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An introduction to set-valued fractional linear programming based on the null set concept

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Abstract Set theory is a generalization of interval theory. However, this theory has shortcomings due to the lack of an inverse element for addition. The concept of null sets was therefore introduced to address this issue. Nonetheless, in set-valued optimization, the use of this concept remains largely insufficient. This article, therefore, introduces a linear fractional set-valued optimization problem, the solution to which is based on the concept of null sets. This concept enables a partial order to be established between sets for simple differences and the Hukuhara difference. On this basis, the notions of optimal and H-optimal solutions have been defined. To solve the proposed set-valued linear fractional optimization problem, it is first transformed into a set-valued linear optimization problem. To make this conversion, we have proposed an adapted version of the Charnes and Cooper method applicable to set-valued linear fractional optimization problems. Subsequently, the obtained set-valued linear optimization problem is transformed into a deterministic linear bi-objective optimization problem using the vectorization technique. To apply a classical method for resolution, the bi-objective problem is converted into a single-objective linear optimization problem using the scalarization technique. Finally, an algorithm has been proposed, and two didactic examples have been solved to better illustrate the steps of the proposed procedure.

Keywords Set-valued, null sets concept, fractional optimization, vectorization, scalarization

AMS 2020 subject classifications 03E04, 03E75, 90C32, 90C48, 90C70

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1. Introduction

Set-valued optimization is a type of optimization in which the coefficients of the objective functions and constraints are sets. This particular optimization problem is a generalization of a specific class of optimization problems where the coefficients are real intervals [2, 20, 13, 16, 29]. In recent years, optimization researchers have dedicated significant attention to this issue. Indeed, D. Kuroiwa [6, 7, 8] and D. Kuroiwa et al. [9] were the first to address this type of optimization problem.
Subsequently, this work generated excitement. We can mention, among others, [17], whose research has contributed to the proposal of numerous concepts in the theory of set comparison. When it [18] was, its work made it possible to transform a set optimization problem into a bi-objective optimization problem using the vectorization technique. The study in [19] focused on the directional derivatives in set optimisation with the set less order relation. The optimality conditions, existence theorems, and non-convex scalarisation in set optimisation problems were studied in [25, 26, 10, 11]. [5] has also designed an algorithm to solve set optimization problems in polyhedral convex games, and [28] studied the convergence of the solution sets for set optimization problems. The well-posed problem issue and the Karush–Kuhn–Tucker conditions in set optimization have been studied in [23] and [24], respectively. The concept of set optimization was used in [27] to model the optimization of photovoltaic

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power plant layout.

The major drawback of all these works lies in manipulating the differences between sets. The elements, or coefficients, of the optimization problem are sets; therefore, the difference of an element does not necessarily equal zero. Inspired by this, Wu[12] introduced the concept of a null set in hyperspace, which consists of all nonempty subsets of a given normed space. He defined two partial orders based on algebraic and Hukuhara differences between any two elements of hyperspace using these concepts. These orders allowed him to solve set optimization problems. In fact, he transformed set optimization problems into classical bi-objective optimization problems. To solve the bi-objective problem, he uses the scalarization technique. His work shows that the optimal solution to the scalarized problem is also the optimal solution to the original set optimization problem.
However, Wu's[12] work is limited to the linear case only. Previously, Wu[14] had used the concept of null set, based on the same principle, to solve interval-valued optimization problems (which are a special case of set-valued problems). Unlike Wu, who uses a linear scalarisation function in his work, A. Moar et al.[3] proposed using a nonlinear scalarisation function in set optimization based on the null set concept. Still based on this concept, Zhang[4] solved multi-valued equilibrium problems. Inspired by Wu's work [12], other researchers such as J. C. Sama and K. Some [21, 22] have proposed using the concept of null set to solve fuzzy nonlinear optimization problems. In all the work carried out, the concept of null set offers another, more efficient possibility in solving certain types of optimization problems whose coefficients and/or variables are sets or fuzzy numbers. Unfortunately, this efficiency has not yet been tested for solving set-valued nonlinear optimization problems. This motivates the present work, which aims to generalise Wu's[12] work in the non-linear case. Therefore, this work proposes an extension to the nonlinear case. This extension will make a significant contribution to the field of set-valued optimization. Specifically, we propose studying a set-valued fractional linear programming case. This choice is because fractional linear optimization problems belong to a class of nonlinear optimization problems with applications in various fields, including planning, management, finance, and engineering. Our reflection attempts to establish the foundations of the theory in the fractional case. For this, first, the linear fractional set optimization problem is transformed into a linear set optimization problem.
Secondly, the set-valued fractional linear programming problem will be transformed into a deterministic bi-objective linear optimization problem using vectorization. Next, we will use a scalarization technique to convert the linear bi-objective problem into a deterministic mono-objective problem. Finally, we will propose an equivalence study between the solutions of the linear bi-objective problem and the initial problem, as well as between the solutions of the linear bi-objective problem and the solutions of the scalarized linear problem.

To better present our results, in Section 2, we will introduce the fundamental elements necessary for understanding the work. Section 3 will present our main results. Two didactic examples will be discussed in Section 4 to demonstrate our method. Section 5 will conclude with a summary.

2. Basic concepts

2.1. Set analysis

This section introduces the concept of a null set and its properties.

Definition 2.1 ([12, 13, 15]). Let $(\mathcal{T}, || \bullet ||)$ be a normed space and $\Xi_{cc}(\mathcal{T})$ be a collection of all compact and convex sets of $\mathcal{T}$. Let $A, B \in \Xi_{cc}(\mathcal{T})$ and ν be real numbers.
Let $\oplus$, $\ominus$, $\odot$ denote the sum, difference, and multiplication between sets, respectively. The following relations hold:

1. $\mathcal{A} \oplus \mathcal{B} = \{a + b \mid a \in \mathcal{A} \quad \text{and} \quad b \in \mathcal{B}\}$,
2. $\nu \odot \mathcal{A} = \{\nu a \mid a \in \mathcal{A}\}$,
3. $\mathcal{A} \ominus \mathcal{B} = A \oplus (-\mathcal{B}) = \{a - b \mid a \in \mathcal{A} \quad \text{and} \quad b \in \mathcal{B}\}$.

Definition 2.2. Let Θ be a continuous linear function on a set $\mathcal{T}$ with coefficients greater than or equal to 1. Let x_1 be the largest value of $\mathcal{T}$ and x_2 be the smallest value of $\mathcal{T}$. The following relations hold:

$$\sup_{\alpha\in\mathcal{T}} \Theta(\alpha) = \Theta(x_1),$$

and

$$\inf_{\alpha\in\mathcal{T}} \Theta(\alpha) = \Theta(x_2).$$

Proposition 2.3 ([12, 13])
Let $(\mathcal{T}, \|\bullet\|)$ be a normed space and Θ be a continuous linear function with coefficients greater than or equal to 1 on $\mathcal{T}$. Let $\mathcal{A},\mathcal{B}\in\Xi_{cc}(\mathcal{T})$ and $\nu\in\mathbb{R}$. Then the following equalities hold:

1.
$$\sup_{\alpha\in\mathcal{A}\oplus\mathcal{B}} \Theta(\alpha) = \sup_{\alpha\in\mathcal{A}}\Theta(\alpha) + \sup_{\alpha\in\mathcal{B}}\Theta(\alpha) \text{ and } \inf_{\alpha\in\mathcal{A}\oplus\mathcal{B}} \Theta(\alpha) = \inf_{\alpha\in\mathcal{A}}\Theta(\alpha) + \inf_{\alpha\in\mathcal{B}}\Theta(\alpha).$$

2.
$$\sup_{\alpha\in\lambda\odot\mathcal{A}} \Theta(\alpha) = \begin{cases} \nu.\sup_{\alpha\in\mathcal{A}}\Theta(\alpha) & \text{if } \nu\geqslant 0,\\ \nu.\inf_{\alpha\in\mathcal{A}}\Theta(\alpha) & \text{if } \nu<0.\end{cases}$$

3.
$$\inf_{\alpha\in\nu\odot\mathcal{A}} \Theta(\alpha) = \begin{cases} \nu.\inf_{\alpha\in\mathcal{A}}\Theta(\alpha) & \text{if } \nu\geqslant 0,\\ \nu.\sup_{\alpha\in\mathcal{A}}\Theta(\alpha) & \text{if } \nu<0.\end{cases}$$

4.
$$\sup_{\alpha\in\mathcal{A}\ominus\mathcal{A}} \Theta(\alpha) = \sup_{\alpha\in\mathcal{A}}\Theta(\alpha) - \inf_{\alpha\in\mathcal{A}}\Theta(\alpha) \text{ if } \inf_{\alpha\in\mathcal{A}\ominus\mathcal{A}} \Theta(\alpha) = \inf_{\alpha\in\mathcal{A}}\Theta(\alpha) - \sup_{\alpha\in\mathcal{A}}\Theta(\alpha).$$

5.
$$\sup_{\alpha\in\nu\odot\mathcal{A}} \Theta(\alpha) + \inf_{\alpha\in\nu\odot\mathcal{A}} \Theta(\alpha) = \nu(\sup_{\alpha\in A}\Theta(\alpha) + \inf_{\alpha\in\mathcal{A}}\Theta(\alpha)).$$

The following is an example of a simple application of Proposition 2.3.

