{|a_n - 0|}{p_1 p_2} = \frac{1/n}{2}.$$

As $n \to \infty$,$\mathrm{E_Y}(a_n, 0, p_1, p_2) \to 0$. For $\epsilon = 0.1$, we need n_0 such that for $n \geq n_0$,

$$\frac{1/n}{2} < 0.1 \implies \frac{1}{2n} < 0.1 \implies n > \frac{1}{0.2} = 5.$$

Thus, $n_0 \approx 5$. For $n \geq 5$, $\mathrm{E_Y}(a_n, 0, p_1, p_2) < 0.1$.

For $p_1 = 1$ and $p_2 = 2$, the sequence $\{a_n = \frac{1}{n}\}$ converges to $a = 0$ in the k-NMS because all three conditions (B, D, E) are satisfied for $n \geq n_0 = 5$.

Lemma 2.17. Assume that $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ is a k-NMS. A sequence $\{a_n\}$ in A converges to a point $a \in \mathrm{A}$ if and only if $\lim_{n\to\infty} \mathrm{B_Y}\left(a_n, a, p_1^k\right) = 1$, $\lim_{n\to\infty} \mathrm{D_Y}\left(a_n, a, p_1^k\right) = 0$, $\lim_{n\to\infty} \mathrm{E_Y}\left(a_n, a, p_1^k\right) = 0$, $p_1, p_2, \ldots, p_k$ are all non-negative.

Definition 2.18. Assume that $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ is a k-NMS, and $\{a_n\}$ be a sequence in A.
(i) $\{a_n\}$ is referred to as a M-Cauchy sequence if for each $\varepsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $\mathrm{B_Y}\left(a_m, a_n, p_1^k\right) > 1 - \varepsilon, \mathrm{D_Y}\left(a_m, a_n, p_1^k\right) < \varepsilon, \mathrm{E_Y}\left(a_m, a_n, p_1^k\right) < \varepsilon$, each $n, m \geq n_0$ and $p_1, p_2, \ldots, p_k$ are all non negative,
(ii) $\{a_n\}$ is referred to as a G-Cauchy sequence if $\lim_{n\to\infty} \mathrm{B_Y}\left(a_n, a_{n+p}, p_1^k\right) = 1$, $\lim_{n\to\infty} \mathrm{D_Y}\left(a_n, a_{n+p}, p_1^k\right) = 0$, $\lim_{n\to\infty} \mathrm{E_Y}\left(a_n, a_{n+p}, p_1^k\right) = 0$, $p_1, p_2, \ldots, p_k, p$ are all non negative.

Example 2.19. Consider the metric space $(\mathbb{R}, d)$, with $\odot$ and $\oplus$ defined as the product t-norm and t-conorm, respectively. Let $\mathrm{d} > 0$ and k be any non negative integer. Define fuzzy sets $\mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}$ on $\mathbb{R}^2 \times (0,\infty)^k$ as follows:

$$\mathrm{B_Y}(\delta, \upsilon, p_1^k) = \frac{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q}}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q} + d(\delta,\upsilon)}, \quad \mathrm{D_Y}(\delta, \upsilon, p_1^k) = \frac{d(\delta,\upsilon)}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q} + d(\delta,\upsilon)}, \quad \mathrm{E_Y}(\delta, \upsilon, p_1^k) = \frac{d(\delta,\upsilon)}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q}},$$

for all $\delta, \upsilon \in \mathbb{R}$ and $p_\mathrm{q} > 0$. These fuzzy sets $\mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}$ form a k-neutrosophic metric on $\mathbb{R}$.

Now, consider the sequence $\mathrm{P}_s = 1 + 1/2 + 1/3 + \cdots + 1/s$ for $s \in \mathbb{N}$. Then, we have:

$$\mathrm{B_Y}(\mathrm{P}_{s+p}, \mathrm{P}_s, p_1^k) = \frac{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q}}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q} + |\mathrm{P}_{s+p} - \mathrm{P}_s|} = \frac{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q}}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q} + \frac{1}{s+1} + \cdots + \frac{1}{s+p}},$$
$$\mathrm{D_Y}(\mathrm{P}_{s+p}, \mathrm{P}_s, p_1^k) = \frac{|\mathrm{P}_{s+p} - \mathrm{P}_s|}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q} + |\mathrm{P}_{s+p} - \mathrm{P}_s|} = \frac{\frac{1}{s+1} + \cdots + \frac{1}{s+p}}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q} + \frac{1}{s+1} + \cdots + \frac{1}{s+p}},$$
$$\mathrm{E_Y}(\mathrm{P}_{s+p}, \mathrm{P}_s, p_1^k) = \frac{|\mathrm{P}_{s+p} - \mathrm{P}_s|}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q}} = \frac{\frac{1}{s+1} + \cdots + \frac{1}{s+p}}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q}}.$$

