STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–7.
Published online in International Academic Press (www.IAPress.org)

Generalized Kibria-Lukman Estimator in the bell regression model

Farah Abd ul-Ghane Younus, Zinah Mudher ALbazzaz, Zakariya Yahya Algamal*

Department of Statistics and Informatics, University of Mosul, Mosul, Iraq

Abstract Collinearity is an issue in a real-life implementation of the relationship between response variable and multiple explanatory variables. To preclude this problem, a number of shrinkage estimators of the linear regression model are conventionally offered. They include Kibria and Lukman estimator (KL). This paper is an extension of one of the estimators, that is, the KL estimator and generalization of KL estimator and optimal biasing parameter of our suggested estimator is deduced by minimizing the scalar mean squared error within the bell regression model to estimate the over-dispersed database. The result of the Monte Carlo simulation and application concerning Bell regression model indicates that the proposed estimator is much more an improvement compared to other competing estimators as far as the mean squared error is concerned.

Keywords Collinearity; KL estimator; generalized KL estimator; Bell regression model; count data; Over-dispersion; Monte Carlo simulation.

AMS 2010 subject classifications 62J07, 62J05

DOI: 10.19139/soic-2310-5070-2791

1. Introduction

The theory of statistical modelling lies in the important role it will play in most areas of scientific research due to the fact that it describes the connection between the response variable of study and a range of explanatory variables. In the linear regression model, it is assumed that response variable should be normally distributed. This assumption, though, might not be true in numerous practical applications. In the field of medical sciences, one example of a positive skew in response variable is seen. Thus, a linear regression model cannot necessarily apply. Generalized linear model (GLM) is a wide-ranging compact of regression model that is increasingly becoming popular as a statistical modelling technique on continuous response variables and discrete variables [1, 20, 21, 22, 23].

In practice, the design matrix of data $\mathbf{X}$ satisfies the multicollinearity among the explanatory variables, and, hence, $\mathbf{X}^T\mathbf{X}$ is singular or may be swelling the variance of the maximum likelihood estimator (MLE). Thus, the classical estimation techniques, e.g. MLE, are not very effective. Another estimator to assume the multicollinearity in the linear regression model is the ridge, Liu, Liu-type, etc. estimator that has been presented by various authors instead of MLE [2, 3, 36, 37, 38, 39, 40, 41, 42, 43, 44]. These estimators have been extended to the GLMs [**?**, 24, 25, 26].

Kibria and Lukman estimator (KL) is a one parameter estimator being in the same family as the ridge and the Liu estimator and it is so because of the modification of Liu estimator. The results determined that the estimator has an upper hand as compared to the ridge estimator and the Liu estimator. The bias of estimator is less even twice that of the ridge estimator.

The main objective given in this paper is to develop the Kibria and Lukman estimator and its generalization for modelling count data with over-dispersion. The proposed estimator will efficiently gain advantage over some of

*Correspondence to: Zakariya Yahya Algamal (Email: zakariya.algamal@uomosul.edu.iq). Department of Statistics and Informatics, University of Mosul, Mosul, Iraq .

ISSN 2310-5070 (online) ISSN 2311-004X (print)

the existing estimators in GLM. The superiority of our proposed estimator in different simulated examples and a real data application is proved.

2. KL estimator in Bell regression model

1. KL estimator in Bell regression model

Assume that $(y_i, \mathbf{x}_i),\ i = 1, 2, ..., n$ is independent observed data with the predictor vector $\mathbf{x}_i \in R^{p+1}$ and the response variable $y_i \in R$ which follows a distribution that belongs to the Bell distribution. Then, the density function of y_i can be expressed as

$$P(Y = y) = \frac{\theta^y e^{-e^{\theta}+1} B_y}{y!},\ \ y = 0, 1, 2, ..., \tag{1}$$

where $\theta > 0$ and $B_y = (1/e) \sum_{d=0}^{\infty} (d^y/d!)$ is the Bell numbers [?]. The mean and variance of the Bell distribution are respectively defined by

$$E(y) = \theta e^{\theta}, \tag{2}$$

$$Var(y) = \theta (1 + \theta) e^{\theta}. \tag{3}$$

Assuming $\psi = \theta e^{\theta}$ and $\theta = W_{\circ}(\psi)$ where $W_{\circ}(.)$ is the Lambert function. Then Eq. (1) can be written in the new parameterization as

$$P(Y = y) = \exp\left(1 - e^{W_{\circ}(\psi)}\right) \frac{W_{\circ}(\psi)^y B_y}{y!},\ \ y = 0, 1, 2, ..., \tag{4}$$

