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A New Result of Statistical Convergence in Neutrosophic Generalized Metric Spaces

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Abstract In this paper, we present the concept of $\mathfrak{G}$- metric space and further generalized to $\mathfrak{G}$-metric of nth order. We define the notion of Neutrosophic Generalized Metric Spaces (NGMS) of order n and present an example to prove this concept. Some characteristics of NGMS are also presented. Additionally we define Statistical Convergent and establish some related concepts.

Keywords $\mathfrak{G}$-metric space, Neutrosophic Generalized Metric Spaces, Statistically Convergent, Statistically Cauchy.

AMS 2010 subject classifications 54E35, 54A40.

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1. Introduction

Zadeh[19] introduced the theory of fuzzy sets in 1965. Following that, fuzzy metric space was described by Kramosil and Michalek [13]. The concept of generalized fuzzy metric spaces was introduced by Singh and Chauhan in 1997 as S-fuzzy metric spaces. The initial idea of fuzzy metric spaces was introduced by Kramosil and Michalek in 1975 and later it was modified by George and Veeramani [7] in 1994. Fuzzy set theory is used in many applications nowadays. Researchers Erceg [5], Grabiec [8], Kaleva

In fuzzy set theory, only membership value was defined. Intuitionistic fuzzy sets was introduced by Atanassov [2]in 1986. J. H. Park introduced the concept of intuitionistic fuzzy metric spaces in 2004. In intuitionistic fuzzy metric spaces along with membership value, non-membership value was included. Floretin Smarandache [9]introduced the concept of neutrosophic sets in 1998. Along with membership, non-membership, indeterminacy concept was included in neutrosophic sets. He has done research in many topics namely neutrosophic probability, set and logic, analytic synthesis & synthetic analysis, interval neutrosophic sets and logic, neutrosophic statistics,single valued neutrosophic graphs, n-valued refined neutrosophic logic and its applications, fuzzy cognitive maps and neutrosophic cognitive maps, complex neutrosophic sets, rough neutrosophic sets etc. Later in 2020, Kirisci and Simsek introduced the notion of neutrosophic metric spaces.

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Metric space is used to measure the distance between any two places. D-metric space is a generalized metric space that was introduced by Dhage [4]. G-metric space is a new class of generalized metric spaces that Mustafa and Sims [14, 15]introduced later in 2005. Numerous writers have conducted research on g-metric spaces, including Choi et al. [3], Jeyaraman et al. [10], and Sun et al. [18]. M. Jeyaraman has done research work on the topic fixed point theory. He has published papers on the topic neutrosophic metric spaces, neutrosophic normed spaces, generalized M-fuzzy metric space, bitopology, generalized fuzzy cone metric spaces,fuzzy topological spaces, G-fuzzy metric spaces, generalized fuzzy metric spaces, non-archimedean generalized intuitionistic fuzzy metric spaces, complex-valued neutrosophic b-metric spaces, anti Fuzzy metric spaces, Hausdorff neutrosophic metric spaces, fuzzy convex metric spaces, M-fuzzy cone metric spaces, statistical convergence etc.

In this study, we have concentrated on statistically convergent sequences and G-metric spaces with n points in neutrosophic metric spaces. We talked about the statistically Cauchy and the uniqueness of statistically convergent sequences. Research on statistical convergence and neutrosophic metric spaces is currently underway. Research on statistical convergence has been conducted by Abazari [1], Fast [6], Jeyaraman et al. [11], Schweizer et al. [16], and Steinhaus [17].

Statistical convergence of order n in neutrosophic metric spaces is significant because it extends classical convergence concepts to handle uncertainty, vagueness, and imprecision by utilizing neutrosophic logic and density measures. It extends the classical concept of statistical convergence to the more general framework of neutrosophic metric spaces, allowing for a broader scope of analysis. The addition of order n introduces a stronger condition for convergence. It allows for a more precise way to quantify how ”close” a sequence is to its limit, accounting for the inherent degrees of uncertainty or ”roughness” in neutrosophic environments.

2. Preliminaries

Definition 2.1. [15] Let $\sum$ be an arbitrary non-empty set and $\mathfrak{G} : \sum^3 \to \mathbb{R}^+$ be a mapping. Then $(\sum, \mathfrak{G})$ is called $\mathfrak{G}$-metric space if, for all $\omega, \rho, \varsigma, \delta \in \sum$, the following conditions hold:
(G-1) $\mathfrak{G}(\omega, \rho, \varsigma) = 0$ if $\omega = \rho = \varsigma$
(G-2) $\mathfrak{G}(\omega, \omega, \rho) > 0$ if $\omega \neq \rho$,
(G-3) $\mathfrak{G}(\omega, \omega, \rho) \leq \mathfrak{G}(\omega, \rho, \varsigma)$ if $\rho \neq \varsigma$,
(G-4) $\mathfrak{G}(\omega, \rho, \varsigma) = \mathfrak{G}(\rho, \varsigma, \omega) = \mathfrak{G}(\omega, \varsigma, \rho) = \dots$ (symmetry in all three variables),
(G-5) $\mathfrak{G}(\omega, \rho, \varsigma) \leq \mathfrak{G}(\omega, \delta, \delta) + \mathfrak{G}(\delta, \rho, \varsigma)$.
In such a case, the function $\mathfrak{G}$ is known as a $\mathfrak{G}$-metric on the set $\sum$.

Example 2.2. Suppose $(\sum, \mathfrak{d})$ is an ordinary metric space.
Define $\mathfrak{G} : \sum^3 \to \mathbb{R}^+$ by $\mathfrak{G}(\omega, \rho, \varsigma) = \frac{1}{2}(\mathfrak{d}(\omega, \rho) + \mathfrak{d}(\rho, \varsigma) + \mathfrak{d}(\omega, \varsigma))$. Then $(\sum, \mathfrak{G})$ is a $\mathfrak{G}$-metric space.

Remark 2.3. Let $(\sum, \mathfrak{G})$ be a $\mathfrak{G}$-metric space.
Define $\mathfrak{d}_{\mathfrak{G}} : \sum^2 \to \mathbb{R}^+$ by $\mathfrak{d}_{\mathfrak{G}}(\omega, \rho) = \frac{1}{3}(\mathfrak{G}(\omega, \rho, \rho) + \mathfrak{G}(\omega, \omega, \rho))$. Then $(\sum, \mathfrak{d}_{\mathfrak{G}})$ is an ordinary metric space.

Definition 2.4. A function $* : [0, 1] \times [0, 1] \to [0, 1]$ referred as the continuous t-norm if,
(1) $*$ is commutative and associative
(2) $\varpi = \varpi * 1$ for any $0 \leq \varpi \leq 1$
(3) for each $0 \leq \varpi_1, \varpi_2, \varpi_3, \varpi_4 \leq 1$, if $\varpi_1 \leq \varpi_3$ and $\varpi_2 \leq \varpi_4$ then $\varpi_1 * \varpi_2 \leq \varpi_3 * \varpi_4$ and
(4) $*$ is continuous.

Definition 2.5. The function $\diamondsuit : [0, 1] \times [0, 1] \to [0, 1]$ is said to be continuous t -conorm if,
(1) $\diamondsuit$ is commutative and associative
(2) $\varpi = \varpi \diamondsuit 0$ for any $0 \leq \varpi \leq 1$
(3) for each $0 \leq \varpi_1, \varpi_2, \varpi_3, \varpi_4 \leq 1$, if $\varpi_1 \leq \varpi_3$ and $\varpi_2 \leq \varpi_4$ then $\varpi_1 \diamondsuit \varpi_2 \leq \varpi_3 \diamondsuit \varpi_4$ and
(4) $\diamondsuit$ is continuous.

Example 2.6. Let $\varpi_1, \varpi_2 \in [0,1]$. Then
(1) $\varpi_1 * \varpi_2 = \min\{\varpi_1, \varpi_2\}$ and $\varpi_1 * \varpi_2 = \varpi_1.\varpi_2$ are continuous t-norms.
(2) $\varpi_1 \diamond \varpi_2 = \max\{\varpi_1, \varpi_2\}$ and $\varpi_1 \diamond \varpi_2 = \min\{\varpi_1 + \varpi_2, 1\}$ are t-conorms that are continuous on $[0,1]$.

