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Zero-Sum Reinsurance and Investment Differential Game under a Geometric Mean Reversion Model

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Abstract This paper investigates a zero-sum stochastic differential game involving a large insurance company and a small insurance company. The large insurance company has sufficient assets to invest in both a risk-free asset and a risky asset. The price process of the risky asset follows the Geometric Mean Reversion (GMR) model and takes into account dividend payments and federal income tax. The small insurance company invests only in the risk-free asset and is subject to federal income tax on the interest earned. The large insurance company seeks to maximize the expected exponential utility of the difference between its surplus and that of the small insurance company to maintain its surplus advantages, while the small insurance company aims to minimize the same quantity to reduce its disadvantages. We establish the corresponding Hamilton-Jacobi-Bellman equations and derive optimal reinsurance-investment and investment-only optimal strategies. Finally, numerical simulations are performed to illustrate our findings.

Keywords Federal income tax, Geometric Mean Reversion model, A zero-sum stochastic differential game, Expected value principle, Optimal investment strategy.

AMS 2010 subject classifications 91A15, 91B70,91G10, 93E15.

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1. Introduction

In recent years, reinsurance and investment problems have gained traction in the mainstream of insurance and actuarial science research. This comes from the necessity for insurers to manage potential significant losses in the insurance market by purchasing reinsurance contracts. Simultaneously, they may choose to invest in the financial market to generate profits, thereby reducing the costs associated with their insurance operations.

In [7], the author modelled the surplus process using a Brownian motion with drift and the risky asset using a geometric Brownian motion, deriving an optimal strategy that maximizes the expected utility of terminal wealth. The study in [2] explored the maximization of the expected exponential utility of terminal wealth and the minimization of the probability of ruin under a no-shorting constraint. The authors derived explicit expressions for the optimal value functions and the corresponding optimal strategies. In [18], the authors investigated an optimal reinsurance and investment problem for an insurer whose surplus process is approximated by a drifted Brownian motion and obtained closed-form expressions for the optimal reinsurance and investment strategies.

In the standard framework to maximize the logarithmic utility of terminal wealth, [31] determined the optimal investment and risk control strategy for an insurer with insider information using the integral forward approach. Several authors have investigated the optimal reinsurance and investment problem in the sense of minimizing the probability of ruin, as explored in [1, 9, 20], and [34]. Others have considered the case of maximizing the utility

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of terminal wealth, as discussed in [37] and [40]. Recent research addressing optimal reinsurance and investment management for insurance companies can be found in [22, 27, 32, 38], and [39].

The studies cited above predominantly address single-agent optimization problems. However, in insurance markets, insurance companies often benchmark their performance against competitors, which significantly influences decision-making processes. Stochastic differential games, whether zero-sum or non-zero-sum, provide a framework to describe competition among two or more insurance companies. The author in [5] studied a non-zero sum game problem between two insurers, where the problem is transformed into finding a regular solution to a quasi-variational inequality. In [4], the author utilized the regularity theory of non-linear partial differential equations to solve a stochastic differential game involving N players. In continuous time, the authors in [19] investigated a non-zero sum Dynkin game with multiple players. In [35], the authors studied the optimal reinsurance investment game, incorporating variance premium principles. Furthermore, [13] studied a nonzero-sum stochastic differential game between two competing insurers, where the premium is calculated using the principle of variance premium. The investment in risky asset is modelled by a constant elasticity of variance (CEV) model. The authors in [29] studied a zero-sum stochastic differential game in which two insurance companies pay out dividends under non-proportional reinsurance. In [26], the authors examined a zero-sum stochastic differential game between a large and a small insurance company, where the large company can invest in both risk-free and risky assets. The risky assets are modeled by the geometric Brownian motion. An extension of this work is presented in [28], where the risky asset is modeled using the constant elasticity of variance (CEV).

Pandemics like COVID-19 and other economically damaging natural disasters, including hurricanes, earthquakes, and wildfires, profoundly impact insurance companies. For example, the authors of [6] provided a comprehensive empirical analysis of how catastrophic risks affect the homeowner insurance market. The work demonstrated how insurers adapt to such risks by increasing insurance rates, resulting in lower loss ratios during catastrophic incidents. Similarly, in [3], the authors investigated the impact of unexpected catastrophic events, including floods, hurricanes, storms, tornadoes, and wildfires, on the profitability of property casualty insurance companies in the United States. In particular, the article suggested significant policy implications for improving the ability of the insurance industry to stabilize its financial and technical performance. Furthermore, the authors in [33] analyzed the insurability of pandemic risk and how future pandemic coverage could be affected by COVID-19, using underwriting policies and scenario analysis.

Life insurance companies have been subject to tax regulations since 1921, as noted by [21], and this includes tax exemption and tax preferred. In addition, the author in [36] revealed that insurance companies were subject to the same taxes as other corporations. Several authors have examined the taxes of life insurance companies up until 1971, but none of them has considered the question of the incidence of this tax. Using firm data from the New York insurance report for the years 1952 through 1965, the author in [23] fitted a linear multiple regression model. The results showed that insurance companies, like other corporations, are subject to federal income taxation. However, specific guidelines apply to insurance companies based on the type of insurance they offer. For example, as noted by [14], taxable income is based on statutory income with similar adjustments. The tax computation begins with statutory pre-tax income from the underwriting and investment exhibit. During the COVID-19 pandemic, several countries have implemented tax relief measures for industries severely affected. However, initiatives to encourage investment and consumption have been more prevalent in non-Organization for Economic Co-operation and Development (OECD) and non-G20 nations. For example, Kenya reduced its corporate income tax rate from 30% to 25%, while Tanzania's tax relief measures were limited [15].

Although most of the research has been done in non-zero sum games, the zero-sum game plays a crucial role in informing market participants about the extent to which one insurance company may have superior information about market conditions compared to another. Moreover, the opposing objectives of the two insurance companies result in control weights that often have opposite signs, rendering them indefinite. In this paper, we study a zero-sum stochastic differential reinsurance and investment game between two insurance companies under the Geometric Mean Reversion (GMR) model. Each insurance company premium process in our study is determined according to the expected value principle. The GMR model is a stochastic process that is well suited for commodity prices, particularly in the long run. In the GMR model, prices tend to revert to their long-term mean or the average marginal cost of the product, which accounts for the returns on venture capital [11, 12]. In our paper, we assume that the surplus processes of insurance companies are approximated by a Brownian motion with drift, and both companies are allowed to purchase proportional reinsurance. In addition, we assume that a large insurance company has a greater initial wealth compared to a smaller insurance company, allowing investment in both risky and risk-free assets. The risk-free asset is described deterministically, while the risky asset is modeled using the Geometric Mean Reversion (GMR) model, which incorporates dividend payments and federal income tax considerations.

As a startup, a small insurance company is in its early stages and typically faces significant risk and uncertainty. While striving to establish itself in the market, the small insurance company prioritizes steady income over aggressive investment during its expansion phase. However, it can still mitigate risk by purchasing proportional reinsurance and must also account for federal income tax on earned interest. We assume that the wealth of the small insurance company increases with the interest rate. Our objective is to examine a zero-sum stochastic differential game in which, on one hand, a large insurance company seeks to maintain its surplus advantage by maximizing the expected exponential utility of the difference between its surplus and that of the small insurance company at the terminal time. On the other hand, the small insurance company aims to minimize the expected exponential utility of this difference, thereby reducing its financial disadvantage. We consider two problems: one involving a reinsurance-investment problem between large and small insurance companies, and the other focusing on investment decisions for the large insurance company alone.

The paper is organized as follows. Section 2 presents the model formulation for the reinsurance-investment problem between a large and a small insurance company. The problem formulation and the verification theorem are provided in Section 3. Section 4 presents the solution to the reinsurance-investment problem, while Section 5 addresses the solution to the investment-only problem for the large insurance company. The numerical analysis of the reinsurance-investment problem is discussed in Section 6. Section 7 concludes the paper.

2. Model formulation

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration with $\mathcal{F} = \mathcal{F}_T$, where T is a fixed time horizon. In what follows, it is assumed that all stochastic processes are adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$ unless otherwise specified. We assume that $B_1(t)$ and $B_2(t)$ (to be introduced later) are correlated with the correlation coefficient ρ_{12}. We assume that $B_3(t)$ (to be introduced later) is independent of $B_1(t)$ and $B_2(t)$.

2.1. Surplus process

Following the framework in [2] and [9], we model the claim process of a large insurance company and a small insurance company, denoted by E_j, $j \in \{1, 2\}$, according to a Brownian motion with a drift as follows:

$$dE_j(t) = a_j dt - b_j dB_j(t), \tag{1}$$

where a_j and b_j are positive constants. $B_j(t)$ are standard Brownian motions, and $Cov[B_1, B_2] = \rho_{12}t$. According to Equation (1), the same market conditions have an impact on the claims made by both insurance companies. Suppose that the premium of the insurance companies $j \in \{1, 2\}$ is calculated according to the expected value principle; i.e $\kappa_j = a_j(1 + \nu_j)$, where $\nu_j > 0$ is the relative safety loading coefficient of the insurance company $j \in \{1, 2\}$. Therefore, the surplus process of insurance companies $j \in \{1, 2\}$ without reinsurance and investment, according to Equation (1), becomes

$$dY_j(t) = \kappa_j dt - dE_j(t) = a_j\nu_j dt + b_j dB_j(t), \qquad j = 1, 2. \tag{2}$$

To mitigate insurance risk, large and small insurance companies can purchase proportionate reinsurance from reinsurance providers. Let $q_j(t)$ for $j \in \{1, 2\}$ represent the value of the risk exposures of the large insurance company and the small insurance company, respectively, satisfying $0 \le q_1(t)$ and $0 \le q_2(t) \le 1$ since they are allowed to purchase proportional reinsurance. When $q_1(t) \in [0, 1]$, it means that the reinsurance company will compensate the insurer for $100(1 - q_1(t))\%$ of the claims at time t, resulting in a net liability of $100q_1(t)\%$ of the original claims for the insurance company. When $q_1(t) \in [0, \infty]$, the insurer can act as a reinsurer for other insurance companies, which we interpret as the acquisition of new business.

