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Images Restoration Based on a New Optimal Parameter to Conjugate Gradient Method

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Abstract Image denoising plays a vital role in numerous image processing applications. This research presents a novel two-phase conjugate gradient method tailored for mitigating impulse noise. The approach leverages a center-weighted median filter, which adaptively identifies noise-affected pixels and applies the conjugate gradient technique to restore them. The method focuses on minimizing a specific functional that maintains edge integrity while reducing noise candidates. One of the key advantages of this technique is its descent-based search mechanism, with the possibility of achieving global convergence through the Wolfe line search conditions. Experimental evaluations demonstrate the method's effectiveness in removing impulse noise using a spectral conjugate gradient approach.

Keywords New Optimal Parameter,Conjugate Gradient Method, Convergence Property, Images Restoration, Numerical Results

AMS 2010 subject classifications 90C30, 65K05, 49M37

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1. Introduction

The objective of this study is to provide a collection of iterative methods for solving optimization problems, using an edge-preserving regularization (EPR) functional as the objective function. To detect and suppress impulsive noise, an Adaptive Median Filter (AMF) [1] is employed. This filter operates based on two key equations that identify potentially corrupted pixels. Let $\bar{y}$ denote the observed noisy image of the original image x, where $\{x_{ij}\}_{i,j=1}^{M,N}$ represents the grayscale intensity values of x. The intermediate image y is obtained in the first phase by applying the adaptive median filter to the noisy image y. The restoration of noisy pixels is then achieved by minimizing the following function:

$$f_\alpha(u) = \sum_{(i,j)\in\mathcal{N}} \left[|u_{i,j} - y_{i,j}| + \frac{\beta}{2}\left(S^1_{i,j} + S^2_{i,j}\right) \right] \tag{1}$$

where $|N|$ is the length of the column vector $u_{i,j} = [u_{i,j}]_{(i,j)\in N}$, β is the regularization parameter and $S^1_{i,j} = 2 \sum_{(m,n)\in \mathrm{P}_{i,j}\cap \mathrm{N}^c} \phi_\alpha(u_{i,j} - y_{m,n}), S^2_{i,j} = \sum_{(m,n)\in \mathrm{P}_{i,j}\cap \mathrm{N}} \phi_\alpha(u_{i,j} - y_{m,n})$ the noise candidate indices set N^c is used to measure the maximum s_{max} and minimum s_{min} of a noisy pixel. The neighborhood of (i,j) is represented by $A = \{1,2,3,......M\} \times \{1,2,3,......N\}$ and $V_{(i,j)} = \left(V_{(i,j)} \cap N^c\right) \cup \left(V_{(i,j)} \cap N\right)$, and an example of such a function is $\phi_\alpha = \sqrt{\alpha + u^2}, \alpha > 0$, an edge-preserving potential function with parameter α. Similar optimization challenges arise when $F_\alpha(u)$ is of the form 1 , with $S^1_{i,j} + S^2_{i,j}$ smooth and $|u_{i,j} - y_{i,j}|$ non-smooth at zero. The

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function is reduced in [2] to a half-quadratic smooth approximation of $F_\alpha(u)$:

$$f_\alpha(u) = \sum_{(i,j)\in N} [2 \times (S^1_{(i,j)} + S^2_{(i,j)}\ )] \tag{2}$$

By employing the conjugate gradient method, it becomes possible to minimize the function $f_\alpha(u)$ where: $Minf_\alpha(u)$, $u \in R^n$ and $f_\alpha(u) : R^n \to\ R$ is a smooth function. The conjugate gradient method, which operates as an iterative optimization algorithm, generates a sequence of solutions according to the following update rule:

$$u_{k+1} = u_k + \alpha_k d_k \tag{3}$$

where d_k represents the search direction and α_k is the step size determined through a reliable and accurate line search. The step size α_k can be computed using the formula:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T Q d_k} \tag{4}$$