As $s \to \infty$, it follows that:

$$\mathrm{B_Y}(\mathrm{P}_{s+p}, \mathrm{P}_s, p_1^k) \to 1, \quad \mathrm{D_Y}(\mathrm{P}_{s+p}, \mathrm{P}_s, p_1^k) \to 0, \quad \mathrm{E_Y}(\mathrm{P}_{s+p}, \mathrm{P}_s, p_1^k) \to 0,$$

for each $p > 0$. Thus, $\{\mathrm{P}_s\}$ is a G-Cauchy sequence but is clearly not an M-Cauchy sequence.
Assume, for contradiction, that $\{\mathrm{P}_s\}$ is an M-Cauchy sequence. Then:

$$\mathrm{B_Y}(\mathrm{P}_m, \mathrm{P}_s, p_1^k) = \frac{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q}}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q} + |\mathrm{P}_m - \mathrm{P}_s|},$$
$$\mathrm{D_Y}(\mathrm{P}_m, \mathrm{P}_s, p_1^k) = \frac{|\mathrm{P}_m - \mathrm{P}_s|}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q} + |\mathrm{P}_m - \mathrm{P}_s|},$$
$$\mathrm{E_Y}(\mathrm{P}_m, \mathrm{P}_s, p_1^k) = \frac{|\mathrm{P}_m - \mathrm{P}_s|}{\mathrm{d}\prod_{\mathrm{q}=1}^k p_\mathrm{q}}.$$

This means that if $\{\mathrm{P}_s\}$ is a Cauchy sequence in the standard metric space $(\mathbb{R}, d)$, then M-Cauchy. It is commonly known, nevertheless, that $|\mathrm{P}_m - \mathrm{P}_s| \approx \ln(m/s)$, which increases dramatically when $m > s$ Consequently, $\{\mathrm{P}_s\}$ is not Cauchy in $(\mathbb{R}, d)$, and as a result, it is not M-Cauchy in the k-NMS on $\mathbb{R}$.

Definition 2.20. Let $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ be a k-NMS.
(i) Each M-Cauchy sequence in A converges to a certain $\mathrm{P} \in \mathrm{A}$, then k-NMS is known as M-complete.
(ii) Each G-Cauchy sequence in A converges to some $\mathrm{P} \in \mathrm{A}$, then k-NMS is known as G-complete.

Remark 2.21. The M-completeness and G-completeness of a k-NMS are equal to the M-completeness and G-completeness of a NMS when $k = 1$.

3. Main Results

Definition 3.1. Let $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ is said to be generalized natural property of a k-NMS if there exist one or more than one parameter(s) $q_i \in \{1, 2, \ldots, k\}$ where $i = 1, 2, \ldots, m$, $m \leq k$ such that

$$\lim_{\lambda_{q_1}, \lambda_{q_2}, \ldots \lambda_{q_m} \to +\infty} \mathrm{B_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_{q_\imath}, \ldots, \lambda_k) = 1,$$
$$\lim_{\lambda_{q_1}, \lambda_{q_2}, \ldots \lambda_{q_m} \to +\infty} \mathrm{D_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_{q_\imath}, \ldots, \lambda_k) = 0,$$
$$\lim_{\lambda_{q_1}, \lambda_{q_2}, \ldots \lambda_{q_m} \to +\infty} \mathrm{E_Y}(\varphi, \iota, \lambda_1, \lambda_2, \ldots, \lambda_{q_\imath}, \ldots, \lambda_k) = 0, \text{for all } \varphi, \iota \in \mathrm{A}.$$

We can see an example of generalized naturalness of k-NMS.

Example 3.2. Let $\mathrm{A}=\mathbb{R}^k$, where k is a positive integer, $w>0$ and $\odot$ and $\oplus$ be the product t-norm. Define a fuzzy set $\mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}$ on $\mathrm{A}^2\times(0,+\infty)^k$ by

$$\lim_{\lambda_1,\lambda_2,\ldots,\lambda_{k-1}\to+\infty} \mathrm{B_Y}\left(\varphi,\iota,\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k\right)=\lim_{\lambda_1,\lambda_2,\ldots,\lambda_{k-1}\to+\infty} w\left[w+\sum_{i=1}^{k-1}\frac{|\varphi_i-\iota_i|}{\lambda_i}\right]^{-1}=1$$

