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Record-Based Reliability Analysis of Weibull Models with Textile Applications

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Abstract Severe operational conditions frequently lead to system failure. One frequent error, though, is that systems can quickly become unstable and stop functioning as intended when operating at extremely high or low levels. This article addresses reliability estimation $R = P(Y < X < Z)$, emphasizing the constraint that the strength (X) must exceed the lower stress (Y) while remaining below the upper stress (Z). Assuming independent Weibull distributions for strength and stresses, reliability estimation of $R = P(Y < X < Z)$ from frequentist and Bayesian perspectives utilizing upper record values is investigated. This study develops Bayesian estimators for the reliability R using three different loss functions: quadratic, linear exponential, and minimum expected. Independent informative (gamma) and non-informative (uniform) priors are assumed, and the corresponding loss functions are incorporated to derive posterior estimates. Bayesian inference for the reliability parameter R is performed using Metropolis–Hastings within a Markov Chain Monte Carlo (MCMC) framework. Further, a detailed simulation study to evaluate the performance of the proposed estimators with MCMC techniques is conducted to facilitate the computation of the posterior estimates. Lastly, to validate the proposed methodologies, the reliability estimates are applied to three real jute fiber datasets. Jute fiber, a biodegradable and cost-effective natural material, is examined for its potential in textile applications. The results highlight its favorable mechanical and thermal properties, indicating that jute fiber is a sustainable and efficient alternative for eco-friendly textile materials.

Keywords Stress-strength reliability; Weibull distribution; upper record values; maximum likelihood estimation; Bayesian inference; Markov Chain Monte Carlo.

AMS 2010 subject classifications 62F10, 62N05, 62F15

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1. Introduction

Record value (RV) and related statistics have long been of interest to statisticians and researchers, as they can be viewed in many daily life situations, such as the temperature of the coldest day ever, the time of the longest tennis game that ever happened, and the height of the longest skyscraper in the world. There are two types of RV, namely, the upper RV (URV) and lower RV (LRV). Nagaraja [1] and Ahsanullah [2] seemed to be the first to introduce the basic concept of RV. An observation X_i is termed as URV if it strictly surpasses all preceding observations. For example, in athletic sports, there is a group of events that involve competitive throwing, with the most prominent being distance throwing events like shot put and javelin in track and field. Similarly, an observation X_i can be viewed as an LRV if it is less than all previous observations. For example, athletic sports include a group of events such as competitive running and swimming, where the shortest durations are recorded.

The stress-strength (SS) reliability offers a fundamental probabilistic framework for system reliability assessment, formally expressed as $R = P(Y < X)$ by evaluating the relationship between component strength

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(X) and applied stress (Y). The system remains operational as long as the stress does not exceed the strength. The SS reliability holds significant importance in reliability engineering, with widespread applications across diverse domains. Important examples of this model in medical, military, and engineering applications have been proposed by Kotz et al. [3]. The foundational concept of SS modeling was first formally introduced by Birnbaum [4], with significant developments contributed by Birnbaum and McCarty [5] and Weerahandi and Johnson [6]. Recently, numerous articles and books have been published on the SS model based on RV. Baklizi [7] investigated the use of Bayesian and likelihood techniques using LRV to SS reliability employing the generalized exponential distribution. Bayesian estimation of the SS reliability applied on URV for one- and two-parameter exponential distributions was examined by Baklizi [8]. Hassan et al. [9, 10] relied on URV utilizing the generalized inverted exponential distribution to estimate SS reliability. The estimation of SS reliability using generalized Rayleigh record data was examined by Pak et al. [11]. For more recent research, the reader can see Khan and Khatoon [12, 13], Chaturvedi and Malhotra [14], Mohamed [15], and Hassan et al. [16]–[18].

Extreme operating conditions can cause severe system failures, whereas failures that fall within specified upper and lower bounds attract less attention. For instance, exceeding blood pressure limitations can be dangerous, and electrical equipment can occasionally malfunction outside permitted power ranges. In the SS reliability model expressed as $R = P(Y < X < Z)$, the strength X usually is constrained by the lower and upper stresses Y and Z, respectively. This kind of interaction can occur across a range of disciplines, including psychology and biology. There are numerous additional real-world applications that are comparable and where dependability assessment is crucial. Placing some electrical equipment above or below a set of power sources can cause them to malfunction or stop working. Similarly, it is not advisable to surpass one's systolic blood pressure threshold limiters. These examples are rather simple and easily understandable and show evident relations among different practical scenarios and thereby are not restricted to just a few examples. This model has many failure rate forms, which is beneficial for a variety of data applications. On the other hand, the RV is preferable because the RV is a different kind of incomplete data that is often seen in a wide range of practical applications, so the RV decreases the time for experiments to achieve the reliability for the strength between lower stress and upper stress.

The concept of the reliability metric $R = P(Y < X < Z)$ was established in foundational work by Chandra and Owen [19], Hlawka [20], and Singh [21]. Building on this foundation, researchers have since developed estimation techniques for R under diverse statistical conditions. For example, Hassan et al. [22] addressed the problem of outliers by examining the estimation of R, assuming that the stress and strength follow Weibull distributions. In a study on the Bayesian estimation of R, Attia and Karam [23] concentrated on situations in which the Dagum distribution is fulfilled by the strength and stress random variables. Abd Elfattah and Taha [24] conducted a significant study on reliability estimation for systems following the inverse Rayleigh distribution, addressing the critical challenge of outlier existence. Raheem et al. [25] considered different methods of estimating the SS model. Yousef and Almetwally [26] proposed a method to infer R for the SS reliability model, utilizing both Bayesian and non-Bayesian estimation approaches. Their work was based on the progressive first failure for the Kumaraswamy distribution. Similarly, the study by Yousef et al. [27] implemented dual analytical approaches (Bayesian and non-Bayesian) to assess the reliability parameter R, based on exponentiated exponential SS assumptions and generalized progressive hybrid censored data. The investigation of R in the context of a generalized inverse exponential distribution based on ranked set sampling (RSS) has been addressed by Hassan et al. [28]. A comprehensive framework for estimating $R = P(Y < X < Z)$ under inverted Kumaraswamy distribution assumptions utilizing RSS methodology was presented by Hassan et al. [29]. Using progressive censoring, estimation of SS reliability was examined by [30, 31]. For more recent studies, see References [32]–[34].

This research estimates the SS model $R = P(Y < X < Z)$, where the component X representing the strength must exceed the stress Y but remain below the stress Z. This work tackles the statistical estimation problem concerning the SS reliability model $R = P(Y < X < Z)$ assuming that X, Y, and Z are drawn from Weibull distributions based on URV. This paper derives its significance from a clearly identified research gap. While the reliability model $R = P(Y < X < Z)$ is known, its specific application to the Weibull distribution in conjunction with URV has not been investigated. Our work fills this void, expanding upon our previous research to provide a novel analysis that advances the literature on reliability.

The methodology is validated using an expanded simulation study; both classical (maximum likelihood) and Bayesian frameworks are extensively examined in this paper, including the use of Markov Chain Monte Carlo (Gibbs sampling for scale parameters and Metropolis-Hastings for the shape parameter). This gives R's statistical inference a comprehensive view. A key advantage of the Bayesian framework is its consideration of both symmetric and asymmetric loss functions, which enables a more complex and context-dependent analysis than a straightforward squared-error (SE) loss function. Also, many researchers did not pay attention to minimum expected loss function (MELO). It's great that three actual datasets on the breaking strength of jute fiber were included. It confirms the suggested models by effectively filling the gap between theoretical methods and practical issues. Accordingly, it contributes to filling an existing knowledge gap and opens avenues for future research. The current work is summarized as follows:

- To determine the SS reliability estimator, the maximum likelihood (ML) estimation approach is employed using the invariance property in view of the URV.
- URV is used to derive Bayesian estimators of R using gamma prior (informative, IF) and uniform prior (non-informative, N-IF).
- The Bayesian estimator of R is obtained under symmetric (SE) and asymmetric loss functions including the linear exponential (LINEX) and the MELO.
- Since there are no explicit forms for the Bayes estimator of R, computing the Bayes estimator using the Markov Chain Monte Carlo (MCMC) approach is considered.
- The estimators' properties are investigated to evaluate and compare the effectiveness of the given estimators using Monte Carlo numerical experiments.
- Three actual datasets are used to illustrate how well the recommended estimating techniques are practically implemented.

This article's remaining sections are organized as follows. Section 2 identifies the mathematical formulation of R. Section 3 discusses how to estimate SS reliability using the ML method based on URV. Bayesian estimators of the reliability R are discussed in Section 4, considering both symmetric and asymmetric loss functions using MCMC techniques. Intensive simulation studies are conducted in Section 5. Section 6 examines the three real data applications of R based on URV. Section 7 presents the concluding remarks and research contributions.

2. Expression of the Model

Reliability engineering, biological studies, and survival analysis frequently employ the Weibull distribution due to its flexibility in modeling various failure patterns and life characteristics. What makes it so valuable is its remarkable flexibility; it can take on many shapes and model increasing, constant, or decreasing failure rates. This versatility means it can be applied in diverse areas like insurance, hydrology, industrial engineering, and weather forecasting. One of the Weibull distribution's most appealing features is that it naturally stems from the extreme value theorem (see [35]). This theoretical underpinning reinforces its significance and wide range of uses, giving it a significant physical interpretation in numerous practical applications (see [36], [37]). The scale parameter λ and shape parameter α characterize the Weibull model having probability density function (PDF) and cumulative density function (CDF), as follows:

$$f(x) = \alpha\lambda x^{\alpha-1}e^{-\lambda x^{\alpha}}; x, \alpha, \lambda > 0, \tag{1}$$

and,

$$F(x;\alpha,\lambda) = 1 - e^{-\lambda x^{\alpha}}, \quad x > 0,\ \alpha, \lambda > 0. \tag{2}$$

Scholars have explored different facets of the Weibull distribution. For instance, Kundu and Gupta [38] evaluated the SS reliability via independent, randomly distributed stress and strength variables in a Weibull model. Pak et al. [39] discussed many techniques for estimating the Weibull distribution when the data gathered are fuzzy numbers. Employing adaptive type-I progressive hybrid censoring, Okasha and Mustafa [40] discussed the E-Bayesian estimate for the Weibull distribution in competing risks. The Weibull distribution's multicomponent SS reliability based on URV was examined by [41]. Jiang [42] studied upon estimating the Weibull distribution using censored samples. Using heavily censored data, the E-Bayesian estimation for the Weibull distribution was covered by [43]. For the Weibull distribution, [44] examined Bayesian and non-Bayesian methods for estimating various entropy measures. Estimation in a Weibull distribution using neoteric ranked set sampling was investigated by [45].