In the proportional reinsurance contract $q_i(t)$, the reinsurance premium is calculated according to the expected value principle, i.e.$(1 + \vartheta_j)a_j(1 - q_j(t))$ where $\vartheta_j > \nu_j$ is the relative safety load coefficient of the reinsurer. With reinsurance, the surplus process of the insurer j becomes

$$dR_j(t) = \big(a_j\nu_j + a_j\vartheta_j - a_j\vartheta_j q_j(t)\big)dt + q_j(t)b_j dB_j(t), \quad j = 1, 2. \tag{3}$$

2.2. The financial market

The financial market consists of a bank account and a stock. The large insurance company has sufficient assets to invest in both a risk-free asset and a risky asset, while the small insurance company invests only in the risk-free asset. The dynamics of the bank account $S_0 = (S_0(t), \ \ 0 \le t \le T)$ at time t is described by the equation

$$\begin{cases} dS_0(t) & = r_0 S_0(t)dt, \\ S_0(0) & = 1. \end{cases} \tag{4}$$

The dynamics of the stock price $S = (S(t), \ \ 0 \le t \le T)$ at time t is described by the GMR model

$$\begin{cases} \mathrm{d}S(t) & = \phi(\alpha - \ln S_t)S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}B_3(t), \\ S_0 & = s_0, \end{cases} \tag{5}$$

where the parameters ϕ, α, σ are positive constants. The coefficient α represents the long-term mean equilibrium, ϕ the speed of convergence to this equilibrium, σ the volatility of the stock, and $B_3(t)$ a standard Brownian motion independent of $B_1(t)$ and $B_2(t)$.

Suppose that the risky asset of the large insurance company pays dividends at a continuous rate, which is proportional to the value of the stock at a constant rate δ, known as the dividend yield. The appropriate expression for the dividend payment is δS. When this dividend is paid over a short period of time, it becomes $\delta S\,\mathrm{d}t$. If the insurance company pays income tax on the dividend, the dynamics for the risky asset becomes:

$$\begin{cases} \mathrm{d}S(t) & = \phi(\alpha + (1 - \lambda_1)\delta - \ln S_t)S(t)\mathrm{d}t + \sigma S(t)\,\mathrm{d}B_3(t), \\ S_0 & = s_0, \end{cases} \tag{6}$$

where λ_1 is the coefficient of income tax rate of the large company.

2.3. Wealth process

Let $\pi(t)$ denote the amount that the large insurance company invests in the risky asset at time t. Then, the remaining wealth $Y_1(t) - \pi(t)$ is invested in the risk-free asset. Let $\{Y_1^{u_1}(t)\}_{t \ge 0}$ be the surplus process of the large insurance company after purchasing reinsurance protection $q_1(t)$ and making investment $\pi(t)$, where $u_1(t) = (\pi(t), q_1(t))$. Then, the wealth process $Y_1^{u_1}(t)$ of the large insurance company with strategy $u_1(t)$ is given by the following dynamics:

$$\begin{cases} dY_1^{u_1}(t) & = \pi(t) \cdot \frac{dS(t)}{S(t)} + (Y_1^{u_1}(t) - \pi(t)) \cdot \frac{dS_0(t)}{S_0(t)} + dR_1(t) \\ & = \Big[r_0 Y_1^{u_1}(t) + \pi(t)\big[\phi(\alpha + (1 - \lambda_1)\delta - \ln S_t) - r_0\big] + \\ & \quad (a_1\nu_1 + a_1\vartheta_1 - a_1\vartheta_1 q_1(t))\Big] dt + \pi(t)\sigma_t dB_3(t) + q_1 b_1 dB_1(t), \\ Y_1^{u_1}(0) & = y_1, \end{cases} \tag{7}$$

where λ_1 is the income tax rate coefficient of a large company.

Since the small insurance company purchases reinsurance and only invests in the risk-free asset, its wealth process

$Y_2^{u_2}(t)$ where $u_2 = \{q_2(t), t \in [0,T]\}$ is governed by the following dynamics:

$$\begin{cases} dY_2^{u_2}(t) &= Y_2^{u_2}(t)\frac{dS_0(t)}{S_0(t)} - (1-\lambda_2)dt + dR_2(t) \\ &= \Big[Y_2^{u_2}(t)r_0 - (1-\lambda_2) + (a_2\nu_2 + a_2\vartheta_2 - a_2\vartheta_2 q_2(t))\Big]dt + q_2(t)b_2 dB_2(t), \\ Y_2^{u_2}(0) &= y_2, \end{cases} \tag{8}$$

where λ_2 is the income tax rate coefficient of the small insurance company.

The large insurance company seeks to maintain its market dominance in terms of its wealth process, while the small insurance company aims to catch up with the large company's wealth. Without loss of generality, we assume $y_1 > y_2$.

Let $Y^{u_1,u_2}(t) := Y_1^{u_1}(t) - Y_2^{u_2}(t)$ denote the difference between the surplus processes of the large and small insurance companies. The dynamics of $Y^{u_1,u_2}(t)$ are given by

$$\begin{cases} dY^{u_1,u_2}(t) &= \Big[r_0 Y^{u_1,u_2}(t) + \pi(t)\phi\big[\alpha + (1-\lambda_1)\delta - \ln S_t\big] - \\ &\qquad (1-\lambda_2) + \Big((a_1\nu_1 + a_1\vartheta_1 - a_1\vartheta_1 q_1(t)) - (a_2\nu_2 + a_2\vartheta_2 \\ &\qquad -a_2\vartheta_2 q_2(t))\Big)\Big]dt + \pi(t)\sigma_t dB_3(t) + q_1 b_1 dB_1(t) - q_2 b_2 dB_2(t), \\ Y^{u_1,u_2}(0) &= y_1 - y_2 = y > 0. \end{cases} \tag{9}$$

Denote $Y_j^{u_1,u_2}(t) = y_j$ for any fixed $t \in [0,T]$. Then, we define the admissible strategy as follows.

Definition 2.1
The strategies $u_1 = \{(\pi(t), q_1(t), t \in [0,T]\}$ and $u_2 = \{q_2(t), t \in [0,T]\}$ are said to be admissible if

(i) $\{u_1(t)\}_{t\in[0,T]}$ and $\{u_2(t)\}_{t\in[0,T]}$ are $\{\mathcal{F}_t\}_{t\in[0,T]}$-progressively measurable processes.

(ii)

$$\Big\{u_1 : \mathbb{E}\Big[\int_0^T (\pi_t^2 + q_1^2(t))\,\mathrm{d}t\Big] < \infty, \quad 0 \le q_1(t)\Big\}.$$
$$\big\{u_2 : 0 \le q_2(t) \le 1\big\}.$$

(iii) $\forall \varrho \in [1, +\infty)$ and $\forall (t,y) \in [0,T] \times \mathbb{R}$, Equation (9) has a unique solution $Y^{u_1,u_2}(t)$ that satisfies $\mathbb{E}_{t,y}\Big[\sup_{s\in[t,T]} |Y^{u_1,u_2}(s)|^{\varrho}\Big] < +\infty$, where $\mathbb{E}_{t,y}[\cdot]$ is the conditional expectation given $Y^{u_1,u_2}(t) = y$.

Let $\Pi \in \{\Pi_1, \Pi_2\}$ denote the admissible set of controls $u \in \{u_1, u_2\}$, and let $\mathcal{U}_S$ denote the set of all admissible strategies.

3. Problem formulation and verification theorem

We consider a utility function $U : \mathbb{R} \to \mathbb{R}$ where U is assumed to be increasing, strictly concave, and satisfies the Inada conditions, that is,

$$\partial_x U(-\infty) = +\infty, \qquad \partial_x U(+\infty) = 0.$$

For a strategy $u \in \{u_1, u_2\}$, we define the value function as follows:

$$G^{u_1,u_2}(t,s,y) = \mathbb{E}[U(Y^{u_1,u_2}(T))|Y^{u_1,u_2}(t) = y, S(t) = s]. \tag{10}$$

The large insurance company seeks to maximize the expected utility of the difference in surplus at the terminal time by adopting a pair of reinsurance and investment strategies to maintain its surplus advantages, while the small insurance company aims to minimize the same quantity to reduce its disadvantages. Suppose that one company's decision is assumed to be completely observed by its opponent.

1. Large insurance company: The objective of the large insurance company is to select the optimal reinsurance investment strategies that maximize the expected payoff $G^{u_1,u_2}(t,s,y)$:

$$\underline{G}(t,s,y) = \sup_{u_1\in\Pi_1}\inf_{u_2\in\Pi_2} G^{u_1,u_2}(t,s,y),$$

where $\underline{G}(t,s,y)$ is the lowest value of the game.

2. Small insurance company: The objective of the small insurance company is select an optimal reinsurance strategy which minimizes the expected payoff $G^{u_1,u_2}(t,s,y)$:

$$\overline{G}(t,s,y) = \inf_{u_2\in\Pi_2}\sup_{u_1\in\Pi_1} G^{u_1,u_2}(t,s,y),$$

where $\overline{G}(t,s,y)$ is the upper value of the game.

A pair of strategy (u_1^*,u_2^*) is said to achieve a Nash equilibrium or equivalently a saddle point for the game if the following inequalities are satisfied for $\forall(u_1,u_2)\in\{\Pi_1,\Pi_2\}$

$$G^{u_1,u_2^*}(t,s,y)\le G^{u_1^*,u_2^*}(t,s,y)\le G^{u_1^*,u_2}(t,s,y) \tag{11}$$

If the game has a saddle point (u_1^*,u_2^*) then it is easy to check that

$$\underline{G}^{u_2^*}(t,s,y) = \overline{G}^{u_1^*}(t,s,y) \tag{12}$$

The value function of the game is as follows:

$$G(t,s,y) = G^{u_1^*,u_2^*}(t,s,y) = \underline{G}^{u_2^*}(t,s,y) = \overline{G}^{u_1^*}(t,s,y) \tag{13}$$

We say that the differential game between the large insurance company and the small insurance company has a value if and only if

$$G(t,s,y) := \underline{G}(t,s,y) = \overline{G}(t,s,y). \tag{14}$$

Let $\mathcal{O}=[0,T]\times\mathbb{R}\times\mathbb{R}$. For any $\Phi(t,s,y)\in\mathcal{C}^{1,2,2}(\mathcal{O})$, we define the following differential operator:

$$\begin{aligned}\mathcal{A}^{u_1,u_2}\Phi(t,s,y) = \Phi_t &+ \Bigg[r_0 y + \pi(t)\Bigg(\phi\big[\alpha+(1-\lambda_1)\delta-\ln s\big]-r_0\Bigg)\\ &-(1-\lambda_2)+\Bigg((a_1\nu_1+a_1\vartheta_1-a_1\vartheta_1 q_1(t))-(a_2\nu_2+a_2\vartheta_2\\ &-a_2\vartheta_2 q_2(t))\Bigg)\Bigg]\Phi_y+\Bigg[\phi(\alpha+(1-\lambda_1)\delta-\ln s)s\Bigg]\Phi_s+\frac{1}{2}\big[\sigma^2 s^2\big]\Phi_{ss}\\ &+\frac{1}{2}\Bigg[\pi^2(t)\sigma^2+q_1^2b_1^2+q_2^2b_2^2-2q_1b_1q_2b_2\rho_{12}\Bigg]\Phi_{yy}+\big[\pi(t)\sigma^2 s\big]\Phi_{sy}.\end{aligned} \tag{15}$$

where Φ_t, Φ_y, Φ_s, Φ_{yy}, Φ_{ss} and Φ_{sy} denote, respectively, the first-order partial derivative with respect to t, the first-order and the second-order partial derivatives with respect to s and y.