as referenced in [3]. To satisfy the Wolfe conditions, the step size α_k must fulfill the following inequalities:

$$f(u_k + \alpha_k d_k) \le\ f(u_k) + \delta\alpha_k g_k^T d_k \tag{5}$$

$$d_k^T g(u_k + \alpha_k d_k) \ge \sigma\ d_k^T g_k \tag{6}$$

where $0 < \delta < \sigma < 1$. For additional details, see references [4, 5]. The selection of the search direction in the conjugate gradient method is governed by the following formula:

$$d_{k+1} = -g_{k+1} + \beta_k d_k \tag{7}$$

where β_k is a scalar. The Fletcher-Reeves approach [6] and the Dai-Yuan (DY) methodology [7] are two examples of distinct formula types. Specifically, [8, 9]. These are some of the numerous guises they might assume:

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \beta_{k+1}^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T y_k} \tag{8}$$

The convergence properties of conjugate gradient algorithms have been extensively investigated in the literature. Zoutendijk [10] is recognized as a pioneer in this area, having established the global convergence of the Fletcher–Reeves (FR) method under accurate line search conditions. Since then, numerous formulations for computing the conjugate gradient coefficient have been developed. These formulations are noted for their strong numerical performance and their ability to guide the algorithm toward a global solution. Recent contributions by Hideaki and Yasushi [11], as well as Basim [12], have further advanced the nonlinear conjugate gradient method. Their research demonstrates significant improvements in numerical efficiency and enhances the potential for global convergence. Based on their studies, the following parameters and formulas have been proposed:

$$\beta_k^{HY} = \frac{\|g_{k+1}\|^2}{2/\alpha_k(f_k - f_{k+1})}, \beta_{k+1}^{B} = \frac{\|g_{k+1}\|^2}{(f_k - f_{k+1})/\alpha_k - g_k^T d_k/2} \tag{9}$$

The application of these methods enables the full exploitation of the advantages and key characteristics inherent in conjugate gradient algorithms. In addition to their robustness, these approaches offer a high degree of computational efficiency. To further improve performance in unconstrained optimization problems, a quadratic model has been proposed, enhancing the original conjugate gradient framework. This modification is designed to optimize the algorithm's efficiency and convergence behavior. Moreover, the development of three-term conjugate gradient (CG) methods has attracted considerable attention in recent research. Significant contributions in this area can be found in [8, 9, 13, 14, 15].

This study begins with a new coefficient conjugate derivation, then presents a fresh perspective on the denominator $d_k^T G v_k$ using the quadratic model. Finally, it concludes with a study of the conjugate gradient method and a convergence analysis. The numerical results of the proposed approach are then shown and compared with the findings of many alternative conjugate gradient methods.

2. A new parameter

This study seeks to formulate novel mathematical expressions. The function f can be assessed through the application of the second-order Taylor series in the following manner:

$$f(x) = f(x_{k+1}) - g_{k+1}^T s_k + \frac{1}{2} s_k^T Q(u_{k+1}) s_k \tag{10}$$

The gradient is the outcome of this:

$$g_{k+1} = g_k + Q(u_{k+1}) s_k \tag{11}$$

The second order curvature may be obtained from (11) in (10), we get:

$$s_k^T Q(u_k) s_k = 2(f_k - f_{k+1}) + 2y_k^T s_k + 2g_k^T s_k \tag{12a}$$

By utilizing some algebra, we can determine:

$$s_k^T Q(u_k) s_k = 1/2 y_k^T s_k + (f_{k+1} - f_k) - g_k^T s_k \tag{12b}$$

The conjugate condition is employed to ascertain the novel parameter. It is imperative to acknowledge that the subsequent information delineates the conjugate condition:

$$d_{k+1}^T Q(u_k) s_k = 0 \tag{13}$$

By using (7), (12b) with (13), we obtain:

$$\beta_k = \frac{\left[\frac{1}{2} + \frac{(f_{k+1} - f_k) - g_k^T s_k}{s_k^T y_k}\right] g_{k+1}^T y_k}{d_k^T y_k} \tag{14}$$

Using exact line search in above equation, we get:

$$\beta_k = \frac{\left[\frac{1}{2} + \frac{(f_{k+1} - f_k) - g_k^T s_k}{s_k^T y_k}\right] \|g_{k+1}\|^2}{d_k^T y_k} \tag{15}$$

The BBE is the name given to this formula. Below is the BBE conjugate gradient algorithm.