Example 2.4. Consider the following sets

$$A_1 = \{\zeta_1(t)/t\in[0,1]\} \text{ and } A_2 = \{\zeta_2(t)/t\in[0,1]\}$$

with ζ_1 and ζ_2 are functions defined as follows:

$$\zeta_1 : [0,1]\to[0,5], \qquad t\mapsto t^2+1,$$

and

$$\zeta_2 : [0,1]\to[0,5], \qquad t\mapsto t+2.$$

Let $\Theta(\alpha) = \nu\alpha$, then we have:

$$\begin{aligned}\sup_{\in A_1}\Theta(\alpha) + \inf_{\alpha\in A_1}\Theta(\alpha) &= \nu\zeta_1(1) + \nu\zeta_1(0)\\ &= 2\nu+\nu\\ &= 3\nu\end{aligned}$$

and

$$\begin{aligned}\sup_{\alpha\in A_2}\Theta(\alpha) + \inf_{\alpha\in A_2}\Theta(\alpha) &= \nu\zeta_2(1) + \nu\zeta_2(0)\\ &= 3\nu+\nu\\ &= 4\nu\end{aligned}$$

Remark 2.5. $\Xi_{cc}(\mathcal{T})$ does not constitute a vector space since for every $\mathcal{A} \in \Xi_{cc}(\mathcal{T})$ and $\mathcal{B} \in \Xi_{cc}(\mathcal{T})$, we have $\mathcal{A} \oplus \mathcal{B} \notin \Xi_{cc}(\mathcal{T})$ and for all real ν, we have $\nu \odot \mathcal{A} \notin \Xi_{cc}(\mathcal{T})$.
Let $\theta_{\mathcal{T}}$ be the zero element of the normed space $\mathcal{T}$, it can be considered as the zero element of $\Xi_{cc}(\mathcal{T})$ since $\mathcal{A} \oplus \{\theta_{\mathcal{T}}\} = \mathcal{A}$. In other words, since $\mathcal{A} \ominus \mathcal{A} \neq \{\theta_{\mathcal{T}}\}$, this means that $\mathcal{A} \ominus \mathcal{A}$ is not the zero element of $\Xi_{cc}(\mathcal{T})$; in other words, the addition of the inverse elements of $\mathcal{A}$ in $\Xi_{cc}(\mathcal{T})$ does not exist.

Example 2.6 ([14]). In the special case of interval arithmetic, setting $\mathcal{A} = \left[a^L, a^U\right]$ with a^L and a^U real numbers such that $a^L \leq a^U$, we have:

$$\begin{aligned}\mathcal{A} \ominus \mathcal{A} = \left[a^L, a^U\right] \ominus \left[a^L, a^U\right] &= \left[a^L, a^U\right] \oplus \left[-a^U, -a^L\right] \\ &= \left[a^L - a^U, a^U - a^L\right]\end{aligned}$$

where $a^U - a^L \geq 0$, which says that the additive inverse element in $\Xi_{cc}(\mathcal{T})$ does not exist. One of the reasons is that the concept of the zero element of $\Xi_{cc}(\mathcal{T})$ is not defined. This also says that $\Xi_{cc}(\mathcal{T})$ cannot form a vector space under the above interval addition and scalar multiplication.

Hence, the following definition.

Definition 2.7 ([12, 13]). The null set of $\Xi_{cc}(\mathcal{T})$ is defined by:

$$\Omega = \{\mathcal{A} \ominus \mathcal{A} \mid \mathcal{A} \in \Xi_{cc}(\mathcal{T})\}. \tag{2.1}$$

It is considered the zeroth element of $\Xi_{cc}(\mathcal{T})$.

Proposition 2.8 ([12])
We have the properties of the null set.

1. If $\omega \in \Omega$ then the zero element $\theta_{\mathcal{T}} \in \omega$. This also says that $\Omega \neq \Xi_{cc}\mathcal{T}$.
2. $\omega \in \Omega$ implies that$-\omega = \omega$.
3. $\nu\Omega = \Omega$ for $\nu \in \mathbb{R}$ whith $\nu \neq 0$.
4. Ω is closed under set addition, that is, $\omega_1 \oplus \omega_2 \in \Omega$ for each $\omega_1, \omega_2 \in \Omega$.

Definition 2.9 ([12, 13, 21, 22]). Let V be an $\mathbb{R}$-vector space. Consider the function $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$, then:

1. $\mathcal{L}$ is additive if and only if $\mathcal{L}(\mathcal{A} \oplus \mathcal{A}) = \mathcal{L}(\mathcal{A}) + \mathcal{L}(\mathcal{A})$.
2. $\mathcal{L}$ is positively homogeneous if and only if $\mathcal{L}(\nu\mathcal{A}) = \nu.\mathcal{L}(\mathcal{A})$ with $\nu \geqslant 0$.
3. $\mathcal{L}$ is positively homogeneous if and only if $\mathcal{L}(\nu\mathcal{A}) = \nu.\mathcal{L}(\mathcal{A})$ if $\nu \geqslant 0$.
4. $\mathcal{L}$ is linear if and only if it is both additive and homogeneous.
5. $\mathcal{L}(\nu^k \odot \mathcal{A}) = \nu^k.\mathcal{L}(\mathcal{A})$ with $\nu \geqslant 0$ and $k > 0$, $\mathcal{L}$ is homogeneous of degree k.

Proposition 2.10 ([12, 13])
Let V be an $\mathbb{R}-$ vector space. Consider the function $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$. Suppose that

1. $-\mathcal{L}(\omega) = \mathcal{L}(-\omega)$ for all $\omega \in \Omega$, then $\mathcal{L}(\omega) = \theta_V$ for all $\omega \in \Omega$ where the zero element of the vector space V is θ_V.
2. $\mathcal{L}(\omega) = \theta_V$ for all $\omega \in \Omega$, and, suppose that $\mathcal{L}$ is additive, then $\mathcal{L}(\mathcal{A} \ominus \mathcal{B}) = \mathcal{L}(\mathcal{A}) - \mathcal{L}(\mathcal{B})$ for all $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$.
3. $\mathcal{L}$ is additive and that the Hukuhara difference $\mathcal{A} \ominus_H \mathcal{B}$ exists for all $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$ then, $\mathcal{L}(\mathcal{A} \ominus_H \mathcal{B}) = \mathcal{L}(\mathcal{A}) - \mathcal{L}(\mathcal{B})$.

Definition 2.11 ([12, 13]). Let $\mathcal{C}$ be a subset of $\Xi_{cc}(\mathcal{T})$ with

$$\mathcal{C} = \left\{C \in \Xi_{cc}(\mathcal{T}) : \sup_{\alpha \in C} \Theta(\alpha) + \inf_{\alpha \in C} \Theta(\alpha) \geq 0\right\}. \tag{2.2}$$

1. $\mathcal{C}$ is is said to be convex if and only if $\nu \odot \mathcal{A} \oplus (1-\nu) \odot \mathcal{B} \in \mathcal{C}$ for all $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$, and $\nu \in [0, 1]$.

2. $\mathcal{C}$ is said to be cone if and only if $\nu \odot \mathcal{A} \in \mathcal{C}$ for $\mathcal{A} \in \mathcal{C}$ and $\nu > 0$.
3. A cone $\mathcal{C}$ is said to be cone convex if and only if it is also convex .

Proposition 2.12 ([12, 13])
Let V be an $\mathbb{R}$-vector space. Consider the function $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$.
Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$. If $\mathcal{L}$ is additive and positively homogeneous, then the set $\mathcal{L}(\mathcal{C}) = \{\mathcal{L}(\mathcal{A}) | \mathcal{A} \in \mathcal{C}\}$ is a convex cone in a vector space V.

Proof
Assume that $\mathcal{C}$ is a convex cone, i.e., if $\mathcal{A}, \mathcal{B} \in \mathcal{C}$, $\nu \odot \mathcal{A} \oplus \mu \odot \mathcal{B} \in \mathcal{C}$ for $\nu > 0, \mu > 0$. Consider the function $\mathcal{L}$ which is additive and positively homogeneous of degree k. We have $\mathcal{L}(\nu \odot \mathcal{A} \oplus \mu \odot \mathcal{B}) \in \mathcal{L}(\mathcal{C})$, then $\nu\mathcal{L}(\mathcal{A}) + \mu\mathcal{L}(\mathcal{B}) \in \mathcal{L}(\mathcal{C})$.
Therefore, $\mathcal{L}(\mathcal{C})$ is a convex cone in the vector space V. □

Proposition 2.13 ([12])
Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$, then $\mathcal{A} \oplus \mathcal{B} \in \mathcal{C}$ for all $\mathcal{A}, \mathcal{B} \in \mathcal{C}$.

Proof
Let $\mathcal{A}, \mathcal{B} \in \mathcal{C}$.
According to the Definition 2.11, we have:

$$\mathcal{A} \in \mathcal{C} \Rightarrow \sup_{\alpha \in \mathcal{A}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{A}} \Theta(\alpha), \qquad (*)$$

$$\mathcal{B} \in \mathcal{C} \Rightarrow \sup_{\alpha \in \mathcal{B}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{B}} \Theta(\alpha). \qquad (**)$$

The relations $(*)$ and $(**)$ give us:

$$\begin{aligned} 0 &\leqslant \sup_{\alpha \in \mathcal{A}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{A}} \Theta(\alpha) + \sup_{\alpha \in \mathcal{B}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{B}} \Theta(\alpha), \\ &= \sup_{\alpha \in \mathcal{A}} \Theta(\alpha) + \sup_{\alpha \in \mathcal{B}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{A}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{B}} \Theta(\alpha), \\ &= \sup_{\alpha \in \mathcal{A} \oplus \mathcal{B}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{A} \oplus B} \Theta(\alpha). \end{aligned}$$

So $\mathcal{A} \oplus \mathcal{B} \in \mathcal{C}$. □

2.2. Preference order

This section presents partial orders and their properties.

Definition 2.14. Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$, for $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$, we define two binary relations on $\Xi_{cc}(\mathcal{T})$ as follows:

1. $\mathcal{A} \preceq \mathcal{B}$ if and only if $\mathcal{B} \ominus \mathcal{A} \in \mathcal{C}$.
2. $\mathcal{A} \preceq_H \mathcal{B}$ if and only if $\mathcal{B} \ominus_H \mathcal{A}$ exists, and $(\mathcal{B} \ominus_H \mathcal{A}) \oplus \omega \in \mathcal{C}$ for all $\omega \in \Omega$.

Proposition 2.15 ([12, 13, 21])
Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$.

1. Assume that $\Omega \subseteq \mathcal{C}$ then, the binary relation $\preceq$ is reflexive.
2. The relation $\preceq$ is transitive.
3. Let $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$ and $\nu > 0$, if $\mathcal{A} \preceq \mathcal{B}$ then, $\nu \odot \mathcal{A} \preceq \nu \odot \mathcal{B}$ i.e. the binary relation $\preceq$ is compatible with multiplication by a scalar.
4. Let $\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{E} \in \Xi_{cc}(\mathcal{T})$, if $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{D} \preceq \mathcal{E}$ then, $\mathcal{A} \oplus \mathcal{D} \preceq \mathcal{B} \oplus \mathcal{E}$. In other words, the binary relation is compatible with set addition.