For any given strategy u_2 by the small insurance company, let $\underline{G}^{u_2}(t,s,y)$ be the optimal expected utility function of a large insurance company, that is,

$$\underline{G}^{u_2}(t,s,y) = \sup_{u_1\in\Pi_1} G^{u_1,u_2}(t,s,y). \tag{16}$$

Then $\underline{G}^{u_2}(t,s,y)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{u_1\in\Pi_1}\mathcal{A}^{u_1,u_2}\underline{G}^{u_2}(t,s,y) = 0. \tag{17}$$

Similarly, for any given strategy u_1 by the large insurance company, let $\overline{G}^{u_1}(t,s,y)$ be the optimal expected utility function of a small company, that is,

$$\overline{G}^{u_1}(t,s,y) = \inf_{u_2 \in \Pi_2} G^{u_1,u_2}(t,s,y). \tag{18}$$

Then $\overline{G}^{u_1}(t,s,y)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\inf_{u_2 \in \Pi_2} \mathcal{A}^{u_1,u_2}\overline{G}^{u_1}(t,s,y) = 0. \tag{19}$$

Definition 3.1
A pair of optimal investment strategies (u_1^*, u_2^*) achieve a Nash equilibrium for the game if the following inequalities are satisfied:
For all $(u_1, u_2) \in \Pi$

$$G^{u_1,u_2^*}(t,s,y) \leq G^{u_1^*,u_2^*}(t,s,y) \leq G^{u_1^*,u_2}(t,s,y). \tag{20}$$

Following (17) and (19), $G^{u_1,u_2}(t,s,y)$ satisfy the following HJB equations

$$\sup_{u_1 \in \Pi_1} \mathcal{A}^{u_1,u_2^*}G^{u_1,u_2^*}(t,s,y) = 0, \qquad 0 \leq t \leq T, \tag{21}$$

$$\inf_{u_2 \in \Pi_2} \mathcal{A}^{u_1^*,u_2}G^{u_1^*,u_2}(t,s,y) = 0 \qquad 0 \leq t \leq T, \tag{22}$$

with the boundary condition $G^{u_1,u_2}(T,s,y) = U(y)$.
The following verification theorem is essential in solving the associated stochastic control problem.

Theorem 3.1
(Verification Theorem) If there exist a continuous function $J^{u_1,u_2}(t,s,y) \in \mathcal{C}^{1,2}(\mathcal{O})$ and a pair of strategy (u_1^*, u_2^*) satisfying

$$u_1^* = \arg\sup_{u_1 \in \Pi_1} \mathcal{A}^{u_1,u_2^*}J^{u_1,u_2^*}(t,s,y). \tag{23}$$

$$u_2^* = \arg\inf_{u_2 \in \Pi_2} \mathcal{A}^{u_1^*,u_2}J^{u_1^*,u_2}(t,s,y). \tag{24}$$

such that for $\forall(t,s,y) \in \mathcal{O}$, $J^{u_1^*,u_2^*}$, (u_1^*, u_2^*) satisfy equations (21) and (22) with the following property

$$\int_0^t \mathbb{E}[(J_y^{u_1,u_2}(w,s,y))^2]dw < \infty. \tag{25}$$

then (u_1^*, u_2^*) is a Nash equilibrium strategy and the value function of the game is $J^{u_1^*,u_2^*}$, which means $G^{u_1^*,u_2^*}(t,s,y) = J^{u_1^*,u_2^*}(t,s,y)$.

Proof
The proof follows the same argument as in [26]. □

4. Optimal reinsurance and investment

Suppose the insurance companies have an exponential utility function

$$U(\gamma) = -\frac{1}{\gamma}e^{-\gamma}, \qquad \gamma > 0, \tag{26}$$

where γ is a constant absolute risk aversion parameter. The exponential utility function plays a key role in insurance mathematics, since it is the only utility function under which the principle of "zero utility" a fair premium that is independent of the level of reserve of an insurance company [17].

The optimal reinsurance and investment strategies of the large insurance company and the reinsurance strategy of the small insurance company in the zero-sum stochastic differential game under the geometric mean reversion (GMR) model with the expected exponential utility are given by the following theorem, where the large insurance company seeks to maximize the expected exponential utility of the difference between the two insurance companies, while the small insurance company aims to minimize the same quantity.

Theorem 4.1
In the context of the GMR model, the equilibrium strategies and the associated value functions for the problem of maximizing the expected exponential utility for the large insurance company while minimizing it for the small insurance company are presented as follows:

(i.) If $\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2(1-\rho_{12})} \geq -\frac{\gamma}{2}$ and $\rho_{12} < 0$, $-\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}} \leq 1$, then the optimal reinsurance strategies of the large insurance company and the small insurance company is as follows:

$$(q_1^*(t), q_2^*(t)) = \big(0, 1\big), \tag{27}$$

$$\pi^*(t) = \left(\frac{[r_0 - \phi(\alpha + (1-\lambda_1)\delta - \ln s]}{\gamma\sigma^2} - sn_s\right) e^{-r_0(T-t)}, \tag{28}$$

and the value function is given by Equation (67).

(ii.) If $\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2(1-\rho_{12})} \geq -\frac{\gamma}{2}$ and $\rho_{12} \geq 0$, then the optimal reinsurance strategies of the large insurance company and the small insurance company is as follows:

$$(q_1^*(t), q_2^*(t)) = \left(\frac{(\vartheta_1 a_1)}{b_1^2} e^{-r_0(T-t)} + \frac{b_2\rho_{12}}{b_1}, 1\right), \tag{29}$$

while the optimal investment strategy of the large insurance company is the same as that of Equation (28) and the corresponding value function is given in equation (78).

(iii.) If $\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2(1-\rho_{12})} \geq -\frac{\gamma}{2}$, $\rho_{12} < 0$ and $e^{r_0 T} < -\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}}$, we have optimal reinsurance strategies of the large insurance company and the small insurance company given as Equation (29) while the optimal investment strategy of the large insurance company is the same as that of Equation (28) and the corresponding value function is given in equation (78).

(iv.) If $\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2(1-\rho_{12})} \geq -\frac{\gamma}{2}$, $\rho_{12} < 0$, $-\left(\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}}\right) > 1$ and $e^{r_0 T} \geq -\left(\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}}\right)$, we have optimal reinsurance strategies of the large insurance company and the small insuance company as follows:

$$(q_1^*(t), q_2^*(t)) = \begin{cases} (0,1) \quad \text{if} \quad 0 \leq t \leq t_1, \\ \left(\frac{\vartheta_1 a_1}{\gamma b_1^2} e^{-r_0(T-t)} + \frac{b_2 \rho_{12}}{b_1}, 1\right) \text{ if } t_1 < t \leq T. \end{cases} \tag{30}$$

v. If $\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2(1-\rho_{12})} < -\frac{\gamma}{2}$ and $e^{r_0 T} < -\frac{2(\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12})}{\gamma b_1 b_2^2(1-\rho_{12})}$, then the optimal equilibrium reinsurance strategies of the large insurance company and the small insurance company are given as follows:

$$(q_1^*(t), q_2^*(t)) = \left(\frac{\vartheta_1 a_1}{\gamma b_1^2} e^{-r_0(T-t)}, 0\right) \quad \text{if} \quad 0 \leq t \leq T. \tag{31}$$

while the optimal investment strategy of the large insirance company is the same as that of Equation (28) and the corresponding value function is given in Equation (93).

vi. If $\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2 (1-\rho_{12})} < -\frac{\gamma}{2}$, $e^{r_0 T} \geq -\frac{2(\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12})}{\gamma b_1 b_2^2 (1-\rho_{12})}$ and $\rho_{12} \geq 0$, then the optimal equilibrium reinsurance strategies of the large insurance company and the small insurance company are given as follows:

$$(q_1^*(t), q_2^*(t)) = \begin{cases} \left(\frac{\vartheta_1 a_1}{\gamma b_1^2} e^{-r_0(T-t)} + \frac{b_2 \rho_{12}}{b_1}, 1\right) & 0 \leq t \leq t_2, \\ \left(\frac{\vartheta_1 a_1}{\gamma b_1^2} e^{-r_0(T-t)}, 0\right) & \text{if} \quad t_2 \leq t \leq T. \end{cases} \tag{32}$$

while the optimal investment strategy of the large insurance company is the same as that in Equation (28) and the corresponding value function is given in Equation (98).

vii. If $\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2 (1-\rho_{12})} < -\frac{\gamma}{2}$, $e^{r_0 T} \geq -\frac{2(\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12})}{\gamma b_1 b_2^2 (1-\rho_{12})}$, $\rho_{12} < 0$, and $-\left(\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}}\right) \leq 1$, then the optimal equilibrium reinsurance strategies of the large insurance company and the small insurance company are given as follows:

$$(q_1^*(t), q_2^*(t)) = \begin{cases} (0, 1) & 0 \leq t \leq t_2, \\ \left(\frac{\vartheta_1 a_1}{\gamma b_1^2} e^{-r_0(T-t)}, 0\right) & \text{if} \quad t_2 \leq t \leq T. \end{cases} \tag{33}$$

while the optimal investment strategy of the large company is the same as that in Equation (28) and the corresponding value function is given in Equation (102).

viii. If $\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2 (1-\rho_{12})} < -\frac{\gamma}{2}$, $e^{r_0 T} \geq -\frac{2(\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12})}{\gamma b_1 b_2^2 (1-\rho_{12})}$, $\rho_{12} < 0$, $-\left(\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}}\right) > 1$, and $e^{r_0 T} < -\left(\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}}\right)$, then the optimal equilibrium reinsurance strategies of the large insurance company and the small insurance company are the same as that of (30), while the optimal investment strategy of the large insurance company is the same as that in Equation (28) and the corresponding value function is given in Equation (102).

xi. If $\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2 (1-\rho_{12})} < -\frac{\gamma}{2}$, $e^{r_0 T} \geq -\frac{2(\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12})}{\gamma b_1 b_2^2 (1-\rho_{12})}$, $\rho_{12} < 0$, $-\left(\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}}\right) > 1$, and $e^{r_0 T} \geq -\left(\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}}\right)$, then the optimal equilibrium reinsurance strategies of the large company and the small insurance company is given as follows:

$$(q_1^*(t), q_2^*(t)) = \begin{cases} \big(0, 1\big) & \text{if} \quad 0 \leq t \leq t_3, \\ \left(\frac{(\vartheta_1 a_1)}{b_1^2} e^{-r_0(T-t)} + \frac{b_2 \rho_{12}}{b_1}, 1\right) & \text{if } t_3 \leq t \leq t_2, \\ \left(\frac{\vartheta_1 a_1}{\gamma b_1^2} e^{-r_0(T-t)}, 0\right) & \text{if} \quad t_2 \leq t \leq T. \end{cases} \tag{34}$$

while the optimal investment strategy of the large insurance company is the same as that in Equation (28) and the corresponding value function is given in Equation (106).