Let the strength $X \sim \text{Weibull}(\alpha, \lambda_1)$, the lower stress $Y \sim \text{Weibull}(\alpha, \lambda_2)$, and the upper stress $Z \sim \text{Weibull}(\alpha, \lambda_3)$, then according to Singh [21], the reliability of SS model $R = P(Y < X < Z)$ is obtained as:

$$R = \int_0^\infty F_Y(x) dF_X(x) - \int_0^\infty F_Y(x) F_Z(x) dF_X(x),$$

where $F_Y(x)$ and $F_Z(x)$ are the CDFs of Y and Z at x, respectively. Then, the reliability function of the SS model $R = P(Y < X < Z)$ for Weibull distribution will be obtained as follows:

$$R = \int_0^\infty \alpha\lambda_1 x^{\alpha-1} e^{-(\lambda_1+\lambda_3)x^\alpha} dx - \int_0^\infty \alpha\lambda_1 x^{\alpha-1} e^{-(\lambda_1+\lambda_2+\lambda_3)x^\alpha} dx.$$

Let $t = x^\alpha$ and $dt = \alpha x^{\alpha-1} dx$. Therefore, R can be expressed as below:

$$\begin{aligned} R &= \lambda_1 \int_0^\infty e^{-(\lambda_1+\lambda_3)t} dt - \lambda_1 \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_3)t} dt \\ &= \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)}. \end{aligned} \tag{3}$$

Note that SS reliability is dependent on the parameters λ_1,λ_2, and λ_3.

3. Maximum Likelihood Estimation of R

The ML estimator of $R = P(Y < X < Z)$ is considered using URVs. Consider three independent Weibull random variables, X, Y, and Z. Let $\underline{r} = (r_0, r_1, ..., r_n)$, $\underline{u} = (u_0, u_1, ..., u_m)$ and $\underline{s} = (s_0, s_1, ..., s_k)$ be the observed URVs of sizes $(n + 1)$, $(m + 1)$, and $(k + 1)$ from Weibull distributions with parameters (α, λ_1), (α, λ_2), and (α, λ_3) respectively. According to Arnold et al. [46], the likelihood function $\underline{r} = (r_0, r_1, ..., r_n)$ from strength X is given by:

$$L_1(\lambda_1, \alpha | \underline{r}) = f(r_n) \prod_{i=0}^{n-1} \frac{f(r_i)}{1 - F(r_i)} = (\alpha\lambda_1)^{n+1} e^{-\lambda_1 r_n^\alpha} \prod_{i=0}^{n} r_i^{\alpha-1}. \tag{4}$$

Similarly, the likelihood functions for $\underline{u} = (u_0, u_1, ..., u_m)$ and $\underline{s} = (s_0, s_1, ..., s_k)$ from Y and Z, respectively, are given by:

$$L_2(\lambda_2, \alpha | \underline{u}) = (\alpha\lambda_2)^{m+1} e^{-\lambda_2 u_m^\alpha} \prod_{j=0}^{m} u_j^{\alpha-1}, \tag{5}$$

and

$$L_3(\lambda_3, \alpha | \underline{s}) = (\alpha\lambda_3)^{k+1} e^{-\lambda_3 s_k^\alpha} \prod_{l=0}^{k} s_l^{\alpha-1}. \tag{6}$$

Therefore, using Equations (4), (5), and (6), the observed URVs $\underline{r}$, $\underline{u}$, and $\underline{s}$ joint likelihood function, denoted by $L\left(\Phi\left|\underline{r}, \underline{u}, \underline{s}\right.\right)$, $\Phi = (\lambda_1, \lambda_2, \lambda_3, \alpha)$, can be expressed as follows:

$$L\left(\Phi\left|\underline{r}, \underline{u}, \underline{s}\right.\right) = \alpha^{n+m+k+3} e^{-(\lambda_1 r_n^{\alpha} + \lambda_2 u_m^{\alpha} + \lambda_3 s_k^{\alpha})} \lambda_1^{n+1} \lambda_2^{m+1} \lambda_3^{k+1} \prod_{l=0}^{k} s_l^{\alpha-1} \prod_{i=0}^{n} r_i^{\alpha-1} \prod_{j=0}^{m} u_j^{\alpha-1}. \tag{7}$$

The natural logarithm of the likelihood function (7), denoted by $\ln L$, is given as:

$$\begin{aligned} ln(L) &= (n+m+k+3)\ln(\alpha) + (n+1)\ln(\lambda_1) + (m+1)\ln(\lambda_2) + (k+1)\ln(\lambda_3) \\ &- \lambda_1 r_n^{\alpha} - \lambda_2 u_m^{\alpha} - \lambda_3 s_k^{\alpha} + (\alpha-1)\left[\sum_{i=0}^{n} r_i + \sum_{j=0}^{m} u_j + \sum_{l=0}^{k} s_l\right]. \end{aligned} \tag{8}$$

The first partial derivatives of the log-likelihood function with respect to λ_1,λ_2, λ_3, and α are given, respectively, by:

$$\frac{\partial \ln(L)}{\partial \lambda_1} = \frac{n+1}{\lambda_1} - r_n^{\alpha}, \quad \frac{\partial \ln(L)}{\partial \lambda_2} = \frac{m+1}{\lambda_2} - u_m^{\alpha}, \quad \frac{\partial \ln(L)}{\partial \lambda_3} = \frac{k+1}{\lambda_3} - s_k^{\alpha}, \tag{9}$$

and,

$$\begin{aligned} &\tfrac{\partial \ln(L)}{\partial \alpha} = \tfrac{n+m+k+3}{\alpha} - \lambda_1 r_n^{\alpha} \ln(r_n) + \sum_{i=0}^{n} \ln(r_i) - \lambda_2 u_m^{\alpha} \ln(u_m) \\ &-\lambda_3 s_k^{\alpha} \ln(s_k) + \sum_{j=0}^{m} \ln(u_j) + \sum_{l=0}^{k} \ln(s_l). \end{aligned} \tag{10}$$

Equating the partial derivatives (9) by zero, the ML estimators of λ_1, λ_2, and λ_3 , denoted by $\hat{\lambda}_1$,$\hat{\lambda}_2$, and $\hat{\lambda}_3$, are acquired as follows :

$$\hat{\lambda}_1 = \frac{n+1}{r_n^{\hat{\alpha}}}, \quad \hat{\lambda}_2 = \frac{m+1}{u_m^{\hat{\alpha}}}, \quad \hat{\lambda}_3 = \frac{k+1}{s_k^{\hat{\alpha}}}.$$

Furthermore, the ML estimator of α symbolized by $\hat{\alpha}$ is acquired by setting Equation (10) by zero and substituting $\hat{\lambda}_1$, $\hat{\lambda}_2$, and $\hat{\lambda}_3$ in Equation (10) gives:

$$\frac{n+m+k+3}{\hat{\alpha}} - \lambda_1 r_n^{\hat{\alpha}} \ln(r_n) + \sum_{i=0}^{n} \ln(r_i) - \lambda_2 u_m^{\hat{\alpha}} \ln(u_m) + \sum_{j=0}^{m} \ln(u_j) - \lambda_3 s_k^{\hat{\alpha}} \ln(s_k) + \sum_{l=0}^{k} \ln(s_l) = 0. \tag{11}$$

Solving Equation (11) numerically using R programming language (R core team [47]) and the maxLik and records packages, we obtain $\hat{\alpha}$. Therefore, the ML estimator of R, symbolized by $\hat{R}$, is obtained by substituting $\hat{\lambda}_1$, $\hat{\lambda}_2$, and $\hat{\lambda}_3$ in Equation (3) as:

$$\hat{R} = \frac{\hat{\lambda}_1 \hat{\lambda}_2}{(\hat{\lambda}_1 + \hat{\lambda}_3)(\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3)}. \tag{12}$$

4. Bayesian Estimation

In this section, the SS reliability $R = P(Y < X < Z)$ is estimated using a Bayesian approach where X, Y, and Z are three independent Weibull random variables with URV measurements. Bayesian estimation is provided using different loss functions such as the SE, LINEX, and MELO.

4.1. Posterior Distribution

According to Aljuaid [48], the conjugate priors of the parameters α, λ_1, λ_2, and λ_3 are assumed to follow independent Gamma distributions in the following manner:

$$\pi_1(\lambda_1) \propto \lambda_1^{a-1} e^{-b\lambda_1}, \quad \pi_2(\lambda_2) \propto \lambda_2^{g-1} e^{-d\lambda_2}, \quad \pi_3(\lambda_3) \propto \lambda_3^{p-1} e^{-q\lambda_3}, \ \pi_4(\alpha) \propto \alpha^{v-1} e^{-w\alpha},$$

where $a, b, g, d, p, q, v, w > 0$ are the hyperparameters. Discussion on the selection of these hyperparameter values is presented in Subsection 4.2. It can be mentioned that the N-IF prior is obtained from IF prior as values of a, b, g, d, p, q, v, w tend to zero. Then, the joint prior distribution for α and λ_1, λ_2, λ_3, will be as follows:

$$\pi(\Phi) \propto \lambda_1^{a-1} \lambda_2^{g-1} \lambda_3^{p-1} \alpha^{v-1} e^{-(b\lambda_1 + d\lambda_2 + q\lambda_3 + w\alpha)} \tag{13}$$

where $\Phi = (\lambda_1, \lambda_2, \lambda_3, \alpha)$. By applying Bayes' theorem, the posterior density of Φ symbolized as $\pi^*(\Phi\,|\underline{r}, \underline{u}, \underline{s})$ is derived through the product of the joint prior density (13) and the likelihood function (7) , yielding the following expression

$$\begin{aligned} &\pi^*(\Phi\,|\underline{r}, \underline{u}, \underline{s}) \propto \pi(\Phi)\, L(\Phi\,|\underline{r}, \underline{u}, \underline{s}) \\ &\propto \alpha^{n+m+k+v+2} {\lambda_1}^{a+n} \prod_{i=0}^{n} r_i^{\alpha-1} {\lambda_2}^{g+m} \prod_{j=0}^{m} u_j^{\alpha-1} {\lambda_3}^{p+k} \prod_{l=0}^{k} s_l^{\alpha-1} \\ &e^{-[(b+r_n^\alpha)\lambda_1 + (d+u_m^\alpha)\lambda_2 + (q+s_k^\alpha)\lambda_3 + w\alpha]}. \end{aligned} \tag{14}$$

The conditional posterior density functions (DFs) of each models λ_1,λ_2, and λ_3 will be considered, respectively, as follows:

$$\lambda_1\,|\alpha\,, data \sim Gamma(a + n + 1, b + r_n^\alpha),$$

$$\lambda_2\,|\alpha\,, data \sim Gamma(g + m + 1, d + u_m^\alpha),$$

and,

$$\lambda_3\,|\alpha\,, data \sim Gamma(p + k + 1, q + s_k^\alpha).$$

It is evident that the conditional posterior DFs of λ_1,λ_2, and λ_3 follows a gamma distribution. Hence, Gibbs sampling is employed to draw samples directly from these conditionals. On the other hand, the conditional posterior of α deviates from this form. Therefore, the Metropolis–Hastings (M–H) algorithm is utilized to obtain samples from the posterior distribution of α within the Gibbs framework.