Proposition 2.16 ([12, 13])
Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$.

1. Suppose that $\Omega \subseteq \mathcal{C}$, then the binary relation $\preceq_H$ is reflexive.
2. The relation $\preceq_H$ is transitive.
3. Let $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$ and $\nu > 0$, if $\mathcal{A} \preceq_H \mathcal{B}$ then $\nu\mathcal{A} \preceq_H \nu\mathcal{B}$ i.e. the binary relation $\preceq_H$ is compatible with scalar multiplication.
4. Let $\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{E} \in \Xi_{cc}(\mathcal{T})$, if $\mathcal{A} \preceq_H \mathcal{B}$ and $\mathcal{D} \preceq_H \mathcal{E}$ then $\mathcal{A} \oplus \mathcal{D} \preceq_H \mathcal{B} \oplus \mathcal{E}$ i.e. the binary relation is compatible with set addition.

Definition 2.17 ([12, 13])**.** Let V be an $\mathbb{R}-$ vector space and $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$. The kernel of $\mathcal{L}$ is defined by:

$$\ker \mathcal{L} = \{\mathcal{A} : \mathcal{L}(\mathcal{A}) = \theta_V\},$$

where θ_V is the zero element of the vector space. It is obvious that

$$\mathcal{L}(\omega) = \theta_V \text{ for all } \omega \in \Omega \text{ if and only if } \Omega \subseteq \ker \mathcal{L}.$$

Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$ and $\mathcal{L}$ be an additive and positively homogeneous function, then $\mathcal{L}(\mathcal{C})$ is a convex cone in an $\mathbb{R}-$vector space V.
So, we can define two binary relations $\preceq$ and $\preceq_H$ on $\mathcal{L}(\Xi_{cc}(\mathcal{T})) \subseteq V$ as follows:

- $\mathcal{L}(\mathcal{A}) \preceq \mathcal{L}(\mathcal{B})$ if and only if $\mathcal{L}(\mathcal{B}) - \mathcal{L}(\mathcal{A}) \in \mathcal{L}(\mathcal{C})$.
- $\mathcal{L}(\mathcal{A}) \preceq_H \mathcal{L}(\mathcal{B})$ if and only if $\mathcal{L}(\mathcal{B}) - \mathcal{L}(\mathcal{A}) \in \mathcal{L}(\mathcal{C})$ and $\mathcal{B} \ominus_H \mathcal{A}$ exists.

Also $\mathcal{L}(\mathcal{A}) \preceq_H \mathcal{L}(\mathcal{B})$ implies that $\mathcal{L}(\mathcal{A}) \preceq \mathcal{L}(\mathcal{B})$.

Proposition 2.18 ([12, 13])
Let V be an $\mathbb{R}$-vector space and $\mathcal{C}$ a convex cone in $\Xi_{cc}(\mathcal{T})$ and let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ be an additive and positively homogeneous function; then we have the following propositions:

1. Assume that $\{\theta_X\} \in \mathcal{C}$, then the binary relation $\preceq_H$ is reflexive in $\mathcal{L}(\Xi_{cc}(\mathcal{T}))$.
2. The binary relation $\preceq_H$ is transitive.
3. The binary relation $\preceq_H$ is compatible with multiplication by a scalar in $\mathcal{L}(\Xi_{cc}(\mathcal{T}))$.
4. The binary relation $\preceq_H$ is compatible with set addition in $\mathcal{L}(\Xi_{cc}(\mathcal{T}))$.

Proposition 2.19 ([12, 13, 22])
Let V be an $\mathbb{R}$-vector space and $\mathcal{C}$ a convex cone in $\Xi_{cc}(\mathcal{T})$ and let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ be an additive and positively homogeneous function.
Suppose that $\Omega \subseteq \ker \mathcal{L}$. Then we have the following hold:

1. $\mathcal{A} \preceq \mathcal{B}$ implies that $\mathcal{L}(\mathcal{A}) \preceq \mathcal{L}(\mathcal{B})$ and, $\mathcal{A} \preceq_H \mathcal{B}$ implies $\mathcal{L}(\mathcal{A}) \preceq_H \mathcal{L}(\mathcal{B})$.
2. Assume $\ker \mathcal{L} \subseteq \mathcal{C}$. Then, $\mathcal{L}(\mathcal{A}) \preceq \mathcal{L}(\mathcal{B})$ implies $\mathcal{A} \preceq \mathcal{B}$, and $\mathcal{L}(\mathcal{A}) \preceq_H \mathcal{L}(\mathcal{B})$ implies that $\mathcal{A} \preceq_H \mathcal{B}$.

3. Main results

This section contains two subsections. The first subsection will propose an extension of the Charnes–Cooper transformation to convert a fractional set-valued optimization problem into a linear set-valued optimization problem. The second subsection will propose a solution algorithm.

3.1. An extension of the Charnes and Cooper transformation for a set-valued fractional linear optimisation problem

Given P, Q and g_j where $j = \overline{1, m}$, set-valued linear functions defined from U to $\Xi_{cc}(\mathcal{T})$, with Q non-zero for a subset of U.

From a formal point of view, a set-valued linear fractional optimization problem with m constraints can be reformulated as follows:

$$\begin{cases} Opt \;\; f(x) = \dfrac{P(x)}{Q(x)} \\ S.t: \\ g_j(x) \preceq 0, \qquad j = \overline{1,m} \\ x \in \mathbb{R}^n_+. \end{cases} \tag{3.1}$$

For some value of x, $Q(x)$ can be equal to zero. To avoid such cases, it is necessary that $\{x \geqslant 0, g(x) \preceq 0 \Rightarrow Q(x) \succ 0\}$ or $\{x \geqslant 0, g(x) \preceq 0 \Rightarrow Q(x) \prec 0\}$.
For convenience, suppose that (3.1) satisfies the following condition:

$$\{x \geqslant 0, g(x) \preceq 0 \Rightarrow Q(x) \succ 0\}.$$

Let $G \subseteq U$ denote the decision set and let $\mathcal{F} = f(G) = \{f(x) |\;\; x \in G\} \subseteq \Xi_{cc}(\mathcal{T})$. We will adapt the Charnes and Cooper method [1] to transform the set-valued fractional linear optimization problem (3.1) into a set-valued linear optimization problem.

3.1.1. Relationship between $\mathcal{L}\left(\dfrac{1}{\mathcal{A}}\right)$ *and* $\dfrac{1}{\mathcal{L}(\mathcal{A})}$

The proposition below establishes an ordering relation between $\mathcal{L}\left(\dfrac{1}{\mathcal{A}}\right)$ and $\dfrac{1}{\mathcal{L}(\mathcal{A})}$. It will make it possible to transform the fractional set optimization problem into a linear set optimization problem by adding an inequality constraint.

Proposition 3.1
Let V be an $\mathbb{R}-$vector space and let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to \mathbb{R}$ be an additive and positively homogeneous function. Let Θ be a linear application with coefficients greater than or equal to 1 on a set $\mathcal{T}$.
Let $\mathcal{A}$ be a nonempty subset of $\Xi_{cc}(\mathcal{T})$ with $\mathcal{A} \succ 0$. Then:

$$\mathcal{L}\left(\frac{1}{\mathcal{A}}\right) \geq \frac{1}{\mathcal{L}(\mathcal{A})}.$$

Proof
$\mathcal{A}$ being a non-empty set of $\Xi_{cc}(\mathcal{T})$, then we denote a the largest value of $\mathcal{A}$ and a' the smallest value of $\mathcal{A}$.
Let $\Theta(\alpha) = \nu\alpha$ with $\nu \geq 1$.

$$\begin{aligned} \mathcal{L}\left(\frac{1}{\mathcal{A}}\right) &= \sup_{\alpha \in \frac{1}{\mathcal{A}}} \Theta(\alpha) + \inf_{\alpha \in \frac{1}{\mathcal{A}}} \Theta(\alpha) \\ &= \Theta(\frac{1}{a'}) + \Theta(\frac{1}{a}) \\ &= \nu\left(\frac{1}{a'} + \frac{1}{a}\right) \\ &= \frac{\nu(a+a')}{aa'} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}(\mathcal{A}) &= \sup_{\alpha \in \mathcal{A}} \Theta(\alpha) + \sup_{\alpha \in \mathcal{A}} \Theta(\alpha) \\ &= \nu(a + a'). \end{aligned}$$

So

$$\frac{1}{\mathcal{L}(\mathcal{A})} = \frac{1}{\nu(a+a')} = \frac{1}{\nu}\left(\frac{1}{a+a'}\right).$$

Let's study the sign of: $\mathcal{L}\left(\frac{1}{\mathcal{A}}\right) - \frac{1}{\mathcal{L}(\mathcal{A})}$. We have:

$$\begin{aligned}\mathcal{L}\left(\frac{1}{A}\right) - \frac{1}{\mathcal{L}(\mathcal{A})} &= \frac{\nu(a+a')}{aa'} - \frac{1}{\nu}\left(\frac{1}{a+a'}\right)\\ &= \frac{\nu(a+a')}{aa'} - \frac{1}{\nu}\left(\frac{1}{a+a'}\right)\\ &= \frac{(\nu(a+a'))^2 - aa'}{\nu aa'(a+a')}.\end{aligned}$$

Since a and a' are positive, then we have: $\nu aa'(a+a') > 0$ so the sign is that of $(\nu(a+a'))^2 - aa'$. Now we know that $(\nu(a+a'))^2 - aa' = \left[\nu(a+a') - \sqrt{aa'}\right]\left[\nu(a+a') + \sqrt{aa'}\right]$. Since a and a' are positive we have $\left[\nu(a+a') + \sqrt{aa'}\right] \geq 0$, so the sign is that of $\left[\nu(a+a') - \sqrt{aa'}\right]$.

