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Modified Bregman extragradient algorithm for equilibrium problems in Banach spaces

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Abstract This paper introduces a modified Bregman extragradient algorithm designed to solve pseudomonotone equilibrium problems in a real reflexive Banach space. The algorithm guarantees weak convergence under mild assumptions and establishes strong convergence under additional conditions. In our proposed algorithm, we utilize two parameters with the Bregman distance and a non-monotonic step size, which is independent of the Bregman Lipschitz constant, to enhance the algorithm's effectiveness. Furthermore, numerical experiments are conducted to validate the performance of the proposed algorithm, demonstrating significant improvements in efficiency compared to traditional algorithms in similar settings.

Keywords Equilibrium problem, extragradient algorithm, Variational inequality, Strong Convergence, Bregman distance

AMS 2010 subject classifications 65K15, 68W10, 47H09.

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1. Introduction

Consider the equilibrium problem (EPb) as follows:

$$\text{find } v \in \mathcal{S} \text{ such that } \quad B(v,t) \geq 0, \forall t \in \mathcal{S}, \tag{1}$$

where $\mathcal{S}$ is a nonempty, closed, and convex subset of a real reflexive Banach space E, $B : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ be a bifunction. We denote by E^* the dual of E. The EPb provides a comprehensive framework that unifies several mathematical concepts, such as optimization problems, variational inequalities, and game theory. Due to their versatility and wide-ranging applications in fields like economics, mechanics, and signal processing (see, e.g., [8, 9]), EPb has been the focus of extensive theoretical and numerical research. In the special case where $B(v,t) := \langle \Psi v, t - v \rangle$ with $\Psi : E \to E$, the EPb coincides with the variational inequality problem $VI(\Psi; \mathcal{S})$ defined as:

$$\text{find } v \in \mathcal{S} \text{ such that } < \Psi v, t - v > \geq 0, \forall t \in \mathcal{S}.$$

On the other hand, if $B(v,t) := h(v) - h(t)$, where $h : \mathcal{S} \to \mathbb{R}$ then the EPb reduces to the optimization problem

$$\min_{v \in \mathcal{S}} h(v) \ s.t. \ v \in \mathcal{S}.$$

Several studies have investigated iterative methods for solving the EPb in Hilbert and Banach spaces, such as the proximal point method ([6, 12]), subgradient extragradient techniques (see, e.g., [1, 18, 14, 15]), the auxiliary problem principle [10], and approaches based on gap functions [11]. Among these methods, the extragradient

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algorithm with a monotonically decreasing step size $\{\varrho_m\}$ has attracted considerable attention for solving the EPb in a real Hilbert space, as proposed by Hieu (see [7]). This method involves solving two optimization problems over a closed convex set at each iteration. The iterative scheme is defined as follows

$$\begin{cases} t_m = \arg\min_{t \in \mathcal{S}} \left(\varrho_m B(v_m, t) + \frac{1}{2} \|t - v_m\|^2 \right), \\ v_{m+1} = \arg\min_{t \in \mathcal{S}} \left(\varrho_m B(t_m, t) + \frac{1}{2} \|t - v_m\|^2 \right), \end{cases}$$

and

$$\varrho_{m+1} = \begin{cases} \min\left\{ \frac{\zeta\left(\|v_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right)}{2M}, \ \varrho_m \right\} & \text{if } M > 0, \\ \varrho_m & \text{otherwise}, \end{cases}$$

where $M = B(v_m, v_{m+1}) - B(v_m, t_m) - B(t_m, v_{m+1})$. Under several conditions, the authors showed that the sequences weakly converge to some solution of the EPb (see [7]).

Recently, Eskandani et al. [5] further developed the extragradient Algorithm in [7], by replacing the Euclidean distance with a so-called Bregman distance to solve the EPb in Banach spaces as follows:

$$\begin{cases} t_m = \arg\min_{t \in \mathcal{S}} \left(\varrho_m B(v_m, t) + D_\varphi(t, v_m) \right), \\ v_{m+1} = \arg\min_{t \in \mathcal{S}} \left(\varrho_m B(t_m, t) + D_\varphi(t, v_m) \right), \end{cases}$$

and

$$\varrho_{m+1} = \begin{cases} \min\left\{ \frac{\zeta(D_\varphi(v_m, t_m) + D_\varphi(v_{m+1}, t_m))}{M}, \ \varrho_m \right\} & \text{if } M > 0, \\ \varrho_m & \text{otherwise}, \end{cases}$$

where $M = B(v_m, v_{m+1}) - B(v_m, t_m) - B(t_m, v_{m+1})$.

Inspired by prior studies, including the classical extragradient algorithms and their modifications, a natural question arises: Can we enhance these methods by introducing a new modified extragradient algorithm with a non-monotonic step size that eliminates the need for prior estimates of the Bregman Lipschitz-like constants in solving EPb within reflexive Banach spaces?

In this paper, we propose a modified Bregman extragradient algorithm designed to solve pseudomonotone EPb in Banach spaces E. The algorithm incorporates two new parameters, utilizes the Bregman distance, and employs a non-monotonic step size that is independent of the Bregman Lipschitz constant, addressing the aforementioned question. The weak convergence of the algorithm is ensured under mild assumptions, while strong convergence is established when the equilibrium bifunction satisfies additional specific conditions. The algorithm associated is of interest for several reasons, but especially that using non-monotonic step size that is independent of the Bregman Lipschitz constants, offering greater flexibility compared to non-increasing monotone step sizes as in [7, 5]. Furthermore, we introduce two parameters specifically designed to enhance the iterative process, resulting in significant improvements in convergence speed and computational efficiency.

This paper is organized as follows: in Section 2, we recall some definitions and important Lemmas used in this paper. In Section 3, we give a new iterative algorithm for solving the EPb with convergence studies. In Section 4 includes numerical experiments to demonstrate the performance of the proposed algorithm on a test problem and compare it with other algorithms. Finally, in Section 5 concludes the paper with a brief summary.

2. Preliminaries

In this section, we provide some definitions, important lemmas, and notions that we will need in the sequel. Throughout this paper, we consider $\mathcal{S}$ to be a nonempty, closed, and convex subset of a reflexive real Banach space E, with its dual space denoted by E^*. The duality pairing between E and E^* is represented by $\langle ., . \rangle$, while the norm is denoted by $\|.\|$ (not necessarily Euclidean). Denote by $\rightharpoonup$ and $\longrightarrow$ the weak convergence and strong convergence, respectively. Let B be a bifunction : $E \times E \to \mathbb{R}$ and $\varphi : E \to \mathbb{R}$ at $v \in \mathcal{S}$ is defined by

$$\partial\varphi(v) := \{v^* \in E^* : h(t) - h(v) \geq < v^*, t - v >, \quad \forall t \in E\}.$$

The function φ is called a Legendre function if it fulfills the following two conditions

- $int(\text{dom } \varphi) \neq \emptyset$ and $\partial\varphi$ is single-valued on its domain;
- $int(\text{dom } \varphi^*) \neq \emptyset$ and $\partial\varphi^*$ is single-valued on its domain.