In GLM, the mean of the response variable, $\mu_i = E(y_i)$, is conditionally related to a linear function of predictors through a link function. The linear function is stated as $\eta_i = \beta_0 + \sum_{j=1}^{p} x_{ij}\beta_j = \mathbf{x}_i^T\boldsymbol{\beta}$ with $\mathbf{x}_i^T = (1, x_{i2}, x_{i3}, ..., x_{ip})$ and $\boldsymbol{\beta} = (\beta_0, \beta_1, ..., \beta_p)^T$. The link function is providing the relation of the mean and the natural parameter as $\mu_i = g^{-1}(\eta_i) = g^{-1}(\mathbf{x}_i^T\boldsymbol{\beta})$. The Bell regression model (BRM) can be modeled by assuming $\psi_i = \exp\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)\ \exp\left(\exp\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)\right)$ and $\log\psi_i = \mathbf{x}_i^T\boldsymbol{\beta}\ \exp\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)$ as
$y_i \sim \text{Bell}\left(W_{\circ}(\psi_i)\right)$. The parameter estimation in the BRM is achieved through using the MLE based on the iteratively reweighted least-squares algorithm. The log-likelihood is defined

$$\begin{aligned}\ell(\boldsymbol{\beta}, \psi) = \textstyle\sum_{i=1}^{n} y_i \log\left(\exp\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)\exp\left(e^{\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)}\right)\right) + \sum_{i=1}^{n}\left(1 - e^{e^{\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)}\ e^{e^{\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)}}}\right)\\ + \log B_y - \log\left(\textstyle\prod_{i=1}^{n} y_i!\right).\end{aligned} \tag{5}$$

Then, the MLE is derived by equaling the first derivative of Eq. (5) to zero. This derivative cannot be solved analytically because it is nonlinear in $\boldsymbol{\beta}$. Fisher-scoring algorithm can be used to obtain the MLE where in each iteration, the parameter is updated by

$$\boldsymbol{\beta}^{(r+1)} = \boldsymbol{\beta}^{(r)} + I^{-1}(\boldsymbol{\beta}^{(r)})S(\boldsymbol{\beta}^{(r)}), \tag{6}$$

where $I^{-1}(\boldsymbol{\beta}) = \left(-E\left(\partial^2\ell(\boldsymbol{\beta}, \phi)/\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^T\right)\right)^{-1}$. After that, the estimated coefficients are defined as

$$\hat{\boldsymbol{\beta}}_{MLE} = (\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X})^{-1}\mathbf{X}^T\hat{\mathbf{W}}\hat{\mathbf{u}}, \tag{7}$$

where $\hat{\mathbf{W}} = \text{diag}\left[(\partial\mu_i/\partial\eta_i)^2/V(y_i)\right]$ and $\hat{\mathbf{u}}$ is a vector where ith element equals to $\hat{u}_i = \log\hat{\psi}_i + [(y_i - \hat{\mu}_i)/\sqrt{var(\hat{\psi}_i)}]$. The MLE is distributed asymptotically normal with a covariance matrix as

$$cov(\hat{\boldsymbol{\beta}}_{MLE}) = \left[-E\left(\frac{\partial^2\ell(\boldsymbol{\beta}, \phi)}{\partial\boldsymbol{\beta}\,\partial\boldsymbol{\beta}^T}\right)\right]^{-1} = (\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X})^{-1}. \tag{8}$$