Remark 4.1
In practice, when the optimal reinsurance strategy of the company satisfies the condition

$$\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2 (1-\rho_{12}^2)} \geq -\frac{\gamma}{2},$$

the insurer retains a reasonable portion of the risk without excessive reliance on reinsurance. This threshold aligns risk retention with tolerance, supports financial stability, and shapes investment choices. A moderately risk-averse insurer retains more risk and invests in higher-yield assets, while highly risk-averse insurers cede more risk and prefer safer investments. The condition ensures an effective balance between risk transfer, retention, and profitability in dynamic reinsurance and investment markets.

Proof
The proof to Theorem 4.1 is given in the Appendix section. □

5. Optimal investment without reinsurance

In this section, only the investment problem is considered. In the investment-only case, the large insurance company is allowed to invest its surplus in both a risk-free asset and a risky asset. The risky asset follows a Geometric Mean Reversion (GMR) model that accounts for dividend payments and federal income tax. However, the company is not allowed to purchase reinsurance. The wealth process of the large insurance company $\bar{Y}^{\pi_1(t)}(t)$ is given by

$$\begin{aligned} d\bar{Y}^{\pi_1(t)}(t) &= \pi_1(t) \cdot \frac{dS(t)}{S(t)} + (\bar{Y}^{\pi_1(t)}(t) - \pi_1(t)) \cdot \frac{dS_0(t)}{S_0(t)} \\ &= r_0 \bar{Y}^{\pi_1(t)}(t) + \pi_1(t)\big[\phi(\alpha + (1-\lambda_1)\delta - \ln S_t) - r_0\big] + \pi_1(t)\sigma_t dB_3(t). \end{aligned} \tag{35}$$

Definition 5.1
A strategy $\bar{u}_1(t) = \{\pi_1(t), t \in [0,T]\}$ is said to be admissible if

(i) $\{\bar{u}_1(t)\}_{t\in[0,T]}$ is $\{\mathcal{F}_t\}$-progressively measurable process.
(ii)

$$\mathbb{E}\left[\int_0^T (\pi_1(t))^2 \, \mathrm{d}t\right] < \infty,$$

(iii) $\forall \varrho \in [1, +\infty)$ and $\forall (t,y) \in [0,T] \times \mathbb{R}$, Equation (35) has a unique solution $\bar{Y}^{\bar{u}_1}(t)$ which satisfies $\mathbb{E}_{t,\bar{y}}\Big[\sup_{s\in[t,T]} |\bar{Y}^{\bar{u}_1}(s)|^{\varrho}\Big] < +\infty$, where $\mathbb{E}_{t,\bar{y}}[\cdot]$ is the conditional expectation given $\bar{Y}^{\bar{u}_1}(t) = \bar{y}$.

We denote the set of all admissible strategies by Π_1. Suppose that a large company is interested in maximizing the expected utility of its wealth at the end of the term T. For a strategy $\pi_1(t)$, we define the value function as follows:

$$\bar{G}^{\pi_1(t)}(t,s,\bar{y}) = \mathbb{E}_{t,s,\bar{y}}[U(\bar{Y}_T^{\pi_1(t)})|\bar{Y}_t = \bar{y}, S(t) = s]. \tag{36}$$

The objective is to find the optimal value function

$$\bar{G}(t,s,\bar{y}) = \sup_{\pi_1 \in \Pi_1} \bar{G}^{\pi_1(t)}(t,s,\bar{y}), \tag{37}$$

and the optimal investment strategy $\pi_1(t)$ such that we have

$$\bar{G}^{\pi_1(t)}(t,s,\bar{y}) = \bar{G}(t,s,\bar{y}). \tag{38}$$

The stochastic optimal control problem (37) is solved by maximizing the performance function (36) subject to the wealth equation (35).

As in the previous sections, we let $\mathcal{O} = [0,T] \times \mathbb{R} \times \mathbb{R}$. We denote $C^{1,2,2}(\mathcal{O})$ as the space of functions F such that F and its partial derivatives F_t, F_s $F_{\bar{y}}$, $F_{\bar{y}\bar{y}}$, F_{ss} and $F_{s\bar{y}}$ are continuous on $\mathcal{O}$. We define a differential operator for any $F(t,s,\bar{y}) \in C^{1,2,2}(\mathcal{O})$:

$$\begin{aligned} \mathcal{L}^{\bar{u}_1} F(t,s,\bar{y}) &= F_t + \left[r_0\bar{y} + \pi_1(t)\Big(\phi\big[\alpha + (1-\lambda_1)\delta - \ln s\big] - r_0\Big)\right] F_{\bar{y}} \\ &+ \Big[\phi(\alpha + (1-\lambda_1)\delta - \ln s)s\Big] F_s + \frac{1}{2}\big[\sigma^2 s^2\big] F_{ss} + \frac{1}{2}\Big[\pi_1^2(t)\sigma^2\Big] F_{\bar{y}\bar{y}} \\ &+ \big[\pi_1(t)\sigma^2 s\big] F_{s\bar{y}}. \end{aligned} \tag{39}$$

From which we obtain an optimal investment strategy as follows:

$$\pi_1^*(t) = -\frac{\big(\phi(\alpha + (1-\lambda_1)\delta - \ln s) - r_0\big)}{\sigma^2\gamma}\frac{F_{\bar{y}}}{F_{\bar{y}\bar{y}}} - s\frac{F_{s\bar{y}}}{F_{\bar{y}\bar{y}}}. \quad (40)$$

We observe that $\pi_1(t)$ depends on the unknown value function F and its partial derivatives.

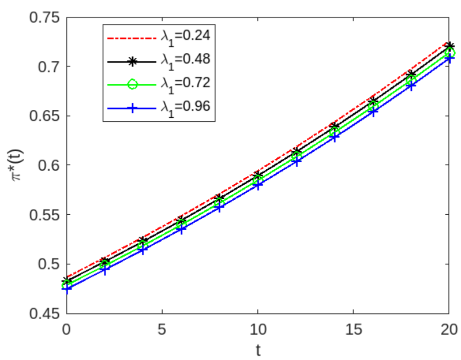
Therefore, substituting (43) into (40), we get the following optimal investment strategy for the large insurance company:

$$\pi_1^*(t) = \left(\frac{\big(\phi[\alpha + (1-\lambda_1)\delta - \ln s] - r_0\big)}{\gamma\sigma^2} - sn_s\right)e^{-r_0(T-t)}. \quad (41)$$

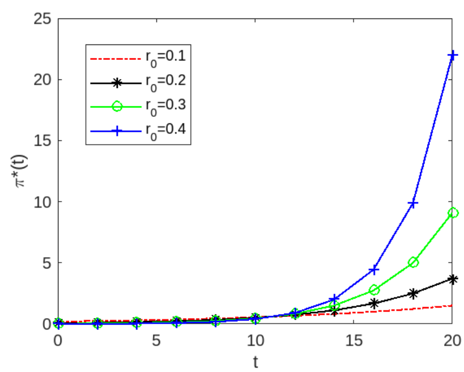
6. Numerical analysis on the reinsurance-investment problem

In this section, we present numerical analyses to illustrate our results. Since the equilibrium reinsurance strategy of the small insurance company is 1 or 0, we focus on analysing the equilibrium reinsurance-investment strategy of the large insurance company. Unless otherwise stated, the following baseline parameters are used throughout the numerical analysis: $r_0 = 0.02$, $T = 10$, $t = 5$, $\delta = 0.1$, $\sigma = 2$, $\gamma = 0.5$, $\phi = 0.43$, $\alpha = 7.55$, $\lambda_1 = 0.30$, $a_1 = 0.1$, $a_2 = 0.2$, $\theta = 1.2$, $\rho_{12} = \pm 0.5$, $b_1 = 1$, $s_t = 0.10$, $n_s = 2$ and $b_2 = 1$.

6.1. Numerical analysis of the optimal investment strategy



(a) The effect of tax rate coefficient λ_1 of income tax on the optimal investment strategy $\pi^*(t)$ of a large company.

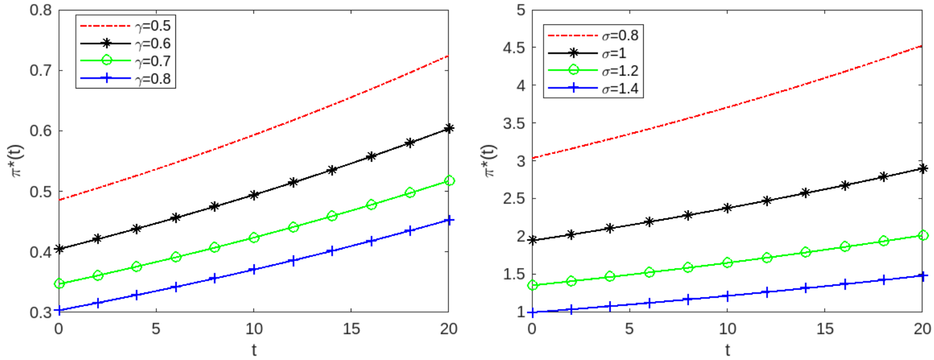


(b) The effect of interest rate r_0 on the optimal investment strategy $\pi^*(t)$ of large insurance company

Figure 1. Optimal investment strategy $\pi^*(t)$ of the large insurance company against (a) tax rate coefficient λ_1 and (b) interest rate r_0.

Figure 1(a) illustrates the sensitivity of the optimal investment strategy $\pi^*(t)$ to the income tax rate coefficient λ_1. As λ_1 increases, the optimal investment $\pi^*(t)$ decreases. This occurs because higher taxes reduce after-tax returns, making some investment opportunities less attractive. Consequently, investors tend to reduce allocations to risky or high-yield assets and favor safer or tax-advantaged options to maintain their desired level of risk-adjusted utility. In effect, higher taxation encourages a more cautious investment approach, diminishing expected wealth in accordance with the investor's risk aversion.

In Figure 1(b), we observe that the optimal investment strategy $\pi(t)$ monotonically increases as the risk-free interest rate r_0 increases. This conclusion seems counterintuitive. As r_0 increases, the return from the risk-free asset becomes more attractive, which would typically lead the insurer to allocate more wealth to the risk-free asset



(a) The effect of γ on the optimal investment strategy $\pi^*(t)$ of a large company.

(b) The effect of σ on the optimal investment strategy $\pi^*(t)$ of a large company.

Figure 2. Optimal investment strategy $\pi^*(t)$ of a large company against (a) γ and (b) σ.