Where $\varphi^* : E^* \to \mathbb{R} \cup \{+\infty\}$ is the Fénchel conjugate function of φ given by

$$\varphi^*(v^*) = \sup\{\langle v, v^*\rangle - \varphi(v) : v \in E\}.$$

A normal cone of $\mathcal{S}$ at $v \in \mathcal{S}$ is defined by

$$N_{\mathcal{S}}(v) := \{v^* \in E^* : < v^*, t - v > \leq 0, \quad \forall t \in \mathcal{S}\}.$$

Definition 2.1
Let $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ be some function.

1. The function φ is called Gâteaux differentiable at a point $v \in int(\text{dom } \varphi)$ if the limit
$$\varphi^\circ(v, t) := \lim_{h \to 0^+} \frac{\varphi(v + ht) - \varphi(v)}{h}, \tag{2}$$
exists for any $t \in E$;
2. φ is Gâteaux differentiable if it is Gâteaux differentiable for every $v \in int(\text{dom } \varphi)$;
3. we say that φ is Fréchet differentiable at $v \in int(\text{dom } \varphi)$ if the limit in (2) is attained uniformly in $\|t\| = 1$;
4. φ is Fréchet differentiable on a subset $\mathcal{S}$ of E if the limit in (2) is attained uniformly for $v \in \mathcal{S}$ and $\|t\| = 1$;
5. The function φ is supercoercive if $\lim_{\|v\| \to \infty} \frac{\varphi(v)}{\|v\|}$;
6. φ is weakly sequentially continuous if $v_m \rightharpoonup v$ implies $\varphi(v_m) \rightharpoonup \varphi(v)$ as $m \to \infty$.

Definition 2.2
([4]) Assume that $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ is Gâteaux differentiable. The Bregman distance with respect to φ is the bifunction

$$D_\varphi : dom(\varphi) \times int(dom(\varphi)) \longrightarrow [0, +\infty),$$

defined by

$$D_\varphi(v, t) := \varphi(v) - \varphi(t) - < \nabla\varphi(t), v - t >, \ \forall\, v \in dom(B),\ t \in int(dom(B)).$$

Unlike standard metrics, the Bregman distance neither exhibits symmetry nor satisfies the triangle inequality. However, it generalizes certain well-known distances. It satisfies the three-point identity:

$$D_\varphi(v, t) + D_\varphi(t, s) - D_\varphi(v, s) = < \nabla\varphi(s) - \nabla\varphi(t), v - t >, \tag{3}$$

and four-point identity

$$D_\varphi(v, t) + D_\varphi(w, s) - D_\varphi(v, s) - D_\varphi(w, t) = < \nabla\varphi(s) - \nabla\varphi(t), v - w >, \tag{4}$$

for any $v, w \in dom\ \varphi$ and $t, s \in int(dom\ \varphi)$.
The Bregman projection ([4]) with respect to φ of $v \in int(dom\ \varphi)$ onto $\mathcal{S}$ is characterized as the unique vector $\pi_{\mathcal{S}}^{\varphi}$ fulfilling

$$\pi_{\mathcal{S}}^{\varphi}(v) := \inf_{t \in \mathcal{S}} (D_\varphi(t, v)).$$

Definition 2.3
([2]) The modulus of total convexity at a point $v \in \text{int}(\text{dom}\ \varphi)$ is defined as a function $\nu_\varphi(v,.) : [0,+\infty) \to [0,\infty]$, given by:

$$\nu_\varphi(v,d) = \inf\{D_\varphi(t,v) : y \in \text{dom}\ g,\ \| t - v \| = d\}.$$

If $\nu_\varphi(v,d)$ is strictly positive for all $d > 0$, the function φ is said to be totally convex at v.
For a non-empty set $\mathcal{S} \subseteq E$, the modulus of total convexity of φ on $\mathcal{S}$ is expressed as:

$$\nu_\varphi(\mathcal{S},d) = \inf\{\nu_\varphi(v,d) : v \in \mathcal{S} \cap \text{int}(\text{dom}\ \varphi)\}.$$

The function φ is referred to as totally convex on bounded subsets if $\nu_g(\mathcal{S},d)$ remains positive for all $d > 0$ and for any bounded, non-empty subset $\mathcal{S}$.

Lemma 2.1
([16]) A uniformly Fréchet differentiable function $\varphi : E \to \mathbb{R}$ that is bounded on bounded subsets of E ensures that $\nabla\varphi$ is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.

Lemma 2.2
[3] The function $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ is totally convex on bounded subsets of E iff for any two sequences $\{v_m\}$ and $\{t_m\}$ in int dom φ and dom φ, respectively, such that the first one is bounded,

$$\lim_{m\to\infty} D_\varphi(t_m,v_m) = 0 \Rightarrow \lim_{m\to\infty} \|t_m - v_m\| = 0.$$

Lemma 2.3
[17] Let the function $\varphi : E \to \mathbb{R}$ be Gâteaux differentiable such that $\nabla\varphi^*$ is bounded on bounded subsets of dom φ^*. Let $v_0 \in E$ and $\{v_m\} \subset dom\ \varphi$. If $D_\varphi(v_0,v_m)$ is bounded, then the sequence $\{v_m\}$ is also bounded.

Theorem 2.1
[21] Let $\varphi : E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then, the following are equivalent:

(i) φ is supercoercive and uniformly convex on bounded subsets of X;
(ii) $\text{dom}\,\varphi^* = E^*$, E^* is bounded on bounded subsets and uniformly smooth on bounded subsets of E^*;
(iii) $\text{dom}\,\varphi^* = E^*$, φ^* is Fréchet differentiable and $\nabla\varphi^*$ is uniformly norm-to-norm continuous on bounded subsets of E^*.

Theorem 2.2
[2] Suppose that $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ is a Legendre function. The function φ is uniformly convex on bounded subsets of E if and only if φ is totally convex on bounded subsets of E.

Definition 2.4
A bifunction B: $E \times E \to \mathbb{R}$ is said to be

1. pseudomonotone on $\mathcal{S}$, i.e.,
$$B(v,t) \geq 0 \Rightarrow B(t,v) \leq 0, \qquad \forall v,t \in \mathcal{S};$$
2. $\gamma-$strongly pseudomonotone on $\mathcal{S}$, i.e., there exists a constant γ such that
$$B(v,t) \geq 0 \Rightarrow B(t,v) \leq -\gamma \|v - t\|^2, \qquad \forall v,t \in \mathcal{S};$$
3. Bregman Lipschitz type continuous on $\mathcal{H}$ with two positive constants L_1 and L_2, i.e.,
$$B(v,t) + B(t,w) \geq B(v,w) - L_1 D_\varphi(t,v) - L_2 D_\varphi(w,t),\ \ \forall t,v,w \in \mathcal{S}.$$

Definition 2.5
The proximal operator $J^{\varphi}_{\tau h}$ of a proper, convex and lower semicontinuous function $h : \mathcal{S} \to \mathbb{R}$ with respect to a Gâteaux differentiable function $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ and a parameter $\tau > 0$ at $v \in E$ is given by

$$J^{\varphi}_{\tau h}(v) := \arg\min_{t\in\mathcal{S}} \left(\tau h(t) + D_{\varphi}(t, v)\right).$$

Lemma 2.4
([20]) Let $\mathcal{S} \subset E$. Consider $h : \mathcal{S} \to \mathbb{R} \cup \{+\infty\}$ as a convex function that is subdifferentiable and lower semicontinuous. Then, t^* is a solution to the following con optimization problem:

$$\min\left\{h(t) : t \in \mathcal{S}\right\},$$

if and only if

$$0 \in \partial h(t^*) + N_{\mathcal{S}}(t^*),$$

where $\partial h(t^*)$, $N_{\mathcal{S}}(t^*)$ are the subdifferential of h and the normal cone of $\mathcal{S}$ at t^*, respectively.