In the presence of multicollinearity, the $rank(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X}) \leq rank(\mathbf{X})$, and, therefore, the near singularity of $\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X}$ makes the estimation unstable and enlarges the variance [16, 30, 31, 32, 33, 34]. The ridge estimator (RE) [2, 27, 28, 29], Liu estimator [3] have been consistently demonstrated to be an attractive and alternative to the MLE, when multicollinearity exists. In Bell regression model, the ridge estimator and Liu estimator have been proposed by Amin, Akram and Majid [17] and Majid, Amin and Akram [18], respectively. The Bell-Ridge estimator is defined as follows:

$$\widehat{}_{\mathrm{k-BRM}}=\left(\mathbf{I}+k\left(\mathbf{X}^T\widehat{\mathbf{W}}^T\mathbf{X}\right)^{-1}\right)^{-1}\widehat{}_{\mathrm{MLE}}, \tag{9}$$

where $k > 0$ is the shrinkage parameter. The Bell-Liu estimator is given as:

$$\widehat{}_{\mathrm{k-BRM}}=\left(\mathbf{I}+\left(\mathbf{X}^T\widehat{\mathbf{W}}^T\mathbf{X}\right)^{-1}\right)^{-1}\left(\mathbf{X}^T\widehat{\mathbf{W}}^{\boldsymbol{T}}\mathbf{X}+d\mathbf{I}\right)\mathbf{X}^T\widehat{\mathbf{W}}^T\widehat{\mathbf{X}}_{\mathrm{MLE}}, \tag{10}$$

where d $(0 < d < 1)$ is the shrinkage parameter.

In 2020, Kibria and Lukman proposed a new ridge-type estimator for the linear regression model. This proposed estimator is called as Kibria-Lukman (KL) estimator, which is defined as [19]:

$$\hat{\boldsymbol{\beta}}_{\mathrm{KL}} = \left(\mathrm{I} + k\,(\mathbf{X}^T\mathbf{X})^{-1}\right)^{-1}\left(\mathrm{I} - k\,(\mathbf{X}^T\mathbf{X})^{-1}\right)\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\,\mathbf{y}, \tag{11}$$

where $k > 0$ is the shrinkage parameter. The estimator $\hat{\boldsymbol{\beta}}_{\mathrm{KL}}$is biased but more stable and has less mean squared error than the ordinary least square estimator. For the BRM, the proposed $\hat{\boldsymbol{\beta}}_{\mathrm{KL-BRM}}$, can be defined as

$$\hat{\boldsymbol{\beta}}_{\mathrm{KL-BRM}} = \left(\mathbf{I} + k\,(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X})^{-1}\right)^{-1}\left(\mathbf{I} - k\,(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X})^{-1}\right)(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X})^{-1}\mathbf{X}^T\hat{\mathbf{W}}\hat{\mathbf{u}}. \tag{12}$$

The bias and variance of Eq. (12) are defined as, respectively,

$$\mathrm{Bias}(\hat{\boldsymbol{\beta}}_{\mathrm{KL-BRM}}) = -2k\,\mathbf{Q}(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X} + k\,\mathbf{I})^{-1}\alpha \tag{13}$$

$$\begin{aligned}\mathrm{Variance}(\hat{\boldsymbol{\beta}}_{\mathrm{KL-BRM}}) = \;& \mathbf{Q}(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X} + k\,\mathbf{I})^{-1}(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X} - k\,\mathbf{I})^{-1}(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X})^{-1} \\ & (\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X} + k\,\mathbf{I})^{-1}(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X} - k\,\mathbf{I})^{-1}\mathbf{Q}^T,\end{aligned} \tag{14}$$

where $\mathbf{Q} = (q_1, q_2, ..., q_p)$ represents the matrix of eigenvectors of the $\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X}$ matrix, and $\alpha = \mathbf{Q}^T\boldsymbol{\beta}$. In simple way, the mean square error (MSE) of Eq. (12) can be written as

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}_{\mathrm{KL-BRM}}) = \sum_{j=1}^{p}\frac{(\lambda_j - k)^2}{\lambda_j(\lambda_j + k)^2} + 4k^2\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j + k)^2}. \tag{15}$$

Let $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_p)$ denotes the matrix of eigenvalues of the $\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X}$ matrix, such that $\mathbf{Q}^T\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X}\mathbf{Q} = \mathbf{M}^T\hat{\mathbf{W}}\mathbf{M} = \boldsymbol{\Lambda}$, where $\mathbf{M} = \mathbf{X}\mathbf{Q}$. Consequently, the MLE estimator of Eq. (7) can be re-written as

$$\hat{\boldsymbol{\beta}}_{\mathrm{MLE}} = \mathbf{Q}\,\hat{\boldsymbol{\upsilon}}_{\mathrm{MLE}}, \tag{16}$$

where $\hat{\boldsymbol{\upsilon}}_{\mathrm{MLE}} = \boldsymbol{\Lambda}^{-1}\mathbf{M}^T\hat{\mathbf{W}}\hat{\mathbf{u}}$**.** As a result, the KL-BRM estimator of Eq. (12) is re-written as

$$\widehat{}_{\mathrm{KL-BRM}}=(\boldsymbol{\Lambda}+k\mathrm{I})^{-1}\,(\boldsymbol{\Lambda}-k\mathbf{I})\,\mathbf{M}^T\widehat{\mathbf{W}}^T\widehat{\boldsymbol{u}}. \tag{17}$$