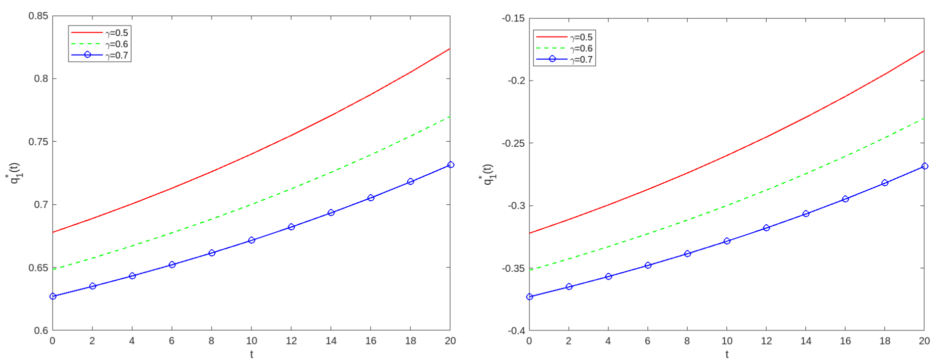
and reduce investment in the risky asset. Therefore, we would expect $\pi(t)$ to decrease with increasing r_0, not to increase.

Figure 2 (a) shows the relationship between the optimal investment strategy $\pi^*(t)$ and the risk aversion coefficient γ. The higher the risk aversion coefficient γ, the more risk averse the insurer. As the risk aversion coefficient γ increases, the insurer will reduce her investments in risky assets to control risk, hence the less $\pi^*(t)$.

Figure 2 (b) shows the effects of volatility σ on the optimal investment strategy $\pi(t)$. The optimal investment strategy $\pi(t)$ decreases with increasing volatility. The higher the volatility, the riskier the risky asset and the less the insurance company will wish to invest in the risky asset.

6.2. Numerical analysis of the optimal reinsurance strategy

The relationship between the risk aversion coefficient γ and the optimal reinsurance strategy $(q_1^*(t))$ is shown in Figure 3, In both scenarios (a) and (b), the optimal reinsurance strategy $q_1^*(t)$ increases with a lower risk aversion coefficient γ. This result suggests that the insurer will purchase less reinsurance, which is sometimes one of the largest expenses for insurers, as he becomes less risk-averse. It is preferable for insurers in a market to rely less on reinsurance, since this could result in an insurer's disproportionately large risk and eventual chance of insolvency.

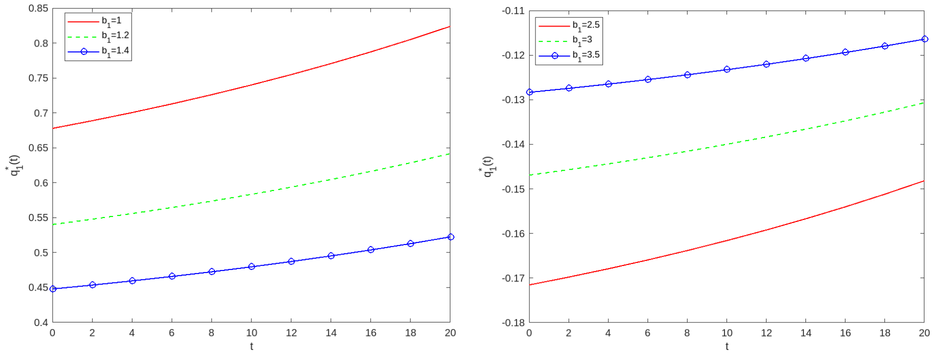


(a) The effect of γ on the optimal reinsurance strategy $q_1^*(t)$ with positive coefficient

(b) The effect of γ on the optimal reinsurance strategy $q_1^*(t)$ with negative coefficient.

Figure 3. Optimal reinsurance strategy $q_1^*(t)$ against γ.

Figure 4 shows the relationship between the optimal reinsurance strategy $q_1^*(t)$ and b_1. In scenario (a) when $\rho_{12} > 0$ the optimal reinsurance strategy $q_1^*(t)$ increases as b_1 decreases. The insurance company will take on more risk independently and buy less reinsurance when b_1 declines. In scenario (b) when $\rho_{12} < 0$ the optimal reinsurance strategy $q_1^*(t)$ decreases from negative values as b_1 increases. The insurance company will buy more reinsurance while taking on less risk on its own when b_1 increases.

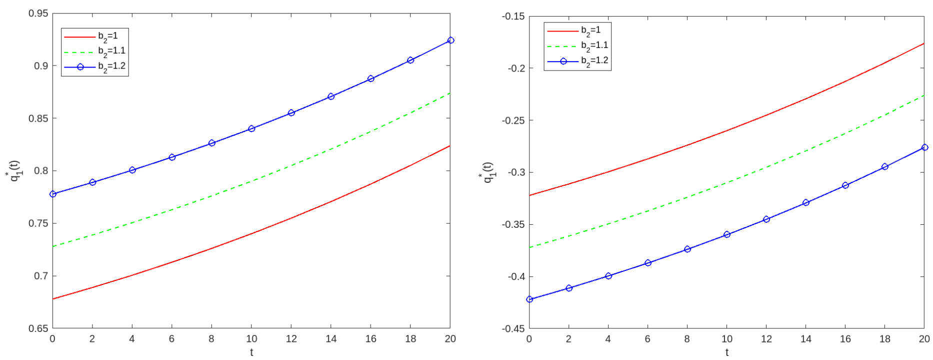


(a) The effect of b_1 on the optimal reinsurance strategy $q_1^*(t)$ with positive coefficient

(b) The effect of b_1 on the optimal reinsurance strategy $q_1^*(t)$ with negative coefficient.

Figure 4. Optimal reinsurance strategy $q_1^*(t)$ against b_1.

Figures 5 (a) and (b) indicate that b_2 has a positive influence on the reinsurance strategy of the large insurance company $q_1^*(t)$ if $\rho_{12} > 0$ and a negative effect if $\rho_{12} < 0$.This can be understood by the fact that the claims processes of the large insurance company and the small insurance company have more serious fluctuations, which cause b_2 to increase when $\rho_{12} > 0$. As a result, the large insurance company will take on greater risk, whereas when $\rho_{12} < 0$, the claim process of the large insurance company fluctuates less and the volatility of the small insurance company increases. So, there will be a lesser risk for the large insurance company.



(a) The effect of b_2 on the optimal reinsurance strategy $q_1^*(t)$ with positive coefficient

(b) The effect of b_2 on the optimal reinsurance strategy $q_1^*(t)$ with negative coefficient.

Figure 5. Optimal reinsurance strategy $q_1^*(t)$ against b_2.

Figure 6 shows that the optimal reinsurance strategy $q_1^*(t)$ increases with an increase in ρ_{12}. If ρ_{12} increases, the amount of money invested in risky assets will decrease, while the cost of paying reinsurance will increase, and the insurance company's retention level will decrease. Therefore, the optimal reinsurance strategy $q_1^*(t)$ increases with increasing ρ_{12}.

Figure 7 shows the effects of ϑ_1 on the large insurance company reinsurance strategy $q_1^*(t)$ with positive and negative correlation coefficients. We see in both (a) and (b) that ϑ_1 is increasing, which produces a greater

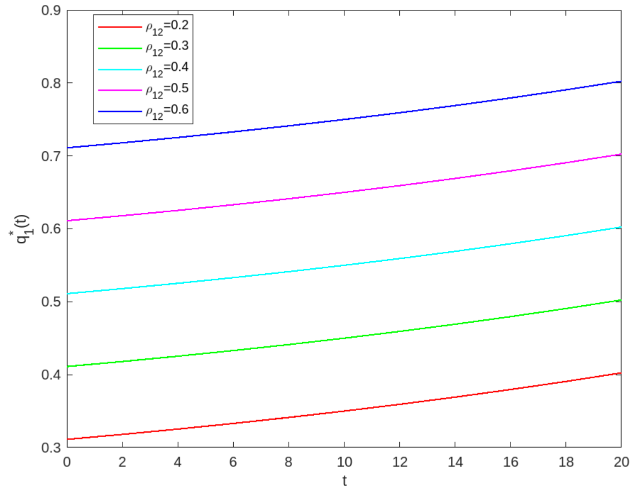
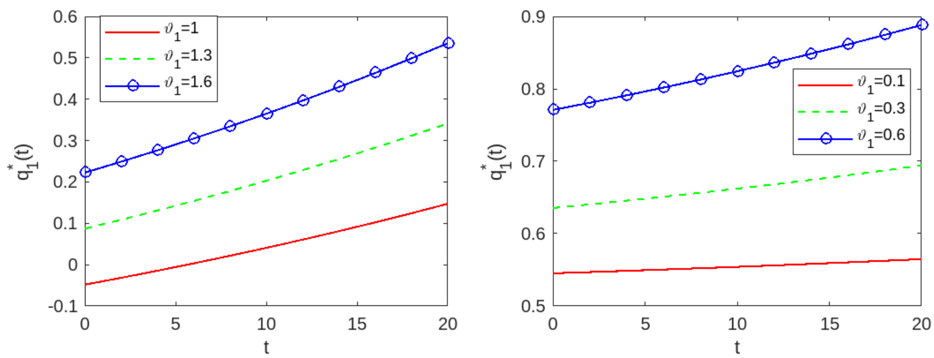


Figure 6. The effect of ρ_{12} on the optimal reinsurance strategy $q_1^*(t)$ with positive coefficient.



(a) The effect of θ on the optimal reinsurance strategy $q_1^*(t)$ with positive coefficient

(b) The effect of ϑ_1 on the optimal reinsurance strategy $q_1^*(t)$ with negative coefficient.

Figure 7. Optimal reinsurance strategy $q_1^*(t)$ against ϑ_1.

reinsurance strategy. This is because the cost of reinsurance will increase as ϑ_1 increases and the large insurance company would rather assume more risk on its own and buy less reinsurance to maintain steady revenue.

7. Conclusion

In this article, we investigate the reinsurance and investment problem for both large and small insurance companies, as well as the investment-only problem for the large insurance company. The large insurance company is assumed to have sufficient assets to invest in both a risk-free asset and a risky asset, with the price process of the risky asset following the Geometric Mean Reversion (GMR) model. In addition, the large insurance company is assumed to pay dividends and federal income tax. The small insurance company is assumed to invest only in the risk-free asset and is subject to federal income tax on the interest earned. Both companies purchase reinsurance, with the reinsurance premium determined by the expected value principle. We first formulate a general zero-sum game, where the large insurance company seeks to maximize the expected exponential utility of terminal wealth to maintain its surplus advantage, while the small insurance company seeks to minimize the same quantity to reduce its disadvantage. We provide the corresponding verification theorem. Then, we solved two cases: the investment-reinsurance case and the investment-only case, deriving closed-form expressions for the optimal strategies and their corresponding value functions. Finally, a numerical analysis was performed to illustrate the impact of model parameters on both the reinsurance-investment and investment-only optimal strategies. The key findings indicate that the tax rate coefficient λ_1 for the large company influences its optimal investment strategy in both scenarios, whether reinsurance is available or not. Furthermore, it was observed that the optimal investment strategy remains unchanged regardless of the presence or absence of reinsurance. Furthermore, the tax rate coefficient λ_2 for the small company does not impact the optimal investment strategy of the large insurance company.