4.2. Hyper-Parameter Elicitation

In practical applications, the elicitation of hyperparameters plays a crucial role in defining appropriate prior distributions. When reliable prior information is available, hyperparameters should be chosen to reflect realistic domain knowledge. For example, by using historical data or expert judgment regarding expected ranges of parameters. In cases where prior knowledge is limited, N-IF priors can be employed to reduce subjective influence while maintaining model flexibility. Furthermore, sensitivity analyses were conducted to ensure that the posterior inferences are robust to reasonable variations in hyperparameter values. These steps help balance prior informativeness with objectivity, leading to more reliable and interpretable Bayesian estimates (BEs) in applied contexts.

When the IF priors are chosen, the hyperparameters will be elicited. The ML estimates for λ_1,λ_2,λ_3 and α will yield these values by equating the estimated mean and variance of $\hat{\Phi}_i = (\hat{\lambda}_{1i}, \hat{\lambda}_{2i}, \hat{\lambda}_{3i}, \hat{\alpha}_i)$, with the mean and variance of the considered Gamma priors, where $i = 1, 2, ..., N$ and N is the number of samples available from the Weibull distribution. For more information (see Dey et al. [49]). Thus, the gamma prior, $\pi(\Phi) \propto \Phi^{m_1-1} e^{-m_2\Phi}$ with mean $E(\Phi) = \frac{m_1}{m_2}$ and variance $V(\Phi) = \frac{m_1}{m_2^2}$, has been incorporated in the present analysis. For our purpose, Φ represents a target parameter of the Weibull distribution with parameters (α, λ_1) , (α, λ_2) and (α, λ_3) respectively, and therefore

- for $\Phi = \lambda_1$, we have $m_1 = a, \ m_2 = b,$
- for $\Phi = \lambda_2$, we have $m_1 = g, \ m_2 = d,$
- for $\Phi = \lambda_3$, we have $m_1 = p, \ m_2 = q,$
- for $\Phi = \alpha$, we have $m_1 = v, \ m_2 = w.$

Now, the hyperparameter values are selected through the following iterative procedure:

1. Initialize the starting parameter value of $\Phi^{(0)} = (\lambda_1, \lambda_2, \lambda_3, \alpha)$.
2. Set iteration index $i = 1$.
3. Based on URVs generate three samples Let $\underline{r} = (r_0, r_1, ..., r_n)$, $\underline{u} = (u_0, u_1, ..., u_m)$, and $\underline{s} = (s_0, s_1, ..., s_k)$ from Weibull distributions with parameters (α, λ_1), (α, λ_2), and (α, λ_3).
4. Obtain Φ by using the ML method (see Section 3), where $\Phi = (\lambda_1, \lambda_2, \lambda_3, \alpha)$.
5. Repeat steps 2–4 N times for $i = 1, 2, \ldots, N$ to get $\hat{\Phi}_i$.
6. Determine the hyperparameters of the Gamma priors by equating the theoretical mean and variance with the empirical mean and variance of gamma distribution $\hat{\Phi}_i$, $i = 1, 2, \ldots, N$.

$$\frac{1}{N}\sum_{i=1}^{N}\hat{\Phi}_i = \frac{m_1}{m_2}, \qquad \frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\Phi}_i - \frac{1}{N}\sum_{i=1}^{N}\hat{\Phi}_i\right)^2 = \frac{m_1}{m_2^2}.$$

To solve the above equations, the estimated hyperparameters turn out to have the following forms:

$$m_1 = \frac{\left(\frac{1}{N}\sum\limits_{i=1}^{N}\hat{\Phi}_i\right)^2}{\frac{1}{N-1}\sum\limits_{i=1}^{N}\left(\hat{\Phi}_i - \frac{1}{N}\sum\limits_{i=1}^{N}\hat{\Phi}_i\right)^2} \qquad m_2 = \frac{\frac{1}{N}\sum\limits_{i=1}^{N}\hat{\Phi}_i}{\frac{1}{N-1}\sum\limits_{i=1}^{N}\left(\hat{\Phi}_i - \frac{1}{N}\sum\limits_{i=1}^{N}\hat{\Phi}_i\right)^2} \tag{15}$$

where $(m_1, m_2) = (a, b), (g, d), (p, q)$, and (w, v).

4.3. Loss Functions

The fundamental role of loss functions lies in quantifying the discrepancy between an estimated parameter value and its actual value, where larger losses indicate poorer estimator performance. Common loss function examples include symmetric (SE) and asymmetric (LINEX, MELO) types. In Bayesian decision theory, the loss incurred can be measured by taking an estimator $\tilde{R}$ of the unknown reliability parameter R. The definition of the SE loss function is as follows:

$$L(R, \tilde{R}) = (R - \tilde{R})^2.$$

Then under SE loss function, R's Bayesian estimator can be obtained via

$$\tilde{R}^{SE} = E[R|\text{data}] = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty R\pi^*(\Phi\,|\underline{r}, \underline{u}, \underline{s})\ \ d(\alpha\lambda_1\lambda_2\lambda_3).$$

The LINEX loss function, which was first presented by Zellner [50], is a very practical asymmetric loss function. The LINEX loss function is defined as follows, presuming that the minimal loss occurs at a particular point $\tilde{R} = R$:

$$L_L(\tilde{R}, R) \propto e^{c(\tilde{R}-R)} - c(\tilde{R} - R) - 1;\ c \neq 0,$$

where $\tilde{R}$ is an estimate of the parameter R and c is the shape parameter. The form of this loss function depends on the value of c. Subsequently, using the LINEX loss function, the Bayes estimator of R is

$$\tilde{R}^L = \frac{-1}{c} ln\left(\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-cR}\pi^*(\Phi\,|\underline{r}, \underline{u}, \underline{s})\ \ d(\alpha\lambda_1\lambda_2\lambda_3)\right).$$

The MELO estimator, introduced by Zellner [50] through minimization of the posterior expectation of a class of generalized quadratic loss functions. The approach was subsequently formalized by Tummala and Sathe[51] as

$$L_M(\tilde{R}, R) = \frac{(\tilde{R} - R)^2}{R^2}.$$

Then, the Bayesian estimator of R under MELO is as follows

$$\tilde{R}^M = \frac{E(R^{-1}\,|data)}{E(R^{-2}\,|data)} = \frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty R^{-1}\pi^*(\Phi\,|\underline{r}, \underline{u}, \underline{s})\ \ d(\alpha\lambda_1\lambda_2\lambda_3)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty R^{-2}\pi^*(\Phi\,|\underline{r}, \underline{u}, \underline{s})\ \ d(\alpha\lambda_1\lambda_2\lambda_3)}.$$

4.4. Markov Chain Monte Carlo

The application of Bayesian statistical inference based on MCMC techniques has risen dramatically in recent years. MCMC is a simulation technique that uses the algorithm for sampling from a distribution through simulation. The most straightforward MCMC method is the M-H algorithm. In situations when direct sampling is challenging, the M-H algorithm is an MCMC technique for extracting a series of random samples from a probability distribution. It is possible to approximate the distribution using this sequence. The M-H algorithm chooses items from a random ”proposal” distribution and uses an acceptance rule to determine whether or not to keep them. At first created by Metropolis et al. [52], the M-H algorithm is a very generic MCMC technique that was later expanded by Hastings [53]. This is how the MCMC process might be stated:

1. Select initial values $\Phi^{(0)}$.
2. Set iteration counter $i = 1$.
3. Simulate $\lambda_1^{(i)}$ from $Gamma(a + n + 1, b + r_n^{\alpha^{(i-1)}})$.
4. Simulate $\lambda_2^{(i)}$ from $Gamma(g + m + 1, d + u_m^{\alpha^{(i-1)}})$
5. Simulate $\lambda_3^{(i)}$ from$Gamma(p + k + 1, q + s_k^{\alpha^{(i-1)}})$
6. To simulate $\alpha^{(i)}$:
   (a) Generate a proposal α^* from $N(\alpha^{(i-1)}, V_\alpha)$.
   (b) Determine the probabilities of acceptance
   $$\rho_\alpha = \min\left[1, \frac{\pi^*(\alpha^*\,|\gamma^{(i-1)})}{\pi^*(\alpha^{(i-1)}\,|\gamma^{(i-1)})}\right], \gamma = \lambda_1, \lambda_2, \lambda_3.$$
   (c) Generate u_1 follow $Uniform(0, 1)$.
   (d) If $u_1 \le \rho_\alpha$, set $\alpha^{(i)} = \alpha^{(*)}$; else, $\alpha^{(i)} = \alpha^{(i-1)}$.
7. Obtain $\Phi^{(i)}$ and $R^{(i)}$
8. $i = i + 1$
9. Perform steps 3 through 8 N times.
   Now, the Bayesian estimates of R under SE, LINEX, and MELO are given, respectively, by

$$\tilde{R}^{SE} = \frac{1}{N - M}\sum_{i=M+1}^{N} R^i, \quad \tilde{R}^L = \frac{-1}{c} ln\left[\frac{1}{N - M}\sum_{i=M+1}^{N}(e^{-c\,R^i})\right];\ c \ne 0,$$

and

$$\tilde{R}^M = \frac{E(R^{-1}\,|\underline{x})}{E(R^{-2}\,|\underline{x})} = \frac{\frac{1}{N-M}\sum_{i=M+1}^{N}(R^i)^{-1}}{\frac{1}{N-M}\sum_{i=M+1}^{N}(R^i)^{-2}},$$

where M represents the burn-in time.

5. Simulation Study

This section employs a Monte Carlo simulation analysis. When X, Y, and Z are three independent Weibull random variables, a simulation technique of the SS reliability based on URVs can be developed using the following processes.