Definition 2.7. [3] Let $\sum$ be a non-empty set. A function $g : \sum^{n+1} \to \mathbb{R}^+$,
where $\sum^n = \prod_{i=1}^n \sum^i$, is called g metric of order n on $\sum$ if the following conditions hold:
($\mathfrak{g}_1$) $g(\omega_0, \omega_1, \ldots, \omega_n) = 0$ iff $\omega_0 = \omega_1 = \cdots = \omega_n$,
($\mathfrak{g}_2$) $g(\omega_0, \omega_1, \ldots, \omega_n) = g(\omega_{\pi(0)}, \omega_{\pi(1)}, \ldots, \omega_{\pi(n)})$ for any permutation π on $\{0, 1, \ldots, n\}$,
($\mathfrak{g}_3$) $g(\omega_0, \omega_1, \ldots, \omega_n) \leq g(\rho_0, \rho_1, \ldots, \rho_n)$ for any $(\omega_0, \omega_1, \ldots, \omega_n), (\rho_0, \rho_1, \ldots, \rho_n) \in \sum^{n+1}$ with $\{\omega_0, \omega_1, \ldots, \omega_n\} \subsetneq \{\rho_0, \rho_1, \ldots, \rho_n\}$,
($\mathfrak{g}_4$) for all $\omega_0, \omega_1, \ldots, \omega_p, \rho_0, \rho_1, \ldots, \rho_q, \varsigma \in \sum$ with $p + q + 1 = n$,
$g(\omega_0, \omega_1, \ldots, \omega_p, \varsigma, \ldots, \varsigma) + g(\rho_0, \rho_1, \ldots, \rho_q, \varsigma, \ldots, \varsigma) \geq g(\omega_0, \omega_1, \ldots, \omega_p, \rho_0, \rho_1, \ldots, \rho_q)$
The tuple $(\sum, g)$ is called g-metric space. For $n = 1$ and $n = 2$, a g-metric reduces to the ordinary metric and $\mathfrak{G}$-metric, respectively.

Example 2.8. [3] Let $(\sum, \mathfrak{d})$ be an ordinary metric space.
Define $g : \sum^{n+1} \to \mathbb{R}^+$ by $g(\omega_0, \omega_1, \ldots, \omega_n) = \max\limits_{0 \leq \mu, v \leq n} \{|\omega_\mu - \omega_v|\}$ for all $\omega_0, \omega_1, \ldots, \omega_n \in \sum$. Then $(\sum, g)$ is a g-metric space.

Definition 2.9. Let $(\Sigma, \mathfrak{g})$ be a metric space and (ω_k) be a sequence in Σ. Then
a) (ω_k) is said to be neutrosophic $\mathfrak{g}$ -Convergent to some $\omega \in \Sigma$, if, $\forall \xi > 0, \exists K \in \mathbb{N}$ such that $\mathfrak{g}_1(\omega_{k_1}, \omega_{k_2}, \ldots, \omega_{k_n}, \omega) < \epsilon, \mathfrak{g}_2(\omega_{k_1}, \omega_{k_2}, \ldots, \omega_{k_n}, \omega) > 1 - \epsilon$,
$\mathfrak{g}_3(\omega_{k_1}, \omega_{k_2}, \ldots, \omega_{k_n}, \omega) > 1 - \epsilon \forall\, k_1, k_2, \ldots, k_n \geq K$
b) (ω_k) is said to be neutrosophic $\mathfrak{g}$ - Cauchy if $\forall\, \epsilon > 0, \exists\, K \in \mathbb{N}$ such that
$\mathfrak{g}_1 \omega_{k_0}, \omega_{k_1}, \omega_{k_2}, \ldots, \omega_{k_n} < \epsilon, \mathfrak{g}_2 \omega_{k_0}, \omega_{k_1}, \omega_{k_2}, \ldots, \omega_{k_n}, \omega > 1 - \epsilon$,
$\mathfrak{g}_3(\omega_{k_0}, \omega_{k_1}, \omega_{k_2}, \ldots, \omega_{k_n}, \omega) > 1 - \epsilon \forall\, k_0, k_1, k_2, \ldots, k_n \geq K$.
A $\mathfrak{g}$ - neutrosophic metric space is said to be complete if every neutrosophic $\mathfrak{g}$ - Cauchy sequence in Σ is neutrosophic $\mathfrak{g}$ -Convergent.

3. Neutrosophic generalized metric spaces (NGMS)

The foundational concept of generalized neutrosophic metric spaces was defined by Kirisci and Simsek. Later many authors including Uddin et. al. introduced specific types of generalized neutrosophic metric spaces.

Definition 3.1. Let $\sum$ be an arbitrary non-empty set, $*$ be continuous t-norm, $\diamond$ and $\circ$ be continuous t-conorm respectively, and (Z, Θ, K) be neutrosophic sets on $\sum^{n+1} \times (0, \infty)$. The seven-tuple $(\sum, Z, \Theta, K, *, \diamond)$ is said to be an neutrosophic generalized metric space (NGMS) of order n if, for all $\varrho, \theta \in (0, \infty)$, the following criteria hold:
(NGMS -1) $Z(\omega_0, \omega_1, \ldots, \omega_n, \varrho) + \Theta(\omega_0, \omega_1, \ldots, \omega_n, \varrho) + K(\omega_0, \omega_1, \ldots, \omega_n, \varrho) \leq 3$ for all $\omega_0, \omega_1, \ldots, \omega_n \in \sum$,
(NGMS-2) $Z(\omega_0, \omega_0, \ldots, \omega_0, \omega_1, \varrho) > 0$ for $\omega_0 \neq \omega_1, \forall\, \omega_0, \omega_1 \in \sum$,
(NGMS-3) $Z(\omega_0, \omega_1, \ldots, \omega_n, \varrho) \geq Z(\rho_0, \rho_1, \ldots, \rho_n, \varrho), \forall\, (\omega_0, \omega_1, \ldots, \omega_n), (\rho_0, \rho_1, \ldots, \rho_n) \in \sum^{n+1}$ with $\{\omega_0, \omega_1, \ldots, \omega_n\} \subsetneq \{\rho_0, \rho_1, \ldots, \rho_n\}$,
(NGMS -4) $Z(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = 1 \Leftrightarrow \omega_0 = \omega_1 = \cdots = \omega_n$,
(NGMS -5) $Z(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = Z(\omega_{\pi(0)}, \omega_{\pi(1)}, \ldots, \omega_{\pi(n)}, \varrho)$ for any permutation π on $\{0, 1, \ldots, n\}$,
(NGMS -6) for all $\omega_0, \omega_1, \ldots, \omega_p, \rho_0, \rho_1, \ldots, \rho_q, \varsigma \in \sum$ with $p + q + 1 = n$,
$Z(\omega_0, \omega_1, \ldots, \omega_p, \varsigma, \ldots, \varsigma, \varrho) * Z(\rho_0, \rho_1, \ldots, \rho_q, \varsigma, \ldots, \varsigma, \theta) \leq Z(\omega_0, \omega_1, \ldots, \omega_p, \rho_0, \rho_1, \ldots, \rho_q, \varrho + \theta)$,
(NGMS -7) $Z(\omega_0, \omega_1, \ldots, \omega_n, .) : (0, \infty) \to [0, 1]$ is a continuous function,
(NGMS -8) $\lim\limits_{\varrho \to \infty} Z(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = 1$ for all $\omega_0, \omega_1, \ldots, \omega_n \in \sum$,
(NGMS -9) $\Theta(\omega_0, \omega_0, \ldots, \omega_0, \omega_1, \varrho) < 1$ for $\omega_0 \neq \omega_1, \forall\, \omega_0, \omega_1 \in \sum$,
(NGMS -10) $\Theta(\omega_0, \omega_1, \ldots, \omega_n, \varrho) \leq \Theta(\rho_0, \rho_1, \ldots, \rho_n, \varrho), \forall\, (\omega_0, \omega_1, \ldots, \omega_n)$,

$(\rho_0, \rho_1, \ldots, \rho_n) \in \sum^{n+1}$ with $\{\omega_0, \omega_1, \ldots, \omega_n\} \subsetneq \{\rho_0, \rho_1, \ldots, \rho_n\}$,
(NGMS -11) $\Theta(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = 0 \Leftrightarrow \omega_0 = \omega_1 = \cdots = \omega_n$,
(NGMS -12) $\Theta(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = \Theta(\omega_{\pi(0)}, \omega_{\pi(1)}, \ldots, \omega_{\pi(n)}, \varrho)$ for any permutation π on $0, 1, \ldots, n$,
(NGMS-13) for all $\omega_0, \omega_1, \ldots, \omega_p, \rho_0, \rho_1, \ldots, \rho_q, \varsigma \in \sum$ with $p + q + 1 = n$
$\Theta(\omega_0, \omega_1, \ldots, \omega_p, \varsigma, \ldots, \varsigma, \varrho) \diamond \Theta(\rho_0, \rho_1, \ldots, \rho_q, \varsigma, \ldots, \varsigma, \theta) \geq \Theta(\omega_0, \omega_1, \ldots, \omega_p, \rho_0, \rho_1, \ldots, \rho_q, \varrho + \theta)$,
(NGMS -14) $\Theta(\omega_0, \omega_1, \ldots, \omega_n, .) : (0, \infty) \to [0, 1]$ is a continuous function,
(NGMS -15) $\lim_{\varrho \to \infty} \Theta(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = 0$ for all $\omega_0, \omega_1, \ldots, \omega_n \in \sum$,
(NGMS -16) $K(\omega_0, \omega_0, \ldots, \omega_0, \omega_1, \varrho) < 1$ for $\omega_0 \neq \omega_1, \forall\ \omega_0, \omega_1 \in \sum$,
(NGMS -17) $K(\omega_0, \omega_1, \ldots, \omega_n, \varrho) \leq K(\rho_0, \rho_1, \ldots, \rho_n, \varrho), \forall\ (\omega_0, \omega_1, \ldots, \omega_n), (\rho_0, \rho_1, \ldots, \rho_n) \in \sum^{n+1}$ with $\{\omega_0, \omega_1, \ldots, \omega_n\} \subsetneq \{\rho_0, \rho_1, \ldots, \rho_n\}$,
(NGMS -18) $K(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = 0 \Leftrightarrow \omega_0 = \omega_1 = \cdots = \omega_n$,
(NGMS -19) $K(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = K(\omega_{\pi(0)}, \omega_{\pi(1)}, \ldots, \omega_{\pi(n)}, \varrho)$ for any permutation π on $\{0, 1, \ldots, n\}$,
(NGMS -20) for all $\omega_0, \omega_1, \ldots, \omega_p, \rho_0, \rho_1, \ldots, \rho_q, \varsigma \in \sum$ with $p + q + 1 = n$,
$K(\omega_0, \omega_1, \ldots, \omega_p, \varsigma, \ldots, \varsigma, \varrho) \diamond K(\rho_0, \rho_1, \ldots, \rho_q, \varsigma, \ldots, \varsigma, \theta)$
$\geq K(\omega_0, \omega_1, \ldots, \omega_p, \rho_0, \rho_1, \ldots, \rho_q, \varrho + \theta)$,
(NGMS -21) $K(\omega_0, \omega_1, \ldots, \omega_n, .) : (0, \infty) \to [0, 1]$ is a continuous function,
(NGMS -22) $\lim_{\varrho \to \infty} K(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = 0$ for all $\omega_0, \omega_1, \ldots, \omega_n \in \sum$.