For $\mathcal{L}\left(\frac{1}{\mathcal{A}}\right) \geq \frac{1}{\mathcal{L}(\mathcal{A})}$ $\left[\nu(a+a') - \sqrt{aa'}\right] \geq 0$, it is necessary that $\left[\nu(a+a') - \sqrt{aa'}\right] \geq 0 \Rightarrow \nu \geq \frac{\sqrt{aa'}}{a+a'}$.
As $\forall a, a' \in \mathbb{R}^+$ justifies that $\frac{\sqrt{aa'}}{a+a'} \leq 1$.
We have $\frac{\sqrt{aa'}}{a+a'} \leq 1 \Rightarrow \sqrt{aa'} \leq a+a'$. So

$$\begin{aligned}\frac{\sqrt{aa'}}{a+a'} \leq 1 &\Rightarrow \sqrt{aa'} \leq a+a'\\ &\Rightarrow aa' \leq a^2 + a'^2 + 2aa'\\ &\Rightarrow 0 \leq a^2 + a'^2 + aa'.\end{aligned}$$

Thus $\nu \geq 1 \geq \frac{\sqrt{aa'}}{a+a'}$, hence $\nu \geq 1$.

So for all $\nu \geq 1$ we have: $\mathcal{L}\left(\frac{1}{\mathcal{A}}\right) \geq \frac{1}{\mathcal{L}(\mathcal{A})}$. Hence the proof. □

3.1.2. Modified Charnes-Cooper method
Consider the following fractional function:

$$f(x) = \frac{P(x)}{Q(x)} = P(x) \odot \frac{1}{Q(x)}.$$

Let $T(x) = \frac{1}{Q(x)}$. We have:

$$\mathcal{L}(T(x)) = \mathcal{L}\left(\frac{1}{Q(x)}\right).$$

By posing

$$t = \mathcal{L}(T(x)) = \mathcal{L}\left(\frac{1}{Q(x)}\right) \geq \frac{1}{\mathcal{L}(Q(x))},$$

We obtain:

$$\begin{cases} t \geq \dfrac{1}{\mathcal{L}(Q(x))} \Rightarrow t\mathcal{L}(Q(x)) \geq 1, \\ y = tx, \\ y = xt \Rightarrow x = \dfrac{y}{t}. \end{cases} \tag{3.2}$$

Substituting in the expression $f(x)$ gives us:

$$\begin{cases} f(x) = P(x)t \Rightarrow f(y,t) = P(\dfrac{y}{t})t, \\ g(\dfrac{y}{t}) \preceq 0. \end{cases} \tag{3.3}$$

The problem becomes:

$$\begin{cases} \min f(y,t) = tP(\dfrac{y}{t}), \\ S.t: \\ g(\dfrac{y}{t}) \preceq 0, j = \overline{1,m}, \\ x \in \mathbb{R}^n_+. \end{cases} \tag{3.4}$$

We have a new form equality constraint: $t\mathcal{L}(Q(\dfrac{y}{t})) \geq 1$.
So, we have the following problem:

$$\begin{cases} \min f(y,t) = P(\dfrac{y}{t})t, \\ S.t: \\ g(\dfrac{y}{t}) \preceq 0, j = \overline{1,m}, \\ t\mathcal{L}(Q(\dfrac{y}{t})) \geq 1, \\ y \in \mathbb{R}^n_+, \quad t > 0. \end{cases} \tag{3.5}$$

Problem (3.5) is a set-valued linear optimization problem. Thus, we have the theorem that guarantees the existence of the solution.

Theorem 3.2
If (y^*, t^*) is a feasible solution to problem (3.5), then x^* is a feasible solution to problem (3.1).

Proof
Assume that (y^*, t^*) is an optimal solution to problem (3.5). We want to prove that x^* is a solution to the problem (3.1).
For all $x \geq 0$, $Q(x) \geq 0$, as (y^*, t^*) is an optimal solution to problem (3.5), then:

$$f(y^*, t^*) > f(y,t) \Rightarrow t^* P(\frac{y^*}{t^*}) > tP(\frac{y}{t}). \tag{3.6}$$

Replacing y^* with t^*x^* and y with tx in equation (3.6) we obtain:

$$t^* P(x^*) > tP(x),$$

As $t^* = \mathcal{L}\left(\dfrac{1}{Q(x^*)}\right)$, we have $T(x^*) = \dfrac{1}{Q(x^*)}$, and $t = \mathcal{L}\left(\dfrac{1}{Q(x)}\right) \Rightarrow T(x) = \dfrac{1}{Q(x)}$.
We obtain a new inequality:

$$\frac{P(x^*)}{Q(x^*)} > \frac{P(x)}{Q(x)} \Rightarrow f(x^*) > f(x).$$

So, x^* is an optimal solution to the problem (3.1). □

3.2. Proposed resolution technique

Let $\mathcal{A}_i$ and $\mathcal{B}_i$ be nonempty sets of a normed vector space $(\mathcal{T}, ||\bullet||)$ for $i = 1, \cdots, n$.
Let:

$$P(x) = x_1 \odot \mathcal{A}_1 \oplus x_2 \odot \mathcal{A}_2 \oplus \cdots \oplus x_n \odot \mathcal{A}_n,$$

and

$$Q(x) = x_1 \odot \mathcal{B}_1 \oplus x_2 \odot \mathcal{B}_2 \oplus \cdots \oplus x_n \odot \mathcal{B}_n.$$

The functions $P(x)$ and $Q(x)$ are set-valued linear functions. In the following, we will only assume $Q(x) \succ 0$. Therefore, we have the following set-valued fractional linear function:

$$f(x) = \frac{x_1 \odot \mathcal{A}_1 \oplus x_2 \odot \mathcal{A}_2 \oplus \cdots \oplus x_n \odot \mathcal{A}_n}{x_1 \odot \mathcal{B}_1 \oplus x_2 \odot \mathcal{B}_2 \oplus \cdots \oplus x_n \odot \mathcal{B}_n}. \tag{3.7}$$

The rest of our work focuses on a minimization problem. A set-valued fractional linear optimization problem can be reformulated as follows:

$$\begin{cases} \min f(x) = \dfrac{x_1 \odot \mathcal{A}_1 \oplus x_2 \odot \mathcal{A}_2 \oplus \cdots \oplus x_n \odot \mathcal{A}_n}{x_1 \odot \mathcal{B}_1 \oplus x_2 \odot \mathcal{B}_2 \oplus \cdots \oplus x_n \odot \mathcal{B}_n}, \\ S.t \\ g_j(x) \preceq 0, \quad j = \overline{1, m}, \\ x \in \mathbb{R}^n_+. \end{cases} \tag{3.8}$$

To solve problem (3.8), follow these steps:

3.2.1. Resolution Steps

Problem (3.8) resolution is solved by following these steps:

Step 1: Using the modified Charnes and Cooper method

The modified Charnes and Cooper method is used to transform the set-valued fractional optimization problem into a set-valued linear optimization problem.

Step 2: Vectorization

It allows the overall optimization problem obtained in Step 1 to be transformed into a deterministic bi-objective optimization problem.

Step 3: Scalarization

This step enables us to transform the deterministic, bi-objective problem obtained in Step 2 into a deterministic linear, mono-objective problem.

Step 4: Resolution

In this step, the deterministic, single-objective, linear problem obtained in Step 3 is solved using any linear programming method. The optimum is found by substituting the minimisers into the initial objective function.

Clearly, the following paragraphs detail the steps described previously.

3.2.2. Step1: Using the modified Charnes and Cooper method

$$f(x) = \frac{x_1 \odot \mathcal{A}_1 \oplus x_2 \odot \mathcal{A}_2 \oplus \cdots \oplus x_n \odot \mathcal{A}_n}{x_1 \odot \mathcal{B}_1 \oplus x_2 \odot \mathcal{B}_2 \oplus \cdots \oplus x_n \odot \mathcal{B}_n},$$

$$= t\left(\frac{y_1}{t} \odot \mathcal{A}_1 \oplus \frac{y_2}{t} \odot \mathcal{A}_2 \oplus \cdots \oplus \frac{y_n}{t} \odot \mathcal{A}_n\right),$$

$$= y_1 \odot \mathcal{A}_1 \oplus y_2 \odot \mathcal{A}_2 \oplus \cdots \oplus y_n \odot \mathcal{A}_n,$$

$$= f(y,t).$$

Transformation of the new constraint:

$$t\mathcal{L}(Q(\frac{y}{t})) = t\mathcal{L}\left(\left(\frac{y_1}{t} \odot \mathcal{B}_1 \oplus \frac{y_2}{t} \odot \mathcal{B}_2 \oplus \cdots \oplus \frac{y_n}{t} \odot \mathcal{B}_n\right)\right),$$

$$= t\left(\sup_{\alpha \in \left(\frac{y_1}{t}\odot\mathcal{B}_1 \oplus \frac{y_2}{t}\odot\mathcal{B}_2 \oplus\cdots\oplus \frac{y_n}{t}\odot\mathcal{B}_n\right)} \Theta(\alpha) + \inf_{\alpha \in \left(\frac{y_1}{t}\odot\mathcal{B}_1 \oplus \frac{y_2}{t}\odot\mathcal{B}_2 \oplus\cdots\oplus \frac{y_n}{t}\odot\mathcal{B}_n\right)} \Theta(\alpha)\right).$$

We obtain a new set-valued linear optimization problem:

$$\begin{cases} \min f(y,t) = y_1 \odot \mathcal{A}_1 \oplus y_2 \odot \mathcal{A}_2 \oplus \cdots \oplus y_n \odot \mathcal{A}_n, \\ S.t: \\ g_j(\frac{y}{t}) \preceq 0, \quad j = \overline{1,m}, \\ t\left(\sup_{\alpha \in \left(\frac{y_1}{t}\odot\mathcal{B}_1 \oplus \frac{y_2}{t}\odot\mathcal{B}_2 \oplus\cdots\oplus \frac{y_n}{t}\odot\mathcal{B}_n\right)} \Theta(\alpha) + \inf_{\alpha \in \left(\frac{y_1}{t}\odot\mathcal{B}_1 \oplus \frac{y_2}{t}\odot\mathcal{B}_2 \oplus\cdots\oplus \frac{y_n}{t}\odot\mathcal{B}_n\right)} \Theta(\alpha)\right) \geq 1, \\ y \in \mathbb{R}^n_+, \quad t > 0. \end{cases} \tag{3.9}$$

Definition 3.3. [12]

1. (y^*, t^*) is an optimal solution to problem (3.9) if and only if $f(y^*, t^*) \in MIN_{\mathcal{C}}(\mathcal{F})$.
2. (y^*, t^*) is an $H-$optimal solution to problem (3.9) if and only if $f(y^*, t^*) \in H - MIN_{\mathcal{C}}(\mathcal{F})$.