Future research could extend this study by examining a scenario in which both companies have the opportunity to invest in risky assets, allowing an analysis of the impact of the income tax coefficient on the small company.

Allowing a small insurer to invest in risky assets, rather than restricting it to risk-free investments, alters its strategic choices and the resulting game-theoretic outcomes. With access to risky assets, the insurer can pursue higher potential returns while managing increased risk, which may reduce its reliance on reinsurance and enhance expected wealth. This expanded strategy space affects the Nash equilibrium, as both small and large insurers now jointly optimize reinsurance and investment decisions, taking into account factors such as the correlation between investment returns and insurance risks, their risk aversion, and expected asset returns. Consequently, equilibrium strategies may adjust, with the small insurer retaining more risk or modifying reinsurance levels to complement investment performance. The associated value functions become more complex, as wealth now depends on stochastic investment returns, increasing both potential utility and variability. Overall, permitting risky investments creates an integrated optimization problem that more accurately reflects real-world insurer behavior, where investment and reinsurance decisions are interconnected and jointly determine optimal strategies and expected outcomes.

In this context, the choice of utility function becomes particularly important. Exponential utility functions are widely applied in reinsurance and investment games because they are mathematically tractable and lead to closed-form solutions. A defining feature is that they imply constant absolute risk aversion (CARA), meaning that an individual's risk aversion does not depend on their wealth level. While this simplifies analysis, it also introduces limitations. In practice, risk aversion often decreases as wealth increases [10], which is better captured by constant relative risk aversion (CRRA) utility forms such as power or logarithmic utility. Exploring CRRA utilities in the context of risky investments would therefore provide a more realistic assessment of optimal strategies for both insurers.

Acknowledgement

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Appendices

A. Proof of Theorem 4.1

For the exponential utility function given in (26), we look for a candidate of the HJB (21) and (22) in the form

$$G(t,s,y) = -\frac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(y-m(t))+n(t,s)]}, \tag{42}$$

with the boundary conditions given by $n(T)=1, m(T)=0$. Taking the partial derivatives of (42), we obtain

$$\begin{aligned} &G_t = -\gamma\big[-r_0\gamma e^{r_0(T-t)}(y-m(t)) - m_t e^{r_0(T-t)} + n_t\big]G, \quad G_y = -\gamma e^{r_0(T-t)}G, \\ &G_s = -\gamma n_s G, \quad G_{ss} = (\gamma^2 n_s^2 - \gamma n_{ss})G, \quad G_{sy} = \gamma^2 e^{r_0(T-t)} n_s G, \quad G_{yy} = \gamma^2 e^{2r_0(T-t)}G. \end{aligned} \tag{43}$$

From (43), we observe that $G_{yy} < 0$, so the infimum in (22) is reached at $q_2^*(t) = 0$ or $q_2^*(t) = 1$. Suppose that

$$\widetilde{q_2}(t) = \left(\frac{-\vartheta_2 a_2 b_1 + \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2 (1-\rho_{12}^2)}\right) \frac{G_y}{G_{yy}}. \tag{44}$$

For a large company, the optimal investment and reinsurance strategies are given as follows:

$$\pi^*(t) = -\frac{\Big(\phi(\alpha + (1-\lambda_1)\delta - \ln s) - r_0\Big)}{\sigma^2} \frac{G_y}{G_{yy}} - \frac{sG_{sy}}{G_{yy}}, \tag{45}$$

and

$$q_1^*(t) = \frac{(\vartheta_1 a_1)}{b_1^2} \frac{G_y}{G_{yy}} + \frac{q_2^*(t) b_2 \rho_{12}}{b_1}. \tag{46}$$

By substituting $\pi^*(t)$, $q_1^*(t)$ and $q_2^*(t)$ into (21) and (22) we obtain the following:

$$\begin{aligned}
&G_t + r_0 y G_y - \frac{1}{2}\frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2 G_y^2}{\sigma^2 G_{yy}} + \\
&\frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]G_y^2}{\sigma^2 G_{yy}} - \frac{s[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]G_y G_{sy}}{G_{yy}} \\
&-\frac{1}{2}\frac{r_0^2 G^2 y}{\sigma^2 G_{yy}} + \frac{s r_0 G_y G_{sy}}{G_{yy}} - (1-\lambda_2)G_y + a_1\nu_1 G_y + a_1\vartheta_1 G_y - \\
&a_1\vartheta_1 q_1^*(t) G_y - a_2\nu_2 G_y - a_2\vartheta_2 G_y + a_2\vartheta_2 q_2^*(t) G_y + \\
&[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]sG_s + \frac{1}{2}\sigma^2 s^2 G_{ss} - \frac{s^2\sigma^2 G_{sy}^2}{G_{yy}} \\
&+\frac{1}{2}q_1^*(t)^2 b_1^2 G_{yy} + \frac{1}{2}q_2^*(t)^2 b_2^2 G_{yy} - q_1^*(t) b_1 q_2^*(t) b_2 \rho_{12} G_{yy} = 0.
\end{aligned} \tag{47}$$

By inserting the derivatives in (43) into (47), we obtain

$$\begin{aligned}
&n_t + r_0 s n_s + \frac{1}{2}s^2\sigma^2\gamma n_s^2 + \frac{1}{2}\sigma^2 s^2 n_{ss} + \frac{1}{2}\frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma\sigma^2} \\
&+ \frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2} + m(t) r_0 e^{r_0(T-t)} - \\
&m_t e^{r_0(T-t)} - (1-\lambda_2)e^{r_0(T-t)} + a_1\nu_1 e^{r_0(T-t)} + a_1\vartheta_1 e^{r_0(T-t)} \\
&- a_1\vartheta_1 q_1^*(t) e^{r_0(T-t)} - a_2\nu_2 e^{r_0(T-t)} - a_2\vartheta_2 e^{r_0(T-t)} + a_2\vartheta_2 q_2^*(t) e^{r_0(T-t)} - \\
&\frac{1}{2}q_1^*(t)^2 b_1^2 \gamma e^{r_0(T-t)} - \frac{1}{2}q_2^*(t)^2 b_2^2 \gamma e^{r_0(T-t)} + q_1^*(t) b_1 q_2^*(t) b_2 \gamma \rho_{12} e^{r_0(T-t)} = 0.
\end{aligned} \tag{48}$$

We solve (21) and (22) in the following cases.

A.1. Case 1

$$\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2 (1-\rho_{12}^2)} \geq -\frac{\gamma}{2}. \tag{49}$$

If

$$\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2 (1-\rho_{12}^2)} \geq -\frac{\gamma}{2},$$

then

$$\widetilde{q_2}(t) < \frac{1}{2}.$$

The optimal equilibrium reinsurance strategies of a large company and a small company are given as follows:

$$q_2^*(t) = 1,$$

and

$$\widetilde{q_1}(t) = \frac{\vartheta_1 a_1}{b_1^2}\frac{G_y}{G_{yy}} + \frac{b_2\rho_{12}}{b_1}. \tag{50}$$

Equation (48) shows that $\widetilde{q_1}(t) \in [0,\infty)$ is equivalent to

$$t \geq t_1 = T - \frac{1}{r_0}\ln\left(\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}}\right). \tag{51}$$

A.1.1. ***Conditions of Case 1*** **Condition 1.**
If $\rho_{12} > 0$ and $e^{r_0 T} < \frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}}$ or when $\rho_{12} < 0$
and $\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \cdot \rho_{12}} \geq 1$, then the optimal reinsurance strategies of a large company and a small company are given by

$$(q_1^*(t), q_2^*(t)) = \big(0, 1\big). \tag{52}$$

By substituting equation (27) into (48), we obtain

$$\begin{aligned}
&n_t + r_0 s n_s + \frac{1}{2}s^2\sigma^2\gamma n_s^2 + \frac{1}{2}\sigma^2 s^2 n_{ss} + \frac{1}{2}\frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma\sigma^2} + \\
&\frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2} + e^{r_0(T-t)}\Big\{m(t)r_0 - m_t - (1-\lambda_2) + a_1\nu_1 + \\
&a_1\vartheta_1 - a_2\nu_2 - a_2\vartheta_2 + a_2\vartheta_2 - \frac{1}{2}b_2^2\gamma\Big\} = 0.
\end{aligned} \tag{53}$$

Equation (53) can be decomposed into two equations by separating variables:

$$m_t - m(t)r_0 + (1-\lambda_2) - a_1\nu_1 - a_1\vartheta_1 - a_2\nu_2 + \frac{1}{2}b_2^2\gamma = 0. \tag{54}$$

$$\begin{aligned}
&n_t + r_0 s n_s + \frac{1}{2}s^2\sigma^2\gamma n_s^2 + \frac{1}{2}\sigma^2 s^2 n_{ss} + \frac{1}{2}\frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma\sigma^2} + \\
&\frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2} = 0.
\end{aligned} \tag{55}$$

Taking into account the boundary condition $m(T) = 0$, the solution to equation (72) is

$$\begin{aligned}
&m(t) = -\frac{1}{r_0}\big[-(1-\lambda_2) + a_1\nu_1 + a_1\vartheta_1 + a_2\nu_2\big][1 - e^{-r_0(T-t)}] + \\
&\frac{1}{2r_0}b_2^2\gamma[1 - e^{-r_0(T-t)}].
\end{aligned} \tag{56}$$

For equation (55), we are going to have the following power transformation.
If we let $n(t,s) = N(t,w)$ where $w = s^{-2}$ then we have the following derivatives.

$$n_t = N_t \qquad n_s = -2s^{-3}N_w, \qquad n_{ss} = 6s^{-4}N_w + 4s^{-6}N_{ww}. \tag{57}$$

with the boundary condition $N(T,w) = 0$. By substituting equation (57) into (55) we have the following equation:

$$\begin{aligned}
&N_t - (2r_0 - 3\sigma^2)wN_w + 2\sigma^2\gamma w^2 N_w^2 + 2\sigma^2 w^2 N_{ww} + \frac{1}{2}\frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma\sigma^2} + \\
&\frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2} = 0.
\end{aligned} \tag{58}$$

We conjecture a solution to (58) as follows

$$N(t,w) = J(t) + K(t)w, \tag{59}$$

where $J(T) = K(T) = 0$, with the following derivatives

$$N_t = J_t + K_t w, \qquad N_w = K(t), \qquad N_{ww} = 0. \tag{60}$$

By substituting the equation (60) into (58) we obtain

$$\begin{aligned} &J_t + K_t w - (2r_0 - 3\sigma^2)wK(t) + 2\sigma^2\gamma w^2 K^2(t) + \frac{1}{2}\frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma\sigma^2} + \\ &\frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2} = 0. \end{aligned} \tag{61}$$