Lemma 2.5
[13] Let $\{a_m\}, \{b_m\}$ and $\{c_m\}$, be positive sequences such that

$$a_{m+1} \leq\ a_m b_m + c_m,\ \forall m \in \mathbb{N}.$$

If $\{b_m\} \subset [1, \infty)$, $\sum\limits_{m=1}^{\infty} (b_m - 1) < \infty$ and $\sum\limits_{m=1}^{\infty} c_m < \infty$. Then $\lim\limits_{m\to\infty} a_m$ exists.

Lemma 2.6
[3] Let $\{a_m\}, \{b_m\}$ be two nonnegative real sequences such that

$$a_{m+1} \leq a_m - b_m.$$

Then, $\lim\limits_{m\to\infty} a_m \in \mathbb{R}$, and $\sum\limits_{m\geq 1} b_m < \infty$.

3. MAIN RESULTS

In this section, by using tow parameters, the Bregman distance and non-monotonic adaptive step size criterion, we propose modified extragradient algorithm for solving the EPb in E. Assume that the solution set of EPb, represented by $EQ(B, \mathcal{S})$, is nonempty.

Algorithm 3.1 (modified Bregman extragradient algorithm for solving the EPb)
Initialization: Given $v_0 \in \mathcal{S},\ \varrho_1 > 0,\ \zeta \in (0, 1),\ \mu \in \left(0, \dfrac{1}{2\zeta}\right)$ and $\tau \in \left[\mu, \frac{1}{\zeta}\right)$. Select the sequences $\{\sigma_m\} \subset [0, \infty)$ and $\{\omega_m\} \subset [1, \infty)$ such that $\sum\limits_{m=1}^{\infty} \sigma_m < \infty$ and $\sum\limits_{m=1}^{\infty} (\omega_m - 1) < \infty$.
Step 1: Compute

$$t_m = \arg\min_{t\in\mathcal{S}} \left(\tau\varrho_m B(v_m, t) + D_{\varphi}(t, v_m)\right) = J^{\varphi}_{\tau\varrho_m B(v_m,.)}(v_m).$$

If $t_m = v_m$, then stop, and t_m is a solution. Otherwise, go to next step.
Step 3: Compute

$$v_{m+1} = \arg\min_{t\in\mathcal{S}} \left(\mu\varrho_m B(t_m, t) + D_{\varphi}(t, v_m)\right) = J^{\varphi}_{\mu\varrho_m B(t_m,.)}(v_m),$$

where

$$\varrho_{m+1} = \begin{cases} \min\left\{\frac{\zeta(D_{\varphi}(t_m,v_m)+D_{\varphi}(v_{m+1},t_m))}{M}, \omega_m\varrho_m + \sigma_m\right\} & \text{if } M > 0, \\ \omega_m\varrho_m + \sigma_m & \text{otherwise}, \end{cases} \tag{5}$$

and $M = B(v_m, v_{m+1}) - B(v_m, t_m) - B(t_m, v_{m+1})$. Set $m := m + 1$ and go to **Step 1**.

Remark 3.1
The observation presented below was extracted from Algorithm 3.1

- When the parameters $\tau = \mu = \omega_m = 1$ and $\sigma_m = 0$, Algorithm 3.1 reduces to the Bregman extragradient algorithm introduced in [5]. The inclusion of the new parameters $(\tau,\ \mu,\ \omega_m,\ \sigma_m)$ significantly improves the numerical performance, yielding better results than the original formulation.
- Furthermore, by setting $\varphi(.) = \frac{1}{2}\left\|.\right\|_2^2$, where $\|.\|_2$ denotes the Euclidean norm, and assuming E is a real Hilbert space, Algorithm 3.1 can be viewed as an extension and enhancement of the method in [7].
- Although similar extragradient algorithms have been studied in Hadamard spaces, such as in [19], our work focuses on Banach spaces and employs Bregman distance with two parameters τ and μ, leading to different approaches in terms of convergence analysis and numerical behavior.

3.1. Weak convergence

For the weak convergence theorem, consider the following assumptions.

Assumption 3.1
Let φ be a function such that

(C_1) φ is a supercoercive and Legendre function which is bounded;
(C_2) φ is uniformly Fréchet differentiable;
(C_3) φ is totally convex on bounded subsets of E;
(C_4) $\nabla\varphi$ is weakly sequentially continuous.

Assumption 3.2
Let B be a bifunction such that

(H_1) The bifunction B is pseudomonotone on $\mathcal{S}$;
(H_2) B is Bregman Lipschitz type continuous on $\mathcal{H}$;
(H_3) $B(v,.)$ is convex and subdifferentiable on $\mathcal{H}$ for each fixed $v \in \mathcal{S}$;
(H_4) for every sequence $\{v_m\} \subset \mathcal{S}$ and $v \in \mathcal{H}$ such that $v_m \rightharpoonup v$ and $\limsup_{m\to\infty} B(v_m, t) \geq 0$, for all $t \in \mathcal{S}$, then $B(v,t) \geq 0$.

It has been proved that under the conditions (H_1)–(H_3), the solution set $EQ(B,\mathcal{S})$ of EPb is closed and convex ([18]).
We begin by proving the following necessary results:

Lemma 3.1
The sequence $\{\varrho_m\}$ created by (5) is well defined and $\lim_{m\to+\infty} \varrho_m$ exists.