Remark 3.4.
By Theoreme 3.2, if (y^*, t^*) is a feasible solution to problem (3.9), then x^* is a feasible solution to problem (3.8).

3.2.3. Step 2: Vectorization
In this step, we use the vectorization technique to transform the problem (3.9) into a bi-objective optimization problem.
Let $(\mathcal{T}, ||\bullet||)$ be a normed vector space, and let Θ be a continuous linear function on $\mathcal{T}$.
$\mathcal{C}$ is a convex cone satisfying $\Omega \subseteq \mathcal{C}$ where

$$\mathcal{C} = \{C \in \Xi_{cc}(\mathcal{T}) / \sup_{\alpha \in C} \Theta(\alpha) + \inf_{\alpha \in C} \Theta(\alpha) \geqslant 0\}.$$

In the following, we will take $V = \mathbb{R}^2$.

Consider $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to \mathbb{R}^2$ and $\mathcal{A}$ a non-empty set in a vector space $(\mathcal{T}, || \bullet ||)$. We have:

$$\mathcal{L}(\mathcal{A}) = \left(-\sup_{\alpha\in\mathcal{A}} \Theta(\alpha) - \inf_{\alpha\in\mathcal{A}} \Theta(\alpha), \sup_{\alpha\in\mathcal{A}} \Theta(\alpha) + \inf_{\alpha\in\mathcal{A}} \Theta(\alpha) \right).$$

Let:

$$f(y,t) = y_1 \odot \mathcal{A}_1 \oplus y_2 \odot \mathcal{A}_2 \oplus \cdots \oplus y_n \odot \mathcal{A}_n \equiv A(y).$$

Then,

$$(\mathcal{L}of)(y,t) = \left(-\sup_{\alpha\in\mathcal{A}(y)} \Theta(\alpha) - \inf_{\alpha\in\mathcal{A}(y)} \Theta(\alpha), \sup_{\alpha\in\mathcal{A}(y)} \Theta(\alpha) + \inf_{\alpha\in\mathcal{A}(y)} \Theta(\alpha) \right),$$

$$= (f_1(y,t), f_2(y,t)).$$

Since $y_i \geqslant 0$ for $i = 1, 2, \cdots, n$, using the previous proposition, we obtain:

$$\sup_{\alpha\in\mathcal{A}(y)} \Theta(\alpha) = \left[\sup_{\alpha\in\mathcal{A}_1} \Theta(\alpha)\right] y_1 + \cdots + \left[\sup_{\alpha\in\mathcal{A}_n} \Theta(\alpha)\right] y_n, \text{ and },$$
$$\inf_{\alpha\in\mathcal{A}(y)} \Theta(\alpha) = \left[\inf_{\alpha\in\mathcal{A}_1} \Theta(\alpha)\right] y_1 + \cdots + \left[\inf_{\alpha\in\mathcal{A}_n} \Theta(\alpha)\right] y_n.$$

So:

$$f_1(y,t) = -\sup_{\alpha\in\mathcal{A}(y)} \Theta(\alpha) - \inf_{\alpha\in\mathcal{A}(y)} \Theta(\alpha),$$
$$= -\left[\sup_{\alpha\in\mathcal{A}_1} \Theta(\alpha) + \inf_{\alpha\in\mathcal{A}_1} \Theta(\alpha)\right] y_1 - \cdots - \left[\sup_{\alpha\in\mathcal{A}_n} \Theta(\alpha) + \inf_{\alpha\in\mathcal{A}_n} \Theta(\alpha)\right] y_n,$$

and

$$f_2(y,t) = \sup_{\alpha\in\mathcal{A}(y)} \Theta(\alpha) + \inf_{\alpha\in\mathcal{A}(y)} \Theta(\alpha),$$
$$= \left[\sup_{\alpha\in\mathcal{A}_1} \Theta(\alpha) + \inf_{\alpha\in\mathcal{A}_1} \Theta(\alpha)\right] y_1 + \cdots + \left[\sup_{\alpha\in\mathcal{A}_n} \Theta(\alpha) + \inf_{\alpha\in\mathcal{A}_n} \Theta(\alpha)\right] y_n.$$

Now, let us transform the set-valued constraints into deterministic constraints.
Let the following constraint be:

$$t \left(\sup_{\alpha\in\left(\frac{y_1}{t}\odot\mathcal{B}_1\oplus\frac{y_2}{t}\odot\mathcal{B}_2\oplus\cdots\oplus\frac{y_n}{t}\odot\mathcal{B}_n\right)} \Theta(\alpha) + \inf_{\alpha\in\left(\frac{y_1}{t}\odot\mathcal{B}_1\oplus\frac{y_2}{t}\odot\mathcal{B}_2\oplus\cdots\oplus\frac{y_n}{t}\odot\mathcal{B}_n\right)} \Theta(\alpha) \right) \geq 1.$$

We have:

$$\sup_{\alpha\in\left(\frac{y_1}{t}\odot\mathcal{B}_1\oplus\frac{y_2}{t}\odot\mathcal{B}_2\oplus\cdots\oplus\frac{y_n}{t}\odot\mathcal{B}_n\right)} \Theta(\alpha) = \frac{y_1}{t} \sup_{\alpha\in\mathcal{B}_1} \Theta(\alpha) + \cdots + \frac{y_n}{t} \sup_{\alpha\in\mathcal{B}_n} \Theta(\alpha), \tag{3.10}$$

and

$$\inf_{\alpha\in\left(\frac{y_1}{t}\odot\mathcal{B}_1\oplus\frac{y_2}{t}\odot\mathcal{B}_2\oplus\cdots\oplus\frac{y_n}{t}\odot\mathcal{B}_n\right)}\Theta(\alpha)=\frac{y_1}{t}\inf_{\alpha\in\mathcal{B}_1}\Theta(\alpha)+\cdots+\frac{y_n}{t}\inf_{\alpha\in\mathcal{B}_n}\Theta(\alpha). \tag{3.11}$$

By adding (3.10) and (3.11), we obtain:

$$\left(\sup_{\alpha\in\left(\frac{y_1}{t}\odot\mathcal{B}_1\oplus\frac{y_2}{t}\odot\mathcal{B}_2\oplus\cdots\oplus\frac{y_n}{t}\odot\mathcal{B}_n\right)}\Theta(\alpha)+\inf_{\alpha\in\left(\frac{y_1}{t}\odot\mathcal{B}_1\oplus\frac{y_2}{t}\odot\mathcal{B}_2\oplus\cdots\oplus\frac{y_n}{t}\odot\mathcal{B}_n\right)}\Theta(\alpha)\right)=$$
$$\left(\sup_{\alpha\in\mathcal{B}_1}\Theta(\alpha)+\inf_{\alpha\in\mathcal{B}_1}\Theta(\alpha)\right)\frac{y_1}{t}+\cdots+\left(\sup_{\alpha\in\mathcal{B}_n}\Theta(\alpha)+\inf_{\alpha\in\mathcal{B}_n}\Theta(\alpha)\right)\frac{y_n}{t}. \tag{3.12}$$

Multiplying t by each member of (3.12) gives us the following result:

$$t\left(\sup_{\alpha\in\left(\frac{y_1}{t}\odot\mathcal{B}_1\oplus\frac{y_2}{t}\odot\mathcal{B}_2\oplus\cdots\oplus\frac{y_n}{t}\odot\mathcal{B}_n\right)}\Theta(\alpha)+\inf_{\alpha\in\left(\frac{y_1}{t}\odot\mathcal{B}_1\oplus\frac{y_2}{t}\odot\mathcal{B}_2\oplus\cdots\oplus\frac{y_n}{t}\odot\mathcal{B}_n\right)}\Theta(\alpha)\right)=$$
$$\left(\sup_{\alpha\in\mathcal{B}_1}\Theta(\alpha)+\inf_{\alpha\in\mathcal{B}_1}\Theta(\alpha)\right)y_1+\cdots+\left(\sup_{\alpha\in\mathcal{B}_n}\Theta(\alpha)+\inf_{\alpha\in\mathcal{B}_n}\Theta(\alpha)\right)y_n.$$

Thus, we obtain the following linear deterministic constraint:

$$\left(\sup_{\alpha\in\mathcal{B}_1}\Theta(\alpha)+\inf_{\alpha\in\mathcal{B}_1}\Theta(\alpha)\right)y_1+\cdots+\left(\sup_{\alpha\in\mathcal{B}_n}\Theta(\alpha)+\inf_{\alpha\in\mathcal{B}_n}\Theta(\alpha)\right)y_n\geq 1.$$

We obtain the following by vectorizing the other constraints:

$$(\mathcal{L}og_j)(\frac{y}{t})=\left(\underline{g}_j(\frac{y}{t}),\overline{g}_j(\frac{y}{t})\right).$$

Let $g_j(\frac{y}{t})=D(\frac{y}{t})$.

$$\underline{g}_j(\frac{y}{t})=-\sup_{\alpha\in D(\frac{y}{t})}\Theta(\alpha)-\inf_{\alpha\in D(\frac{y}{t})}\Theta(\alpha)\quad and\quad \overline{g}_j(\frac{y}{t})=\sup_{\alpha\in D(\frac{y}{t})}\Theta(\alpha)+\inf_{\alpha\in D(\frac{y}{t})}\Theta(\alpha),$$

and

$$\mathcal{L}(0)=(0,0).$$

We know that;

$$\begin{aligned} g_j(\frac{y}{t})\preceq 0 &\Rightarrow \mathcal{L}(g_j(\frac{y}{t}))\leqslant\mathcal{L}(0),\\ &\Rightarrow\left(\underline{g}_j(\frac{y}{t}),\overline{g}_j(\frac{y}{t})\right)\leqslant(0,0),\\ &\Rightarrow \underline{g}_j(\frac{y}{t})\leqslant 0\ \ and\ \ \overline{g}_j(\frac{y}{t})\leqslant 0.\end{aligned}$$

As $\underline{g}_j(\frac{y}{t})\leqslant\overline{g}_j(\frac{y}{t})$ then, $\underline{g}_j(\frac{y}{t})\leqslant\overline{g}_j(\frac{y}{t})\leqslant 0$. So, let us consider the constraints $\overline{g}_j(\frac{y}{t})\leqslant 0$ with $j=\overline{1,m}$.