Further, we call the tuple (Z, Θ, K), the neutrosophic generalized metric space (in short, NGMS) on $\sum$.

Example 3.2. Let $(\sum, g)$ be a g-metric space with order n. For $\varrho > 0$, define
$Z(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = \frac{\varrho}{\varrho + g(\omega_0, \omega_1, \ldots, \omega_n)}$, $\Theta(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = \frac{g(\omega_0, \omega_1, \ldots, \omega_n)}{\varrho + g(\omega_0, \omega_1, \ldots, \omega_n)}$,
$K(\omega_0, \omega_1, \ldots, \omega_n, \varrho) = \frac{g(\omega_0, \omega_1, \ldots, \omega_n)}{\varrho}$,
where $\varpi_1 * \varpi_2 = \varpi_1 \cdot \varpi_2$, $\varpi_1 \diamond \varpi_2 = \min\{\varpi_1 + \varpi_2, 1\}$ and $\varpi_1 \circ \varpi_2 = \min\{\varpi_1 + \varpi_2, 1\}$ for all $\varpi_1, \varpi_2 \in [0, 1]$. Then $(\sum, Z, \Theta, K, *, \diamond)$ is a NGMS.

Proof
We only show that (Z, Θ, K) satisfies the conditions (NGMS-6), (NGMS-13) and (NGMS-20) the rest follows easily.
(NGMS-6): Let $\varrho, \theta > 0$ and $\omega_0, \omega_1, \ldots, \omega_p, \rho_0, \rho_1, \ldots, \rho_q, \varsigma \in \sum$ with $p + q + 1 = n$. Then

$$g(\omega_0, \omega_1, \ldots, \omega_p, \rho_0, \rho_1, \ldots, \rho_q) \leq g(\omega_0, \omega_1, \ldots, \omega_p, \varsigma, \ldots, \varsigma) + g(\rho_0, \rho_1, \ldots, \rho_q, \varsigma, \ldots, \varsigma) \tag{3.2.1}$$

Now

$$\begin{aligned}
&Z(\omega_0, \omega_1, \ldots, \omega_p, \varsigma, \ldots, \varsigma, \varrho) * Z(\rho_0, \rho_1, \ldots, \rho_q, \varsigma, \ldots, \varsigma, \theta) \\
&= \frac{\varrho}{\varrho + g(\omega_0, \omega_1, \ldots, \omega_p, \varsigma, \ldots, \varsigma)} \cdot \frac{\theta}{\theta + g(\rho_0, \rho_1, \ldots, \rho_q, \varsigma, \ldots, \varsigma)} \\
&\leq \frac{\varrho\theta}{\varrho\theta + \varrho.g(\rho_0, \rho_1, \ldots, \rho_q, \varsigma, \ldots, \varsigma) + \theta.g(\omega_0, \omega_1, \ldots, \omega_p, \varsigma, \ldots, \varsigma)} \\
&= \frac{1}{1 + \frac{g(\rho_0, \rho_1, \ldots, \rho_q, \varsigma, \ldots, \varsigma)}{\theta} + \frac{g(\omega_0, \omega_1, \ldots, \omega_p, \varsigma, \ldots, \varsigma)}{\varrho}} \\
&\leq \frac{1}{1 + \frac{g(\rho_0, \rho_1, \ldots, \rho_q, \varsigma, \ldots, \varsigma)}{\varrho + \theta} + \frac{g(\omega_0, \omega_1, \ldots, \omega_p, \varsigma, \ldots, \varsigma)}{\varrho + \theta}} \\
&= \frac{\varrho + \theta}{\varrho + \theta + g(\rho_0, \rho_1, \ldots, \rho_q, \varsigma, \ldots, \varsigma) + g(\omega_0, \omega_1, \ldots, \omega_p, \varsigma, \ldots, \varsigma)}
\end{aligned}$$

Therefore using the Equation (3.2.1), it follows that

$$\begin{aligned}
&Z(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma,\varrho) * Z(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma,\theta)\\
&\le \frac{\varrho+\theta}{\varrho+\theta+g(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q)}\\
&= Z(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q,\varrho+\theta)
\end{aligned}$$

(NGMS-13): As above, select ϱ,θ,p and q. Then

$$\begin{aligned}
&g(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q) \le g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma) + g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)\\
&\Rightarrow 1+\frac{\varrho+\theta}{g(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q)}\\
&\ge 1+\frac{\varrho+\theta}{g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma)+g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)}\\
&\Rightarrow \frac{g(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q)}{\varrho+\theta+g(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q)}\\
&\le \frac{g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma)+g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)}{\varrho+g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma)+\theta+g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)}\\
&\le \frac{g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma)}{\varrho+g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma)}+\frac{g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)}{\theta+g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)}
\end{aligned}$$

Therefore,
$\Theta(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q,\varrho+\theta) \le \Theta(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma,\varrho) + \Theta(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma,\theta)$.
Since, $\Theta(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q,\varrho+\theta) \le 1$, we have

$$\begin{aligned}
\Theta(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q,\varrho+\theta) &\le \min\{\Theta(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma,\varrho)+\Theta(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma,\theta),1\}\\
&= \Theta(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma,\varrho)\diamond\Theta(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma,\theta)
\end{aligned}$$

(NGMS-20): As above, select ϱ,θ,p and q. Then

$$\begin{aligned}
g(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q) &\le g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma) + g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)\\
\Rightarrow 1+\frac{\varrho+\theta}{g(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q)} &\ge 1+\frac{\varrho+\theta}{g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma)+g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)}\\
\Rightarrow \frac{g(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q)}{\varrho+\theta+g(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q)} &\le \frac{g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma)+g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)}{\varrho+g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma)+\theta+g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)}\\
&\le \frac{g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma)}{\varrho+g(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma)}+\frac{g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)}{\theta+g(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma)}
\end{aligned}$$

Therefore,
$K(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q,\varrho+\theta) \le K(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma,\varrho) + K(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma,\theta)$.
Since, $K(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q,\varrho+\theta) \le 1$, we have

$$\begin{aligned}
K(\omega_0,\omega_1,\dots,\omega_p,\rho_0,\rho_1,\dots,\rho_q,\varrho+\theta) &\le \min\{K(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma,\varrho)+K(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma,\theta),1\}\\
&= K(\omega_0,\omega_1,\dots,\omega_p,\varsigma,\dots,\varsigma,\varrho)\circ K(\rho_0,\rho_1,\dots,\rho_q,\varsigma,\dots,\varsigma,\theta)
\end{aligned}$$

The above example is also true for
$\varpi_1 * \varpi_2 = \min\{\varpi_1,\varpi_2\}, \varpi_1\diamond\varpi_2 = \max\{\varpi_1,\varpi_2\}$ and $\varpi_1\circ\varpi_2 = \max\{\varpi_1,\varpi_2\}$ for all $\varpi_1,\varpi_2 \in [0,1]$.
Since the above metric space $(\sum, Z, \Theta, K, *, \diamond)$ is induced by the g-metric, known as standard NGMS. □

Proposition 3.3. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS, where $\varrho * \varrho = \varrho, \varrho \diamond \varrho = \varrho$ and $\varrho \circ \varrho = \varrho$ for all $\varpi \in [0,1]$. Then the following hold:

(a) $Z\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) \geq Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{p-1}}\right)$,