Proof
Since B fulfills (H_2), it follows that

$$\frac{\zeta\left(D_\varphi(t_m, v_m) + D_\varphi(v_{m+1}, t_m)\right)}{\left(B(v_m, v_{m+1}) - B(v_m, t_m) - B(t_m, v_{m+1})\right)} \geq \frac{\zeta\left(D_\varphi(t_m, v_m) + D_\varphi(v_{m+1}, t_m)\right)}{\left(L_1 D_\varphi(t_m, v_m) + L_2 D_\varphi(v_{m+1}, t_m)\right)} \geq \frac{\zeta}{\max\{L_1, L_2\}}.$$

This, in addition to the expression (5), gives $\varrho_{m+1} \geq \min\left\{\frac{\zeta}{\max\{L_1,L_2\}}, \varrho_m\right\}$. Moreover $\varrho_m \geq \min\left\{\frac{\zeta}{\max\{L_1,L_2\}}, \varrho_1\right\}$. In contrast, it becomes clear from expression (5) that

$$\varrho_{m+1} \leq \omega_m \varrho_m + \sigma_m, \forall m \geq 1.$$

It follows from conditions on $\{\omega_m\}$, $\{\sigma_m\}$ and Lemma 2.5 that $\lim_{m\to+\infty} \varrho_m$ exists. Since $\min\left\{\frac{\zeta}{\max\{L_1,L_2\}}, \varrho_1\right\}$ is the lower boundary of $\{\varrho_m\}$, then $\lim_{m\to+\infty} \varrho_m := \varrho > 0$. □

Remark 3.2
From the definition of $\{t_m\}$ and Lemma 2.4, we have

$$0 \in \partial\left(\tau\varrho_m B(v_m,t) + D_\varphi(t,v_m)\right)(t_m) + N_{\mathcal{S}}(t_m).$$

This implies that there exists $w \in \partial B(v_m,t_m)$ and $\eta \in N_{\mathcal{S}}(t_m)$, such that

$$\tau\varrho_m w + \nabla\varphi(t_m) - \nabla\varphi(v_m) + \eta = 0.$$

Thus, we have from the definition of $N_{\mathcal{S}}$

$$\begin{aligned}&< \nabla\varphi(v_m) - \nabla\varphi(t_m), t - t_m >= \tau\varrho_m < w, t - t_m > + < \eta, t - t_m >,\\ &\leq \tau\varrho_m < w, t - t_m >, \ \forall t \in \mathcal{S}.\end{aligned}$$

Since, $w \in \partial B(v_m,t_m)$, we have

$$< w, t - t_m > \leq B(v_m,t) - B(v_m,t_m), \ \forall t \in \mathcal{S}.$$

From the last two inequalities, we get

$$< \nabla\varphi(v_m) - \nabla\varphi(t_m), t - t_m > \leq \tau\varrho_m\left(B(v_m,t) - B(v_m,t_m)\right), \ \forall t \in \mathcal{S}. \tag{6}$$

If $v_m = t_m$, then from (6) and τ, $\varrho_m > 0$, we obtain $B(t_m,t) \geq 0$, for all $t \in \mathcal{S}$. Thus $t_m \in EQ(B,\mathcal{S})$.

Lemma 3.2
Let $\{v_m\}$ and $\{t_m\}$ be the two sequences generated by Algorithm 1. Fix $v^* \in EQ(B,\mathcal{S})$. Then

$$D_\varphi(v^*,v_{m+1}) \leq D_\varphi(v^*,v_m) - \mu\left(\frac{1}{\tau} - \frac{\zeta\varrho_m}{\varrho_{m+1}}\right)D_\varphi(t_m,v_m) - \mu\left(\frac{1}{\tau} - \frac{\zeta\varrho_m}{\varrho_{m+1}}\right)D_\varphi(v_{m+1},t_m).$$

Proof
According to the definition of $\{v_{m+1}\}$ and Remark 3.2, one has

$$< \nabla\varphi(v_m) - \nabla\varphi(v_{m+1}), t - v_{m+1} > \leq \mu\varrho_m\left(B(t_m,t) - B(t_m,v_{m+1})\right), \forall t \in \mathcal{S}. \tag{7}$$

In particular, substituting $t = v_{m+1}$ in (6), we get

$$< \nabla\varphi(v_m) - \nabla\varphi(t_m), v_{m+1} - t_m > \leq \tau\varrho_m\left(B(v_m,v_{m+1}) - B(v_m,t_m)\right), \tag{8}$$

Addining (7) with (8)

$$\begin{aligned}\tau\mu\varrho_m\big(B(v_m,v_{m+1}) - B(v_m,t_m) - B(t_m,v_{m+1})\big) \geq\ & \mu\big(\langle\nabla\varphi(v_m) - \nabla\varphi(t_m), v_{m+1} - t_m\rangle\big)\\ &+ \tau\big(\langle\nabla\varphi(v_m) - \nabla\varphi(v_{m+1}), t - v_{m+1}\rangle\big)\\ &+ \tau\mu\varrho_m B(t_m,t), \quad \forall t \in \mathcal{S}.\end{aligned} \tag{9}$$

From the definition of ϱ_m, we have

$$\left(B(v_m,v_{m+1}) - B(v_m,t_m) - B(t_m,v_{m+1})\right) \leq \frac{\zeta}{\varrho_{m+1}}\left(D_\varphi(t_m,v_m) + D_\varphi(v_{m+1},t_m)\right). \tag{10}$$

By Bregman three point identity (3), it follows that

$$< \nabla\varphi(v_m) - \nabla\varphi(t_m), v_{m+1} - t_m >= D_\varphi(v_{m+1},t_m) + D_\varphi(t_m,v_m) - D_\varphi(v_{m+1},v_m), \tag{11}$$

and

$$< \nabla\varphi(v_m) - \nabla\varphi(v_{m+1}), t - v_{m+1} >= D_\varphi(t, v_{m+1}) + D_\varphi(v_{m+1}, v_m) - D_\varphi(t, v_m). \tag{12}$$

Applying (10), (11) and (12) into (9), we obtain

$$\begin{aligned}\frac{\tau\mu\zeta\varrho_m}{\varrho_{m+1}}\left(D_\varphi(t_m, v_m) + D_\varphi(v_{m+1}, t_m)\right) \geq \mu\left(D_\varphi(v_{m+1}, t_m) + D_\varphi(t_m, v_m) - D_\varphi(v_{m+1}, v_m)\right)\\ + \tau\left(D_\varphi(t, v_{m+1}) + D_\varphi(v_{m+1}, v_m) - D_\varphi(t, v_m)\right)\\ + \tau\mu\varrho_m B(t_m, t), \quad \forall t \in \mathcal{S}.\end{aligned}$$

Then

$$\begin{aligned}D_\varphi(t, v_{m+1}) \quad \leq \quad & D_\varphi(t, v_m) - D_\varphi(v_{m+1}, v_m) + \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\left(D_\varphi(t_m, v_m) + D_\varphi(v_{m+1}, t_m)\right)\\ & -\frac{\mu}{\tau}\left(D_\varphi(v_{m+1}, t_m) + D_\varphi(t_m, v_m) - D_\varphi(v_{m+1}, v_m)\right)\\ & +\mu\varrho_m B(t_m, t) \quad (\forall t \in \mathcal{S}).\end{aligned} \tag{13}$$

Therefore, it follows from relation (13) that

$$\begin{aligned}D_\varphi(t, v_{m+1}) \leq D_\varphi(t, v_m) - \left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right)D_\varphi(t_m, v_m) - \left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right)D_\varphi(v_{m+1}, t_m)\\ - \left(1 - \frac{\mu}{\tau}\right)D_\varphi(v_{m+1}, v_m) + \mu\varrho_m B(t_m, t) \quad (\forall t \in \mathcal{S}).\end{aligned}$$