3. Global convergence

This section aims to analyze the algorithm's global convergence properties. Initially, we establish the following:

1. On the given set $\Omega = \{u : u \in R^n, f(u) \le f(u_1)\}$, $f(u)$ is bounded from below.
2. The following inequality is met by $L > 0$, $\tau, \upsilon \in R^n$ since the derivative $\nabla \mathrm{f}(u)$ is Lipschitz continuous:

$$\|g(\tau) - g(\upsilon)\| \le \|\tau - \upsilon)\|, \forall \tau, \upsilon \in R^n \tag{16}$$

See [16, 17].

Theorem 1
If we apply a novel method to generate $\{x_k\}\, and \, \{d_k\}$, we obtain:

$$d_{k+1}^T g_{k+1} < 0 \text{ and } d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k. \tag{17}$$

Proof
Obviously, $d_k = -g_k$ is required for $d_1^T g_1 < 0$. For any k, the $d_k^T g_k < 0$ should be considered. One can easily acquire it from (7) and (15):

$$d_{k+1}^T g_{k+1} = -g_{k+1}^T g_{k+1} + \beta_k d_k^T g_{k+1} = -\beta_k \frac{(s_k^T y_k)^2}{(1/2 s_k^T y_k + (f_{k+1} - f_k) - s_k^T g_k)} + \beta_k d_k^T g_{k+1} \tag{18}$$

that ensures:

$$d_{k+1}^T g_{k+1} = \beta_k [d_k^T g_{k+1} - \frac{(s_k^T y_k)^2}{(1/2 s_k^T y_k + (f_{k+1} - f_k) - s_k^T g_k)}] \tag{19}$$

The result is acquired by utilizing (11) and (19)

$$d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k \tag{20}$$

Definitely $d_k^T g_k < 0$, this results in:

$$d_{k+1}^T g_{k+1} < 0 \tag{21}$$

The proof is completed. □

In order to analyze the overall convergence properties of the conjugate gradient method, a comprehension of the Zoutendijk condition [10] is necessary.

Lemma 1
If we assume that both (1) and (2) are true, that α_k fulfill the Wolfe conditions, as well as d_k being descent direction, then:

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty \tag{22}$$

Theorem 2
Assuming the premises and lemma 1 are true and $\{u_k\}$ is a new sequence, then:

$$\lim_{k \to \infty} \inf \|g_k\| = 0 \tag{23}$$

Proof
Equation (23) is untrue by contradiction. We can identify a $r > 0$ such that, for each k:

$$\|g_{k+1}\| > r \tag{24}$$

One can get the following outcome by squaring the search duration as $d_{k+1} + g_{k+1} = \beta_k \, d_k$ on both sides:

$$\|d_{k+1}\|^2 + \|g_{k+1}\|^2 + 2 d_{k+1}^T g_{k+1} = (\beta_k)^2 \|d_k\|^2 \tag{25}$$

When (20) is applied to (25), the results are obtained:

$$\|d_{k+1}\|^2 = \frac{(d_{k+1}^T g_{k+1})^2}{(d_k^T g_k)^2} \|d_k\|^2 - 2 d_{k+1}^T g_{k+1} - \|g_{k+1}\|^2 \tag{26}$$