$\Theta\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) \leq \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{p-1}}\right)$,

$K\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) \leq K\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{p-1}}\right)$,

(b) $Z\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) \geq Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{n-p}}\right)$,

$\Theta\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) \leq \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{n-p}}\right)$,

$K\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) \leq K\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{n-p}}\right)$,

Proof
(a): By using the condition (NGMS-6), we get

$$\begin{aligned} Z\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) &\geq Z\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-1)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right) * Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right) \\ &\geq Z\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-2)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right) * Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right) * Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right) \end{aligned}$$

Since $\varrho * \varrho = \varrho$ for all $\varpi \in [0,1]$, we have

$$\begin{aligned} Z\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) &\geq Z\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-2)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right) * Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right) \\ &\geq Z\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-3)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^3}\right) * Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^3}\right) * Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right) \\ &\geq Z\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-3)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^3}\right) * Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^3}\right) \\ &\vdots \end{aligned}$$

$$\geq Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{p-1}}\right) * Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{p-1}}\right)$$
$$= Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{p-1}}\right)$$

By using the condition (NGMS-13), we get

$$\Theta\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) \leq \Theta\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-1)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right) \diamond \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right)$$
$$\leq \Theta\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-2)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right) \diamond \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right) \diamond \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right)$$

Since $\varrho \diamond \varrho = \varrho$ for all $\varpi \in [0,1]$, we have

$$\Theta\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) \leq \Theta\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-2)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right) \diamond \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right)$$
$$\leq \Theta\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-3)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^3}\right) \diamond \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^3}\right) \diamond \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right)$$
$$\leq \Theta\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-3)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^3}\right) \diamond \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^3}\right)$$
$$\vdots$$
$$\leq \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{p-1}}\right) \diamond \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{p-1}}\right)$$
$$= \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^{p-1}}\right)$$

By using the condition (NGMS-20), we get

$$K\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) \leq K\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-1)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right) \circ K\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right)$$
$$\leq K\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-2)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right) \circ K\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right) \circ K\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right)$$

Since $\varrho \circ \varrho = \varrho$ for all $\varpi \in [0,1]$, we have

$$K\left(\underbrace{\omega, \omega, \ldots, \omega}_{p-\text{times}}, \varsigma, \ldots, \varsigma, \varrho\right) \leq K\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-2)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right) \circ K\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right)$$
$$\leq K\left(\underbrace{\omega, \omega, \ldots, \omega}_{(p-3)-\text{times}}, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^3}\right) \circ K\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^3}\right) \circ K\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^2}\right)$$

$$\leq K\left(\underbrace{\omega,\omega,\ldots,\omega}_{(p-3)-\text{times}},\varsigma,\ldots,\varsigma,\frac{\varrho}{2^3}\right)\circ K\left(\omega,\varsigma,\ldots,\varsigma,\frac{\varrho}{2^3}\right)$$

$$\vdots$$

$$\leq K\left(\omega,\varsigma,\ldots,\varsigma,\frac{\varrho}{2^{p-1}}\right)\circ K\left(\omega,\varsigma,\ldots,\varsigma,\frac{\varrho}{2^{p-1}}\right)$$
$$= K\left(\omega,\varsigma,\ldots,\varsigma,\frac{\varrho}{2^{p-1}}\right)$$

(b) By using (a), we have

$$Z\left(\underbrace{\omega,\omega,\ldots,\omega}_{p-\text{times}},\varsigma,\ldots,\varsigma,\varrho\right)=Z\left(\underbrace{\varsigma,\ldots,\varsigma}_{(n+1-p)-\text{times}},\omega,\omega,\ldots,\omega,\varrho\right)\geq Z\left(\varsigma,\omega,\omega,\ldots,\omega,\frac{\varrho}{2^{n-p}}\right),$$

$$\Theta\left(\underbrace{\omega,\omega,\ldots,\omega}_{p-\text{times}},\varsigma,\ldots,\varsigma,\varrho\right)=\Theta\left(\underbrace{\varsigma,\ldots,\varsigma}_{(n+1-p)-\text{times}},\omega,\omega,\ldots,\omega,\varrho\right)\leq \Theta\left(\varsigma,\omega,\omega,\ldots,\omega,\frac{\varrho}{2^{n-p}}\right) \text{ and}$$

$$K\left(\underbrace{\omega,\omega,\ldots,\omega}_{p-\text{times}},\varsigma,\ldots,\varsigma,\varrho\right)=K\left(\underbrace{\varsigma,\ldots,\varsigma}_{(n+1-p)-\text{times}},\omega,\omega,\ldots,\omega,\varrho\right)\leq K\left(\varsigma,\omega,\omega,\ldots,\omega,\frac{\varrho}{2^{n-p}}\right)$$

□

Definition 3.4. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS and $\omega_0 \in \sum$. The open ball with center ω_0 and radius $\sigma \in (0,1)$ in relation $\varrho > 0$, is the set
$B_{\omega_0}^{(Z,\Theta,K)}(\varrho,\sigma) = \left\{ \begin{array}{c} \rho \in \sum : Z(\omega_0,\rho,\rho,\ldots,\rho,\varrho) > 1-\sigma, \\ \Theta(\omega_0,\rho,\rho,\ldots,\rho,\varrho) < \sigma \text{ and } K(\omega_0,\rho,\rho,\ldots,\rho,\varrho) < \sigma \end{array} \right\}.$

Remark 3.5. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS. Define $\mathcal{T}^{(Z,\Theta,K)} = \{\Omega \subset \sum :$ for each $\omega \in \Omega, \exists\ \sigma \in (0,1)$ and $\varrho > 0$ such that $B_{\omega}^{(Z,\Theta,K)}(\varrho,\sigma) \subset \Omega\}$. Then $\mathcal{T}^{(Z,\Theta,K)}$ is a topology on $\sum$ induced by (Z,Θ,K). Clearly, the set $\left\{B_{\omega}^{(Z,\Theta,K)}\left(\frac{1}{n},\frac{1}{n}\right)\right\}$ is a local base at $\omega \in \sum$ hence the topology $\mathcal{T}^{(Z,\Theta,K)}$ is first countable. Also, we notice that every open ball is an open set in the topology $\mathcal{T}^{(Z,\Theta,K)}$.

Proposition 3.6. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS. Suppose $Z(\omega_0,\omega_1,\ldots,\omega_n,\varrho) > 1-\varpi$, $\Theta(\omega_0,\omega_1,\ldots,\omega_n,\varrho) < \varpi$ and $K(\omega_0,\omega_1,\ldots,\omega_n,\varrho) < \varpi$ for some $\varpi \in (0,1)$ and $\varrho > 0$.
If $n(\{\omega_0,\omega_1,\ldots,\omega_n\}) \geq 3$, then $\omega_\mu \in B_{\omega_0}^{(Z,\Theta,K)}(\varrho,\varpi)$ for each $\mu \in 0,1,\ldots,n$.

Proof
Let $n(\{\omega_0,\omega_1,\ldots,\omega_n\}) \geq 3$. Clearly, for each $u \in \{0,1,\ldots,n\}$ we have $\{\omega_0,\omega_\mu,\ldots,\omega_\mu\} \subsetneq \{\omega_0,\omega_1,\ldots,\omega_n\}$. Therefore, by using (NGMS-3), (NGMS-10) and (NGMS-17), we find

$$Z(\omega_0,\omega_\mu,\ldots,\omega_\mu,\varrho) \geq Z(\omega_0,\omega_1,\ldots,\omega_n,\varrho) > 1-\varpi,$$
$$\Theta(\omega_0,\omega_\mu,\ldots,\omega_\mu,\varrho) \leq \Theta(\omega_0,\omega_1,\ldots,\omega_n,\varrho) < \varpi \text{ and}$$
$$K(\omega_0,\omega_\mu,\ldots,\omega_\mu,\varrho) \leq K(\omega_0,\omega_1,\ldots,\omega_n,\varrho) < \varpi$$

Thus $\omega_\mu \in B_{\omega_0}^{(Z,\Theta,K)}(\varrho,\varpi)$ for each $u \in \{0,1,\ldots,n\}$. □

Lemma 3.7. Let $(\sum, Z, \Theta, K, , *, \diamond, \circ)$ be a NGMS, where $\varpi * \varpi = \varpi, \varpi \diamond \varpi = \varpi$ and $\varpi \circ \varpi = \varpi$ for all $\varpi \in [0,1]$. Then $(\sum, \mathcal{T}^{(Z,\Theta,K)})$ is Hausdorff.