Thus, we can reformulate the deterministic linear optimization problem as follows:

$$(P_v)\begin{cases} \min(\mathcal{L}of)(y,t) = (f_1(y,t), f_2(y,t)) \\ \overline{g}_j(\frac{y}{t}) \leq 0, j = \overline{1,m} \\ \left(\sup_{\alpha\in\mathcal{B}_1}\Theta(\alpha) + \inf_{\alpha\in\mathcal{B}_1}\Theta(\alpha)\right) y_1 + \cdots + \left(\sup_{\alpha\in\mathcal{B}_n}\Theta(\alpha) + \inf_{\alpha\in\mathcal{B}_n}\Theta(\alpha)\right) y_n \geq 1 \\ y \in \mathbb{R}^n_+, t > 0. \end{cases} \tag{3.13}$$

Definition 3.5. [12]

1. (y^*, t^*) is an optimal solution to problem (3.13) if and only if $(\mathcal{L}of)(y^*, t^*) \in MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$.
2. (y^*, t^*) is an optimal solution to problem (3.13) if and only if $(\mathcal{L}of)(y^*, t^*) \in H - MIN_{L(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$.

The following propositions guarantee the concepts of minimal elements and $H-$minimal elements of $\mathcal{F}$ and of $\mathcal{L}(\mathcal{F})$.

Proposition 3.6
Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to \mathbb{R}^2$ be an additive and positively homogeneous function from $\Xi_{cc}(\mathcal{T})$ to a vector space $\mathbb{R}^2$ and let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$.
Let $\mathcal{F}$ be a subset of $\Xi_{cc}(\mathcal{T})$ and let (y^*, t^*) be an $H-$ optimal solution to the problem (3.9). Assume that $\Omega \subseteq \ker\mathcal{L} \subseteq \mathcal{C}$. Then, $f(y^*, t^*) \in H - MIN_{\mathcal{C}}(\mathcal{F})$ if and only if $\mathcal{L}(f(y^*, t^*)) \in H - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$.

Proof
Suppose that (y^*, t^*) is an H-optimal solution to problem (3.9).
let $f(y^*, t^*) \in H - MIN_{\mathcal{C}}(\mathcal{F})$. Let us assume that it exists $B \in \mathcal{L}(\mathcal{F})$ such that $B \preceq_H \mathcal{L}(f(y^*, t^*))$, where $B = \mathcal{L}(f(y,t))$ for $f(y,t) \in \mathcal{F}$; i.e. $\mathcal{L}(f(y,t)) \preceq_H \mathcal{L}(f(y^*, t^*))$. We want to show that $\mathcal{L}(f(y^*, t^*)) \preceq_H \mathcal{L}(f(y,t)) = B$. We see that $f(y,t) \preceq_H f(y^*, t^*)$, that is to say also that $f(y^*, t^*) \preceq_H f(y,t)$ by the definition of the minimal element.
Using the Proposition 2.19, we have $\mathcal{L}(f(y^*, t^*)) \preceq_H \mathcal{L}(f(y,t)) = B$. Which shows that $\mathcal{L}(f(y^*, t^*))$ is an H-minimal element of $\mathcal{L}(\mathcal{F})$ with respect to the convex cone of $\mathcal{L}(\mathcal{C})$.
Conversely, for $\mathcal{L}(f(y^*, t^*)) \in H - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$, we assume that there exists $f(y,t) \in \mathcal{F}$ such that $f(y,t) \preceq_H f(y^*, t^*)$. We want to assert that $f(y^*, t^*) \preceq_H f(y,t)$. Using the Proposition 2.19, we have $\mathcal{L}(f(y,t)) \preceq_H \mathcal{L}(f(y^*, t^*))$. Since $\mathcal{L}(f(y^*, t^*)) \in H - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$, we have $\mathcal{L}(f(y^*, t^*)) \preceq_H \mathcal{L}(f(y,t))$. By the Proposition 2.19, we have $f(y^*, t^*) \preceq_H f(y,t)$. Which shows that $f(y^*, t^*)$ is a minimal element of $\mathcal{F}$. □

Proposition 3.7
Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to \mathbb{R}^2$ be an additive and positively homogeneous function from $\Xi_{cc}(\mathcal{T})$ to a vector space $\mathbb{R}^2$ and let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$.
Let $\mathcal{F}$ be a subset of $\Xi_{cc}(\mathcal{T})$ and (y^*, t^*) an optimal solution to the problem (3.9). Assume that $\Omega \subseteq \ker\mathcal{L} \subseteq \mathcal{C}$. Then, $f(y^*, t^*) \in MIN_{\mathcal{C}}(\mathcal{F})$ if and only if $\mathcal{L}(f(y^*, t^*)) \in MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$.

Proof
Assume that (y^*, t^*) is an optimal solution to problem (3.9).
Let $f(y^*, t^*) \in MIN_{\mathcal{C}}(\mathcal{F})$. Assume that there exists $B \in \mathcal{L}(\mathcal{F})$ such that $B \preceq \mathcal{L}(f(y^*, t^*))$, where $B = \mathcal{L}(f(y,t))$ for $f(y,t) \in \mathcal{F}$; that is, we have $\mathcal{L}(f(y,t)) \preceq \mathcal{L}(f(y^*, t^*))$.We want to affirm that $\mathcal{L}(f(y^*, t^*)) \preceq \mathcal{L}(f(y,t)) = B$.
We see that $f(y,t) \preceq f(y^*, t^*)$, by the definition of the minimal element, is also to say that $f(y^*, t^*) \preceq f(y,t)$.
Using the Proposition 2.19, we have $\mathcal{L}(f(y^*, t^*)) \preceq \mathcal{L}(f(y,t)) = B$. Which shows that $\mathcal{L}(f(y^*, t^*))$ is a minimal element of $\mathcal{L}(\mathcal{F})$ with respect to the convex cone of $\mathcal{L}(\mathcal{C})$.
Conversely, for $\mathcal{L}(f(y^*, t^*)) \in MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$, we assume that there exists $f(y,t) \in \mathcal{F}$ such that $f(y,t) \preceq f(y^*, t^*)$. We want to assert that $f(y^*, t^*) \preceq f(y,t)$. Using the Proposition 2.19, we have $\mathcal{L}(f(y,t)) \preceq \mathcal{L}(f(y^*, t^*))$. Since $\mathcal{L}(f(y^*, t^*)) \in MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$, we have $\mathcal{L}(f(y^*, t^*)) \preceq \mathcal{L}(f(y,t))$. By Proposition 2.19, we have $f(y^*, t^*) \preceq f(y,t)$. Which shows that $f(y^*, t^*)$ is a minimal element of $\mathcal{F}$. □

The following result guarantees the equivalence of optimal and $H-$optimal solutions of problems (3.13) and (3.9).

Proposition 3.8
Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to \mathbb{R}^2$ be an additive and positively homogeneous function from $\Xi_{cc}(\mathcal{T})$ to a vector space $\mathbb{R}^2$ and let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$. Assume that $\Omega \subseteq ker\mathcal{L} \subseteq \mathcal{C}$.

1. (y^*, t^*) is an optimal solution to problem (3.9) if and only if (y^*, t^*) is an optimal solution to problem (3.13).
2. (y^*, t^*) is an H-optimal solution to problem (3.9) if and only if (y^*, t^*) is an H-optimal solution to problem (3.13).

Proof

1. Suppose that (y^*, t^*) is an H-optimal solution to problem (3.9). Let $A^* = f(y^*, t^*)$. Let $\mathcal{F}$ be a subset of $\Xi_{cc}(\mathcal{T})$ and $A^* \in \mathcal{F}$. As $\Omega \subseteq ker\mathcal{L} \subseteq \mathcal{C}$, we have $A^* \in MIN_{\mathcal{C}}(\mathcal{F})$ which implies that $\mathcal{L}(A^*) \in MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$ i.e. $\mathcal{L}(f(y^*, t^*)) \in MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$. Using Definition 3.5, (y^*, t^*) is an optimal solution to problem (3.13).
   Conversely, (y^*, t^*) is an optimal solution to problem (3.13) by definition of the minimum elements, $\mathcal{L}(f(y^*, t^*)) \in MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$. Using Proposition 3.7 we have: $f(y^*, t^*) \in MIN_{\mathcal{C}}(\mathcal{F})$ and by definition (y^*, t^*) is a solution to problem (3.9).
2. Suppose that (y^*, t^*) is an $H - optimal$ solution to problem (3.9). Let $A^* = f(y^*, t^*)$.
   Let $\mathcal{F}$ be a subset of $\Xi_{cc}(\mathcal{T})$ and $A^* \in \mathcal{F}$, since $\Omega \subseteq ker\mathcal{L} \subseteq \mathcal{C}$ we have $A^* \in H - MIN_{\mathcal{C}}(F)$ which implies that $\mathcal{L}(A^*) \in H - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$ i.e. $\mathcal{L}(f(y^*, t^*)) \in H - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$. Using Definition 3.5, (y^*, t^*) is an optimal solution to problem (3.13).
   Conversely, (y^*, t^*) is an optimal solution to problem (3.13) by definition of the H- minima elements, $\mathcal{L}(f(y^*, t^*)) \in H - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$. Using Proposition 3.6 we have: $f(y^*, t^*) \in H - MIN_{\mathcal{C}}(\mathcal{F})$ and by definition (y^*, t^*) is a solution to problem (3.9).