Divided (26) by $(d_{k+1}^T g_{k+1})^2$, then result is:

$$\begin{aligned} \frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} &= \frac{\|d_k\|^2}{(d_k^T g_k)^2} - \frac{\|g_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} - \frac{2}{d_{k+1}^T g_{k+1}} \\ &\leq \frac{\|d_k\|^2}{(d_k^T g_k)^2} - \left(\frac{\|g_{k+1}\|}{d_{k+1}^T g_{k+1}} + \frac{1}{\|g_{k+1}\|^2} \right) + \frac{1}{\|g_{k+1}\|^2} \\ &\leq \frac{\|d_k\|^2}{(d_k^T g_k)^2} + \frac{1}{\|g_{k+1}\|^2} \end{aligned} \tag{27}$$

As a result, we found:

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} \le \sum_{i=1}^{k+1} \frac{1}{\|g_i\|^2} \tag{28}$$

Suppose $c_1 > 0$ has $\|g_k\| \ge c_1$ for every $k \in n$. Then:

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} < \frac{k+1}{c_1^2} \tag{29}$$

Ultimately, we have:

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} = \infty \tag{30}$$

Similarly,$\lim_{k\to\infty} \inf\|g_k\| = 0$ holds according to Lemma1. This is not the only outcome that may be obtained using other formulas. See [8]. □

4. Numerical Results

In this investigation, we provide substantiation regarding the efficacy of the BBE algorithm in the attenuation of salt-and-pepper impulse noise (2). The BBE methodology utilizes the following parameters: and Table 1 delineates the research flow diagram, whereas Table 1 grants access to the original test imagery. The simulations are conducted on a personal computing device employing MATLAB 2015a. The performance of the BBE technique is assessed in juxtaposition with the FR method. It is imperative to underscore that the principal aim of this study is to ascertain the rate at which carbon emissions can be mitigated. The evaluation of image quality is conducted through the Signal-to-Noise Ratio (SNR):

$$\text{PSNR} = 10 \cdot \log_{10} \left(\frac{255^2}{\frac{1}{MN} \sum_{i,j} \left(u_{i,j}^r - u_{i,j}^*\right)^2} \right) \tag{31}$$

The $u_{i,j}^*$ represents the pixel values in the original image, while the $u_{i,j}^r$ represents the pixel values in the restored image. Both methods employ the same criteria to determine the stopping point.

$$\frac{|f(u_k) - f(u_(k-1))|}{|f(u_k)|} \le 10^{-4} \text{ and } \|f(u_k)\| \le 10^{-4} \left(1 + |f(u_k)|\right) \tag{32}$$

The findings of the examinations are presented in Table 1, below. Table 1 includes the peak signal-to-noise ratio, abbreviated as PSNR, as well as the number of iterations, abbreviated as NI, and the number of function evaluations, abbreviated as NF. Many references have addressed the subject of improvement from different points of view, as is clear in fact [18, 19, 20, 21, 22], in order to strengthen the direct background of the test.

Table 1. Numerical results of FR and BBE algorithms.

Image	Noise level r(%)	FR-Method			BBE-Method		
		NI	NF	PSNR	NI	NF	PSNR
Le	50	82	153	30.5529	62	65	30.4901
	70	81	155	27.4824	64	67	27.0102
	90	108	211	22.8583	69	73	22.9429
Ho	50	52	53	30.6845	42	43	35.0424
	70	63	116	31.2564	52	53	30.9988
	90	111	214	25.2870	65	67	24.7525
El	50	35	36	33.9129	32	33	33.8427
	70	38	39	31.8640	38	39	31.8269
	90	65	114	28.2019	52	53	28.2381
c512	50	59	87	35.5359	36	51	35.4844
	70	78	142	30.6259	46	55	30.7098
	90	121	236	24.3962	65	71	24.9006

The table shows that the performance of the suggested algorithms is better than the FR approach in terms of the number of function evaluations, the peak signal-to-noise ratio, and the number of iterations.