Proof
Let $\omega, \rho \in \sum$ so that $\omega \neq \rho$.
Then $0 < Z(\omega, \rho, \ldots, \rho, \varrho) < 1$ and $0 < \Theta(\omega, \rho, \ldots, \rho, \varrho) < 1$ and $0 < K(\omega, \rho, \ldots, \rho, \varrho) < 1$ for all $\varrho > 0$.
Let $\varpi_1 = Z(\omega, \rho, \ldots, \rho, \varrho)$, $\varpi_2 = \Theta(\omega, \rho, \ldots, \rho, \varrho)$ and $\varpi_3 = K(\omega, \rho, \ldots, \rho, \varrho)$.
Choose $\varpi \in (0,1)$ such that $\varpi = \max\{\varpi_1, 1-\varpi_2, 1-\varpi_3\}$. For given $\varpi < \varpi_0 < 1$, there are $\varpi_4, \varpi_5, \varpi_6 \in (0,1)$ such that $\varpi_4 > \varpi_0$ and $(1-\varpi_5) < 1-\varpi_0$ and $(1-\varpi_6) < 1-\varpi_0$.
Put $\varpi_7 = \max\{\varpi_4, \varpi_5, \varpi_6\}$. Now, consider the open balls
$B_\omega^{(Z,\Theta,K)}\left(\frac{\varrho}{2^n}, 1-\varpi_7\right)$, $B_\rho^{(Z,\Theta,K)}\left(\frac{\varrho}{2^n}, 1-\varpi_7\right)$ and $B_\kappa^{(Z,\Theta,K)}\left(\frac{\varrho}{2^n}, 1-\varpi_7\right)$.
We show that $B_\omega^{(Z,\Theta,K)}\left(\frac{\varrho}{2^n}, 1-\varpi_7\right) \cap B_\rho^{(Z,\Theta,K)}\left(\frac{\varrho}{2^n}, 1-\varpi_7\right) \cap B_\kappa^{(Z,\Theta,K)}\left(\frac{\varrho}{2^n}, 1-\varpi_7\right) = \emptyset$.
Let on contrary, $\varsigma \in B_\omega^{(Z,\Theta,K)}\left(\frac{\varrho}{2^n}, 1-\varpi_7\right) \cap B_\rho^{(Z,\Theta,K)}\left(\frac{\varrho}{2^n}, 1-\varpi_7\right) \cap B_\kappa^{(Z,\Theta,K)}\left(\frac{\varrho}{2^n}, 1-\varpi_7\right)$.
Then $\varpi_1 = Z(\omega, \rho, \ldots, \rho, \varrho) \geq Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right) * Z\left(\varsigma, \rho, \ldots, \rho, \frac{\varrho}{2}\right)$.
By using Proposition (3.3), we get $Z\left(\varsigma, \rho, \ldots, \rho, \frac{\varrho}{2}\right) \geq Z\left(\rho, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right)$. Consequently,

$$\begin{aligned}\varpi_1 = Z(\omega, \rho, \ldots, \rho, \varrho) &\geq Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right) * Z\left(\rho, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right)\\ &\geq Z\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right) * Z\left(\rho, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right)\\ &\geq \varpi_7 * \varpi_7 \geq \varpi_4 * \varpi_4 > \varpi_0 > \varpi_1.\end{aligned}$$

Also $\varpi_2 = \Theta(\omega, \rho, \ldots, \rho, \varrho) \leq \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right) \diamond \Theta\left(\varsigma, \rho, \ldots, \rho, \frac{\varrho}{2}\right)$.
Again, by using proposition(3.3), we get $\Theta\left(\varsigma, \rho, \ldots, \rho, \frac{\varrho}{2}\right) \leq \Theta\left(\rho, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right)$.
Hence,

$$\begin{aligned}\varpi_2 = \Theta(\omega, \rho, \ldots, \rho, \varrho) &\leq \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right) \diamond \Theta\left(\rho, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right)\\ &\leq \Theta\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right) \diamond \Theta\left(\rho, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right)\\ &\leq (1-\varpi_7) \diamond (1-\varpi_7)\\ &\leq (1-\varpi_5) \diamond (1-\varpi_5) < (1-\varpi_0) < \varpi_2.\end{aligned}$$

Also $\varpi_3 = K(\omega, \rho, \ldots, \rho, \varrho) \leq K\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right) \diamond K\left(\varsigma, \rho, \ldots, \rho, \frac{\varrho}{2}\right)$.
Again, by using proposition (3.3), we get $K\left(\varsigma, \rho, \ldots, \rho, \frac{\varrho}{2}\right) \leq K\left(\rho, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right)$.
Hence,

$$\begin{aligned}\varpi_3 = K(\omega, \rho, \ldots, \rho, \varrho) &\leq K\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2}\right) \diamond K\left(\rho, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right)\\ &\leq K\left(\omega, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right) \diamond K\left(\rho, \varsigma, \ldots, \varsigma, \frac{\varrho}{2^n}\right)\\ &\leq (1-\varpi_7) \diamond (1-\varpi_7)\\ &\leq (1-\varpi_6) \diamond (1-\varpi_6) < (1-\varpi_0) < \varpi_3.\end{aligned}$$

Thus, we have a contradiction and so $(\sum, \mathcal{T}^{(Z,\Theta,K)})$ is Hausdorff. □

Definition 3.8. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS. A sequence (ω_k) in $\sum$ is said to be convergent to some $\omega \in \sum$ with respect to the (Z, Θ, K) if, for every $\varpi \in (0,1)$ and $\varrho > 0$, $\exists$ $n_0 \in \mathbb{N}$ such that
$Z(\omega, \omega_{\mu_1}, \ldots, \omega_{\mu_n}, \varrho) > 1-\varpi$, $\Theta(\omega, \omega_{\mu_1}, \ldots, \omega_{\mu_n}, \varrho) < \varpi$ and $K(\omega, \omega_{\mu_1}, \ldots, \omega_{\mu_n}, \varrho) < \varpi$
for all $\mu_1, \mu_2, \ldots, \mu_n \geq n_0$.

Theorem 3.9. Let $(\sum, Z, \Theta, K, *, \diamond)$ be an NGMS and $\mathcal{T}^{(Z,\Theta,K)}$ be the topology on $\sum$. Then a sequence $(\omega_k) in \sum$ is convergent to ω iff $Z(\omega_k, \omega_k, \ldots, \omega_k, \omega, \varrho) \to 1$, $\Theta(\omega_k, \omega_k, \ldots, \omega_k, \omega, \varrho) \to 0$ and $K(\omega_k, \omega_k, \ldots, \omega_k, \omega, \varrho) \to 0$ as $k \to \infty$ for every $\varrho > 0$.

Proof
Assume that (ω_k) converges to ω.
Then, for every $\varrho > 0$ and $\varpi \in (0,1), \exists\, q_0 \in \mathbb{N}$ such that $\omega_k \in B_\omega^{(Z,\Theta,K)}(\varrho,\varpi),\ \forall\ \ k \geq q_0$.
Consequently, for all $k \geq q_0$, we obtain
$Z(\omega,\omega_k,\omega_k,\ldots,\omega_k,\varrho) > 1-\varpi$, $\Theta(\omega,\omega_k,\omega_k,\ldots,\omega_k,\varrho) < \varpi$ and $K(\omega,\omega_k,\omega_k,\ldots,\omega_k,\varrho) < \varpi$.
This implies that
$1 - Z(\omega,\omega_k,\omega_k,\ldots,\omega_k,\varrho) < \varpi$, $\Theta(\omega,\omega_k,\omega_k,\ldots,\omega_k,\varrho) < \varpi$ and $K(\omega,\omega_k,\omega_k,\ldots,\omega_k,\varrho) < \varpi$.
As a result, we find
$Z(\omega_k,\omega_k,\ldots,\omega_k,\omega,\varrho) \to 1$, $\Theta(\omega_k,\omega_k,\ldots,\omega_k,\omega,\varrho) \to 0$ and $K(\omega_k,\omega_k,\ldots,\omega_k,\omega,\varrho) \to 0$ as $k \to \infty$.
Conversely, suppose that
$Z(\omega_k,\omega_k,\ldots,\omega_k,\omega,\varrho) \to 1, \Theta(\omega_k,\omega_k,\ldots,\omega_k,\omega,\varrho) \to 0$ and $K(\omega_k,\omega_k,\ldots,\omega_k,\omega,\varrho) \to 0$ as $k \to \infty$ for every $\varrho > 0$.
Then, for given $\varpi \in (0,1), \exists\, q_0 \in \mathbb{N}$ so that $1 - Z(\omega,\omega_k,\omega_k,\ldots,\omega_k,\varrho) < \varpi$,
$\Theta(\omega,\omega_k,\omega_k,\ldots,\omega_k,\varrho) < \varpi$ and $K(\omega,\omega_k,\omega_k,\ldots,\omega_k,\varrho) < \varpi\ \forall\ \ k \geq q_0$.
Hence $\omega_k \in B_\omega^{(Z,\Theta,K)}(\varrho,\varpi),\ \forall\ \ k \geq q_0$. Thus (ω_k) is convergent to ω. □

Definition 3.10. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS. A sequence (ω_k) in $\sum$ is stated to be Cauchy with respect to the (Z,Θ,K) if, for every $\varpi \in (0,1)$ and $\varrho > 0, \exists\, q_0 \in \mathbb{N}$ such that $Z(\omega_{\mu_0},\omega_{\mu_1},\ldots,\omega_{\mu_n},\varrho) > 1-\varpi$, $\Theta(\omega_{\mu_0},\omega_{\mu_1},\ldots,\omega_{\mu_n},\varrho) < \varpi$ and $K(\omega_{\mu_0},\omega_{\mu_1},\ldots,\omega_{\mu_n},\varrho) < \varpi$ for all $\mu_0,\mu_1,\ldots,\mu_n \geq q_0$. A NGMS $(\sum, Z, \Theta, K, *, \diamond)$ is considered to be complete if all of the Cauchy sequence (ω_k) in $\sum$ is convergent.