$$\lim_{\lambda_1,\lambda_2,\ldots,\lambda_{k-1}\to+\infty} \mathrm{D_Y}\left(\varphi,\iota,\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k\right)=\lim_{\lambda_1,\lambda_2,\ldots,\lambda_{k-1}\to+\infty} w\left[w+\sum_{i=1}^{k-1}\frac{|\varphi_i-\iota_i|^2}{\lambda_i^2}\right]^{-1}=0$$

$$\lim_{\lambda_1,\lambda_2,\ldots,\lambda_{k-1}\to+\infty} \mathrm{E_Y}\left(\varphi,\iota,\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k\right)=\lim_{\lambda_1,\lambda_2,\ldots,\lambda_{k-1}\to+\infty} w\left[w+\sum_{i=1}^{k-1}\frac{|\varphi_i-\iota_i|}{\lambda_i^2}\right]^{-1}=0$$

for all $\varphi=(\varphi_1,\varphi_2,\ldots,\varphi_{k-1}),\iota=(\iota_1,\iota_2,\ldots,\iota_{k-1})\in\mathrm{A}$. Then $(\mathrm{A},\mathrm{B_Y},\mathrm{D_Y},\mathrm{E_Y},\odot,\oplus)$ has a generalized natural property of k-NMS.

Remark 3.3. If we put $m=1$ in Definition (3.1), the space reduces into q natural k-NMS.

Definition 3.4. Assume that $(\mathrm{A},\mathrm{B_Y},\mathrm{D_Y},\mathrm{E_Y},\odot,\oplus)$ is a k-NMS. Consider a mapping $\mathrm{P}:\mathrm{A}\to\mathrm{A}$ is called a Tirado-type k-neutrosophic contraction mapping if there exists a constant δ in $(0,1)$ such that, for every φ,ι in A with $\varphi\neq\iota$, and for parameters $\lambda_1,\lambda_2,\ldots,\lambda_k>0$,These disparities are true:

$$1-\mathrm{B_Y}\left(\mathrm{P}(\varphi),\mathrm{P}(\iota),\lambda_1^k\right)\leq\delta(1-\mathrm{B_Y}\left(\varphi,\iota,\lambda_1^k\right),\mathrm{D_Y}\left(\mathrm{P}(\varphi),\mathrm{P}(\iota),\lambda_1^k\right)\leq\delta\mathrm{D_Y}\left(\varphi,\iota,\lambda_1^k\right),\mathrm{E_Y}\left(\mathrm{P}(\varphi),\mathrm{P}(\iota),\lambda_1^k\right)\leq\delta\mathrm{E_Y}\left(\varphi,\iota,\lambda_1^k\right).$$

Here, λ_1^k represents the vector of parameters $(\lambda_1,\lambda_2,\ldots,\lambda_k)$, and the functions $\mathrm{B_Y}$, $\mathrm{D_Y}$, and $\mathrm{E_Y}$ are the neutrosophic metric functions defined on A. Under these conditions, we can establish a fixed-point result for P using the Tirado-type k-neutrosophic contraction principle.

Theorem 3.5. Let $\mathrm{P}:\mathrm{A}\to\mathrm{A}$ be a Tirado-type k-neutrosophic contraction mapping and $(\mathrm{A},\mathrm{B_Y},\mathrm{D_Y},\mathrm{E_Y},\odot,\oplus)$ be a G-complete k-NMS. Consequently, P has a unique fixed point.

Proof
Choose $\varphi_0\in\mathrm{A}$ be any arbitrary point. Construct a sequence $\{\varphi_n\}$ by Picard iteration method $\varphi_n=\mathrm{P}\varphi_{n-1}$ for each n in $\mathbb{N}\cup\{0\}$. It is necessary to demonstrate that this sequence is G-Cauchy. For anyone n in $\mathbb{N}\cup\{0\}$, we have

$$\begin{aligned}1-\mathrm{B_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)&\leq\delta(1-\left(\mathrm{B_Y}\left(\varphi_{n-1},\varphi_n,\lambda_1^k\right)\right)),\\ \mathrm{D_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)&\leq\delta\left(\mathrm{D_Y}\left(\varphi_{n-1},\varphi_n,\lambda_1^k\right)\right),\\ \mathrm{E_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)&\leq\delta\left(\mathrm{E_Y}\left(\varphi_{n-1},\varphi_n,\lambda_1^k\right)\right).\end{aligned}$$