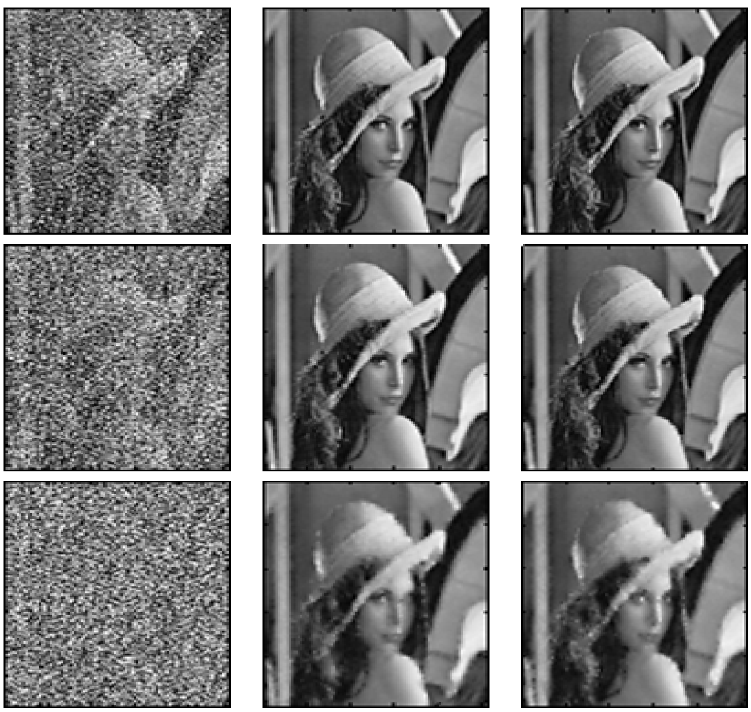

Figure 1. The results of the FR and ew algorithms for the 256×256 Lena picture are shown

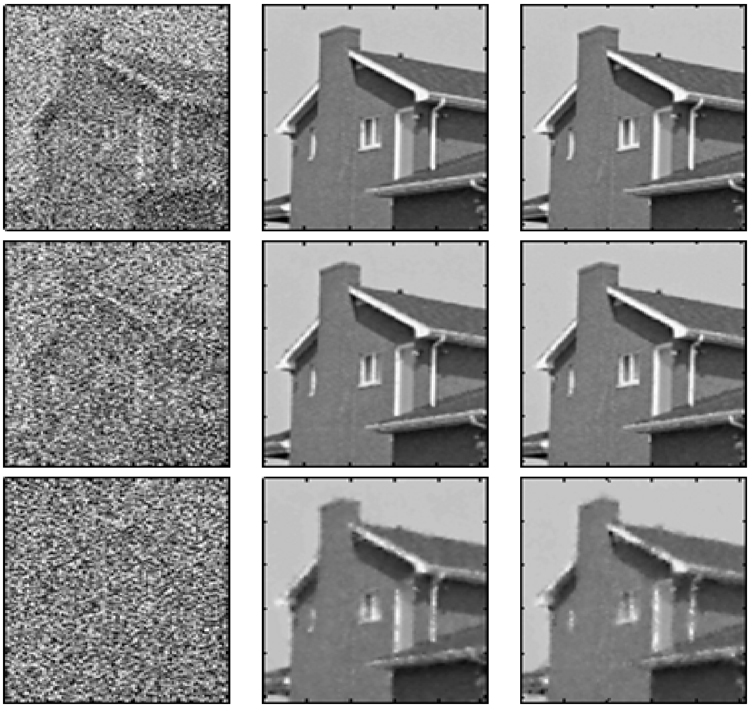

Figure 2. The results of the FR and New algorithms for the 256×256 House picture are shown