We are going to have the following splitting equations from

$$J_t + \frac{1}{2}\frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma\sigma^2} + \frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2} = 0. \tag{62}$$

$$K_t - (2r_0 - 3\sigma_1^2)K(t) = 0. \tag{63}$$

$$2\sigma^2\gamma K^2(t) = 0. \tag{64}$$

We obtain the solutions to (62), (63) and (64) by taking the following boundary conditions:
$J(T) = K(T) = 0$.

$$J(t) = \left[\frac{1}{2}\frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma\sigma^2} + \frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2}\right](T-t). \tag{65}$$

$$K(t) = 0. \tag{66}$$

Taking the boundary condition into account, we obtain

$$G(t,s,y) = -\frac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(y-m_1(t)) + J_1(t) + K(t)s^{-2}]}. \tag{67}$$

where

$$\begin{aligned} m_1(t) = &-\frac{1}{r_0}\big[-(1-\lambda_2) + a_1\nu_1 + a_1\vartheta_1 + a_2\nu_2\big][1 - e^{-r_0(T-t)}] \\ &+ \frac{1}{2r_0}b_2^2\gamma[1 - e^{-r_0(T-t)}]. \end{aligned} \tag{68}$$

$$J_1(t) = \left[\frac{1}{2}\frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma\sigma^2} + \frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2}\right](T-t). \tag{69}$$

and

$$K(t) = 0. \tag{70}$$

Condition 2.
If $\rho_{12} > 0$ and $\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}} \geq 1$ or when $\rho_{12} < 0$ and $e^{r_0 T} < \frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}} \geq 1$, then the optimal reinsurance strategies of a large company and a small company are given by

$$(q_1^*(t), q_2^*(t)) = \left(\frac{(\vartheta_1 a_1)}{b_1^2}e^{-r_0(T-t)} + \frac{b_2\rho_{12}}{b_1}, 1\right). \tag{71}$$

By substituting equation (71) into (48), we obtain the following :

$$\begin{aligned}&n_t + r_0 s n_s + \frac{1}{2}s^2\sigma^2\gamma n_s^2 + \frac{1}{2}\sigma^2 s^2 n_{ss} + \frac{1}{2}\frac{\phi^2(\alpha+(1-\lambda_1)\delta-\ln s)^2}{\gamma\sigma^2}\\&+\frac{r_0[\phi(\alpha+(1-\lambda_1)\delta-\ln s)]}{\gamma\sigma^2}+\frac{1}{2}\frac{r_0^2}{\gamma\sigma^2}-\frac{a_1^2\vartheta_1^2}{b_1^2}+\\&e^{r_0(T-t)}\Bigg\{m(t)r_0-m_t-(1-\lambda_2)+a_1\nu_1+a_1\vartheta_1-a_2\nu_2-\frac{a_1\vartheta_1 b_2\rho_{12}}{b_1}-\\&\frac{1}{2}\frac{a_1^2\vartheta_1^2\gamma e^{-2r_0(T-t)}}{b_1^2}+\frac{1}{2}b_2^2\gamma(\rho_{12}^2-1)\Bigg\}=0.\end{aligned} \tag{72}$$

Equation (72) can be decomposed in the following equations

$$\begin{aligned}&m_t-m(t)r_0-(1-\lambda_2)-a_1\nu_1-a_1\vartheta_1+a_2\nu_2+\frac{a_1\vartheta_1 b_2\rho_{12}}{b_1}+\\&\frac{1}{2}\frac{a_1^2\vartheta_1^2\gamma e^{-2r_0(T-t)}}{b_1^2}-\frac{1}{2}b_2^2\gamma(\rho_{12}^2-1)=0.\end{aligned} \tag{73}$$

and

$$\begin{aligned}&n_t + r_0 s n_s + \frac{1}{2}s^2\sigma^2\gamma n_s^2 + \frac{1}{2}\sigma^2 s^2 n_{ss} + \frac{1}{2}\frac{\phi^2(\alpha+(1-\lambda_1)\delta-\ln s)^2}{\gamma\sigma^2}+\\&\frac{r_0[\phi(\alpha+(1-\lambda_1)\delta-\ln s)]}{\gamma\sigma^2}+\frac{1}{2}\frac{r_0^2}{\gamma\sigma^2}-\frac{a_1^2\vartheta_1^2}{b_1^2}=0.\end{aligned} \tag{74}$$

Taking the boundary condition $m(T)=0$ into account, the solution to equation (73) is as follows;

$$\begin{aligned}m(t)=&-\frac{1}{r_0}\Bigg[(1-\lambda_2)+a_1\nu_1+a_1\vartheta_1-a_2\nu_2+\frac{a_1\vartheta_1 b_2\rho_{12}}{b_1}+\\&\frac{1}{2}b_2^2\gamma(\rho_{12}^2-1)\Bigg][1-e^{-r_0(T-t)}]+\frac{1}{2}\frac{a_1^2\vartheta_1^2\gamma}{r_0 b_1^2}[e^{2r_0(T-t)}-e^{r_0(T-t)}].\end{aligned} \tag{75}$$

Equation (74) can be solved also by power transformation method and we obtain the following by taking the following boundary conditions:
$J(T)=K(T)=0.$

$$\begin{aligned}J(t)=&\Bigg[\frac{1}{2}\frac{\phi^2(\alpha+(1-\lambda_1)\delta-\ln s)^2}{\gamma\sigma^2}+\frac{r_0[\phi(\alpha+(1-\lambda_1)\delta-\ln s)]}{\gamma\sigma^2}\\&+\frac{1}{2}\frac{r_0^2}{\gamma\sigma^2}-\frac{a_1^2\vartheta_1^2}{b_1^2}\Bigg](T-t).\end{aligned} \tag{76}$$

$$K(t)=0. \tag{77}$$

Taking the boundary condition into account, we obtain

$$G(t,s,y)=-\frac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(y-m_2(t))+J_2(t)+K(t)s^{-2}]}, \tag{78}$$

where

$$\begin{aligned}m_2(t)=&-\frac{1}{r_0}\Bigg[(1-\lambda_2)+a_1\nu_1+a_1\vartheta_1-a_2\nu_2+\frac{a_1\vartheta_1 b_2\rho_{12}}{b_1}+\frac{1}{2}b_2^2\gamma(\rho_{12}^2-1)\Bigg]\\&[1-e^{-r_0(T-t)}]+\frac{1}{2}\frac{a_1^2\vartheta_1^2\gamma}{r_0 b_1^2}[e^{2r_0(T-t)}-e^{r_0(T-t)}].\end{aligned} \tag{79}$$

and

$$J_2(t) = \left[\frac{1}{2}\frac{\phi^2(\alpha+(1-\lambda_1)\delta-\ln s)^2}{\gamma\sigma^2} + \frac{r_0[\phi(\alpha+(1-\lambda_1)\delta-\ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2} - \frac{a_1^2\vartheta_1^2}{b_1^2}\right](T-t). \tag{80}$$

and

$$K(t) = 0. \tag{81}$$

Condition 3.
If $\rho_{12} < 0$ and $1 < \frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}}$ and $\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}} \leq e^{r_0 T}$, then the optimal reinsurance strategies of a large company and a small company are given as follows:

$$(q_1^*(t), q_2^*(t)) = \begin{cases} (0,1) \quad \text{if} \quad 0 \leq t \leq t_1, \\ \left(\frac{\vartheta_1 a_1}{\gamma b_1^2} e^{-r_0(T-t)} + \frac{b_2\rho_{12}}{b_1}, 1\right) \text{ if } t_1 \leq t \leq T. \end{cases} \tag{82}$$

Since $G(t, y, s)$ is continuous at $t = t_1$ and taking the Boundary conditions into account, we obtain the following value condition

$$G(t,s,y) = \begin{cases} -\frac{1}{\gamma} e^{-\gamma[e^{r_0(T-t)}(y-m_3(t))+J_3(t)+K(t)s^{-2}]}, & 0 \leq t \leq t_1, \\ -\frac{1}{\gamma} e^{-\gamma[e^{r_0(T-t)}(y-m_2(t))+J_2(t)+K(t)s^{-2}]}, & t_1 \leq t \leq T, \end{cases} \tag{83}$$

where

$$m_3(t) = -\frac{1}{r_0}\left[-(1-\lambda_2) + a_1\nu_1 + a_1\vartheta_1 + a_2\nu_2\right][1-e^{-r_0(t_1-t)}] + \frac{1}{2r_0} b_2^2\gamma[e^{-r_0(t_1-t)} - e^{-r_0(T-t)}]. \tag{84}$$

and

$$J_3(t) = \left[\frac{1}{2}\frac{\phi^2(\alpha+(1-\lambda_1)\delta-\ln s)^2}{\gamma\sigma^2} + \frac{r_0[\phi(\alpha+(1-\lambda_1)\delta-\ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2}\right](T-t_1). \tag{85}$$

and $K(t) = 0$, $m_2(t)$ and $J_2(t)$ are given by equations (79) and (80)

A.2. Case 2

If

$$\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2 (1-\rho_{12})} < -\frac{\gamma}{2},$$

and

$$e^{r_0 T} < -\frac{2(\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12})}{\gamma b_1 b_2^2 (1-\rho_{12})},$$

then

$$\tilde{q_2}(t) < \frac{1}{2}.$$

The optimal equilibrium reinsurance strategies of a large company and a small company are given as follows:

$$(q_1^*(t), q_2^*(t)) = \left(\frac{\vartheta_1 a_1}{\gamma b_1^2} e^{-r_0(T-t)}, 0\right) \quad \text{if} \quad 0 \le t \le T. \tag{86}$$

Equation (48) can be simplified as follows:

$$\begin{aligned} & n_t + r_0 s n_s + \frac{1}{2} s^2 \sigma^2 \gamma n_s^2 + \frac{1}{2} \sigma^2 s^2 n_{ss} + \\ & \frac{1}{2} \frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma \sigma^2} + \frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma \sigma^2} \\ & + \frac{1}{2} \frac{r_0^2}{\gamma \sigma^2} - \frac{a_1^2 \vartheta_1^2}{b_1^2} + e^{r_0(T-t)} \Big\{ m(t) r_0 - m_t - (1-\lambda_2) + a_1 \nu_1 + \\ & a_1 \vartheta_1 - a_2 \nu_2 - a_2 \vartheta_2 - \frac{1}{2} \frac{a_1^2 \vartheta_1^2 \gamma e^{-2r_0(T-t)}}{b_1^2} \Big\} = 0. \end{aligned} \tag{87}$$