Noting that $\frac{\mu}{\tau} \in (0, 1]$ then, we have

$$\begin{aligned}D_\varphi(t, v_{m+1}) \leq & D_\varphi(t, v_m) - \left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right)D_\varphi(t_m, v_m)\\ & - \left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right)D_\varphi(v_{m+1}, t_m) + \mu\varrho_m B(t_m, t), \quad \forall t \in \mathcal{S}.\end{aligned} \tag{14}$$

Let $t = v^* \in EQ(B, \mathcal{S})$. Therefore, from the pseudo monotonicity of B, we have $B(v^*, t_m) \geq 0$. Thus, $B(t_m, v^*) \leq 0$. Hence from (14), we get

$$\begin{aligned}D_\varphi(v^*, v_{m+1}) \quad \leq \quad & D_\varphi(v^*, v_m) - \mu\left(\frac{1}{\tau} - \frac{\zeta\varrho_m}{\varrho_{m+1}}\right)D_\varphi(t_m, v_m)\\ & -\mu\left(\frac{1}{\tau} - \frac{\zeta\varrho_m}{\varrho_{m+1}}\right)D_\varphi(v_{m+1}, t_m).\end{aligned} \tag{15}$$

□

Lemma 3.3
The sequences $\{v_m\}$ and $\{t_m\}$ generated by Algorithm 3.1 are bounded.

Proof
From Lemma 3.1, one knows that $\lim_{m\to+\infty}\frac{\varrho_m}{\varrho_{m+1}} = 1$. This together with the assumptions on the parameters $\zeta \in (0, 1)$, $\mu \in \left(0, \frac{1}{2\zeta}\right)$ and $\tau \in \left[\mu, \frac{1}{\zeta}\right)$ yields that

$$\lim_{m\to\infty}\left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right) = \mu\left(\frac{1}{\tau} - \zeta\right) > 0.$$

Let $\epsilon \in \left(0, \mu\left(\frac{1}{\tau} - \zeta\right)\right)$. Consequently, there exists $m_0 \in \mathbb{N}$ satisfying

$$\left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right) > \epsilon > 0,\ \forall m \geq m_0. \tag{16}$$

From Lemma 3.2, we have

$$D_\varphi(v^*, v_{m+1}) \leq D_\varphi(v^*, v_m) - \epsilon\left(D_\varphi(t_m, v_m) + D_\varphi(v_{m+1}, t_m)\right), \tag{17}$$

which take the form

$$a_{m+1} \leq a_m - b_m, \tag{18}$$

where

$$\begin{cases} a_m = D_\varphi(v^*, v_m), \\ b_m = \epsilon\left(D_\varphi(t_m, v_m) + D_\varphi(v_{m+1}, t_m)\right). \end{cases}$$

Thus, from Lemma 2.6, it follows that the limit of a_m and $\lim_{m\to\infty} b_m = 0$ for all $m \geq 0$. Hence, from the definition of b_m, we have

$$\lim_{m\to\infty} D_\varphi(t_m, v_m) = \lim_{m\to\infty} D_\varphi(v_{m+1}, t_m) = 0. \tag{19}$$

From Lemma 2.2 , we conclude that

$$\lim_{m\to+\infty} \|t_m - v_m\| = \lim_{m\to+\infty} \|v_{m+1} - t_m\| = 0. \tag{20}$$

Consequently,

$$\lim_{m\to+\infty} \|v_{m+1} - v_m\| = 0.$$

Therefore, from Lemma 2.3 we have

$$\lim_{m\to+\infty} \|\nabla\varphi(v_{m+1}) - \nabla\varphi(v_m)\| = 0. \tag{21}$$

From Theorems 2.1 and 2.2, φ^* is bounded on bounded subsets of E^* and hence $\nabla\varphi^*$ is also bounded on bounded subsets of E^*. From this, (17) and Lemma 2.3, the sequence $\{v_n\}$ is bounded. As a result, $\{t_m\}$ is also bounded. □

Now, we prove that the sequences $\{v_m\}$ and $\{t_m\}$ generated by Algorithm 3.1 converge weakly to an element $v^* \in EQ(B, \mathcal{S})$.

Theorem 3.1
Let Assumptions 3.1–3.2 be satisfied. Then for each $v^* \in EQ(B, \mathcal{S}) \neq \emptyset$, the sequences $\{v_m\}$ and $\{t_m\}$ generated by Algorithm 1, converge weakly to v^*.

Proof
To show that $\{v_n\}$ converges to a solution of EPb, it is left to prove that any cluster point of $\{v_n\}$ belongs to $EQ(B, \mathcal{S})$. Let $\bar{v}$ be a cluster point of $\{v_m\}$. Hence $\{v_m\}$ is bounded, there exists a subsequence $\{v_{m_k}\}$ of $\{v_m\}$ such that $v_{m_k} \rightharpoonup \bar{v}$ as $k \to \infty$. From (20), we also have $t_{m_k} \rightharpoonup \bar{v}$. Next, we show that $\bar{v} \in EQ(B, \mathcal{S})$. Letting $m = m_k$ in (14) and using $\left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_{m_k}}{\varrho_{m_k+1}}\right) > 0$, we have

$$\tau\mu\varrho_{m_k} B(t_{m_k}, t) \geq D_\varphi(t, v_{m_k+1}) - D_\varphi(t, v_{m_k}). \tag{22}$$

Passing to the limit in (22), we obtain

$$\begin{aligned}
\limsup_{k\to\infty} \tau\mu\varrho_{m_k} B(t_{m_k}, t) &\geq \limsup_{k\to\infty}\left(D_\varphi(t, v_{m_k+1}) - D_\varphi(t, v_{m_k})\right), \\
&\geq \limsup_{k\to\infty}\left(D_\varphi(t, v_{m_k+1}) - D_\varphi(t, v_{m_k}) - D_\varphi(v_{m_k}, v_{m_k+1})\right), \\
&= \limsup_{k\to\infty} < \nabla\varphi(v_{m_k}) - \nabla\varphi(v_{m_k+1}), t - v_{m_k} > .
\end{aligned}$$

It follows from (21), boundedness of $\{v_m\}$, the parameters $\tau, \mu, \varrho_{m_k} \geq 0$ and Condition (H_4) that

$$0 \leq \lim_{k\to\infty} \sup B(t_{m_k}, t) \leq B(\bar{v}, t), \quad (\forall t \in \mathcal{S})\,. \tag{23}$$

Then $\bar{v} \in EQ(B, \mathcal{S})$. By utilizing equations (3.1) and (16), it follows that $\{v_m\}$ is Bregman monotone with respect to $EQ(B, \mathcal{S})$. Then $\{v_m\}$ converges weakly to a point in $\mathcal{S}$([5, Lemmas 10]). Consequently, the desired result is obtained by applying [5, Lemmas 11,12]. □

3.2. Strong convergence

Next, we examine the strong convergence of the algorithm, which guarantees that the iterates converge in a stronger sense than weak convergence. The specific assumption required for strong convergence will be outlined in the following.