By doing this repeatedly, we get

$$\begin{aligned}1-\mathrm{B_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)&\leq\delta^n(1-\left(\mathrm{B_Y}\left(\varphi_0,\varphi_1,\lambda_1^k\right)\right)),\\ \mathrm{D_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)&\leq\delta^n\left(\mathrm{D_Y}\left(\varphi_0,\varphi_1,\lambda_1^k\right)\right),\\ \mathrm{E_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)&\leq\delta^n\left(\mathrm{E_Y}\left(\varphi_0,\varphi_1,\lambda_1^k\right)\right).\end{aligned}\tag{3.5.1}$$

for each $n\in N$. As $n\to+\infty$ and since $\delta\in(0,1)$, we conclude by (3.5.1),

$$\begin{aligned}\lim_{n\to+\infty}1-\left(\mathrm{B_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)\right)&\leq\lim_{n\to+\infty}\delta^n(1-\left(\mathrm{B_Y}\left(\varphi_0,\varphi_1,\lambda_1^k\right)\right)),\\ \lim_{n\to+\infty}\left(\mathrm{D_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)\right)&\leq\lim_{n\to+\infty}\delta^n\left(\mathrm{D_Y}\left(\varphi_0,\varphi_1,\lambda_1^k\right)\right),\\ \lim_{n\to+\infty}\left(\mathrm{E_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)\right)&\leq\lim_{n\to+\infty}\delta^n\left(\mathrm{E_Y}\left(\varphi_0,\varphi_1,\lambda_1^k\right)\right).\end{aligned}$$

$$\lim_{n\to+\infty}1-\mathrm{B_Y}\left(\varphi_n,\varphi_{n+1},r_1^k\right)=0,\ \lim_{n\to+\infty}\mathrm{D_Y}\left(\varphi_n,\varphi_{n+1},r_1^k\right)=0,\ \lim_{n\to+\infty}\mathrm{E_Y}\left(\varphi_n,\varphi_{n+1},r_1^k\right)=0.$$

That is,

$$\lim_{n\to+\infty} \mathrm{B_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)=1,\ \lim_{n\to+\infty} \mathrm{D_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)=0,\ \lim_{n\to+\infty} \mathrm{E_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)=0. \tag{3.5.2}$$

for all $\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k>0$. For each $n\in\mathbb{N}$ and $p>0$,

$$\begin{aligned}
\mathrm{B_Y}\left(\varphi_n,\varphi_{n+p},\lambda_1^k\right) &\geq \mathrm{B_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1,\lambda_2,\ldots,\frac{\lambda_q}{2},\ldots,\lambda_k\right)\odot \mathrm{B_Y}\left(\varphi_{n+1},\varphi_{n+p},\lambda_1,\lambda_2,\ldots,\frac{\lambda_q}{2},\ldots,\lambda_k\right),\\
\mathrm{D_Y}\left(\varphi_n,\varphi_{n+p},\lambda_1^k\right) &\leq \mathrm{D_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1,\lambda_2,\ldots,\frac{\lambda_q}{2},\ldots,\lambda_k\right)\oplus \mathrm{D_Y}\left(\varphi_{n+1},\varphi_{n+p},\lambda_1,\lambda_2,\ldots,\frac{\lambda_q}{2},\ldots,\lambda_k\right),\\
\mathrm{E_Y}\left(\varphi_n,\varphi_{n+p},\lambda_1^k\right) &\leq \mathrm{E_Y}\left(\varphi_n,\varphi_{n+1},\lambda_1,\lambda_2,\ldots,\frac{\lambda_q}{2},\ldots,\lambda_k\right)\oplus \mathrm{E_Y}\left(\varphi_{n+1},\varphi_{n+p},\lambda_1,\lambda_2,\ldots,\frac{\lambda_q}{2},\ldots,\lambda_k\right).
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{B_Y}\left(\varphi_n,\varphi_{n+p},\lambda_1^k\right) \geq\ & \mathrm{B}^2_{\mathrm{Y}_q}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)\odot \mathrm{B}^{2^2}_{\mathrm{Y}_q}\left(\varphi_{n+1},\varphi_{n+2},\lambda_1^k\right)\odot\ldots\\
&\odot \mathrm{B}^{2^{p-1}}_{\mathrm{Y}_q}\left(\varphi_{n+p-2},\varphi_{n+p-1},\lambda_1^k\right)\odot\\
&\odot \mathrm{B}^{2^{p}}_{\mathrm{Y}_q}\left(\varphi_{n+p-1},\varphi_{n+p},\lambda_1^k\right),\\
\mathrm{D_Y}\left(\varphi_n,\varphi_{n+p},\lambda_1^k\right) \leq\ & \mathrm{D}^2_{\mathrm{Y}_q}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)\oplus \mathrm{D}^{2^2}_{\mathrm{Y}_q}\left(\varphi_{n+1},\varphi_{n+2},\lambda_1^k\right)\oplus\ldots\\
&\oplus \mathrm{D}^{2^{p-1}}_{\mathrm{Y}_q}\left(\varphi_{n+p-2},\varphi_{n+p-1},\lambda_1^k\right)\oplus\\
&\oplus \mathrm{D}^{2^{p}}_{\mathrm{Y}_q}\left(\varphi_{n+p-1},\varphi_{n+p},\lambda_1^k\right),\\
\mathrm{E_Y}\left(\varphi_n,\varphi_{n+p},\lambda_1^k\right) \leq\ & \mathrm{E}^2_{\mathrm{Y}_q}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)\oplus \mathrm{E}^{2^2}_{\mathrm{Y}_q}\left(\varphi_{n+1},\varphi_{n+2},\lambda_1^k\right)\oplus\ldots\\
&\oplus \mathrm{E}^{2^{p-1}}_{\mathrm{Y}_q}\left(\varphi_{n+p-2},\varphi_{n+p-1},\lambda_1^k\right)\oplus\\
&\oplus \mathrm{E}^{2^{p}}_{\mathrm{Y}_q}\left(\varphi_{n+p-1},\varphi_{n+p},\lambda_1^k\right).
\end{aligned} \tag{3.5.3}$$