Definition 3.11. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS. Then every convergent sequence (ω_k) is Cauchy in $\sum$.

4. Statistical convergence and Cauchy in NGMS

Statistical convergence of order n defines a sequence's which converge to a limit using a modified density based on the power of the index, where the set of indices for which the terms deviate from the limit has n-natural density zero. For a sequence (ω_k) and a limit ω, this means the proportion of terms $|\omega_k - \omega| \geq \epsilon$ within the first n terms approaches zero as n goes to infinity.
Our goal in this part is to investigate the concept of statistical convergence of sequences in the NGMS $(\sum, Z, \Theta, K, *, \diamond)$. Let $\Gamma \subseteq \mathbb{N}$. $\mathfrak{d}(\Gamma)$ is the asymptotic (or natural) density of the set Γ, is defined as:
$\mathfrak{d}(\Gamma) = \lim_{k\to\infty} \frac{1}{k}|\{n \leq k : n \in \Gamma\}|$ provided the limit exists. Here, $|\Xi|$ denotes the cardinality of the set Ξ. A real sequence (ω_k) is called statistically convergent to $p \in \mathbb{R}$ if $\mathfrak{d}(\{k \in \mathbb{N} : |\omega_k - p| > \xi\}) = 0$ holds for every $\xi > 0$ and the limit is denoted by $\omega_k \xrightarrow{st} p$ (see [6], [17]).

Definition 4.1. Consider the n-product of $\mathbb{N}$,
i.e., $\mathbb{N}^n = \prod_{u=1}^n \mathbb{N}^u$. Let $\Psi \subseteq \mathbb{N}^n$ and $\Psi(k) = \{(\mu_1,\mu_2,\ldots,\mu_n) \in \Psi : \mu_1,\mu_2,\ldots,\mu_n \leq k\}$. Then, the n-dimensional asymptotic density of the set Ψ is defined as $\mathfrak{d}_n(\Psi) = \lim_{k\to\infty} \frac{n!}{k^n}|\Psi(k)|$.
For a subset $\Gamma \subseteq \mathbb{N}$, the n-dimensional asymptotic density of the set Γ is defined as
$\mathfrak{d}_n(\Gamma) = \lim_{k\to\infty} \frac{n!}{k^n}|\Gamma(k)|$, where $\Gamma(k) = \{(k_1,k_2,\ldots,k_n) \in \Gamma^n : k_1,k_2,\ldots,k_n \leq k\}$.

Definition 4.2. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS. A sequence (ω_k) in $\sum$ is statistically convergent to a $\omega \in \sum$ with respect to (Z,Θ,K) if, for every $\varpi \in (0,1)$ and $\varrho > 0$,

$$\mathfrak{d}_n\left(\left\{\begin{array}{c}(\mu_1,\mu_2,\ldots,\mu_n) \in \mathbb{N}^n : Z(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\varrho) \leq 1-\varpi \text{ or} \\ \Theta(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\varrho) \geq \varpi \text{ or } K(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\varrho) \geq \varpi\end{array}\right\}\right) = 0$$

In such a case, we write $\omega_k \xrightarrow{st-(Z,\Theta,K)} \omega$ or $st - \lim_k \omega_k = \omega$.

Lemma 4.3. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS. Suppose (ω_k) is a sequence in $\sum$. Then, for given $\varrho > 0$ and $\varpi \in (0,1)$, the following are equivalent:
(a) $st - \lim_k \omega_k = \omega$.
(b) $\mathfrak{d}_n\left(\left\{\begin{array}{c}(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) > 1 - \varpi, \\ \Theta(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi \text{ and } K(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi\end{array}\right\}\right) = 1$
(c) $\mathfrak{d}_n(\{(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) \leq 1 - \varpi\}) = 0$,
$\mathfrak{d}_n(\{(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : \Theta(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) \geq \varpi\}) = 0$ and
$\mathfrak{d}_n(\{(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : K(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) \geq \varpi\}) = 0$
(d) $\mathfrak{d}_n(\{(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) > 1 - \varpi\}) = 1$,
$\mathfrak{d}_n(\{(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : \Theta(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi\}) = 1$ and
$\mathfrak{d}_n(\{(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : K(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi\}) = 1$.

Proof
(a) ⇒ (b):
By definition $st - \lim \omega_k = \omega$ means that for every $\varpi > 0$, the set of indices where the neutrosophic closeness fails has density zero.
This directly implies that with density one, the conditions $Z > 1 - \varpi, \Theta < \varpi$ and $K < \varpi$ hold simultaneously.
Hence (b) follows.
(b) ⇒ (c):
Condition (b) states that the set where $Z > 1 - \varpi, \Theta < \varpi, K < \varpi$ has density one.
Its complement (where $Z \leq 1 - \varpi$ or $\Theta \geq \varpi$ or $K \geq \varpi$) therefore has density zero.
This gives the three seperate conditions listed in (c).
(c) ⇒ (d):
If the density of the "bad sets" (where $Z \leq 1 - \varpi$, or $\Theta \geq \varpi$, or $K \geq \varpi$) is zero, then the density of the corresponding "good sets" must be one.
Thus, we obtain the three equalities in (d).
(d) ⇒ (a):
If with density one we have $Z > 1 - \varpi, \Theta < \varpi$ and $K < \varpi$, then the sequence satisfies the neutrosophic convergence conditions almost everywhere in the statistical sense.
This is exactly the definition of statistical convergence in NGMS.
Hence (a) follows.
Therefore, all four statements are equivalent. □

Theorem 4.4. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS. Suppose (ω_k) is a sequence in $\sum$ such that $st - \lim_k \omega_k = \omega$.
Then, for any $\varpi \in (0,1)$ and $\varrho > 0$, we have
$\mathfrak{d}_n(\{(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : \omega_{\mu_v} \notin B_\omega^{(Z,\Theta,K)}(\varrho, \varpi) \text{ for every } v \in \{1, 2, \ldots, n\}\}) = 0$

Proof
For given $\varpi \in (0,1)$ and $\varrho > 0$, set
$\Gamma(\varrho, \varpi) = \left\{\begin{array}{c}(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) > 1 - \varpi, \\ \Theta(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi \text{ and } K(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi\end{array}\right\}$ and
$Y(\varrho, \varpi) = \left\{(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : \omega_{\mu_v} \in B_\omega^{(Z,\Theta,K)}(\varrho, \varpi) \text{ for every } v \in 1, 2, \ldots, n\right\}$.
Since $st - \lim_k \omega_k = \omega$, so $\mathfrak{d}_n(\Gamma(\varrho, \varpi)) = 1$. Let $(\mu_1, \mu_2, \ldots, \mu_n) \in \Gamma(\varrho, \varpi)$.
Then $Z(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) > 1 - \varpi$, $\Theta(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi$ and $K(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi$.
By using Proposition (3.6), we can conclude that $\omega_{\mu_v} \in B_\omega^{(Z,\Theta,K)}(\varrho, \varpi)$ for every $v \in \{1, 2, \ldots, n\}$.
Therefore $(\mu_1, \mu_2, \ldots, \mu_n) \in Y(\varrho, \varpi)$ and so $\Gamma(\varrho, \varpi) \subseteq Y(\varrho, \varpi)$.
This implies that $\mathfrak{d}_n(Y(\varrho, \varpi)) = 1$. Hence the result follows. □

Theorem 4.5. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS. If a sequence (ω_k) in $\sum$ is convergent to $\omega \in \sum$ then $st - \lim_k \omega_k = \omega$.

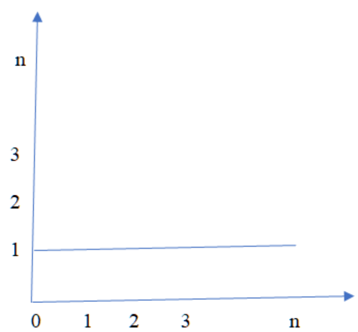
Proof
Suppose that (ω_k) is convergent to $\omega \in \sum$. Then, for every $\varpi \in (0,1)$ and $\varrho > 0, \exists\ t_0 \in \mathbb{N}$ so that $Z(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) > 1 - \varpi, \Theta(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi$ and $K(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi\ \forall\ \mu_1, \mu_2, \ldots, \mu_n \geq q_0$.
Define

$$\Gamma(q) = \left\{ \begin{array}{c} (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : \mu_1, \mu_2, \ldots, \mu_n \leq q, Z(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) > 1 - \varpi, \\ \Theta(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi \text{ and } K(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \varrho) < \varpi \end{array} \right\}$$

Clearly, $|\Gamma(q)| \geq \binom{q-q_0}{n}$. $\Rightarrow \lim_{q\to\infty} \frac{n!}{q^n} |\Gamma(q)| \geq \lim_{q\to\infty} \frac{n!}{q^n} \binom{q-q_0}{n} = 1$.