Assumption 3.3
Assume the following conditions

(H'$_1$) The bifunction B is $\gamma-$strongly pseudomonotone on $\mathcal{S}$;
(H$_2$) B is Bregman Lipschitz type continuous on $\mathcal{H}$;
(H$_3$) $B(v,.)$ is convex and subdifferentiable on $\mathcal{H}$ for each fixed $v \in \mathcal{S}$;
(H$_4$) for all bounded sequences v_m and t_m in $\mathcal{S}$,

$$\|v_m - t_m\| \to 0 \Rightarrow B(v_m, t_m) \to 0.$$

Theorem 3.2
Let Assumption 3.3 and (C_1, C_2, C_3) in Assumptions 3.1 be satisfied. Then. for each $v^* \in EQ(B, \mathcal{S}) \neq \emptyset$, the sequences $\{v_m\}$ and $\{t_m\}$ generated by Algorithm 3.1, converges strongly to v^*.

The proof of strong convergence for Algorithm 3.1 is based on the same reasoning as in [5].

Proof
As shown in Theorem 3.1, all cluster points of the sequence $\{v_m\}$ are elements of $EQ(B, \mathcal{S})$. Now, consider arbitrary subsequences $\{v_m\}$ and $\{v_{m_k}\}$ of $\{v_{m_n}\}$ that converge strongly to p and q, respectively. From the expression (4), it follows that

$$< p - q, \nabla\varphi(v_{m_k}) - \nabla\varphi(v_{m_n}) >= D_\varphi(p, v_{m_k}) - D_\varphi(q, v_{m_n}) - D_\varphi(p, v_{m_n}) - D_\varphi(q, v_{m_k}).$$

According to (17), $\lim_{m\to\infty} D_\varphi(p, v_m)$ and $\lim_{m\to\infty} D_\varphi(q, v_m)$ exist. By utilizing this fact, Lemma 2.1, and letting $m \to \infty$, it follows that $p = q$. Hence, the sequence $\{v_m\}$ converges strongly to a point in $EQ(B, \mathcal{S})$. Next, we show that if $v_{m_k} \rightharpoonup \bar{v}$, then $v_{m_k} \to \bar{v}$. Assume that $v_{m_k} \rightharpoonup \bar{v}$. Therefore, by (20), $t_{m_k} \rightharpoonup \bar{v}$. Substituting $t = \bar{v}$ into (7), we get

$$\begin{aligned}
0 &\leq \mu\varrho_{m_k}\Big(B(t_{m_k}, \bar{v}) - B(t_{m_k}, v_{m_k})\Big) - \langle \nabla\varphi(v_{m_k}) - \nabla\varphi(v_{m_k+1}), v_{m_k+1} - \bar{v}\rangle, \\
&= \mu\varrho_{m_k}\Big(B(t_{m_k}, \bar{v}) - B(t_{m_k}, v_{m_k+1})\Big) + \langle \nabla\varphi(v_{m_k}) - \nabla\varphi(v_{m_k+1}), v_{m_k+1} - \bar{v}\rangle, \\
&\leq \mu\varrho_{m_k}\Big(B(t_{m_k}, \bar{v}) - B(t_{m_k}, v_{m_k+1})\Big) + \|\nabla\varphi(v_{m_k}) - \nabla\varphi(v_{m_k+1})\|\,\|v_{m_k+1} - \bar{v}\|.
\end{aligned}$$

Using (20), (21), Lemma 3.1, condition H$_4$, and the boundedness of $\{v_m\}$, it follows that

$$\lim_{k\to\infty} \inf B(t_{m_k}, \bar{v}) \geq 0. \tag{24}$$

Given that $B(t_{m_k}, \bar{v}) \geq 0$, there exists a constant γ such that $B(t_{m_k}, \bar{v}) \leq -\gamma\|t_{m_k} - \bar{v}\|^2$. Combining this with (24), we conclude that

$$0 \leq \liminf_{k\to\infty} B(t_{m_k}, \bar{v}) \leq \liminf_{k\to\infty} \left(-\gamma\|t_{m_k} - \bar{v}\|^2\right) \leq -\gamma \left(\limsup_{k\to\infty} \|t_{m_k} - v\|^2\right) \leq 0.$$

Consequently, $t_{m_k} \to \bar{v}$, and therefore, $v_{m_k} \to \bar{v}$. □

4. Illustrative experiments

The numerical results are presented in this section to demonstrate the performance of our proposed algorithm. All the programs were implemented in MATLAB (R2023a) on a Intel(R) Core(TM) i5-8265U CPU @ 1.60 GHz 1.80 GHz with RAM 8.00 GB.

Example 4.1
Consider the enhanced version of the Nash–Cournot oligopolistic equilibrium model [18]. Assume there are n companies that manufacture the same commodity. Let v represent a vector where each element v_i specifies the quantity of the commodity generated by the company i. The price function P for each individual company is defined as:

$$P_i(S) = \phi_i - \psi_i S, \quad \text{where } \phi_i > 0,\ \psi_i > 0, \text{ and } S = \sum_{i=1}^{m} v_i.$$

The function of income $F_i(v)$ is given by:

$$F_i(x) = P_i(S)v_i - t_i(v_i),$$

where $t_i(v_i)$ is the value tax and fee for producing x_i.
The strategy framework is given by:

$$\mathcal{C} := \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n, \quad \text{where } \mathcal{C}_i = [v_i^{\min}, v_i^{\max}].$$

Each firm strives to achieve its optimum profit by taking into account the amount of demand based on the production of other companies.
A point $p^* \in \mathcal{C}$ is an equilibrium point of the model if

$$F_i(p^*) \geq F_i(p^*[v_i]), \quad \forall x_i \in \mathcal{C}_i,\ \forall i = 1, 2, \ldots, n,$$

where $p^*[v_i]$ is the vector obtained from p^* by replacing the i-th component with v_i.
Define

$$\varphi(v,t) := -\sum_{i=1}^{n} F_i(v[t_i]), \quad \text{and} \quad B(v,t) := \varphi(v,t) - \varphi(v,v),$$

and the problem becomes:

$$\text{Find } p^* \in \mathcal{C} \text{ such that } B(p^*, t) \geq 0, \quad \forall y \in \mathcal{C}.$$

Consider the bifunction $B : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ defined in the context of the Nash-Cournot equilibrium model as follows:

$$B(v,t) = \langle Pv + Qt + q, t - v\rangle, \quad \forall\, v, t \in \mathcal{S},$$

where $\mathcal{S} \subset \mathbb{R}^n$ is the feasible set, $q \in \mathbb{R}^n$ is a given vector, P and $Q \in \mathbb{R}^{n\times n}$ are matrices, where Q is symmetric positive semidefinite, and $Q - P$ is symmetric negative semidefinite, ensuring that B is monotone.