5.1. Numerical Investigation

1. The values of the Weibull distribution parameters in the SS model $(\lambda_1, \lambda_2, \lambda_3, \alpha)$are determined as follows:
   (a) $\alpha = 0.1,\ \lambda_1 = 4,\ \lambda_2 = 15,\ \lambda_3 = 1,$
   (b) $\alpha = 0.1,\ \lambda_1 = 3,\ \lambda_2 = 15.6,\ \lambda_3 = 0.5,$
   (c) $\alpha = 0.1,\ \lambda_1 = 2,\ \lambda_2 = 16,\ \lambda_3 = 0.2,$
   (d) $\alpha = 0.1,\ \lambda_1 = 2,\ \lambda_2 = 36,\ \lambda_3 = 0.1.$
2. The mean and variance of the ML estimates (MLEs) are equated to the mean and variance of the Gamma priors using Equation (15) to produce the elicited priors, as discussed in Section 4.
3. The datasets are determined as follows:
   - 10,000 generate sample sizes of 5 and 10 URV.
   - 100,000 generate a sample size of 15 URV.
4. The URV sizes are determined as m for lower stress Y, n for strength X, and k for upper stress Z. The URV sizes considered as $(n, m, k) = (5, 5, 5), (10, 5, 5), (5, 10, 10), (10, 5, 10), (10, 10, 10), (15, 5, 10), (15, 15, 15)$
5. Values of $c = -0.5, 0.5, -1, 1, -2$, and 2 are considered in the LINEX loss function.
6. $N = 10{,}000$ is the number of replications considered, and 20% of the samples are discarded as a burn-in.
7. Random samples of the Weibull distribution with sizes n, m, and k are created by generating three uniformly distributed random samples within $(0, 1)$ interval. Then, they are transformed using the quantile of the Weibull distribution function.
8. The Weibull distribution parameters are estimated using the ML estimation and M-H algorithm with the Gibbs sampler in MCMC for Bayesian estimation techniques.
9. Different measures of performance are computed including the Abias and the MSE for each estimation technique, which is defined by

$$\text{MSE} = \frac{1}{N}\sum_{i=1}^{N}\left(\tilde{R}_i - R\right)^2, \qquad \text{Abias} = \frac{1}{N}\sum_{i=1}^{N}\left|\tilde{R}_i - R\right|.$$

5.2. Numerical Findings

The simulation results yield a number of significant findings and conclusions. The full description of these results is reported in Tables 4– 9, and they are summarized in Tables 1–3. Tables 4– 9 present the complete simulation study results, including Abias and MSE for MLEs and BEs. These results cover both IF and N-IF priors under all loss functions, across different values of R and varying numbers of records. On the other hand, Tables 1 and 2 offer a brief summary of the MSE for MLEs and BEs (under both IF and N-IF priors for all loss functions) specifically when $R = 0.6$ across different record numbers. Table 3 specifically displays the numerical MSE results for BEs under the LINEX loss function for selected values of the parameter c. Finally, the numerical results are visually depicted in Figures 1- 8.

5.2.1. *Results based on Tables1–3 and Tables 4–9*

1. Using various loss functions, the BEs of SS reliability R based on IF prior demonstrate superior performance compared to the MLEs across all SS reliability R for all record numbers, as shown in Table 2, Tables 4– 9, and Figure 1 left panel.
2. The BEs of SS reliability R based on N-IF prior show closeness to the MLEs across all SS reliability R and sample sizes under different loss functions, as shown in Table 2, Tables 4– 9, and Figure 2 left panel).

Table 1. MSE of R under MLE and BEs under different loss functions with IF prior at $R = 0.6$.

(n, m, k)	MLE	Bayesian Estimation (IF Prior)			
		SE	LINEX		MELO
			$c = -0.5$	$c = 0.5$	
(5,5,5)	0.01612	0.00717	0.00709	0.00725	0.00620
(10,10,10)	0.00609	0.00164	0.00162	0.00167	0.00133
(15,15,15)	0.00337	0.00023	0.00023	0.00023	0.00029
(5,10,10)	0.00723	0.00101	0.00102	0.00100	0.00137
(10,5,10)	0.01195	0.00346	0.00350	0.00342	0.00406
(10,5,5)	0.00815	0.00111	0.00109	0.00113	0.00085
(15,5,10)	0.01510	0.00568	0.00572	0.00564	0.00632

Table 2. MSE of R under MLE and BEs under different loss functions with N-IF prior at $R = 0.6$.

(n, m, k)	MLE	Bayesian Estimation (N-IF Prior)			
		SE	LINEX		MELO
			$c = -0.5$	$c = 0.5$	
(5,5,5)	0.01612	0.01215	0.01193	0.01238	0.01143
(10,10,10)	0.00609	0.00574	0.00575	0.00573	0.00671
(15,15,15)	0.00337	0.00350	0.00352	0.00347	0.00426
(5,10,10)	0.00723	0.00781	0.00793	0.00770	0.01169
(10,5,10)	0.01195	0.01606	0.01646	0.01567	0.02638
(10,5,5)	0.00815	0.00776	0.00782	0.00770	0.01125
(15,5,10)	0.01510	0.02041	0.02091	0.01992	0.03275

Table 3. MSE of R under BEs with IF prior under LINEX loss function for different c values at $R = 0.6$.

(n, m, k)	Bayesian Estimation (LINEX Loss Function)					
	$c = -0.5$	$c = 0.5$	$c = -1$	$c = 1$	$c = -2$	$c = 2$
(5,5,5)	0.00709	0.00725	0.01074	0.01115	0.00731	0.00810
(10,10,10)	0.00162	0.00167	0.00022	0.00021	0.00148	0.00163
(15,15,15)	0.00023	0.00023	0.00020	0.00021	0.00024	0.00021
(5,10,10)	0.00102	0.00100	0.00120	0.00123	0.00034	0.00029
(10,5,10)	0.00350	0.00342	0.00823	0.00799	0.00715	0.00686
(10,5,5)	0.00109	0.00113	0.00061	0.00066	0.00089	0.00074
(15,5,10)	0.00572	0.00564	0.01505	0.01468	0.00468	0.00444

3. The BEs of SS reliability R based on IF prior outperform those based on N-IF prior for all SS reliability R and all record numbers under different loss functions, as shown in Tables 1–2, Tables 4– 9 and Figures 5–6 in the left panel.
4. For SS reliability R=0.6, the MSE of both MLEs and BEs using N-IF and IF prior declines as the record numbers increase, as shown in Tables 4– 9, and Figures 1–2.
5. The BEs of R for IF prior outperform MLEs for large record numbers $(n, m, k) = (10, 10, 10)$, $(15, 15, 15)$ by producing smaller MSE, as shown in Figure 1.

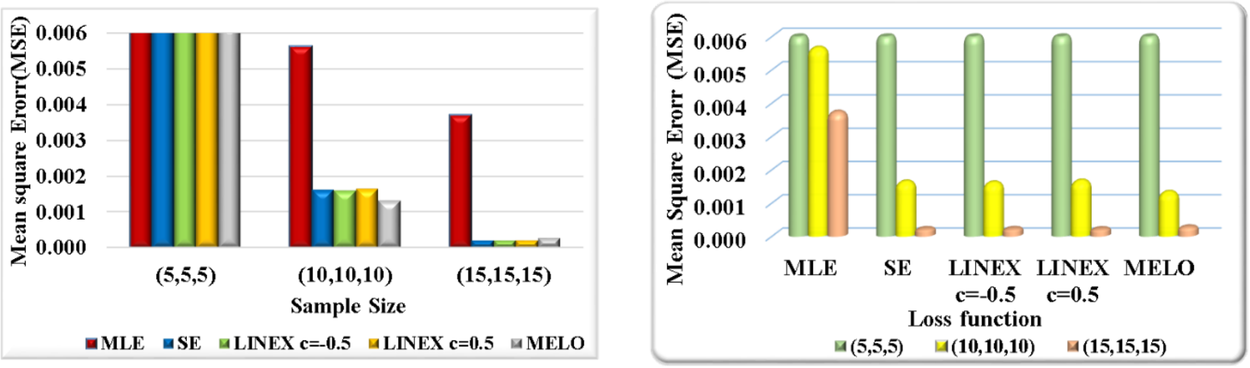


Figure 1. MSE of MLE and BEs under different loss functions with IF prior for various sample sizes at $R = 0.6$.

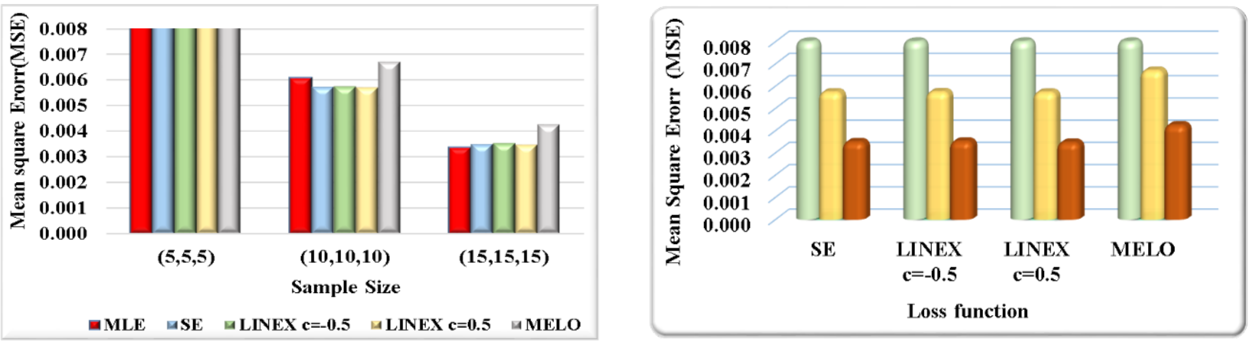
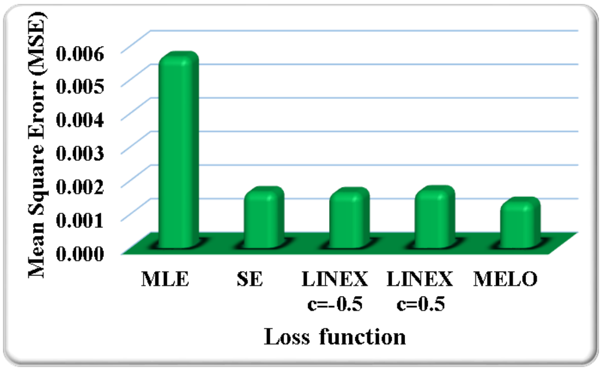
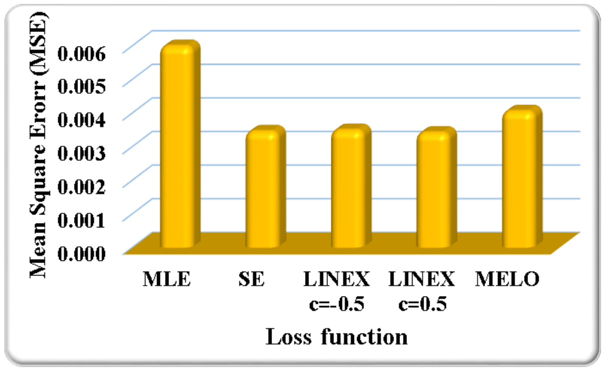


Figure 2. MSE of MLE and BEs under different loss functions with N-IF prior for various sample sizes at $R = 0.6$.