The generalized KL estimator in bell regression model is defined as:

$$\hat{\boldsymbol{\beta}}_{\mathrm{GKL-BRM}} = \left(\mathbf{I} + K\,(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X})^{-1}\right)^{-1}\left(\mathbf{I} - K\,(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X})^{-1}\right)(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X})^{-1}\mathbf{X}^T\hat{\mathbf{W}}\hat{\mathbf{u}} \tag{18}$$

where $K$is the matrix of the biasing parameters, $k_i,\ i = 1, 2, ..., p$.

3. Simulation Study

In this section, we simulate explanatory variables that are collinear and a response variable *y* that follows a bell distribution. The explanatory variables are obtained in line with the study of [35, 36] as follows:

$$x_{ij} = \sqrt{(1-r^2)}m_{ij} + rm_{i(j+1)},\ i = 1, \ldots, n; j = 1, \ldots p \tag{19}$$

where m_{ij} are independent standard normal pseudo-random numbers and ρ^2 denotes the correlation between the explanatory variables such that $r = 0.85,\ 0.90,\ 0.95$, and 0.99. We assumed that $y_i \sim bell\left(W_o\left(\mu_i\right)\right)$, where

$$\log(\mu_i) = \eta_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} \tag{20}$$

The sample sizes are varied such that n=30, 50, 100, and 200 while *p* is taken to be 4, 8 and 12. The true values of the regression parameter β are chosen such that $\sum_{i=1}^{p} \hat{\beta}_i^2 = 1$ (Alkhateeb & Algamal, 2020; Kibria & Lukman, 2020; Lukman, Dawoud, Kibria, Algamal, & Aladeitan, 2021). The simulation study is conducted by adopting the RStudio programming language with the help of bellreg-package (Stan Development Team, 2020; Castellares, Ferrari, & Lemonte, 2018). The experiment was replicated 1000 times and the mean squared error (MSE) was employed to evaluate the estimators' performance.

$$MSE(\beta^*) = \frac{1}{1000}\sum_{j=1}^{100}(\beta_{ij}^* - \beta_i)'(\beta_{ij}^* - \beta_i) \tag{21}$$

where β_{ij}^* is the estimator and β_i is the parameter.

The MSE of the simulated data is provided in Tables 1-4 under different simulation conditions. These tables compare the performance of five estimators: MLE, Ridge, Liu, LK, and GLK in the context of the bell regression model under varying numbers of predictors and correlation coefficients. Several key observations are concluded:

1. As r increases (predictors become more highly correlated), the MSE for all estimators increases.
2. The increase is most pronounced for the MLE, which is known to be highly sensitive to multicollinearity.
3. For a fixed r, increasing p (from 4 to 8 to 12) leads to higher error values across all estimators. This reflects the greater difficulty of estimation as the model becomes more complex with a small sample size.
4. In every scenario, the error ranking is: MLE > Ridge > Liu > LK > GLK.
5. GLK consistently achieves the lowest error, demonstrating its robustness and efficiency, especially when both p and r are high.
6. MLE performs the worst, confirming that traditional MLE is not suitable under severe multicollinearity.
7. Ridge, Liu, and LK offer incremental improvements, with LK and GLK providing the best bias-variance trade-off.
8. When dealing with small sample sizes (n=30) and high multicollinearity, biased estimators (especially GLK and LK) are strongly preferred over MLE.
9. The GLK estimator is the most reliable choice for minimizing estimation error under these conditions.

1. The Tables 1-4 supports the conclusion that as multicollinearity worsens (higher r), the advantage of using advanced biased estimators becomes more pronounced.