for all $\lambda_1,\lambda_2,\ldots,\lambda_k>0$. Letting the limit as $n\to+\infty$ and by using (3.5.2), we have

$$\lim_{n\to+\infty} \mathrm{B}^a_{\mathrm{Y}_q}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)=1,\ \lim_{n\to+\infty} \mathrm{D}^a_{\mathrm{Y}_q}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)=0,\ \lim_{n\to+\infty} \mathrm{E}^a_{\mathrm{Y}_q}\left(\varphi_n,\varphi_{n+1},\lambda_1^k\right)=0,$$

for all $\lambda_1,\lambda_2,\ldots,\lambda_k>0$ and $a>0$. This inequality (3.5.3) yields

$$\begin{aligned}
\lim_{n\to+\infty} \mathrm{B_Y}\left(\varphi_n,\varphi_{n+p},\lambda_1^k\right) &\geq 1\odot 1\odot 1\odot\ldots\odot 1=1,\\
\lim_{n\to+\infty} \mathrm{D_Y}\left(\varphi_n,\varphi_{n+p},\lambda_1^k\right) &\leq 0\oplus 0\oplus 0\oplus\ldots\oplus 0=0,\\
\lim_{n\to+\infty} \mathrm{E_Y}\left(\varphi_n,\varphi_{n+p},\lambda_1^k\right) &\leq 0\oplus 0\oplus 0\oplus\ldots\oplus 0=0.
\end{aligned} \tag{3.5.4}$$

For all $\lambda_1,\lambda_2,\ldots,\lambda_k,p>0$. Hence, in A, the sequence $\{\varphi_n\}$ is a G-Cauchy sequence. There exists $u\in\mathrm{A}$ such that the sequence $\{\varphi_n\}$ converges to u since the space $(\mathrm{A},\mathrm{B_Y},\mathrm{D_Y},\mathrm{E_Y},\odot,\oplus)$ is G-complete.

$$\lim_{n\to+\infty} \mathrm{B_Y}\left(\varphi_n,u,\lambda^k\right)=1,\ \lim_{n\to+\infty} \mathrm{D_Y}\left(\varphi_n,u,\lambda_1^k\right)=0,\ \lim_{n\to+\infty} \mathrm{E_Y}\left(\varphi_n,u,\lambda_1^k\right)=0. \tag{3.5.5}$$

for all $\lambda_1,\lambda_2,\ldots,\lambda_k>0$. For a self-map P, we now need to demonstrate that u is a fixed point.

$$\begin{aligned}
1-\mathrm{B_Y}\left(\varphi_{n+1},\mathrm{P}u,\lambda_1^k\right) &= 1-\mathrm{B_Y}\left(\mathrm{P}\varphi_n,\mathrm{P}u,\lambda_1^k\right)\leq\delta\left(1-\mathrm{B_Y}\left(\varphi_n,u,\lambda_1^k\right)\right),\\
\mathrm{D_Y}\left(\varphi_{n+1},\mathrm{P}u,\lambda_1^k\right) &= \mathrm{D_Y}\left(\mathrm{P}\varphi_n,\mathrm{P}u,\lambda_1^k\right)\leq\delta\left(\mathrm{D_Y}\left(\varphi_n,u,\lambda_1^k\right)\right),\\
\mathrm{E_Y}\left(\varphi_{n+1},\mathrm{P}u,\lambda_1^k\right) &= \mathrm{E_Y}\left(\mathrm{P}\varphi_n,\mathrm{P}u,\lambda_1^k\right)\leq\delta\left(\mathrm{E_Y}\left(\varphi_n,u,\lambda_1^k\right)\right).
\end{aligned}$$