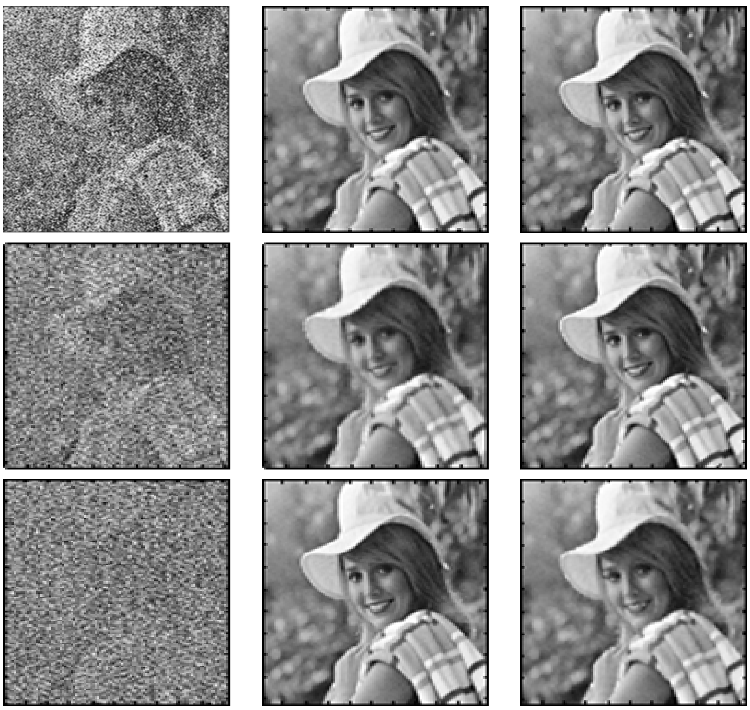

Figure 3. The results of the FR and New algorithms for the 256×256 Elaine picture are shown

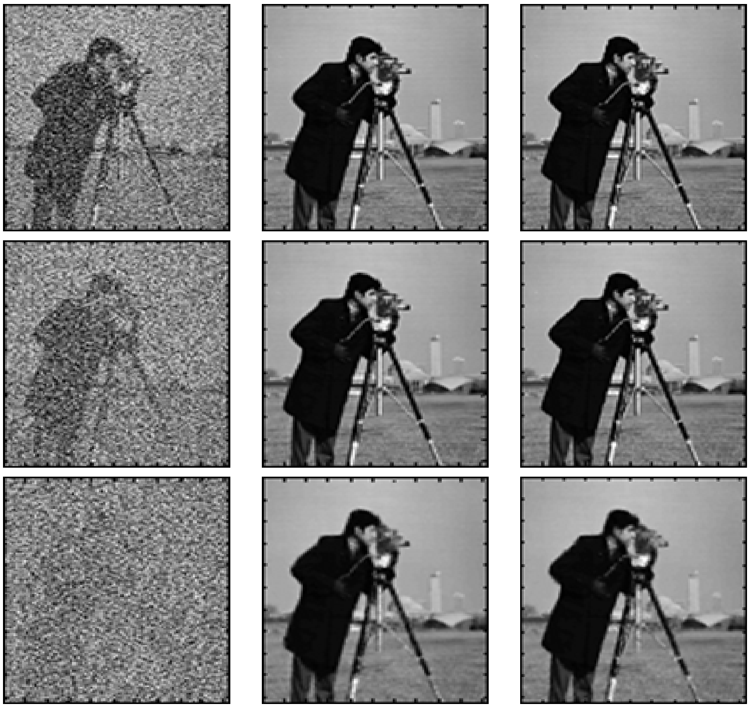

Figure 4. The results of the FR and New algorithms for the 256×256 Cameraman picture are shown

5. Conclusions

We introduced a revised conjugate gradient formula incorporating a new optimal parameter, which is called BBE conjugate gradient method. Our objective was to enhance the precision of our findings. By utilizing the search criteria, we discovered the global convergence of the Wolfe line. Simulations suggest that BBE have the potential to significantly decrease the number of function evaluations and iterations without compromising the image quality.

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