6. For all record numbers, the BEs of R consistently perform in the neighborhood of other MLEs, as shown in Figure 2 in the left panel.
7. For SS reliability $R = 0.6$, as record numbers rise, the MSE of BEs based on N-IF decreases, as in Figure 2 in the right panel.
8. Figure 3 (left panel) shows that for equal record values $(n, m, k) = (10, 10, 10)$, at the actual value of $R = 0.6$, the BEs of R under MELO are superior to the others under the loss function in terms of producing fewer MSE. On the other hand, Figure 3 (right panel) shows that for unequal record numbers $(n, m, k) = (10, 5, 10)$ at the actual value of $R = 0.6$, the BEs of R under SE have the fewest MSE.
9. The BEs of SS reliability R based on IF prior outperform the BEs of SS reliability R based on N-IF prior using a variety of loss functions, in the right and left panels as demonstrated in Figure 4 at $R = 0.6$ and $(n, m, k) = (10, 10, 10)$.
10. Figure 4 (left panel) shows that for equal record values $(n, m, k) = (10, 10, 10)$ at an actual value of $R = 0.6$, when compared to the other loss functions, the BEs of R based on N-IF prior under MELO provide the largest MSE. On the other hand, the BEs of R based on IF prior under MELO demonstrate superior performance from other loss functions in terms of producing bigger MSE, in Figure 4 (right panel).
11. Using various loss functions, the BEs of SS reliability R based on IF prior demonstrate superior performance compared to the BEs of SS reliability R based on N-IF prior in the right and left panels (see Figure 5 for $(n, m, k) = (10, 5, 10)$ and $R = 0.6$).
12. For unequal record values $(n, m, k) = (10, 5, 10)$ at actual value of $R = 0.6$, Figure 5 (left panel) shows that the BEs of R based on N-IF prior under MELO are the worst among the other loss functions in terms of producing bigger MSE.

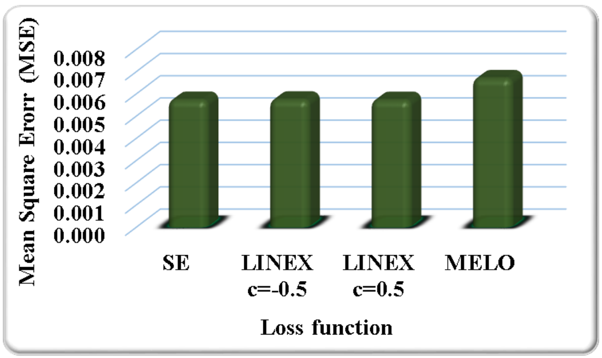


(a) $(n, m, k) = (10, 10, 10)$

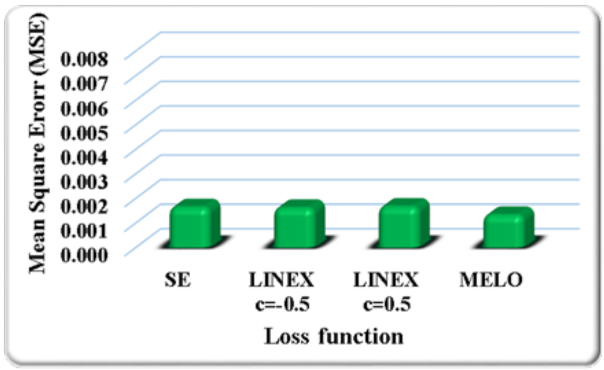


(b) $(n, m, k) = (10, 5, 10)$

Figure 3. MSE of R under MLE and different loss functions with IF prior for different sample sizes at $R = 0.6$.

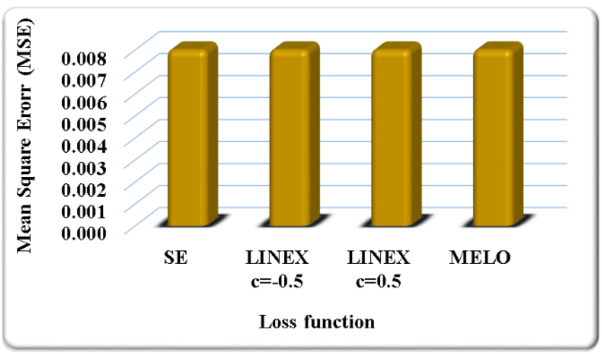


(a) N-IF prior

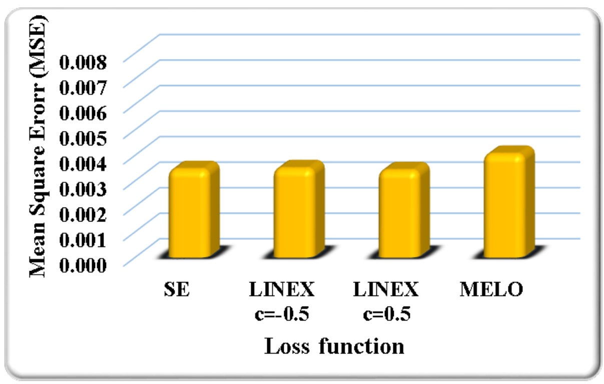


(b) IF prior

Figure 4. MSE of R under different loss functions with $(n, m, k) = (10, 10, 10)$ for $R = 0.6$.



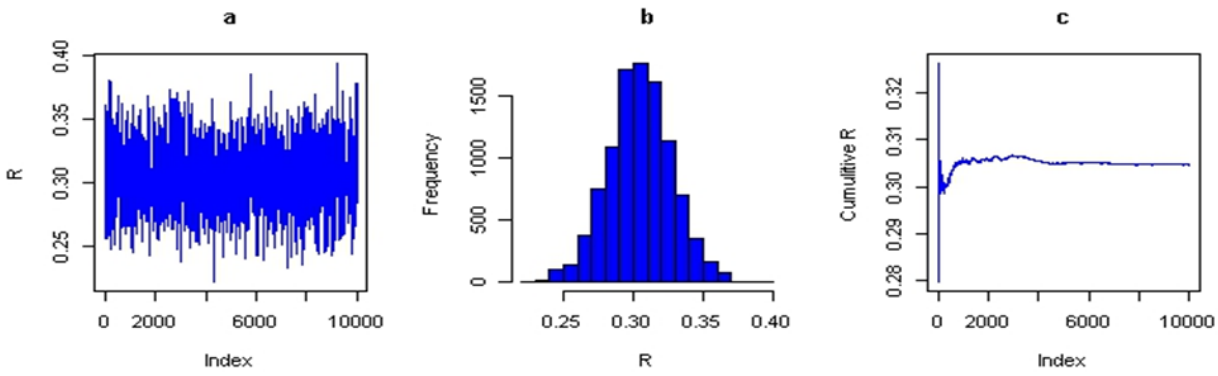
(a) N-IF prior



(b) IF prior

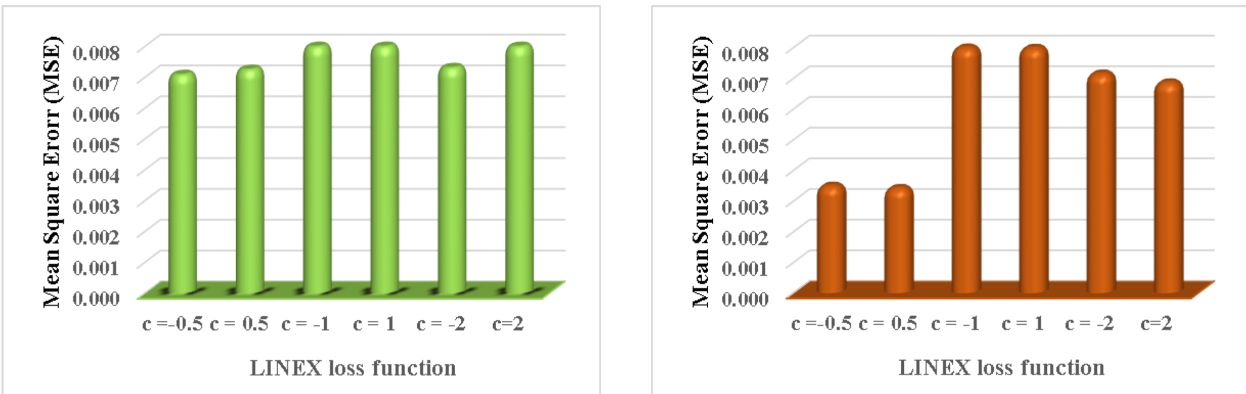
Figure 5. MSE of R under different loss functions with $(n, m, k) = (10, 5, 10)$ for $R = 0.6$.

13. A trace plot utilizing various SS reliability estimations is shown in Figure 6. Diagnostic plots indicate good convergence, as the generated posterior values closely match the theoretical posterior DF. Also, the MCMC chains exhibit stability.



(a) Trace Plot (b) Histogram Plot (c) Convergence Plot

Figure 6. MCMC diagnostic plots for SS reliability under SE loss function with IF prior at $R = 0.6$.



(a) $(n, m, k) = (5, 5, 5)$ (b) $(n, m, k) = (10, 5, 10)$

Figure 7. MSE of BEs under different LINEX loss functions with IF prior for varying c values at $R = 0.6$.

14. The effect of starting values and ensure that the Markov chains reached their stationary distributions. The choice of burn-in length involves a trade-off: a short burn-in may retain bias from initial states, whereas an excessively long one increases computational cost without improving inference. Similarly, thinning was considered to reduce autocorrelation among successive samples; however, excessive thinning can unnecessarily decrease the effective sample size. To verify convergence, we employed visual inspection of trace plots, which indicated satisfactory convergence and reliable posterior estimates.
15. For comparison between different values for c in the LINEX loss function such as $c = (-0.5, 0.5), (-1, 1), (-2, 2)$, Figure 7 (left panel) shows that for equal record values (n, m, k) = (5,5,5) at an actual value of R = 0.6, the BEs of R under the LINEX loss function at $c = (-0.5, 0.5)$ superior to the LINEX loss function at $c = (-1, 1), (-2, 2)$ in terms of producing fewer MSE. On the other hand, Figure 7 (right panel) shows that for unequal record numbers $(n, m, k) = (10, 5, 10)$ at the actual value of $R = 0.6$, the BEs of R under the LINEX loss function at $c = (-0.5, 0.5)$ superior to the LINEX loss function at $c = (-1, 1), (-2, 2)$ in terms of producing a lower MSE, as shown in Table 3.