Letting the limit as $n \to +\infty$ and by using (3.5.5),

$$\lim_{n\to+\infty} 1 - \mathrm{B_Y}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right) = 0, \quad \lim_{n\to+\infty} \mathrm{D_Y}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right) = 0, \quad \lim_{n\to+\infty} \mathrm{E_Y}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right) = 0.$$

$$\lim_{n\to+\infty} \mathrm{B_Y}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right) = 1, \quad \lim_{n\to+\infty} \mathrm{D_Y}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right) = 0, \quad \lim_{n\to+\infty} \mathrm{E_Y}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right) = 0.$$

that is,

$$\mathrm{B_Y}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right) = 1, \mathrm{D_Y}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right) = 0, \mathrm{E_Y}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right) = 0, \tag{3.5.6}$$

for all $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathrm{B_Y}\left(u, \mathrm{P}u, \lambda_1^k\right) &\geq \mathrm{B_Y^2}\left(u, \varphi_{n+1}, \lambda_1^k\right) \odot \mathrm{B_Y^2}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right), \\
\mathrm{D_Y}\left(u, \mathrm{P}u, \lambda_1^k\right) &\leq \mathrm{D_Y^2}\left(u, \varphi_{n+1}, \lambda_1^k\right) \oplus \mathrm{D_Y^2}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right), \\
\mathrm{E_Y}\left(u, \mathrm{P}u, \lambda_1^k\right) &\leq \mathrm{E_Y^2}\left(u, \varphi_{n+1}, \lambda_1^k\right) \oplus \mathrm{E_Y^2}\left(\varphi_{n+1}, \mathrm{P}u, \lambda_1^k\right).
\end{aligned}$$

Letting the limit as $n \to +\infty$ and using (3.5.5) and (3.5.6), we have

$$\mathrm{B_Y}\left(u, \mathrm{P}u, \lambda_1^k\right) = 1, \mathrm{D_Y}\left(u, \mathrm{P}u, \lambda_1^k\right) = 0, \mathrm{E_Y}\left(u, \mathrm{P}u, \lambda_1^k\right) = 0,$$

for all $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$. This suggests that u is a fixed point of P, or that $\mathrm{P}u = u$.
Let v be an additional fixed point. of P such that $u \neq v$ for uniqueness. Then $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$ exist in such a way that

$$\mathrm{B_Y}\left(u, v, \lambda_1^k\right) < 1, \mathrm{D_Y}\left(u, v, \lambda_1^k\right) \geq 0, \mathrm{E_Y}\left(u, v, \lambda_1^k\right) \geq 0,$$

Now,

$$\begin{aligned}
\delta(1 - \mathrm{B_Y}\left(u, v, \lambda_1^k\right)) &\geq 1 - \mathrm{B_Y}\left(\mathrm{P}u, \mathrm{P}v, \lambda_1^k\right) = 1 - \mathrm{B_Y}\left(u, v, \lambda_1^k\right), \\
\delta \mathrm{D_Y}\left(u, v, \lambda_1^k\right) &\geq \mathrm{D_Y}\left(\mathrm{P}u, \mathrm{P}v, \lambda_1^k\right) = \mathrm{D_Y}\left(u, v, \lambda_1^k\right), \\
\delta \mathrm{E_Y}\left(u, v, \lambda_1^k\right) &\geq \mathrm{E_Y}\left(\mathrm{P}u, \mathrm{P}v, \lambda_1^k\right) = \mathrm{E_Y}\left(u, v, \lambda_1^k\right).
\end{aligned}$$

which implies that $\delta \geq 1$, which are the contradiction. The fixed point of P must therefore be unique, meaning that $u = v$. □