Equation (87) can be decomposed in the following equations

$$m_t - m(t) r_0 + (1-\lambda_2) - a_1 \nu_1 - a_1 \vartheta_1 + a_2 \nu_2 + a_2 \vartheta_2 + \frac{1}{2} \frac{a_1^2 \vartheta_1^2 \gamma e^{-2r_0(T-t)}}{b_1^2} = 0. \tag{88}$$

and

$$\begin{aligned} & n_t + r_0 s n_s + \frac{1}{2} s^2 \sigma^2 \gamma n_s^2 + \frac{1}{2} \sigma^2 s^2 n_{ss} + \frac{1}{2} \frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma \sigma^2} + \\ & \frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma \sigma^2} + \frac{1}{2} \frac{r_0^2}{\gamma \sigma^2} - \frac{a_1^2 \vartheta_1^2}{b_1^2} = 0. \end{aligned} \tag{89}$$

Taking the boundary condition $m(T) = 0$ into account, the solution to equation (88) is as follows;

$$\begin{aligned} m(t) = & -\frac{1}{r_0}\Big[-(1-\lambda_2) + a_1 \nu_1 + a_1 \vartheta_1 - a_2 \nu_2 - a_2 \vartheta_2 \Big] [1 - e^{-r_0(T-t)}] \\ & + \frac{1}{2} \frac{a_1^2 \vartheta_1^2 \gamma}{r_0 b_1^2} [e^{2r_0(T-t)} - e^{r_0(T-t)}]. \end{aligned} \tag{90}$$

Equation (89) can be solved also by power transformation method and we obtain the following by taking the following boundary conditions: $J(T) = K(T) = 0$.

$$\begin{aligned} J(t) = & \Big[\frac{1}{2} \frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma \sigma^2} + \frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma \sigma^2} + \\ & \frac{1}{2} \frac{r_0^2}{\gamma \sigma^2} - \frac{a_1^2 \vartheta_1^2}{b_1^2} \Big] (T-t). \end{aligned} \tag{91}$$

$$K(t) = 0. \tag{92}$$

Taking the boundary condition into account, we obtain

$$G(t, s, y) = -\frac{1}{\gamma} e^{-\gamma[e^{r_0(T-t)}(y - m_4(t)) + J_4(t) + K(t) s^{-2}]}, \tag{93}$$

where

$$m_4(t) = -\frac{1}{r_0}\left[-(1-\lambda_2) + a_1\nu_1 + a_1\vartheta_1 - a_2\nu_2 - a_2\vartheta_2\right][1 - e^{-r_0(T-t)}] + \frac{1}{2}\frac{a_1^2\vartheta_1^2\gamma}{r_0 b_1^2}[e^{2r_0(T-t)} - e^{r_0(T-t)}]. \tag{94}$$

$$J_4(t) = \left[\frac{1}{2}\frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma\sigma^2} + \frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2} - \frac{a_1^2\vartheta_1^2}{b_1^2}\right](T-t). \tag{95}$$

and

$$K(t) = 0. \tag{96}$$

A.3. Case 3

If

$$\frac{\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12}}{b_1 b_2^2(1-\rho_{12})} < -\frac{\gamma}{2},$$

and

$$e^{r_0 T} \geq -\frac{2(\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12})}{\gamma b_1 b_2^2(1-\rho_{12})},$$

Then the equilibrium optimal reinsurance strategies of a small company are given as follows:

$$q_2^*(t)) = \begin{cases} 1, & \text{for} \quad 0 \leq t \leq t_2. \\ 0 & \text{for} \quad t_2 \leq t \leq T, \end{cases}$$

where

$$t_2 = T - \frac{1}{r_0}\ln\left(\frac{2(\vartheta_2 a_2 b_1 - \vartheta_1 a_1 b_2 \rho_{12})}{\gamma b_1 b_2^2(1-\rho_{12})}\right).$$

Let $t_3 = \min(t_1, t_2)$.

A.3.1. ***Conditions of Case 3*** **Condition 1.**

If $\rho_{12} > 0$ when $0 < t \leq t_2$, Then the optimal reinsurance strategies of a large company and a small company are expressed as those in equation (71) and when $t_2 < t \leq T$, $q_1^*(t)$ and $q_2^*(t)$ are the same as those in (86)

$$(q_1^*(t), q_2^*(t)) = \begin{cases} \left(\frac{(\vartheta_1 a_1)}{b_1^2}e^{-r_0(T-t)} + \frac{b_2\rho_{12}}{b_1}, 1\right) & 0 \leq t \leq t_2, \\ \left(\frac{\vartheta_1 a_1}{\gamma b_1^2}e^{-r_0(T-t)}, 0\right) & \text{if} \quad t_2 \leq t \leq T. \end{cases} \tag{97}$$

Similarly, the expression of the value function is given as follows:

$$G(t,s,y) = \begin{cases} -\frac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(y - m_5(t)) + J_5(t) + K(t)s^{-2}]}, & 0 \leq t \leq t_1, \\ -\frac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(y - m_4(t)) + J_4(t) + K(t)s^{-2}]}, & t_1 \leq t \leq T, \end{cases} \tag{98}$$

where

$$m_5(t) = -\frac{1}{r_0}\left[(1-\lambda_2) + a_1\nu_1 + a_1\vartheta_1\right][1-e^{-r_0(T-t)}] - \left(\frac{a_1\vartheta_1 b_2\rho_{12} + a_2\nu_2 b_1}{r_0 b_1} + \frac{1}{2}b_2^2\gamma(\rho_{12}^2 - 1)\right)[1-e^{-r_0(t_2-t)}] + \frac{1}{2}\frac{a_1^2\vartheta_1^2\gamma}{r_0 b_1^2}[e^{2r_0(T-t)} - e^{r_0(T+t-2t_2)}]. \tag{99}$$

$$J_5(t) = \left[\frac{1}{2}\frac{\phi^2(\alpha+(1-\lambda_1)\delta-\ln s)^2}{\gamma\sigma^2} + \frac{r_0[\phi(\alpha+(1-\lambda_1)\delta-\ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2} - \frac{a_1^2\vartheta_1^2}{b_1^2}\right](T-t). \tag{100}$$

$K(t) = 0$ while $m_4(t)$ and $J_4(t)$ are given by equations (94) and (95)

Condition 2.
If $\rho_{12} < 0$ and $-\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}} \leq 1$, when $0 \leq t \leq t_2$, the optimal reinsurance strategies of the large and small companies are expressed as those in equation (52) and when $t_2 \leq t \leq T$, then the optimal reinsurance strategies are the same as those in equation (86):

$$(q_1^*(t), q_2^*(t)) = \begin{cases} (0,1), & \text{if} \quad 0 \leq t \leq t_2, \\ \left(\frac{\vartheta_1 a_1}{\gamma b_1^2}e^{-r_0(T-t)}, 0\right) & \text{if} \quad t_2 \leq t \leq T. \end{cases} \tag{101}$$

where the value function is given by

$$G(t,s,y) = \begin{cases} -\frac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(y-m_6(t))+J_6(t)+K(t)s^{-2}]}, & 0 \leq t \leq t_2, \\ -\frac{1}{\gamma}e^{-\gamma[e^{r_0(T-t)}(y-m_4(t))+J_4(t)+K(t)s^{-2}]}, & t_2 \leq t \leq T. \end{cases} \tag{102}$$

where

$$m_6(t) = -\frac{1}{r_0}\big[-(1-\lambda_2) + a_1\nu_1 + a_1\vartheta_1 + a_2\nu_2\big][1-e^{-r_0(T-t)}] + \frac{1}{2r_0}b_2^2\gamma[1-e^{-r_0(t_2-t)}]. \tag{103}$$

$$J_6(t) = \left[\frac{1}{2}\frac{\phi^2(\alpha+(1-\lambda_1)\delta-\ln s)^2}{\gamma\sigma^2} + \frac{r_0[\phi(\alpha+(1-\lambda_1)\delta-\ln s)]}{\gamma\sigma^2} + \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2}\right](T-t_2). \tag{104}$$

also $K(t) = 0$, $m_4(t)$ and $J_4(t)$ are given by equations (94) and (95).

Condition 3.
If $\rho_{12} < 0$ and $-\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}} > 1$ and $e^{r_0 T} \geq -\frac{\vartheta_1 a_1}{\gamma b_1 b_2 \rho_{12}} > 1$, when $0 \leq t \leq t_3$, then the optimal reinsurance strategies of a large company and a small company are the same as those in equation (52), when $t_3 \leq t \leq t_2$ the optimal reinsurance strategies are expressed as those in equation (71) and when $t_2 \leq t \leq T$, the optimal reinsurance strategies are shown in equation (86) as follows;

$$(q_1^*(t), q_2^*(t)) = \begin{cases} \big(0, 1\big) & \text{if} \quad 0 \le t \le t_3, \\ \left(\frac{(\vartheta_1 a_1)}{b_1^2} e^{-r_0(T-t)} + \frac{b_2 \rho_{12}}{b_1}, 1\right) & \text{if} \quad t_3 \le t \le t_2, \\ \left(\frac{\vartheta_1 a_1}{\gamma b_1^2} e^{-r_0(T-t)}, 0\right) & \text{if} \quad t_2 \le t \le T. \end{cases} \tag{105}$$

The value function is given as follows:

$$G(t, s, y) = \begin{cases} -\frac{1}{\gamma} e^{-\gamma[e^{r_0(T-t)}(y - m_7(t)) + J_7(t) + K(t)s^{-2}]} & 0 \le t \le t_3, \\ -\frac{1}{\gamma} e^{-\gamma[e^{r_0(T-t)}(y - m_5(t)) + J_5(t) + K(t)s^{-2}]} & t_3 \le t \le t_2, \\ -\frac{1}{\gamma} e^{-\gamma[e^{r_0(T-t)}(y - m_4(t)) + J_4(t) + K(t)s^{-2}]} & t_2 \le t \le T. \end{cases} \tag{106}$$

where

$$\begin{aligned} m_7(t) = -\frac{1}{r_0}\big[-(1-\lambda_2) + a_1\nu_1 + a_1\vartheta_1 + a_2\nu_2\big][1 - e^{-r_0(T-t)}] + \\ \frac{1}{2r_0} b_2^2 \gamma [1 - e^{-r_0(t_2 - t)}]. \end{aligned} \tag{107}$$

$$\begin{aligned} J_7(t) = \Bigg[\frac{1}{2}\frac{\phi^2(\alpha + (1-\lambda_1)\delta - \ln s)^2}{\gamma\sigma^2} + \frac{r_0[\phi(\alpha + (1-\lambda_1)\delta - \ln s)]}{\gamma\sigma^2} + \\ \frac{1}{2}\frac{r_0^2}{\gamma\sigma^2}\Bigg](T - t_3). \end{aligned} \tag{108}$$

where $K(t) = 0$, $m_4(t)$, $J_4(t)$, $m_5(t)$ and $J_5(t)$ are given by equations (94), (95), (99) and (100).

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