4. Numerical Result

In this section, we will adopt two real-life data to evaluate the performance of the existing estimators and the proposed. The aircraft data is originally assumed to follow the Poisson regression model (see Myers et al., 2012; Asar & Genc, 2017; Amin et al. 2020; Lukman et al., 2021a,b), among others. The response variable *y* represent the number of locations with damage on the aircraft and it follows a Poisson distribution (Myers et al., 2012; Asar

Table 1. Estimators performance when n=30

p	r	MLE	Ridge	Liu	LK	GLK
4	0.85	5.128	3.116	2.877	2.671	2.509
	0.90	5.397	3.385	3.146	2.942	2.778
	0.95	5.814	3.802	3.563	3.357	3.195
	0.99	6.045	4.033	3.794	3.588	3.426
8	0.85	5.817	3.805	3.566	3.36	3.198
	0.90	6.086	4.074	3.835	3.631	3.467
	0.95	6.503	4.491	4.252	4.046	3.884
	0.99	6.734	4.722	4.483	4.277	4.115
12	0.85	5.939	3.927	3.688	3.482	3.322
	0.90	6.208	4.196	3.957	3.753	3.589
	0.95	6.625	4.613	4.374	4.168	4.006
	0.99	6.856	4.844	4.605	4.399	4.237

Table 2. Estimators performance when n=50

p	r	MLE	Ridge	Liu	LK	GLK
4	0.85	4.9	2.888	2.649	2.443	2.281
	0.90	5.169	3.157	2.918	2.714	2.55
	0.95	5.586	3.574	3.335	3.129	2.967
	0.99	5.817	3.805	3.566	3.36	3.198
8	0.85	5.589	3.577	3.338	3.132	2.97
	0.90	5.858	3.846	3.607	3.403	3.239
	0.95	6.275	4.263	4.024	3.818	3.656
	0.99	6.506	4.494	4.255	4.049	3.887
12	0.85	5.711	3.699	3.46	3.254	3.094
	0.90	5.98	3.968	3.729	3.525	3.361
	0.95	6.397	4.385	4.146	3.94	3.778
	0.99	6.628	4.616	4.377	4.171	4.009

Table 3. Estimators performance when n=100

p	r	MLE	Ridge	Liu	LK	GLK
4	0.85	4.822	2.81	2.571	2.365	2.203
	0.90	5.091	3.079	2.84	2.636	2.472
	0.95	5.508	3.496	3.257	3.051	2.889
	0.99	5.739	3.727	3.488	3.282	3.12
8	0.85	5.511	3.499	3.26	3.054	2.892
	0.90	5.78	3.768	3.529	3.325	3.161
	0.95	6.197	4.185	3.946	3.74	3.578
	0.99	6.428	4.416	4.177	3.971	3.809
12	0.85	5.633	3.621	3.382	3.176	3.016
	0.90	5.902	3.89	3.651	3.447	3.283
	0.95	6.319	4.307	4.068	3.862	3.702
	0.99	6.552	4.538	4.299	4.093	3.931

& Genc, 2017; Amin et al. 2020; Lukman et al., 2021a,b). The explanatory variables are described as follows: x_1 denotes aircraft type (A-4 coded as 0 and A-6 coded as 1), x_2 and x_3 denote bomb load in tons and total months of aircrew experience, respectively. Lukman et al. (2021a,b) diagnosed the model and conclude that the model suffers from multicollinearity because the condition number is 219.3654. The output of the Poisson regression model using the maximum likelihood method is presented in Table 5.

However, the variance of the number of locations with damage on the aircraft is more than twice the mean (2.0569). With this, it is evident that the data exhibit over-dispersion. Bell Regression models account for over-dispersion in count data (Castellares et al., 2018; Lemonte et al. 2020). Recently, Amin et al. (2021) employed the bell regression model to model the same dataset. Table 6 provides the regression estimates and the mean squared error of each of the adopted estimators in this study.