The two matrices P, Q are generated randomly (Generate two random orthogonal matrices O_1 and O_2 using the `RandOrthMat` function. Create diagonal matrices A_1 and A_2 with values within $[0, 2]$ and $[-2, 0]$, respectively. Define $B_1 = O_1 A_1 O_1^T$ (positive semi-definite) and $B_2 = O_2 A_2 O_2^T$ (negative semi-definite). Set $Q = B_1 + B_1^T$, $T = B_2 + B_2^T$, and $P = Q - T$. Randomly generate the vector q with elements in the range $[-1, 1]$). We use the same stopping rule $D_m = \|t_m - v_m\|^2 \leq 10^{-6}$. In the numerical results presented in the following tables, 'Iter.' represents the number of iterations, while 'CPU(s)' denotes the execution time in seconds. The set $\mathcal{S}$ is given by:

$$\mathcal{S} = \{v \in \mathbb{R}^n : -5 \leq v_i \leq 5, \quad i = 1, 2, \ldots, n\}.$$

The optimization subproblems in these examples have been solved by FMINCON optimization toolbox in MATLAB software. In all experiments, we selected the parameters for Algorithm 3.1 as follows: $\varrho_0 = 0.1, \zeta = 0.1$, $\tau = 4.5$, $\omega_m = \dfrac{1}{20\,(m+1)^{1.1}}$ and $\sigma_m = \dfrac{1}{(m+1)^3}$. To investigate the sensitivity of the proposed algorithm to parameter choices, we performed a series of experiments varying the value of μ. The results, illustrated in Fig. 1, show how μ influences the convergence behavior and computational performance of the algorithm. Furthermore, the performance of Algorithm 3.1 was evaluated for different Bregman distances and different values n $(60, 120, 180, 240)$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be defined by

i $\varphi(v) := -\sum_{i=1}^{n} v_i \log(v_i)$.

ii $\varphi(v) := \frac{1}{2}\|v\|^2$.

iii $\varphi(v) := \sum_{i=1}^{n} \log(v_i)$.

Additionally, we establish that the corresponding Bregman distances can be expressed as

i $D_\varphi(v, t) := \sum_{i=1}^{n} \left(v_i \log\left(\frac{v_i}{t_i}\right) + t_i - v_i\right)$, which is called the Kullback-Leibler distance(shortly denoted by KLD);

ii $D_\varphi(v, t) := \frac{1}{2}\|v - t\|^2$, which is called the squared Euclidean distance(denoted by SED);

iii $D_\varphi(v, t) := \sum_{i=1}^{n} \left(\log\left(\frac{v_i}{t_i}\right) + \frac{v_i}{t_i} - 1\right)$, which is called Itakura-Saito distance (ISD).

The numerical results shown in Fig. 2 and Table 1 indicate that the proposed algorithm achieves superior performance when the Bregman distance is chosen as the Kullback-Leibler distance (KLD).

Table 1. Comparison of iterations and CPU time for different dim.

n	**KLD**		**SED**		**ISD**	
	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)
60	12	3.78	31	1.79	82	28.01
120	10	2.77	29	2.13	98	29.02
180	9	3.01	31	4.36	113	45.79
240	11	4.87	33	7.73	128	70.71

Finally, the Algorithm 3.1 (shortly Alg. 1), was compared with the explicit extragradient Algorithm suggested by Hieu et al. [7] (shortly, EEG Alg), the improved extragradient Algorithm introduced by Zeghad et al. [22] (shortly, IISE Alg) and the Bregman explicit extragradient Algorithm proposed by Eskandani et al. [5] (shortly, BEEG Alg) to assess its efficiency and effectiveness. The control parameters of all algorithms are choose as follows:

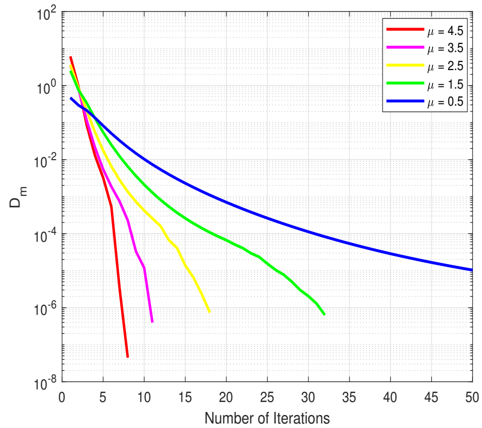

Figure 1. Numerical behavior of Algorithm 3.1 with $\mu \in \{0.5, 1.5, 2.5, 3.5, 4.5\}$.

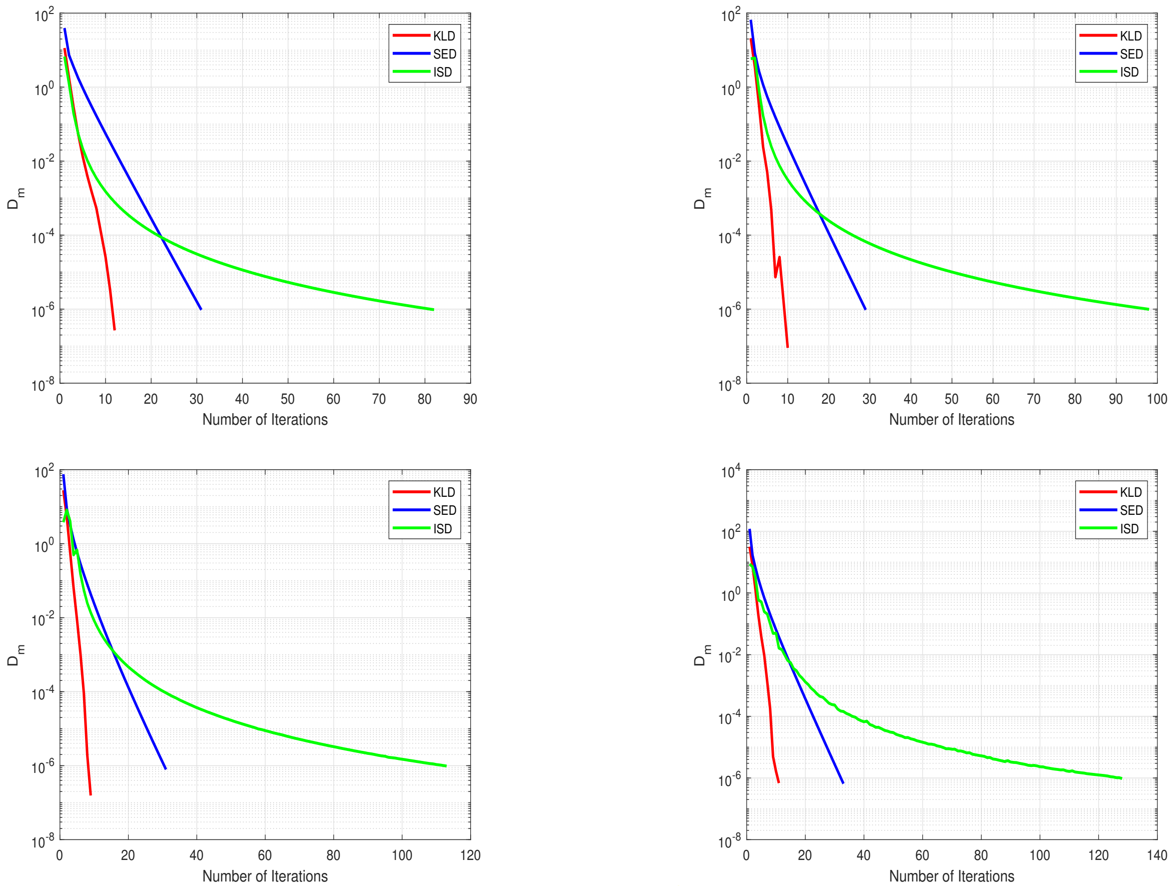

Figure 2. Example 4.1, Top Left: n = 60; Top Right: n= 120, Bottom Left: n = 180; Bottom Right: n = 240.