Example 3.6. Let $\mathrm{A} = [0,1]^2$ and (A, ϖ) be a standard metric, and let $\odot$ and $\oplus$ be the product of t-norm, $w > 0$, and $k \in \mathbf{Z}^+$. Define a membership function $\mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y} : \mathrm{A}^2 \times (0, +\infty)^k \to [0,1]$ by

$$\begin{aligned}
\mathrm{B_Y}(\varphi, \iota, r_1, r_2, r_3) &= \frac{w}{w + \varpi(\varphi, \iota)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)}, \\
\mathrm{D_Y}(\varphi, \iota, r_1, r_2, r_3) &= \frac{\varpi(\varphi, \iota)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)}{w + \varpi(\varphi, \iota)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)}, \\
\mathrm{E_Y}(\varphi, \iota, r_1, r_2, r_3) &= \frac{\varpi(\varphi, \iota)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)}{w}.
\end{aligned}$$

for all $\varphi = (\varphi_1, \varphi_2), \iota = (\iota_1, \iota_2) \in \mathrm{A}$ and $r_1, r_2, r_3 \in (0, +\infty)$. Then $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ is a G-complete 3-NMS. In addition,

$$\lim_{r_1, r_2 \to +\infty} \mathrm{B_Y}(\varphi, \iota, r_1, r_2, r_3) = 1, \lim_{r_1, r_2 \to +\infty} \mathrm{D_Y}(\varphi, \iota, r_1, r_2, r_3) = 0, \lim_{r_1, r_2 \to +\infty} \mathrm{E_Y}(\varphi, \iota, r_1, r_2, r_3) = 0.$$

For all $\varphi, \iota \in \mathrm{A}$ and $r_3 > 0$, the space $(\mathrm{A}, \mathrm{B_Y}, \mathrm{D_Y}, \mathrm{E_Y}, \odot, \oplus)$ is a generalized natural 3-NMS. Establish a mapping $\mathrm{P} : \mathrm{A} \to \mathrm{A}$ by

$$\mathrm{P}(\varphi_1, \varphi_2) = \left(\frac{\varphi_1}{2}, \frac{\varphi_2}{2}\right), \quad \text{for all } (\varphi_1, \varphi_2) \in \mathrm{A}.$$

Consider $\varphi = (\varphi_1, \varphi_2), \iota = (\iota_1, \iota_2) \in \mathrm{A}$ and $r_1, r_2 > 0$, we have:

$$\begin{aligned}\delta(1-\mathrm{B_Y}(\varphi,\iota,r_1,r_2,r_3)) &= \delta\left(1-\left(\frac{w}{w+(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}\right)\right)\\ &= \delta\left(1-\left(\frac{wr_1r_2}{wr_1r_2+(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)(r_1+r_2)}\right)\right).\end{aligned}$$

When $\delta \in (\frac{wr_1r_2}{2wr_1r_2+(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)(r_1+r_2)}, 1)$, we get:

$$\begin{aligned}\delta(1-\mathrm{B_Y}(\varphi,\iota,r_1,r_2,r_3)) &\le \left(1-\frac{2wr_1r_2}{2wr_1r_2+(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)(r_1+r_2)}\right)\\ &= \frac{\left(\frac{|\varphi_1-\iota_1|}{2}+\frac{|\varphi_2-\iota_2|}{2}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}{w+\left(\frac{|\varphi_1-\iota_1|}{2}+\frac{|\varphi_2-\iota_2|}{2}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}\\ &= 1-\mathrm{B_Y}(\mathrm{P}\varphi,\mathrm{P}\iota,r_1,r_2,r_3).\end{aligned}$$

Consider,

$$\delta(\mathrm{D_Y}(\varphi,\iota,r_1,r_2,r_3)) = \delta\frac{\varpi(\varphi,\iota)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}{w+\varpi(\varphi,\iota)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)} = \delta\frac{(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}{w+(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}.$$

When $\delta \in (\frac{(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)(r_1+r_2)}{2wr_1r_2+(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)(r_1+r_2)}, 1)$, we get:

$$\begin{aligned}\delta\mathrm{D_Y}(\varphi,\iota,r_1,r_2,r_3) &\ge \frac{(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}{2w+(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}\\ &= \frac{\left(\frac{|\varphi_1-\iota_1|}{2}+\frac{|\varphi_2-\iota_2|}{2}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}{w+\left(\frac{|\varphi_1-\iota_1|}{2}+\frac{|\varphi_2-\iota_2|}{2}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}\\ &= \mathrm{D_Y}(\mathrm{P}\varphi,\mathrm{P}\iota,r_1,r_2,r_3).\end{aligned}$$

And,

$$\delta\mathrm{E_Y}(\varphi,\iota,r_1,r_2,r_3) = \delta\frac{\varpi(\varphi,\iota)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}{w} = \delta\frac{(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}{w}.$$