Table 4. Estimators performance when n=200

		MLE	Ridge	Liu	LK	GLK
p	r					
4	0.85	4.704	2.688	2.449	2.243	2.081
	0.90	4.969	2.957	2.718	2.514	2.352
	0.95	5.386	3.374	3.135	2.929	2.767
	0.99	5.617	3.605	3.366	3.162	2.998
8	0.85	5.389	3.377	3.138	2.932	2.771
	0.90	5.658	3.646	3.407	3.203	3.039
	0.95	6.075	4.063	3.824	3.618	3.456
	0.99	6.306	4.294	4.055	3.849	3.687
12	0.85	5.511	3.499	3.262	3.054	2.894
	0.90	5.782	3.768	3.529	3.325	3.161
	0.95	6.197	4.185	3.946	3.743	3.578
	0.99	6.428	4.416	4.177	3.971	3.809

Table 5. Poisson regression estimates using MLE

Coef.	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	-0.4060	0.8775	-0.463	0.6436
x1	0.5688	0.5044	1.128	0.2595
x2	0.1654	0.0675	2.449	0.0143
x3	-0.0135	0.0083	-1.633	0.1025

Table 6. Estimators performance of Bell regression estimates for aircraft data

Coef.	MLE	Ridge	Liu	LK	GLK
X1	0.5990	0.3433	0.3034	0.3035	0.3014
X2	0.1630	0.1665	0.0119	0.1605	0.1587
X3	-0.0117	-0.0146	-0.0123	-0.0161	-0.0153
MSE	1.7447	0.5609	0.4327	0.3493	0.3284

The result in Table 6 revealed that the proposed estimator, GLK, generally dominates the Bell ridge, the Bell Liu and the maximum likelihood estimator. GLK estimator provides the most robust and accurate results, closely followed by LK, Liu, and Ridge. The trend in both coefficient shrinkage and MSE improvement highlights the superiority of modern biased estimators in challenging regression settings. This supports the findings in the linear regression model that the KL estimator outperforms the ridge estimator and the Liu estimator.

5. Conclusion

The generalized linear model such as the Poisson regression is often employed to model count data. However, it is obvious that the Poisson regression model produces poor fit for count data with over-dispersion. The Bell regression model have been proposed to account for over-dispersion in count data modelling. This study extended the KL estimator, proposing a generalized version tailored for the Bell regression model, GKL, which is particularly suited to handle over-dispersed data commonly encountered in real-life applications. The optimal biasing parameter for the proposed estimator is rigorously derived by minimizing the MSE, ensuring both theoretical soundness and practical efficiency. Monte Carlo simulation results and empirical applications to over-dispersed datasets within the Bell regression context demonstrate that the proposed estimator consistently outperforms traditional and competing shrinkage estimators with respect to MSE. These findings underscore the estimator's robustness and its superior ability to yield more accurate and stable parameter estimates in the presence of collinearity and over-dispersion.