Table 4. Bayesian estimation with different loss functions and MLE with R = 0.6

(n,m,k)		MLE	Bayesian estimation							
			Non Informative prior				Informative prior			
			SE	LINEX		MELO	SE	LINEX		MELO
				c = -0.5	c = 0.5			c = -0.5	c = 0.5	
(5,5,5)	**Abias**	0.07938	0.04907	0.04637	0.05173	0.00766	0.08237	0.08187	0.08288	0.07617
	MSE	0.01612	0.01215	0.01193	0.01238	0.01143	0.00717	0.00709	0.00725	0.00620
(10,10,10)	**Abias**	0.01291	0.00347	0.00493	0.00202	0.02529	0.03490	0.03451	0.03528	0.02992
	MSE	0.00609	0.00574	0.00575	0.00573	0.00671	0.00164	0.00162	0.00167	0.00133
(15,15,15)	**Abias**	0.00500	0.01476	0.01577	0.01376	0.02945	0.00741	0.00762	0.00719	0.01036
	MSE	0.00337	0.00350	0.00352	0.00347	0.00426	0.00023	0.00023	0.00023	0.00029
(5,10,10)	**Abias**	0.00278	0.02933	0.03145	0.02723	0.06431	0.00723	0.00808	0.00639	0.01938
	MSE	0.00723	0.00781	0.00793	0.00770	0.01169	0.00101	0.00102	0.00100	0.00137
(10,5,10)	**Abias**	0.06448	0.09314	0.09548	0.09080	0.13668	0.05719	0.05752	0.05686	0.06217
	MSE	0.01195	0.01606	0.01646	0.01567	0.02638	0.00346	0.00350	0.00342	0.00406
(10,5,5)	**Abias**	0.01404	0.01270	0.01520	0.01021	0.05367	0.03013	0.02977	0.03049	0.02540
	MSE	0.00815	0.00776	0.00782	0.00770	0.01125	0.00111	0.00109	0.00113	0.00085
(15,5,10)	**Abias**	0.09431	0.12067	0.12287	0.11847	0.16353	0.07471	0.07497	0.07444	0.07885
	MSE	0.01510	0.02041	0.02091	0.01992	0.03275	0.00568	0.00572	0.00564	0.00632

Table 5. **Bayesian estimation with different loss functions and MLE with R = 0.7**

(n,m,k)		MLE	Bayesian estimation							
			Non Informative prior				Informative prior			
			SE	LINEX		MELO	SE	LINEX		MELO
				c = -0.5	c = 0.5			c = -0.5	c = 0.5	
(5,5,5)	**Abias**	0.04516	0.01639	0.01386	0.01887	0.01922	0.06312	0.06268	0.06355	0.05831
	MSE	0.00970	0.00829	0.00829	0.00831	0.01015	0.00432	0.00427	0.00438	0.00376
(10,10,10)	**Abias**	0.00064	0.01559	0.01700	0.01418	0.03418	0.69956	0.69955	0.69957	0.69944
	MSE	0.00483	0.00508	0.00514	0.00502	0.00644	0.00051	0.00052	0.00051	0.00063
(15,15,15)	**Abias**	0.01551	0.02517	0.02615	0.02419	0.03775	0.69990	0.69990	0.69990	0.69987
	MSE	0.00373	0.00414	0.00420	0.00408	0.00514	0.00017	0.00018	0.00017	0.00021
(5,10,10)	**Abias**	0.01194	0.03788	0.03995	0.03582	0.06796	0.69960	0.69959	0.69961	0.69943
	MSE	0.00665	0.00807	0.00826	0.00789	0.01221	0.00047	0.00048	0.00046	0.00064
(10,5,10)	**Abias**	0.07753	0.10649	0.10900	0.10399	0.14733	0.69679	0.69674	0.69684	0.69605
	MSE	0.01311	0.01841	0.01895	0.01788	0.02958	0.00328	0.00334	0.00323	0.00402
(10,5,5)	**Abias**	0.01035	0.03681	0.03930	0.03434	0.07370	0.69886	0.69887	0.69885	0.69900
	MSE	0.00690	0.00810	0.00831	0.00789	0.01322	0.00121	0.00120	0.00123	0.00107
(15,5,10)	**Abias**	0.10565	0.13273	0.13517	0.13031	0.17386	0.69449	0.69442	0.69457	0.69342
	MSE	0.01819	0.02452	0.02516	0.02389	0.03780	0.00558	0.00566	0.00551	0.00665

Table 6. **Bayesian estimation with different loss functions and MLE with R = 0.799**

(n,m,k)		MLE	Bayesian estimation							
			Non Informative prior				Informative prior			
			SE	LINEX		MELO	SE	LINEX		MELO
				c = -0.5	c = 0.5			c = -0.5	c = 0.5	
(5,5,5)	Abias	0.00758	0.01711	0.01928	0.01499	0.04521	0.02220	0.02198	0.02243	0.01993
	MSE	0.00591	0.00681	0.00698	0.00665	0.01026	0.00060	0.00059	0.00061	0.00051
(10,10,10)	Abias	0.01742	0.03166	0.03287	0.03046	0.04595	0.01185	0.01207	0.01164	0.01411
	MSE	0.00402	0.00492	0.00502	0.00481	0.00650	0.00032	0.00033	0.00031	0.00038
(15,15,15)	Abias	0.02585	0.03452	0.03535	0.03369	0.04399	0.00712	0.00722	0.00702	0.00810
	MSE	0.00369	0.00430	0.00438	0.00423	0.00529	0.00010	0.00011	0.00010	0.00012
(5,10,10)	Abias	0.02666	0.04927	0.05108	0.04749	0.07251	0.01689	0.01714	0.01664	0.01951
	MSE	0.00568	0.00777	0.00800	0.00754	0.01159	0.00039	0.00040	0.00038	0.00049
(10,5,10)	Abias	0.08282	0.10882	0.11120	0.10647	0.14274	0.06559	0.06607	0.06510	0.07108
	MSE	0.01296	0.01829	0.01886	0.01774	0.02810	0.00473	0.00480	0.00467	0.00550
(10,5,5)	Abias	0.03461	0.05807	0.06031	0.05587	0.08818	0.02048	0.02062	0.02033	0.02197
	MSE	0.00736	0.00994	0.01027	0.00962	0.01575	0.00045	0.00046	0.00045	0.00052
(15,5,10)	Abias	0.10867	0.13383	0.13621	0.13147	0.16891	0.08000	0.08025	0.07975	0.08282
	MSE	0.01804	0.02437	0.02505	0.02371	0.03613	0.00655	0.00659	0.00651	0.00701

Table 7. **Bayesian estimation with different loss functions and MLE with R = 0.899**

(n,m,k)		MLE	Bayesian estimation							
			Non Informative prior				Informative prior			
			SE	LINEX		MELO	SE	LINEX		MELO
				c = -0.5	c = 0.5			c = -0.5	c = 0.5	
(5,5,5)	Abias	0.00396	0.02053	0.02184	0.01926	0.03570	0.01622	0.01609	0.01634	0.01508
	MSE	0.00300	0.00418	0.00431	0.00405	0.00621	0.00035	0.00034	0.00035	0.00035
(10,10,10)	Abias	0.00941	0.02371	0.02440	0.02302	0.03091	0.01159	0.01150	0.01168	0.01080
	MSE	0.00228	0.00302	0.00309	0.00296	0.00378	0.00023	0.00022	0.00023	0.00021
(15,15,15)	Abias	0.02267	0.03201	0.03252	0.03152	0.03703	0.00456	0.00451	0.00460	0.00413
	MSE	0.00220	0.00286	0.00290	0.00281	0.00337	0.00006	0.00006	0.00006	0.00006
(5,10,10)	Abias	0.02754	0.04991	0.05111	0.04873	0.06357	0.01309	0.01324	0.01293	0.01453
	MSE	0.00383	0.00612	0.00629	0.00594	0.00849	0.00027	0.00027	0.00027	0.00031
(10,5,10)	Abias	0.04997	0.07778	0.07938	0.07621	0.09714	0.01067	0.01084	0.01050	0.01223
	MSE	0.00610	0.01029	0.01061	0.00998	0.01480	0.00027	0.00027	0.00026	0.00031
(10,5,5)	Abias	0.02546	0.05043	0.05192	0.04899	0.06793	0.00976	0.00989	0.00963	0.01092
	MSE	0.00736	0.00994	0.01027	0.00962	0.01575	0.00045	0.00046	0.00045	0.00052
(15,5,10)	Abias	0.10867	0.13383	0.13621	0.13147	0.16891	0.08000	0.08025	0.07975	0.08282
	MSE	0.01804	0.02437	0.02505	0.02371	0.03613	0.00655	0.00659	0.00651	0.00701

Table 8. **Bayesian estimation under different loss function with c= -1,1 and MLE , at R=0.6.**

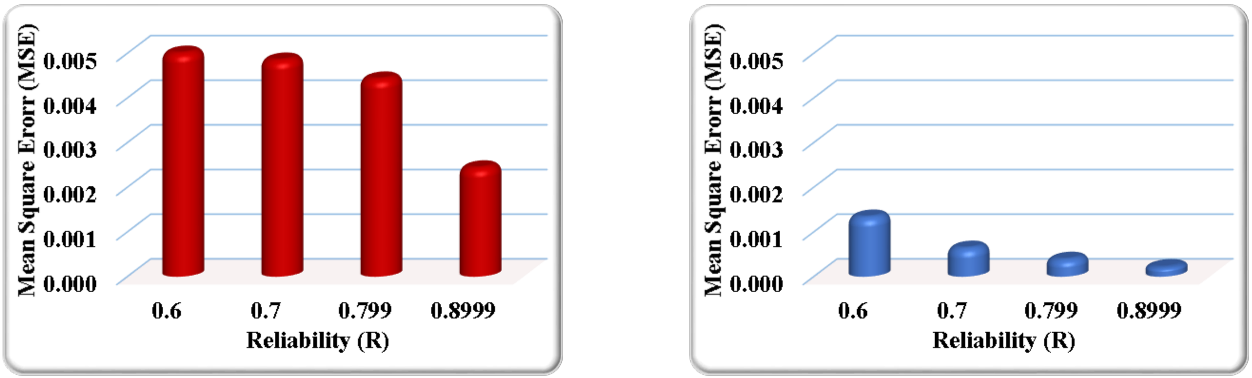
(n,m,k)		MLE	Bayesian estimation			
			SE	LINEX		MELO
				c = -1	c = 1	
(5,5,5)	Abias	0.07493	0.10222	0.10121	0.10323	0.09621
	MSE	0.01581	0.01094	0.01074	0.01115	0.00976
(10,10,10)	Abias	0.01655	0.00518	0.00577	0.00459	0.00923
	MSE	0.00549	0.00022	0.00022	0.00021	0.00028
(15,15,15)	Abias	0.00789	0.00434	0.00390	0.00477	0.00140
	MSE	0.00362	0.00020	0.00020	0.00021	0.00019
(5,10,10)	Abias	0.00439	0.00514	0.00326	0.00700	0.00816
	MSE	0.00667	0.00121	0.00120	0.00123	0.00131
(10,5,10)	Abias	0.06399	0.08908	0.08976	0.08839	0.09458
	MSE	0.01141	0.00811	0.00823	0.00799	0.00912
(10,5,5)	Abias	0.01632	0.02289	0.02229	0.02348	0.01898
	MSE	0.00755	0.00063	0.00061	0.00066	0.00047
(15,5,10)	Abias	0.09629	0.12142	0.12219	0.12066	0.12800
	MSE	0.01556	0.01487	0.01505	0.01468	0.01651

Table 9. **Bayesian estimation under different loss function with c= -2,2 and MLE , at R=0.6.**

(n,m,k)		MLE	Bayesian estimation			
			SE	LINEX		MELO
				c = -2	c = 2	
(5,5,5)	Abias	0.07286	0.08476	0.08236	0.08711	0.07741
	MSE	0.01445	0.00770	0.00731	0.00810	0.00655
(10,10,10)	Abias	0.01326	0.03717	0.03619	0.03816	0.03402
	MSE	0.00555	0.00156	0.00148	0.00163	0.00134
(15,15,15)	Abias	0.00477	0.01198	0.01263	0.01132	0.01422
	MSE	0.00337	0.00023	0.00024	0.00021	0.00029
(5,10,10)	Abias	0.00975	0.00669	0.00846	0.00493	0.01282
	MSE	0.00732	0.00031	0.00034	0.00029	0.00044
(10,5,10)	Abias	0.06999	0.08317	0.08403	0.08230	0.08658
	MSE	0.01209	0.00701	0.00715	0.00686	0.00759
(10,5,5)	Abias	0.01260	0.02635	0.02781	0.02490	0.03157
	MSE	0.00725	0.00081	0.00089	0.00074	0.00112
(15,5,10)	Abias	0.09348	0.06671	0.06762	0.06581	0.07016
	MSE	0.01551	0.00456	0.00468	0.00444	0.00503

5.2.2. Results Based on Tables 4–9

1. For all actual values of R, the MSE for MLEs consistently decreases with increasing R across unequal record sizes $(n, m, k) = (10, 5, 5)$ and $(5, 10, 10)$.