□

3.2.4. Step 3: Scalarization
Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to \mathbb{R}^2$ be an additive and positively homogeneous function from $\Xi_{cc}(\mathcal{T})$ in a vector space $\mathbb{R}^2$ and let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$. Then $\overline{\mathcal{C}} \equiv \mathcal{L}(\mathcal{C})$ is a convex cone of $\mathbb{R}^2$. Let V' denote the set of linear functions from $\mathbb{R}^2$ to $\mathbb{R}$. The dual cone of $\overline{\mathcal{C}}$ is defined by:

$$\overline{\mathcal{C}}_{V'} = \{\phi \in V' / \phi(\overline{c}) \geqslant 0 \ \ \overline{c} \in \overline{\mathcal{C}}\}. \tag{3.14}$$

Definition 3.9. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$. The scalarization function is defined by:

$$\phi(x, y) = \lambda_1 x + \lambda_2 y + k,$$

where $k > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$ are constants and $\lambda_1 < \lambda_2$.

λ_1 and λ_2 can be interpreted as preference weights provided by the decision maker, and k is a constant that can be ignored.

The following is obtained by applying this scalarizing function:

$$\begin{aligned}\phi((\mathcal{L}of)(y,t)) &= \phi\left(-\sup_{\alpha\in\mathcal{A}(y)}\Theta(\alpha)-\inf_{\alpha\in\mathcal{A}(y)}\Theta(\alpha),\ \sup_{\alpha\in\mathcal{A}(y)}\Theta(\alpha)+\inf_{\alpha\in\mathcal{A}(y)}\Theta(\alpha)\right),\\ &= (\lambda_2-\lambda_1)\left[\sup_{\alpha\in\mathcal{A}(y)}\Theta(\alpha)+\inf_{\alpha\in\mathcal{A}(y)}\Theta(\alpha)\right]+k,\\ &= (\lambda_2-\lambda_1)\left[\left(\sup_{\alpha\in\mathcal{A}_1}\Theta(\alpha)+\inf_{\alpha\in\mathcal{A}_1}\Theta(\alpha)\right)x_1+\cdots+\left(\sup_{\alpha\in\mathcal{A}_n}\Theta(\alpha)+\inf_{\alpha\in\mathcal{A}_n}\Theta(\alpha)\right)x_n\right]+k,\\ &> 0.\end{aligned}$$

Since $\lambda_1<\lambda_2$ and $k>0$, and since $\phi((\mathcal{L}of)(y,t))>0$, then $\phi\in\overline{\mathcal{C}}$. We obtain the following problem:

$$\begin{cases}\min\phi((\mathcal{L}of)(y,t)),\\ S.t:\\ \overline{g}_j(\frac{y}{t})\leq 0, j=\overline{1,m},\\ \left(\sup_{\alpha\in\mathcal{B}_1}\Theta(\alpha)+\inf_{\alpha\in\mathcal{B}_1}\Theta(\alpha)\right)y_1+\cdots+\left(\sup_{\alpha\in\mathcal{B}_n}\Theta(\alpha)+\inf_{\alpha\in\mathcal{B}_n}\Theta(\alpha)\right)y_n\geq 1,\\ y\in\mathbb{R}^n_+, t>0.\end{cases} \tag{3.15}$$

Problem (3.21) is a deterministic linear optimization problem, and we have the following theorem.
The following result establishes the equivalence between the optimal and H-optimal solutions to problems (3.21) and (3.9).

Theorem 3.10
Let the function $\mathcal{L}:\Xi_{cc}(\mathcal{T})\to\mathbb{R}^2$ be an additive and positively homogeneous function, and let $\mathcal{C}$ be a convex cone satisfying $\{\theta_X\}\in\mathcal{C}$.
If $\phi\in\overline{\mathcal{C}}_{V'}$ and an element $(y^*,t^*)\in G$ such that:

$$\phi((\mathcal{L}of)(y^*,t^*))\leqslant\phi((\mathcal{L}of)(y,t))\ \ for\ \ (y,t)\in G, \tag{3.16}$$

then, (y^*,t^*) is both an optimal solution and an H-optimal solution to the problem (3.9).

Proof
Assume that (y^*,t^*) is not an optimal or H-optimal solution to problem (3.21) then, there exists $(y,t)\in G$ such that $((\mathcal{L}of)(y,t))\preceq_H((\mathcal{L}of)(y^*,t^*))$ and $((\mathcal{L}of)(y^*,t^*))\npreceq_H((\mathcal{L}of)(y,t))$. If $((\mathcal{L}of)(y^*,t^*))=((\mathcal{L}of)(y,t))$ then this contradicts $((\mathcal{L}of)(y^*,t^*))\npreceq_H((\mathcal{L}of)(y,t))$.
For this purpose, we have $\theta_V\notin((\mathcal{L}of)(y^*,t^*))-((\mathcal{L}of)(y,t))\in\mathcal{L}(\mathcal{C})=\overline{\mathcal{C}}$. Since $\phi\in\overline{\mathcal{C}}^o_{V'}$, we obtain:

$$\phi((\mathcal{L}of)(y^*,t^*))-\phi((\mathcal{L}of)(y,t))=\phi(((\mathcal{L}of)(y^*,t^*))-((\mathcal{L}of)(y,t)))>0, \tag{3.17}$$

which is a contradiction. So (y^*,t^*) is an H-optimal solution to problem (3.13). Using the previous proposition, it follows that (y^*,t^*) is an H-optimal solution to problem (3.9).
On the other hand, considering the binary relation $\preceq$ and using the previous proposition, one can similarly demonstrate that (y^*,t^*) is an optimal solution to the problem (3.9). □

3.2.5. Step 4: Resolution The resolution is done using any linear programming optimization method. Indeed, at this stage, the problem concerned is linear mono-objective. Considering the previous steps, the obtained minimizers will be the minimizers of the original problem.

Algorithm 1 Solving a Linear Fractional Set Optimisation Problem

1. **Enter the starting problem and the scalars λ_1 and λ_2 such that $\lambda_1 < \lambda_2$ and $k \geq 0$.**
2. **Using the modified Charnes and Cooper method:**
   Introduce the new variables y, and t such that $x = \dfrac{y}{t}$, where $t > 0$, and construct the following new problem:

$$\begin{cases} \min f(y,t) = P(\frac{y}{t})t \\ g(\frac{y}{t}) \preceq 0, j = \overline{1,m}; \\ t\mathcal{L}(Q(\frac{y}{t})) \geq 1 \\ y \in \mathbb{R}^n_+, \quad t > 0. \end{cases} \tag{3.18}$$

3. **Vectorization of functions:**
   Let Θ be a linear application of the form $\Theta(\alpha) = \nu\alpha$ with $\nu \geq 1$

$$\begin{cases} \min(\mathcal{L}of)(y,t) = (f_1(y,t), f_2(y,t)) \\ \overline{g}_j(\frac{y}{t}) \leq 0, j = \overline{1,m}; \\ t\mathcal{L}(Q(\frac{y}{t})) \geq 1 \\ y \in \mathbb{R}^n_+, t > 0. \end{cases} \tag{3.19}$$

4. **Scalarization function: :**
   Construct a scalar function, $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ from $\mathcal{L}(f(x))$, depending on λ_1 and λ_2:

$$\phi(x,y) = \lambda_1 x + \lambda_2 y + k. \tag{3.20}$$

5. **Scalarization of objective functions: :**
   Use the equation (3.20) to convert the problem (3.19) as follows:

$$\begin{cases} \min \phi((\mathcal{L}of)(y,t)) = \lambda_1 f_1(y,t) + \lambda_2 f_2(y,t) + k \\ \overline{g}_j(\frac{y}{t}) \leq 0, j = \overline{1,m}; \\ t\mathcal{L}(Q(\frac{y}{t})) \geq 1 \\ y \in \mathbb{R}^n_+, t > 0. \end{cases} \tag{3.21}$$

6. **Resolution**
   Use any linear programming method to determine the value of y^* and t^*.
7. **Iteration on the parameters:** If (y^*, t^*) does not belong to the admissible set G or the set of efficient solutions G^H, do:
   (a) If $(y^*, t^*) \in G$ ou $(y^*, t^*) \in G^H$, then:
   $$(y^*, t^*)$$
   is an optimal solution.
   (b) Else
   Update parameters λ_1, λ_2.
   (c) Return to step 4 to recalculate the scalarizing problem.
8. **Exit :** Display the optimal solution. $x^* = \dfrac{y^*}{t^*}$
9. **End.**

3.3. Algorithm

All steps can be summarized in the Algorithm1:
The complexity of the method is exponential. Indeed,

- for the Charnes and Cooper method, we transform the original problem into an equivalent one. The complexity of this transformation is constant. These are the elementary operations. It can be estimated as $O(n)$, where n denotes the degree or number of vertices necessary for the operation at the objective function level.
- for vectorization, we apply functions $\sup$ and $\inf$. These are simple comparisons in a set. The complexity of $\sup \Theta(\alpha)$ is $O(|A|)$ where $|A| = card\{A\} = n$ with n the number of elements.
  So $\sup_{\alpha \in A} \Theta(\alpha) + \inf_{\alpha \in A} \Theta(\alpha)$ is of complexity $2O(n)$ and this sum is repeated n times at the level of the objective function. Thus, $\min f(y,t)$ has a worst-case complexity of $O(n^2)$, where n denotes the degree or number of summation at the numerator.
  At the constraint level, we have $O(mn)$ where m is the summation number of the denominator. Therefore, the complexity of vectorization is $\max\left\{O(n^2), O(mn)\right\}$.
- for scalarization, considering a bi-objective problem where $f \in \mathbb{R}^2$, then we will have two sums to perform on the objective function, so the complexity is: $\max\left\{2O(n^2), O(mn)\right\}$.
- for the simplex algorithm, it is an algorithm for exploring points, which are actually the decision variables. This requires $2^n - 1$ operations with n the number of variables. Therefore, the worst-case complexity is exponential, i.e., $O(2^n)$.
- Conclusion:
  If the process is repeated k times until the optimal solution is obtained, the complexity is estimated to be:

$$\max\left\{2kO(n^2), kO(mn), kO(2^n)\right\} = kO(2^n).$$

4. Some didactic examples

This section provides examples of applications. These specific didactic cases will help us to understand the theory presented in this work.

Example 4.1. A sustainable agriculture project involving technical and economic uncertainties.
An agricultural cooperative manages a 10-hectare farm and wishes to develop a sustainable farming system there. The cooperative plans to divide the area between two crops.

- x: hectares devoted to the traditional crop of wheat.
- y: hectares devoted to the innovative crop of lentils.