Consequently, $\lim_{q\to\infty} \frac{n!}{q^n} |(\Gamma(q))^c| = 0$. This implies that $st - \lim_k \omega_k = \omega$.
To demonstrate that the converse of the above mentioned theorem (4.5) is not true, we give the following example. □

Example 4.6. Consider the NGMS $(\sum, Z, \Theta, K, *, \diamond)$ defined in Example (3.2),
where $\sum = \mathbb{R}$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}^+$ such that $g(\omega_0, \omega_1, \ldots, \omega_n) = \max_{0\leq\mu,v\leq n} \{|\omega_\mu - \omega_v|\}$. Now, define the sequence (ω_k) in $\mathbb{R}$ by $\omega_k = \left\{ \begin{array}{ll} k, & \text{if } k = \mu^3, \\ 1, & \text{otherwise} \end{array} \right\} \mu \in \mathbb{N}$. Then, (ω_k) is statistically convergent to 1, but not convergent.



A statistically convergent sequence does not have to be convergent because statistical convergence allows for a set of "bad" terms to have a vanishingly small density, while classical convergence requires all terms to eventually get arbitrarily close to the limit.
Convergence: is strict; every term must get close to the limit.
Statistical convergence: is more forgiving, focusing on the overall trend. It allows for the existence of a "negligible" proportion of terms that deviate from the limit, as long as these deviations become infinitely rare over time.

Theorem 4.7. Let $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS. Suppose (ω_k) is a sequence in $\sum$ such that (ω_k) is statistically convergent. Then $st - \lim_k \omega_k$ is unique.

Proof
Assume that $st - \lim_k \omega_k = \omega$ and $st - \lim_k \omega_k = \varsigma$. We need to show that $\omega = \varsigma$. For given $\varpi \in (0,1)$, choose $\varpi_1 \in (0,1)$ such that $(1 - \varpi_1) * (1 - \varpi_1) > 1 - \varpi$, $\varpi_1 \diamond \varpi_1 < \varpi$ and $\varpi_1 \circ \varpi_1 < \varpi$.
For given $\varpi > 0$, consider the following sets:

$$\Gamma(\varrho, \varpi_1) = \left\{ \begin{array}{c} (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z\left(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \frac{\varrho}{2}\right) \leq 1 - \varpi_1 \text{ or} \\ \Theta\left(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \frac{\varrho}{2}\right) \geq \varpi_1 \text{ or } K\left(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \frac{\varrho}{2}\right) \geq \varpi_1 \end{array} \right\}$$

$$\Xi(\varrho, \varpi_1) = \left\{ \begin{array}{c} (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z\left(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \frac{\varrho}{2}\right) \leq 1 - \varpi_1 \text{ or} \\ \Theta\left(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \frac{\varrho}{2}\right) \geq \varpi_1 \text{ or } K(\omega, \omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \frac{\varrho}{2}) \geq \varpi_1 \end{array} \right\}$$

$$\Gamma^c(\varrho,\varpi_1)=\left\{\begin{array}{c}(\mu_1,\mu_2,\ldots,\mu_n)\in\mathbb{N}^n: Z\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)>1-\varpi_1 \text{ and}\\ \Theta\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)<\varpi_1 \text{ and } K\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)<\varpi_1\end{array}\right\}$$

$$\Xi^c(\varrho,\varpi_1)=\left\{\begin{array}{c}(\mu_1,\mu_2,\ldots,\mu_n)\in\mathbb{N}^n: Z\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)>1-\varpi_1 \text{ and}\\ \Theta\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)<\varpi_1 \text{ and } K\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)<\varpi_1\end{array}\right\}$$

Since $st-\lim_k \omega_k=\omega$ and $st-\lim_k \omega_k=\varsigma$, we have $\mathfrak{d}_n(\Gamma(\varrho,\varpi_1))=0$ and $\mathfrak{d}_n(\Xi(\varrho,\varpi_1))=0$.
Also, by Lemma (4.3), we have $\mathfrak{d}_n(\Gamma^c(\varrho,\varpi_1)=\mathfrak{d}_n(\Xi^c(\varrho,\varpi_1)=1$. Thus $\mathfrak{d}_n(\Gamma(\varrho,\varpi_1)\cup\Xi(\varrho,\varpi_1))=0$, implies $\mathfrak{d}_n((\Gamma(\varrho,\varpi_1)\cup\Xi(\varrho,\varpi_1))^c)=\mathfrak{d}_n(\Gamma(\varrho,\varpi_1)^c\cap\Xi(\varrho,\varpi_1)^c)=1$.
Let $(\mu_1,\mu_2,\ldots,\mu_n)\in\Gamma(\varrho,\varpi_1)^c\cap\Xi(\varrho,\varpi_1)^c$.
Then, by using (NGMS-6), (NGMS-3) and the part (3) of Definition (2.4), we get

$$\begin{aligned}Z(\omega,\varsigma,\ldots,\varsigma,\varrho)&\geq Z\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)*Z\left(\omega_{\mu_n},\varsigma,\varsigma,\ldots,\varsigma,\frac{\varrho}{2}\right)\\&\geq Z\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)*Z\left(\varsigma,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)\\&\geq(1-\varpi_1)*(1-\varpi_1)\\&>1-\varpi.\end{aligned}$$

Also, by using (NGMS -13), (NGMS -10) and part (3) of Definition (2.5), we have

$$\begin{aligned}\Theta(\omega,\varsigma,\ldots,\varsigma,\varrho)&\leq \Theta\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)\diamond\Theta\left(\omega_{\mu_n},\varsigma,\varsigma,\ldots,\varsigma,\frac{\varrho}{2}\right)\\&\leq \Theta\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)\diamond\Theta\left(\varsigma,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)\\&\leq\varpi_1\diamond\varpi_1\\&<\varpi.\end{aligned}$$

Also, by using (NGMS -20), (NGMS -17) and part (3) of Definition (2.5), we have

$$\begin{aligned}K(\omega,\varsigma,\ldots,\varsigma,\varrho)&\leq K\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)\circ K\left(\omega_{\mu_n},\varsigma,\varsigma,\ldots,\varsigma,\frac{\varrho}{2}\right)\\&\leq K\left(\omega,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)\circ K\left(\varsigma,\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\frac{\varrho}{2}\right)\\&\leq\varpi_1\circ\varpi_1\\&<\varpi.\end{aligned}$$

Since $\varpi\in(0,1)$ is arbitrary, we conclude that
$Z(\omega,\varsigma,\ldots,\varsigma,\varrho)=1$, $\Theta(\omega,\varsigma,\ldots,\varsigma,\varrho)=0$ and $K(\omega,\varsigma,\ldots,\varsigma,\varrho)=0\ \forall\ \varrho>0$. Hence $\omega=\varsigma$. □

Definition 4.8. Let $(\sum,Z,\Theta,K,*,\diamond)$ be a NGMS. A sequence (ω_k) in $\sum$ is statistically Cauchy with respect to (Z,Θ,K) if, for every $\varpi\in(0,1)$ and $\varrho>0,\exists\ N=N(\varpi)\in\mathbb{N}$ such that

$$\mathfrak{d}_n\left(\left\{\begin{array}{c}(\mu_1,\mu_2,\ldots,\mu_n)\in\mathbb{N}^n: Z\left(\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\omega_N,\varrho\right)\leq 1-\varpi \text{ or}\\ \Theta\left(\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\omega_N,\varrho\right)\geq\varpi \text{ or } K\left(\omega_{\mu_1},\omega_{\mu_2},\ldots,\omega_{\mu_n},\omega_N,\varrho\right)\geq\varpi\end{array}\right\}\right)=0$$

Theorem 4.9. Let $(\sum,Z,\Theta,K,*,\diamond)$ be a NGMS and (ω_k) to be a sequence in $\sum$ such that (ω_k) is statistically convergent. Then (ω_k) is statistically Cauchy.

Proof
Let $st-\lim_k\omega_k=\omega$. For given $\varpi\in(0,1)$, select $\varpi_1\in(0,1)$ so that $(1-\varpi_1)*(1-\varpi_1)>1-\varpi,\varpi_1\diamond\varpi_1<\varpi$ and $\varpi_1\circ\varpi_1<\varpi$.