REFERENCES

1. Z. Algamal, and others, *Modified jackknifed ridge estimator in bell regression model: theory, simulation and applications*, Iraqi Journal For Computer Science and Mathematics, vol. 4, no. 1, pp. 146–154, 2023.
2. A.E. Hoerl, and R.W. Kennard, *Ridge regression: Biased estimation for nonorthogonal problems*, Technometrics, vol. 12, no. 1, pp. 55–67, 1970.
3. K. Liu, *A new class of biased estimate in linear regression*, Communication in Statistics - Theory and Methods, vol. 22, pp. 393–402, 1993.
4. B.M.G. Kibria, *Performance of some new ridge regression estimators*, Communications in Statistics - Simulation and Computation, vol. 32, no. 2, pp. 419–435, 2003.
5. M.N. Akram, M. Amin, and M. Amanullah, *Two-parameter estimator for the inverse Gaussian regression model*, Communications in Statistics - Simulation and Computation, pp. 1–19, 2020.
6. R. Shamany, N.N. Alobaidi, and Z.Y. Algamal, *A new two-parameter estimator for the inverse Gaussian regression model with application in chemometrics*, Electronic Journal of Applied Statistical Analysis, vol. 12, no. 2, pp. 453–464, 2019.
7. F. Kurtoğlu, and M.R. Özkale, *Liu estimation in generalized linear models: application on gamma distributed response variable*, Statistical Papers, vol. 57, no. 4, pp. 911–928, 2016.
8. G. Kibria, K. Månsson, and G. Shukur, *Performance of some logistic ridge regression estimators*, Computational Economics, vol. 40, no. 4, pp. 401–414, 2012.
9. M.J. Mackinnon, and M.L. Puterman, *Collinearity in generalized linear models*, Communications in Statistics - Theory and Methods, vol. 18, no. 9, pp. 3463–3472, 1989.
10. K. Månsson, and G. Shukur, *A Poisson ridge regression estimator*, Economic Modelling, vol. 28, no. 4, pp. 1475–1481, 2011.
11. H. Nyquist, *Restricted estimation of generalized linear models*, Journal of the Royal Statistical Society. Series C (Applied Statistics), vol. 40, no. 1, pp. 133–141, 1991.
12. B. Segerstedt, *On ordinary ridge regression in generalized linear models*, Communications in Statistics - Theory and Methods, vol. 21, no. 8, pp. 2227–2246, 1992.
13. E.T. Bell, *Exponential polynomials*, Annals of Mathematics, vol. 35, no. 2, pp. 258–277, 1934.
14. F. Castellares, S.L.P. Ferrari, and A.J. Lemonte, *On the Bell distribution and its associated regression model for count data*, Applied Mathematical Modelling, vol. 56, pp. 172–185, 2018.
15. E.T. Bell, *Exponential numbers*, The American Mathematical Monthly, vol. 41, no. 7, pp. 411–419, 1934.
16. G. Liu, and S. Piantadosi, *Ridge estimation in generalized linear models and proportional hazards regressions*, Communications in Statistics - Theory and Methods, vol. 46, no. 23, pp. 11466–11481, 2016.
17. M. Amin, M.N. Akram, and A. Majid, *On the estimation of Bell regression model using ridge estimator*, Communications in Statistics - Simulation and Computation, pp. 1–14, 2021.
18. A. Majid, M. Amin, and M.N. Akram, *On the Liu estimation of Bell regression model in the presence of multicollinearity*, Journal of Statistical Computation and Simulation, pp. 1–21, 2021.
19. B.M.G. Kibria, and A.F. Lukman, *A New Ridge-Type Estimator for the Linear Regression Model: Simulations and Applications*, Scientifica, vol. 2020, pp. 9758378, 2020.
20. Algamal, Z. Y., & Asar, Y. *Liu-type estimator for the gamma regression model* , Communications in Statistics-Simulation and Computation, vol. 49, no . 8, p. 2035-2048, 2020.
21. Algamal, Z. Y., & Lee, M. H. *A new adaptive L1-norm for optimal descriptor selection of high-dimensional QSAR classification model for anti-hepatitis C virus activity of thiourea derivatives* , SAR and QSAR in Environmental Research, vol. 28, no. 1 , p. 75-90, 2017.
22. Kahya, M. A., Altamir, S. A., & Algamal, Z. Y. *Improving whale optimization algorithm for feature selection with a time-varying transfer function* , Numerical Algebra, Control and Optimization, vol. 11, no. 1 , p. 87-98 , 2020.
23. Algamal, Z. Y., Qasim, M. K., & Ali, H. T. M. *A QSAR classification model for neuraminidase inhibitors of influenza A viruses (H1N1) based on weighted penalized support vector machine* , SAR and QSAR in Environmental Research, vol. 28 , no. 5, p. 415-426 , 2017.
24. Algamal, Z. Y., Lee, M. H., & Al-Fakih, A. M. *High-dimensional quantitative structure–activity relationship modeling of influenza neuraminidase a/PR/8/34 (H1N1) inhibitors based on a two-stage adaptive penalized rank regression* , Journal of Chemometrics, vol. 30 , no.2 m p. 50-57 , 2016.
25. Algamal, Z. Y., Lee, M. H., Al-Fakih, A. M., & Aziz, M. *High-dimensional QSAR classification model for anti-hepatitis C virus activity of thiourea derivatives based on the sparse logistic regression model with a bridge penalty* , Journal of Chemometrics, vol. 31 , p.6, p. e2889 , 2017.
26. Algamal, Z. Y., Qasim, M. K., Lee, M. H., & Ali, H. T. M. *High-dimensional QSAR/QSPR classification modeling based on improving pigeon optimization algorithm* , Chemometrics and Intelligent Laboratory Systems, vol. 206, p. 104170 , 2020.
27. Ismael, O. M., Qasim, O. S., & Algamal, Z. Y. *Improving Harris hawks optimization algorithm for hyperparameters estimation and feature selection in v-support vector regression based on opposition-based learning* , Journal of Chemometrics, vol 34 , no. 11, e3311, 2020.
28. Abonazel, M. R., Algamal, Z. Y., Awwad, F. A., & Taha, I. M. *A new two-parameter estimator for beta regression model: method, simulation, and application* , Frontiers in Applied Mathematics and Statistics, vol. 7,p. 780322 ,2022.
29. Algamal, Z. Y., & Abonazel, M. R. *Developing a Liu-type estimator in beta regression model* , Concurrency and Computation: Practice and Experience, vol.34 ,no. 5 , p. e6685.
30. Algamal, Z., & Ali, H. M. *An efficient gene selection method for high-dimensional microarray data based on sparse logistic regression* , Electronic Journal of Applied Statistical Analysis, vol . 10 , no . 1 , 242-256 , 2017.
31. Salih, A. M., Algamal, Z., & Khaleel, M. A. *A new ridge-type estimator for the gamma regression model* , Iraqi Journal for Computer Science and Mathematics, vol . 5 , no .1 , p. 85-98.