(a) MLE vs actual R (b) MELO vs actual R

Figure 8. MSE of R under MLE and MELO with IF prior for different R values at $(n, m, k) = (10, 10, 10)$.

2. Across unequal sample sizes $(n, m, k) = (10, 5, 5)$ at all R, the Abias for MLEs and BEs under IF prior consistently diminishes with increasing R as presented in Tables 4–9.
3. Figure 8 (left panel) illustrates that for all actual values of R and $(n, m, k) = (10, 10, 10)$, as R gets closer to 1, the MSE of the MLEs falls.
4. The MSE of the MELO shows an inverse relationship with the actual value of R at $(n, m, k) = (10, 10, 10)$ as represented in Figure 8 (right panel).
5. Across all the equally tested sample sizes, the MSE for both MLEs and BEs using both N-IF and IF prior consistently diminishes with an increase in the actual value of R, as presented in Tables 4–9.
6. When working with equal number of record values, BEs of R under IF prior derived using the MELO function consistently yield smaller MSE, outperforming other loss functions, such as
   - At $R = 0.6$ for $(n, m, k) = (5, 5, 5)$ and $(10, 10, 10)$.
   - At $R = 0.799$ for $(n, m, k) = (5, 5, 5)$.
   - At $R = 0.8999$ for $(n, m, k) = (10, 10, 10)$ and $(15, 15, 15)$.
7. Tables A.1–A.6 show that Abias for BEs based on N-IF and IF priors consistently decreases with increasing record numbers across all R values, irrespective of the loss function employed.
8. Tables A.1–A.6 show that for all equal record numbers, the Abias for both MLEs and BEs (N-IF and IF priors) consistently decrease as R increases.
9. The MSE of MLEs and BEs under both N-IF and IF prior consistently decreases with an increase in record numbers across all SS reliability R, as shown in Tables 4–9.

6. Data Application

For demonstration purposes, the analysis of a real dataset is described in this section.

6.1. Data Description

Data from three jute fiber breaking strengths are examined. First, we will introduce the jute fiber, a natural plant fiber obtained from the bark of jute plants. It is one of the most widely used vegetable fibers after cotton, extracted from the stalks of the jute plant, grown mostly in India, Bangladesh, and parts of China. Its appearance is long, soft, shiny, and golden-brown in color. Because jute fiber is primarily composed of cellulose and lignin, it is robust, long-lasting, and biodegradable. Jute fiber properties are high tensile strength, breathability and lightweightness, biodegradability and recyclability, and moderate moisture regain (absorbs water easily). Jute fiber has a lot of uses, such as sacks, ropes, carpets, mats, curtains, rugs, geotextiles (for soil erosion control), and eco-friendly packaging

Table 10. The breaking strength of jute fibre data sets

Z (5 mm)	566.31	270.79	516.48	823.03	226.53	367.70	185.42	441.87	618.57	546.11
	268.20	315.33	809.23	218.86	583.97	304.84	129.08	537.45	496.28	167.87
	306.99	178.25	370.02	168.20	554.61	360.80	260.97	254.29	495.51	187.68
X (10 mm)	693.73	704.66	323.83	778.17	123.06	637.66	383.43	151.48	108.94	50.16
	671.49	183.16	257.44	727.23	291.27	101.15	376.42	163.40	141.38	700.74
	262.90	353.24	422.11	43.93	590.48	212.13	303.90	506.60	530.55	177.25
Y (20 mm)	71.46	419.02	284.64	585.57	456.60	113.85	187.85	688.16	662.66	45.58
	578.62	756.70	594.29	166.49	99.72	707.36	765.14	187.13	145.96	350.70
	547.44	116.99	375.81	581.60	119.86	48.01	200.16	36.75	244.53	83.55

Table 11. Descriptive statistics of the breaking strength of jute fibre data

Data	Min	Q1	Median	Mean	Q3	Max	DKS	PVKS
Z	129.08	233.47	338.07	384.37	532.21	823.03	0.11522	0.7787
X	43.93	166.86	313.87	365.73	575.50	778.17	0.10590	0.8548
Y	36.75	117.71	264.59	340.74	580.86	765.14	0.14907	0.4727

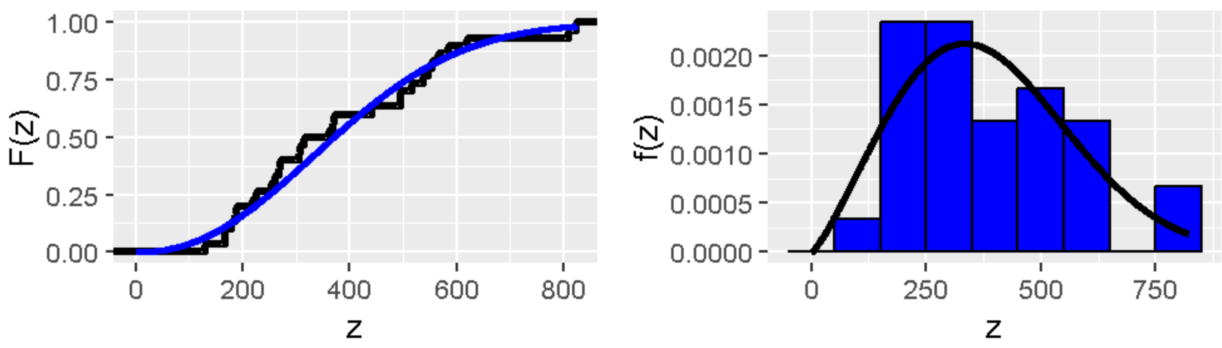
as an alternative to plastic.

Data from three jute fibre breaking strengths at 5, 10, and 20 mm are examined and represented in Table 10. These datasets were extracted from Xia et al. [54]. Then, URVs are highly relevant in textile engineering because they capture the strongest observations in a sequence of fiber strength measurements. This is important since many applications require identifying and utilizing fibers that can withstand exceptionally high stresses. URVs are important for material benchmarking; they highlight the maximum tensile strengths achieved under testing, which can be used to benchmark the best-performing fibers or production batches. Also, the URVs are important for process improvement; in manufacturing, upper records reflect technological improvements (e.g., better retting, spinning, or finishing of jute fibers). Each new upper record can signal a significant enhancement in quality. For specialized applications, such as industrial textiles, ropes, and composites, knowing the URV for strength capacity helps define safety factors and predict the upper performance limits of the material. For reliability, by studying the pattern of upper records, engineers can estimate how often stronger fibers appear and how likely future batches are to exceed current performance levels. So, URVs are highly relevant in textile engineering because they capture the strongest observations in a sequence of fiber strength measurements. This is important since many applications require identifying and utilizing fibers that can withstand exceptionally high stresses.

6.2. Goodness-of-fit test and likelihood ratio test

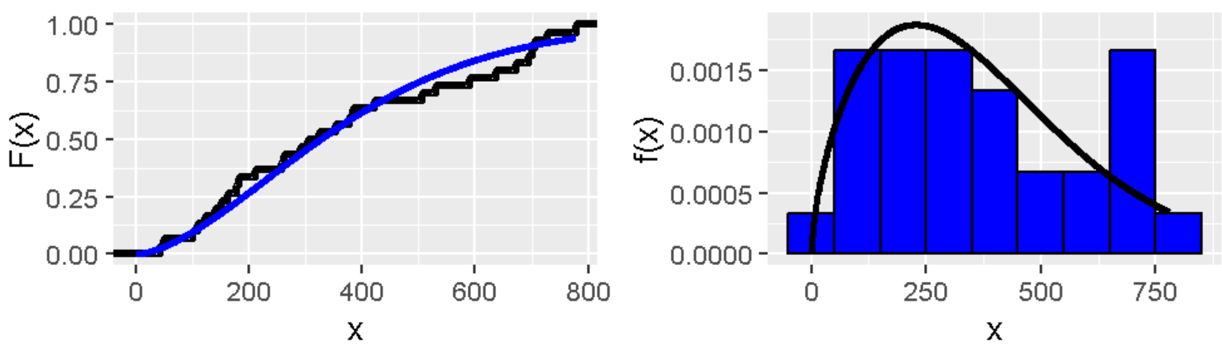
Fitting of the Weibull distribution is examined to this data. Distance of Kolmogorov–Smirnov (DKS) and p-values (PVKS) are computed for three jute fibre breaking strength data sets. Some descriptive statistics for the three real data sets, iincluding minimum and maximum values, mean, first and third quartiles, median, DKS, and PVKS, are given in Table 11. The value of PVKS suggests that these three data sets can be well modelled by the Weibull distribution. For three sets of specimens exposed to three different jute fibre breaking strengths at 5, 10, and 20 mm, respectively, Figures 9, 10, and 11 demonstrate that the histogram plot and the actual, theoretical CDFs almost coincide.

The previous analysis assumed that the stress and strength random variables followed the Weibull distribution with a common shape parameter (α). Consequently, when estimating the reliability with actual data, we must also verify whether their shape parameters are equal, as mentioned below.