For $\varrho > 0$, consider the following sets:

$$\Phi(\varpi_1) = \left\{ \begin{array}{c} (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega, \frac{\varrho}{2}\right) \leq 1 - \varpi_1 \text{ or} \\ \Theta\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega, \frac{\varrho}{2}\right) \geq \varpi_1 \text{ or } K\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega, \frac{\varrho}{2}\right) \geq \varpi_1 \end{array} \right\} \text{ and}$$

$$\Phi(\varpi_1)^c = \left\{ \begin{array}{c} (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega, \frac{\varrho}{2}\right) > 1 - \varpi_1 \text{ and} \\ \Theta\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega, \frac{\varrho}{2}\right) < \varpi_1 \text{ and } K\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega, \frac{\varrho}{2}\right) < \varpi_1 \end{array} \right\}.$$

Since $st - \lim_k \omega_k = \omega$, so $\mathfrak{d}_n(\Phi(\varpi_1)) = 0$ and $\mathfrak{d}_n(\Phi(\varpi_1)^c) = 1$. Let $(v_1, v_2, \ldots, v_n) \in \Phi(\varpi_1)^c$. Then, we have $Z\left(\omega_{v_1}, \omega_{v_2}, \ldots, \omega_{v_n}, \omega, \frac{\varrho}{2}\right) > 1 - \varpi_1$, $\Theta\left(\omega_{v_1}, \omega_{v_2}, \ldots, \omega_{v_n}, \omega, \frac{\varrho}{2}\right) < \varpi_1$ and $K\left(\omega_{v_1}, \omega_{v_2}, \ldots, \omega_{v_n}, \omega, \frac{\varrho}{2}\right) < \varpi_1$. Fix $v_k \in \mathbb{N}$, for some $k \in \{1, 2, \ldots, n\}$.
Then

$$\begin{aligned} Z\left(\omega_{v_k}, \omega, \ldots, \omega, \frac{\varrho}{2}\right) &\geq Z\left(\omega_{v_1}, \omega_{v_2}, \ldots, \omega_{v_n}, \omega, \frac{\varrho}{2}\right) > 1 - \varpi_1 \\ \Theta\left(\omega_{v_k}, \omega, \ldots, \omega, \frac{\varrho}{2}\right) &\leq \Theta\left(\omega_{v_1}, \omega_{v_2}, \ldots, \omega_{v_n}, \omega, \frac{\varrho}{2}\right) < \varpi_1 \text{ and} \\ K\left(\omega_{v_k}, \omega, \ldots, \omega, \frac{\varrho}{2}\right) &\leq K\left(\omega_{v_1}, \omega_{v_2}, \ldots, \omega_{v_n}, \omega, \frac{\varrho}{2}\right) < \varpi_1. \end{aligned}$$

For $(\mu_1, \mu_2, \ldots, \mu_n) \in \Phi(\varpi_1)^c$, we have

$$\begin{aligned} Z(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho) &\geq Z\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega, \frac{\varrho}{2}\right) * Z\left(\omega_{v_k}, \omega, \ldots, \omega, \omega, \frac{\varrho}{2}\right) \\ &> (1 - \varpi_1) * (1 - \varpi_1) > 1 - \varpi, \\ \Theta(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho) &\leq \Theta\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega, \frac{\varrho}{2}\right) \diamond \Theta\left(\omega_{v_k}, \omega, \ldots, \omega, \omega, \frac{\varrho}{2}\right) \\ &< \varpi_1 \diamond \varpi_1 < \varpi \text{ and} \\ K(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho) &\leq K\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega, \frac{\varrho}{2}\right) \circ K\left(\omega_{v_k}, \omega, \ldots, \omega, \omega, \frac{\varrho}{2}\right) \\ &< \varpi_1 \circ \varpi_1 < \varpi. \end{aligned}$$

This implies that

$$\Phi(\varpi_1)^c \subseteq \left\{ \begin{array}{c} (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho\right) > 1 - \varpi, \\ \Theta(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho) < \varpi \text{ and } K(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho) < \varpi \end{array} \right\}$$

Consequently,

$$\mathfrak{d}_n(\Phi(\varpi_1)^c) \leq \mathfrak{d}_n\left(\left\{ \begin{array}{c} (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho) > 1 - \varpi, \\ \Theta(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho) < \varpi \text{ and } K(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho) < \varpi \end{array} \right\}\right).$$

Therefore,

$$\mathfrak{d}_n\left(\left\{ \begin{array}{c} (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho\right) > 1 - \varpi, \\ \Theta\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho\right) < \varpi \text{ and } K\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{v_k}, \varrho\right) < \varpi \end{array} \right\}\right) = 1$$

and thus

$$\mathfrak{d}_n\left(\left\{ \begin{array}{c} (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n : Z\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{\nu_k}, \varrho\right) \leq 1 - \varpi, \\ \Theta\left(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{\nu_k} \varrho\right) \geq \varpi \text{ or } K(\omega_{\mu_1}, \omega_{\mu_2}, \ldots, \omega_{\mu_n}, \omega_{\nu_k} \varrho) \geq \varpi \end{array} \right\}\right) = 0.$$

This completes the proof of the theorem. □

Theorem (4.9)'s converse statement is untrue. To illustrate this, let's examine the following example.

Example 4.10. Let $\sum = (0, 1]$ and $(\sum, Z, \Theta, K, *, \diamond)$ be a NGMS which is defined in Example(4.6). Consider the sequence (ω_k) defined by $\omega_k = \left\{ \begin{array}{ll} 1, & \text{if } k = \mu^3, \\ \frac{1}{k}, & \text{otherwise} \end{array} \right\} \mu \in \mathbb{N}$. In this case (ω_k) is not statistically convergent, but it is statistically Cauchy.

5. Difference between fuzzy sets, intuitionistic fuzzy sets and NGMS

Feature	Fuzzy Metric Space (FMS)	Intuitionistic Fuzzy Metric Space (IFMS)	Neutrosophic Generalized Metric Space (NGMS)
Underlying idea	Based on Zadeh's fuzzy sets (1965) and fuzzy distance concepts	Extends FMS by considering both membership and non-membership values	Extends IFMS by incorporating truth, indeterminacy, and falsity functions with generalized distance
Degree of uncertainty	Handles uncertainty via membership function	Handles uncertainty with membership and non-membership (hesitancy allowed)	Handles uncertainty with three independent components: truth, indeterminacy, and falsity
Metric representation	$M(x,y,t)$ gives degree of closeness between x, y at time t	Pair $(M(x,y,t), N(x,y,t))$ represents membership and non-membership closeness	It has membership, non-membership and indeterminacy values $Z(\alpha, \alpha_{u_1}, \alpha_{u_2}, \ldots, \alpha_{u_n}, \eta) \leq 1 - \zeta$ $\Theta(\alpha, \alpha_{u_1}, \alpha_{u_2}, \ldots, \alpha_{u_n}, \eta) \geq \zeta$ $K(\alpha, \alpha_{u_1}, \alpha_{u_2}, \ldots, \alpha_{u_n}, \eta) \geq \zeta$

Example 5.1. For a cancer patient, in fuzzy sets we can assign the value the possibility to have cancer.
In an intutionistic fuzzy sets, we can assign the value for possibility to have cancer and also the possibility not to have cancer.
In neutrosophic sets, we can assign the value for possibility to have cancer, the possibility not to have cancer and also the value that the patient may or may not have cancer (not able to decide whether the patient has cancer. ie some symptoms shows that the patient have cancer and some symptoms shows that the patient may not have cancer).

6. Applications

In medical diagnostics, data is often incomplete, conflicting and imprecise with different physicians or tests providing varying degrees of certainty and uncertainty. Consider a new medical AI designed to diagnose a rare disease based on a stream of patient data, including biomarker measurements, clinical symptoms and lab result. The system is still in its testing phase and the data is known to be noisy and inconsistent due to variations in measurement equipment and human input. The AI analyzes a sequence of data points, x_j, from different patients to determine a trend converging toward a known state, x. This convergence indicates the system is learning to identify the disease's "fingerprint".

Application of statistical convergence of order n

The patient data is complex and uncertain. For each patient data point, x_j, a neutrosophic metric can be defined based on three values:

1. Truth(T): The degree to which the diagnosis appears correct based on a specific set of strong indicators.
2. Indeterminacy(I): The degree to which the data is vague or contradictory. For instance, some symptoms align with the disease while others do not.
3. Falsity(F): The degree to which the data points to a different, known diagnosis.

The neutrosophic metric on the sequence of diagnoses, (T_j, I_j, F_j) captures the uncertainty.

Medical imaging and image processing:

Neutrosophic sets and their associated metric spaces are used to handle the ambiguity and uncertainty inherent in medical images, which can be noisy or low-contrast. Statistical convergence is applied to analyze sequences of images for:

Denoising: Algorithms that reduce noise while preserving important image features.
Segmentation: Methods for automatically detecting objects or regions of interest, such as tumors in MRI scans.
Classification: Techniques that assist in the automated diagnosis of medical conditions, such as skin lesions from dermoscopy images.

7. Conclusion

In this study, we have proved that statistically convergent sequence has a unique limit. Also, statistically convergent sequence is Cauchy. Applications of statistical convergence of order n in neutrosophic metric spaces are focusing on uncertain information and mathematical operators within these spaces to solve problems in various uncertain settings. This involves using neutrosophic methods to handle vagueness, indeterminacy, and inconsistency in real-world data, providing functional tools to identify the diseases in medical field in Image processing.

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