- Alg. 1: $\varphi(v) = -\sum_{i=1}^{n} v_i \log(v_i)$.

- EEG Alg : $\varrho_0 = 0.1$, $\zeta = 0.1$.
- IISE Alg : $\varrho_0 = 0.1$, $\zeta = 0.1$, $\mu = 0.7$, $\gamma = 0.2$, $\omega_m = \dfrac{1}{20\,(m+1)^{1.1}}$, $\sigma_m = \dfrac{1}{(m+1)^3}$.
- BEEG Alg : $\varrho_0 = 0.1$, $\zeta = 0.1$ and $\varphi(v) = -\sum_{i=1}^{n} v_i \log(v_i)$.

We test the algorithms for different values of n $(n = 50, 100, 150, 200, 500)$. The numerical results for all algorithms are presented in Fig. 3 and Table 2. It can be observed that our algorithm (Algorithm 3.1) outperforms EEG Alg, IISE Alg and BEEG Alg in terms of the number of iterations (Iter.) and execution time in seconds (CPU(s)), while achieving the same tolerance.

Table 2. Comparison of iterations and CPU time for different dim.

n	Alg. 1		EEG Alg		IISE Alg		BEEG Alg	
	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)
50	11	2.45	33	0.57	21	0.42	75	13.92
100	9	2.74	41	1.80	33	1.66	85	26.09
150	8	3.13	50	4.57	36	3.60	80	31.83
200	7	3.19	43	7.03	33	6.77	91	43.72

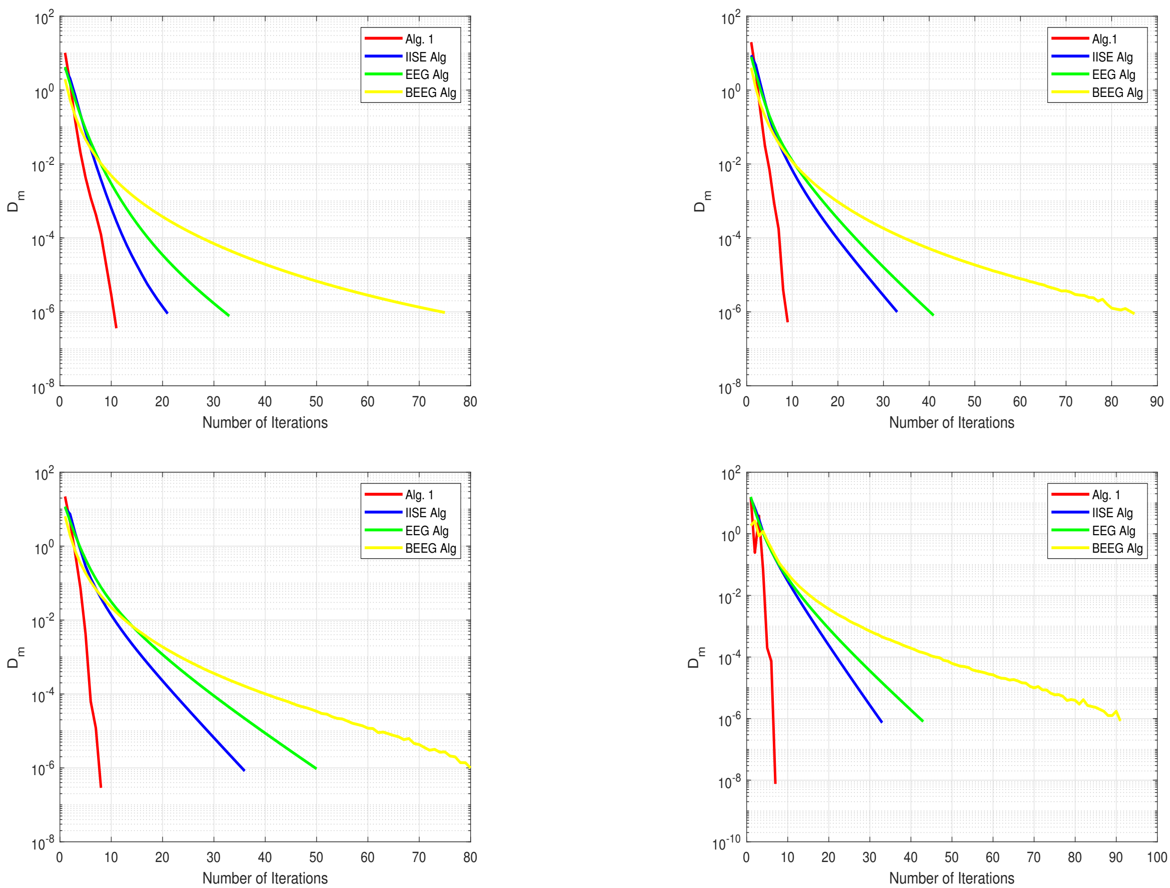

Figure 3. Example 4.1 , Top Left: n = 50; Top Right: n= 100, Bottom Left: n = 150; Bottom Right: n = 200.

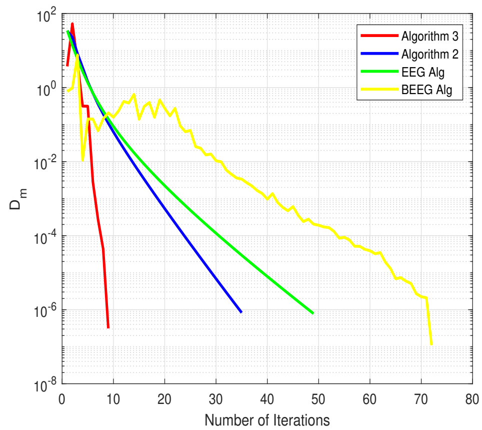

Figure 4. Numerical behavior of all Algorithms $n = 500$.

5. Conclusions

This study proposed a modified Bregman extragradient algorithm for solving pseudomonotone equilibrium problems in reflexive Banach spaces. By introducing two parameters, the Bregman distance, and a non-monotonic step size independent of the Lipschitz constant, the algorithm demonstrated enhanced convergence and computational efficiency. Theoretical analysis established both weak and strong convergence under specific conditions, while numerical experiments validated its superior performance compared to traditional methods.

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