The objective is not to maximize gross revenue, nor to minimize cost, but to maximize the economic efficiency of the project:

$$\text{efficiency} = \frac{\text{Income} \quad \text{income}}{\text{Total} \quad \text{cost}}. \tag{4.1}$$

This ratio indicates how much income is generated for every euro invested, which is a key indicator of financial sustainability.

In addition to variable income and costs per hectare, there are also fixed income and costs, which depend on the economic and institutional scenario and are therefore uncertain.

Rather than using fixed values or probability laws, the cooperative uses a discrete scenario approach based on three realistic cases.

1. Pessimistic factors include unfavourable climatic conditions, low prices, high costs and low subsidies.
2. Average factors include normal conditions and the subsidy standar,

3. Optimistic factors included good conditions, strong demand, high subsidies and controlled costs.

Each scenario assigns a specific value to each parameter, including fixed terms. Each set contains the following elements: $\{\text{Pessimistic}, \ \text{Average}, \ \text{Optimistic}\}$.

This model enables us to prepare for the worst-case scenario, represent the most likely situation, and identify the best potential outcome.
The parameters of the problem are as follows:

	Wheat	Lentils	Constant terms
Income per ha $Keuro$	$A_1 = \{6, 8, 10\}$	$A_2 = \{7, 9, 11\}$	-
Cost per ha $Keuro$	$B_1 = \{6, 5, 4\}$	$B_2 = \{5, 4, 3\}$	-
Fixed income $Keuro$	-	-	$A_0 = \{4, 5, 6\}$
Fixed cost $Keuro$	-	-	$B_0 = \{2, 3, 4\}$

We aim to maximise the revenue/cost ratio, which is an affine fractional function:

$$\begin{cases} \max f(x_1, x_2) = \dfrac{A_1 x_1 \oplus A_2 x_2 \oplus A_0}{B_1 x_1 \oplus B_2 x_2 \oplus B_0} \\ x_1 + x_2 \leq 10 \\ x_1 \geq 0, x_2 \geq 0. \end{cases} \tag{4.2}$$

We consider the linear application $\Theta(\alpha) = \alpha$ and the constants $\lambda_1 = \dfrac{1}{2}$ and $\lambda_2 = \dfrac{3}{2}$, $k = 0$.

- Step 1: modified Charnes and Cooper transformation

$$\begin{cases} \max f(y, t) = A_1 y_1 \oplus A_2 y_2 \oplus A_0 \\ t\mathcal{L}\left(B_1(y_1/t)\right) \oplus B_2(y_2/t) \oplus B_0) = 1 \\ y_1 + y_2 \leq 10t \\ y_1 \geq 0, y_2 \geq 0, t > 0. \end{cases} \tag{4.3}$$

- Step 2: Vectorization

$$\begin{cases} \min \mathcal{L}(f(y, t)) = \mathcal{L}(A_1 y_1 \oplus A_2 y_2 \oplus A_0) \\ t\mathcal{L}\left(B_1(y_1/t) \oplus B_2(y_2/t) \oplus B_0\right) = 1 \\ y_1 + y_2 \leq 10t \\ y_1 \geq 0, y_2 \geq 0, t > 0. \end{cases} \tag{4.4}$$

$$\begin{aligned} &\mathcal{L}(f(y, t)) = \mathcal{L}(A_1 y_1 \oplus A_2 y_2 \oplus A_0) \\ &= \left[\left(\sup_{\alpha \in A_1} \Theta(\alpha) + \inf_{\alpha \in A_1} \Theta(\alpha) \right) y_1 + \left(\sup_{\alpha \in A_2} \Theta(\alpha) + \inf_{\alpha \in A_2} \Theta(\alpha) \right) y_2 + \left(\sup_{\alpha \in A_0} \Theta(\alpha) + \inf_{\alpha \in A_0} \Theta(\alpha) \right) t, \right. \\ &\left. - \left(\sup_{\alpha \in A_1} \Theta(\alpha) + \inf_{\alpha \in A_1} \Theta(\alpha) \right) y_1 - \left(\sup_{\alpha \in A_2} \Theta(\alpha) + \inf_{\alpha \in A_2} \Theta(\alpha) \right) y_2 - \left(\sup_{\alpha \in A_0} \Theta(\alpha) + \inf_{\alpha \in A_0} \Theta(\alpha) \right) t \right] \\ &= \left[(10+8)\, y_1 + (11+9)\, y_2 + (6+5)\, t, -(10+8)\, y_1 - (11+9)\, y_2 - (6+5)\, t \right] \\ &= (18y_1 + 20y_2 + 11t, -18y_1 - 20y_2 - 11t) \end{aligned}$$

We do the same:

$$\begin{aligned} t\mathcal{L}\left(B_1(y_1/t) \oplus B_2(y_2/t) \oplus B_0\right) &= \left(\sup_{\alpha \in B_1} \Theta(\alpha) + \inf_{\alpha \in B_1} \Theta(\alpha) \right) y_1 + \left(\sup_{\alpha \in B_2} \Theta(\alpha) + \inf_{\alpha \in B_2} \Theta(\alpha) \right) y_2 + \\ &\quad \left(\sup_{\alpha \in B_0} \Theta(\alpha) + \inf_{\alpha \in B_0} \Theta(\alpha) \right) t \\ &= 11y_1 + 9y_2 + 7t. \end{aligned}$$

So the problem becomes:

$$\begin{cases} \max \mathcal{L}(f(y,t)) = (18y_1 + 20y_2 + 11t, -18y_1 - 20y_2 - 11t) \\ 11y_1 + 9y_2 + 7t = 1 \\ y_1 + y_2 \leq 10t \\ y_1 \geq 0, y_2 \geq 0, t > 0. \end{cases} \tag{4.5}$$

- Step 3: Scalarization

$$\begin{cases} \max \phi(\mathcal{L}(f(y,t))) = (\lambda_2 - \lambda_1)(18y_1 + 20y_2 + 11t) \\ 11y_1 + 9y_2 + 7t = 1 \\ y_1 + y_2 \leq 10t \\ y_1 \geq 0, y_2 \geq 0, t > 0. \end{cases} \tag{4.6}$$

Which implies

$$\begin{cases} \max \phi(\mathcal{L}(f(y,t))) = 18y_1 + 20y_2 + 11t \\ 11y_1 + 9y_2 + 7t = 1 \\ y_1 + y_2 \leq 10t \\ y_1 \geq 0, y_2 \geq 0, t > 0. \end{cases} \tag{4.7}$$

Thus, Problem (4.7) is a linear optimization problem.

- Step 4: Resolution
With the Dantzig simplex algorithm, we have the solution: $y_1^* = 0$, $y_2^* = \dfrac{10}{97}$ and $t^* = \dfrac{1}{97}$.

By substitution, we have: $x_1^* = 0$, $x_2^* = 10$. So the cooperative can cultivate 10 hectares of lentils and $f^* = \{\frac{74}{34}, \frac{95}{43}, \frac{116}{52}\}$.

Example 4.2.
Consider the following interval-valued fractional optimization problem[29]:

$$\begin{cases} \max f(x) = \dfrac{[3,5]x_1 \oplus [1,4]x_2 \oplus [7,11]}{[\frac{1}{2},2]x_1 \oplus [1,2]x_2 \oplus [4,6]} \\ \\ x_1 + 3x_2 \leqslant 30 \\ -x_1 + 2x_2 \leqslant 5 \\ x_1 \geq 0; x_2 \geq 0, \end{cases} \tag{4.8}$$

and the linear application $\Theta(\alpha) = \alpha$ and the constants $\lambda_1 = \dfrac{1}{2}$ and $\lambda_2 = \dfrac{3}{2}$, $k = 0$: We have:

- modified Charnes and Cooper transformation

$$\begin{cases} \max f(x,t) = [3,5]y_1 \oplus [1,4]y_2 \oplus [7,11]t \\ t\mathcal{L}([\frac{1}{2},2]\frac{y_1}{t} \oplus [1,2]\frac{y_2}{t} \oplus [4,6]) = 1 \\ y_1 + 3y_2 \leqslant 30t \\ -y_1 + 2y_2 \leqslant 5t \\ y_1 \geq 0, y_2 \geq 0, t > 0. \end{cases} \tag{4.9}$$

- Vectorization

$$\begin{cases} \max \mathcal{L}(f(y,t)) = (8y_1 + 5y_2 + 18t, -8y_1 - 5y_2 - 18t) \\ \frac{5}{2}y_1 + 3y_2 + 10t = 1 \\ y_1 + 3y_2 \leqslant 30t \\ -y_1 + 2y_2 \leqslant 5t \\ y_1 \geq 0, y_2 \geq 0, t > 0. \end{cases} \tag{4.10}$$

- Scalization

$$\begin{cases} \max \phi(\mathcal{L}(f(y,t))) = 8y_1 + 5y_2 + 18t \\ \frac{5}{2}y_1 + 3y_2 + 10t = 1 \\ y_1 + 3y_2 \leqslant 30t \\ -y_1 + 2y_2 \leqslant 5t \\ y_1 \geq 0, y_2 \geq 0, t > 0. \end{cases} \tag{4.11}$$

- Resolution Using the Dantzig simplex method: $y_1 = \dfrac{6}{17}$, $y_2 = 0$, $t = \dfrac{1}{85}$.
  By substitution we obtain: $x_1 = 30$, $x_2 = 0$ and $f^* = [\frac{97}{66}, \frac{161}{19}]$.

5. Conclusion

This paper proposed a formulation of a set-valued linear fractional optimization problem. This formulation is a generalization of the interval-valued optimization theory. A resolution technique based on the modified Charnes and Cooper linearization technique, vectorization, and scalarization was proposed using the concept of null sets. To better describe the steps of the method, an algorithm and two didactic examples were presented. The results of this work could be used to model and solve certain real-life problems of an imprecise nature, particularly in economics, to measure the efficiency of systems expressed in the form of relationships between technical and/or economic criteria. However, this work has certain limitations, namely a lack of comparative studies, an absence of test problems in the literature, and a lack of case studies. These elements could be new avenues of research and will make good contributions to the existing literature. Also, the study of sensitivity and phi, and the study of the problem in infinite dimensions can be studied in the following.

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