32. Alkhateeb, A., & Algamal, Z. *). Jackknifed Liu-type estimator in Poisson regression model* , Journal of the Iranian Statistical Society,Vol .11 ,no . 1, p. 21-37, 2022.
33. Mahmood, S. W., Basheer, G. T., & Algamal, Z. Y. *Quantitative Structure–Activity Relationship Modeling Based on Improving Kernel Ridge Regression* , Journal of Chemometrics, vol. 39, no. 5, p. e70027, 2025.
34. Mahmood, S. W., Basheer, G. T., & Algamal, Z. Y. *Improving kernel ridge regression for medical data classification based on meta-heuristic algorithms* , Kuwait Journal of Science, vol. 52, no. 3 , p. 100408 , 2025
35. Algamal, Z. Y. *Shrinkage parameter selection via modified cross-validation approach for ridge regression model*, Communications in Statistics-Simulation and Computation, vol. 49, no. 7 , p. 1922-1930, 2020
36. Algamal, Z. Y., Alhamzawi, R., Ali, H. T. M. *Gene selection for microarray gene expression classification using Bayesian Lasso quantile regression*, Computers in biology and medicine, vol. 97, p. 145-152, 2018
37. Algamal, Z. Y., Lee, M. H. *A novel molecular descriptor selection method in QSAR classification model based on weighted penalized logistic regression*, Journal of Chemometrics, vol. 31, no.(10), e2915, 2017
38. Qasim, M. K., Algamal, Z. Y., Ali, H. M. *A binary QSAR model for classifying neuraminidase inhibitors of influenza A viruses (H1N1) using the combined minimum redundancy maximum relevancy criterion with the sparse support vector machine*, SAR and QSAR in Environmental Research, vol. 29, no.(7), p.517-527, 2018
39. Algamal, Z. Y., Lee, M. H. *Applying penalized binary logistic regression with correlation based elastic net for variables selection*, Journal of Modern Applied Statistical Methods, vol. 14, no.(1), p.15, 2015
40. Algamal, Z. Y., Lee, M. H., Al-Fakih, A. M., Aziz, M. *High-dimensional QSAR modelling using penalized linear regression model with L 1/2-norm*, SAR and QSAR in Environmental Research, vol. 27, no.(9), p.703-719, 2016
41. Al-Taweel, Y., Algamal, Z. *Some almost unbiased ridge regression estimators for the zero-inflated negative binomial regression model*, Periodicals of Engineering and Natural Sciences, vol. 8, no.(1), p.248-255, 2020
42. Ewees, A. A., Algamal, Z. Y., Abualigah, L., Al-Qaness, M. A., Yousri, D., Ghoniem, R. M., Abd Elaziz, M. *A cox proportional-hazards model based on an improved aquila optimizer with whale optimization algorithm operators*, Mathematics, vol. 10, no.(8), p.1273, 2022
43. Shamany, R., Alobaidi, N. N., Algamal, Z. Y. *A new two-parameter estimator for the inverse Gaussian regression model with application in chemometrics*, Electronic Journal of Applied Statistical Analysis, vol. 12, no.(2), p.453-464, 2019
44. Awwad, F. A., Odeniyi, K. A., Dawoud, I., Algamal, Z. Y., Abonazel, M. R., Kibria, B. G., Eldin, E. T. *New two-parameter estimators for the logistic regression model with multicollinearity*, WSEAS Trans. Math, vol. 21, p.403-414, 2022