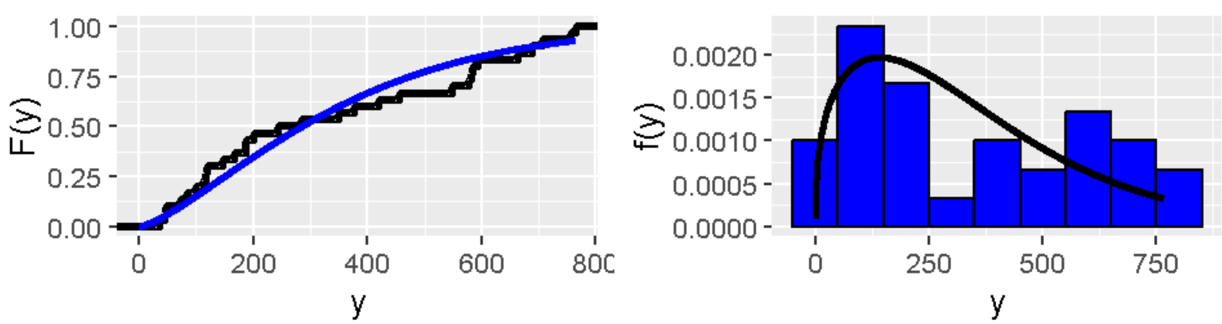
(a) Theoretical and empirical CDFs (b) Histogram Plot

Figure 9. Theoretical and empirical CDFs and histogram plot of data set I (stress Z).



(a) Theoretical and empirical CDFs (b) Histogram Plot

Figure 10. Theoretical and empirical CDFs and histogram plot of data set II (strenght X).



(a) Theoretical and empirical CDFs (b) Histogram Plot

Figure 11. Theoretical and empirical CDFs and histogram plot of data set III (stress Y).

Assume that $X \sim$ Weibull(λ_1, α_1), $Y \sim$ Weibull(λ_2, α_2) and $Z \sim$ Weibull(λ_3, α_3), the conforming log-likelihood value is $L_1 = -41.8699$. At this point, we have completed the following hypothesis tests:

$$H_0 : \alpha_1 = \alpha_2 = \alpha_3; \quad H_1 : \alpha_1 \neq \alpha_2 \neq \alpha_3.$$

In this case, the likelihood ratio test value is $-2(L_0 - L_1) = -5.45921$ with a degree of freedom of 2. Furthermore, the chi-squared test has a p-value of two, and because this p-value is more than 0.05, the null hypothesis cannot be rejected. As a result, it H_0 is a reasonable assumption in this case.

6.3. Verification of Data Accuracy

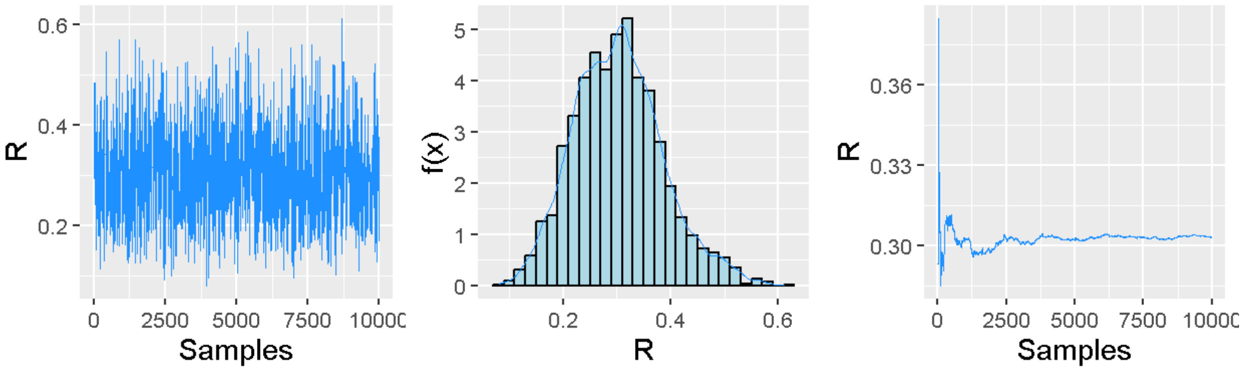
In textile engineering, fiber failure is strongly governed by stress thresholds. The strength X denotes the inherent fiber strength, which varies due to natural variability in fiber diameter, crystallinity, or the presence of defects. The lower stress Y represents the minimum stress that a fiber must withstand to avoid premature breakage under handling or operational loads, while the upper stress Z reflects the maximum stress level beyond which structural degradation, permanent deformation, or thermal damage occurs. Therefore, the model $R = P(Y < X < Z)$ directly links shifts in the statistical properties of fiber strength X and the design thresholds Y and Z to known degradation mechanisms (e.g., moisture-induced plasticization, hydrolytic or microbiological attack, and thermal embrittlement). Some environmental conditions act as natural stresses in jute fiber, such as temperature and relative humidity, each playing an independent role in determining fiber quality and flexibility. The optimal temperature range for maintaining the strength and suppleness of the fibers is between 20 and 30 °C. When the temperature drops below 15 °C, the fibers become brittle and prone to breakage, while prolonged exposure to temperatures above 30 °C gradually reduces their quality. Regarding relative humidity, the ideal range is 60–70%, which ensures a proper balance between strength and flexibility. If the humidity falls below 50%, the fibers become dry and fragile, whereas exceeding 80% increases the risk of mold growth and leads to a loss of their natural luster.

Let X be the jute fiber strength (thickness, toughness, and crystallinity) with parameter λ_1, Y is lower stress limits (lower limit for temperature and humidity) with parameter λ_2, and Z is upper stress limits (upper limit for temperature and humidity) with parameter λ_3. In general λ_1, λ_2, and λ_3 have mainly effect on R, the reliability measure R increases as the fiber strength increases. Moreover, variations in the lower and upper pressure limits directly affect the range of R; the range of R expands when the upper pressure limit increases, also when the lower pressure limit decreases.

Consequently, in the context of jute fiber applications, the reliability measure $R = P(Y < X < Z)$ has a clear practical interpretation. For instance, an estimate $R = 0.3073$, It means there is about a 30.73% probability that the fiber's strength will fall between its lower and upper stress thresholds, i.e., it can withstand the applied load without failing prematurely (too weak) or becoming unstable (too brittle). In real-world terms, this value quantifies how often the fiber is expected to operate safely within its functional stress range, capturing the likelihood that the material performs as intended under normal conditions of use (e.g., in textiles exposed to tension, humidity, and heat). This provides an engineering perspective by linking the statistical model to the actual reliability of fibers under stress thresholds, thereby offering useful insights for textile engineering applications. Referring to the above real data, the BE of R is 0.3073 while the MLE of R is 0.29314 with a standard error of 0.12124 and the corresponding standard error of 0.00094.

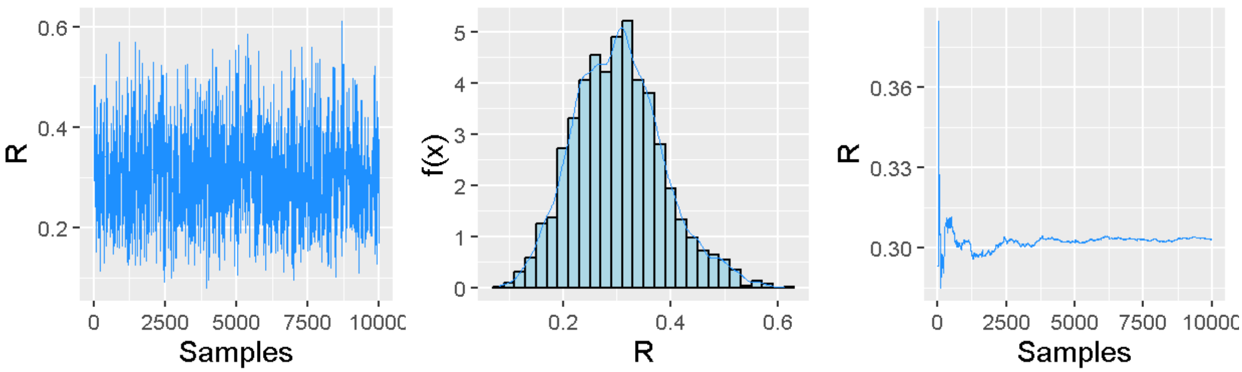
To monitor the convergence of the MCMC outputs, Figures 12 and 13 display the trace plot of the posterior distributions of the SS reliability and α estimate. The MCMC process's degree of convergence is depicted in this figure. The histograms for the marginal posterior densities of the R and α estimates based on 10,000 chain values are also displayed. As seen in Figure (12, 13), the estimates unequivocally demonstrate that each generated posterior is symmetric with regard to the theoretical posterior DFs.

In this study, a common shape parameter α is assumed for the strength (X) and the lower and upper stresses (Y) and (Z). This assumption, while simplifying the analytical derivation and allowing a closed-form expression for



(a) Trace Plot (b) Histogram Plot (c) Convergence Plot

Figure 12. *The MCMC plots for SS using real data.*



(a) Trace Plot (b) Histogram Plot (c) Convergence Plot

Figure 13. *The MCMC plots for α using real data.*

R, is also supported by practical considerations. In many reliability and material strength applications—such as those involving homogeneous materials like jute fibers—the underlying failure mechanisms and environmental conditions are often similar across the different stress and strength variables. Consequently, the Weibull distributions describing these variables may reasonably share a similar shape parameter, reflecting comparable variability patterns. However, we acknowledge that this assumption may not always hold in heterogeneous or complex systems where the failure processes differ significantly. The impact of relaxing this assumption could be investigated in future work, potentially through models allowing different shape parameters for X, Y, and Z.

7. Concluding Remarks

This article discusses the reliability estimation of $R = P(Y < X < Z)$ in this article, with particular attention to the need for strength to (X) be less than upper-bound stress (Z) and more than lower-bound stress Y. The study explores the estimation of the reliability function $R = P(Y < X < Z)$ from both the frequentist and Bayesian perspectives, assuming independent Weibull-distributed stresses and strength. Three different loss functions, including SE, LINEX, and MELO, are employed to derive the corresponding estimators within the Bayesian framework. IF and N-IF priors are incorporated, and the M-H technique is used to derive Bayesian estimates of R. The performance of the suggested estimators is evaluated by a thorough simulation study, which is enhanced by the use of MCMC techniques to calculate the Bayes estimates.

To demonstrate practical applicability, the methodologies are validated on three real datasets of jute fibre strength. The results underscore the effectiveness of the derived estimators in evaluating component reliability within the SS model, offering valuable insights for reliability analysis in engineering and materials science. When estimating SS reliability (R), the BEs prove superior to MLEs, particularly when considering various loss functions. The accuracy of both MLEs and BEs, measured by MSE proves with a higher number of records. Furthermore, the MSE for both methods consistently drops as the actual value of R becomes higher. Notably, Bayesian estimates of R derived under the MELO loss function consistently achieve a lower MSE, outperforming estimates from other loss functions. Future research may explore the use of advanced methods such as Hamiltonian Monte Carlo and variational inference to enhance sampling efficiency and allow for a more comprehensive comparison with the proposed approach.

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