When $\delta \in \left(\frac{2w}{w+(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)(r_1+r_2)}, 1\right)$, the inequality becomes:

$$\begin{aligned}\delta\mathrm{E_Y}(\varphi,\iota,r_1,r_2,r_3) &\ge \frac{(|\varphi_1-\iota_1|+|\varphi_2-\iota_2|)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}{2w}\\ &= \frac{\left(\frac{|\varphi_1-\iota_1|}{2}+\frac{|\varphi_2-\iota_2|}{2}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}{w}\\ &= \mathrm{E_Y}(\mathrm{P}\varphi,\mathrm{P}\iota,r_1,r_2,r_3).\end{aligned}$$

Thus, P is a Tirado-type k-neutrosophic contraction mapping. Therefore, $(0,0)$ is the unique fixed point of the self-map P.

4. Conclusion

This study established a fixed-point theorem under G-completeness by introducing Tirado-type k-fuzzy contraction mappings and k-NMS. These findings fill in gaps in the literature on metric space and generalize current theories. The usefulness of the framework is demonstrated by an example. Future research will examine practical uses in mathematical modeling, optimization, and decision-making. It will also broaden the theoretical framework to incorporate dynamic or higher-dimensional neutrosophic systems.

REFERENCES

1. K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20(1), 87–96, 1986.
2. A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, 64(3), 395-399, 1994.
3. D. Gopal, W. Sintunavarat, A. S. Ranadive, S. Shukla, *The investigation of k-fuzzy metric spaces with the first contraction principle in such spaces*, Soft Computing, pp. 1-9, 2023, https://doi. org/10.1007/s00500-023-07946-y.
4. H. Huang, B. Carić, T. Došenović, D. Rakić, M. Brdar, *Fixed-point theorems in fuzzy metric spaces via fuzzy f-contraction*, Mathematics, 9(6), 661, 2021.
5. M. Jeyaraman, M. Pandiselvi, D. Poovaragavan, *Some Results on Multidimensional Fixed Point Theorems in Partially Ordered Generalized Intuitionistic Fuzzy Metric Spaces*, Annals of Communications in Mathematics, 7 (2), 186-204, 2024.
6. M. Jeyaraman, D. Poovaragavan, J. Johnsy, *Fixed Point Theorem for Quasi-Contraction Maps on Generalized Intuitionistic Fuzzy Metric Spaces*, Bulletin of Kerala Mathematics Association, 16(2), 127-139, 2022.
7. J. Johnsy, M. Jeyaraman, *Fixed point theorems for $(\psi - \phi)$-contractions in generalized neutrosophic metric spaces*, Bulletin of Mathematical Analysis and Applications, 16(1), 13–25, 2024.
8. M. Kirisci, N. Simsek, *Neutrosophic metric spaces*, Math. Sci., 14(3), 1–11, 2020.
9. I. Kramosil, J. Michálek, *Fuzzy metrics and statistical metric spaces*, Kybernetika, 11(5),336-344, 1975.
10. D. Poovaragavan, M. Jeyaraman, *Multidimensional common fixed point theorems in V-fuzzy metric spaces*, Bulletin of the International Mathematical Virtual Institute, 12(2), 219–226, 2022.
11. D. Poovaragavan, M. Jeyaraman, *Some fixed point results in V-fuzzy metric space using rational contraction mapping*, J. Math. Comput. Sci., 12 (2022), Article ID 75.
12. J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos Solitons Fractals, 22, 1039–1046, 2004.
13. U. D. Patel, V. Chandra, M. Savatović, S. Radenović, *On k-fuzzy metric spaces with applications*, Nonlinear Analysis: Modelling and Control, 30(1), 1-22, 2025.
14. S. Sowndrarajan, M. Jeyaraman, F. Smarandache, *Fixed point results for contraction theorems in neutrosophic metric spaces*, Neutrosophic Sets and Systems, 3(6), 308–318, 2020.
15. N. Simsek, M. Kirisci, *Fixed point theorems in neutrosophic metric spaces*, Math. Sci., 10, 221–230, 2019.
16. F. Smarandache, *Neutrosophy: Neutrosophic probability, set, and logic*, ProQuest Information and Learning, 1998.
17. R. Vasuki, P. Veeramani, *Fixed point theorems and Cauchy sequences in fuzzy metric spaces*, Fuzzy Sets and Systems, 135(3), 415-417, 2003.
18. D. Wardowski, *Fuzzy contractive mappings and fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, 222, 108-114, 2013.
19. L. A. Zadeh, *Fuzzy sets*, Information and Control, 8, 338-353, 1965.