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Some Properties of the Central Local Metric Dimension on Corona Product Graphs

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Abstract The central local metric dimension is a concept where a local metric set contains all central vertices. This concept was introduced in 2023. This concept is related to distance in a graph. In real life, there are so many applications of local metric dimension and central vertices. If a vital object is represented as a central vertex in a graph, its placement can utilize the concept of a central vertex, allowing people to easily reach it. Suppose the vital objects are health services, education facilities, train stations, and water stations. The government can use the central local metric dimension concepts to optimize transportation infrastructure management and create good transportation governance for these vital objects. Let G be a connected graph with vertex set $V(G)$ and order n. A central vertex in G is a vertex with the shortest distance to any other vertex in G, and all central vertices in G are represented in a set that is a central set, denoted by $S(G)$. Let W be a local metric set of G, if $S(G) \subseteq W$, then W is a central local metric set of G. The cardinality of the central local metric set with minimal cardinality is called the central local metric dimension of G. This paper presents some properties of the central local metric dimension of $G \odot H$. The results show that the elements of the central set of $G \odot H$ are vertices in $V(G \odot H)$ that correspond to the central set of G. Since in $G \odot H$, for all $x_{0i}, y_j^i \in V(G \odot H)$ applies $d(x_{0i}, y_j^i) = 1$, then there is no intersection between the central set and the local basis set of it.

Keywords Accessible transportation, central vertex, distance, eccentricity, optimization, properties, vital object

AMS 2010 subject classifications 05C12

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1. Introduction

Graph theory is a branch of discrete mathematics that has undergone rapid development in recent years. Let G be a graph with a vertex set $V(G)$, an edge set $E(G)$, and an order n. If there are vertices u and v in G so that u is adjacent to v, then we denote it by $u \sim v$ or simply $uv \in E(G)$. The distance between those two vertices is $d(u, v)$ [1]. The sum of all vertices that are connected to a vertex v is called the degree of v or $deg(v)$. The maximal degree of any vertex in a graph G is denoted by $\Delta(G)$, and the minimal degree of any vertex in a graph G is denoted by $\delta(G)$. Lemma 1.1 describes the degree limitation of a vertex in a graph G.

Lemma 1.1
If x is a vertex in a graph G with order n, then $0 \leq \delta(G) \leq deg(x) \leq \Delta(G) \leq n - 1$ [2].

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The eccentricity of the vertex $u \in V(G)$, denoted by $e(v)$, is the largest $d(u,v)$ for all $v \in V(G)$, and the radius of G, denoted by $rad(G)$, is the smallest eccentricity of all vertices in G. A vertex $u \in V(G)$ is called a central vertex of G if $e(u) = rad(G)$. A central set of G, denoted by $S(G)$, is a set whose its elements are all the central vertices of G or $S(G) = \{s|e(s) = rad(G), s \in V(G)\}$ [3].

Let W be a subset of $V(G)$, $W = \{w_1, w_2, \ldots, w_k\}$, where $k \leq n$, and the metric code of a vertex $x \in V(G)$ with respect to W is the k-vector $r(u|W) = (d(u,w_1), d(u,w_2), ..., d(u,w_k))$, for all $u \in V(G)$. W is called a local metric set if every pair of $u \sim v$ in G has a distinct metric code with respect to W, that is, $r(u|W) \neq r(v|W)$. The local metric set with minimum cardinality is called the local basis set of G, and its cardinality is called the local metric dimension of G, denoted by $lmd(G)$ [4]. Okamoto et al. in [4] presented a characterization of all non-trivial connected graphs of order n having local metric dimension 1, $n-1$, or $n-2$ and gave bounds for the local metric dimension of a graph. Some essential properties of a local metric set supporting our main results are summarized in the following lemma.

Lemma 1.2
Given a connected graph G and $U \subseteq V(G)$, if there is a subset of U that is a local metric set of G, then U is also a local metric set [4].

Lemma 1.3
Let G be a connected graph. If $W \subseteq V(G)$, then for every $v_i, v_j \in W$ with $i \neq j$, $r(v_i|W) \neq r(v_j|W)$ [1].

Listiana et al. in [3] introduced the concept of the central local metric dimension of a graph as shown in Definition 1.1. The boundaries for the central local metric dimension of a graph G and some properties of the central local metric dimension of some graphs with the same diameter and radius are also presented in [3], as shown in Theorem 1.1. In [2], Listiana et al. found that a central vertex of $K_1 + H$ is a single vertex in K_1, so we have two possibilities of $lmd_s(K_1 + H)$. In addition, those results are presented in Theorem 1.2 and Theorem 1.3.

Definition 1.1
Let $W \subseteq V(G)$ be a local metric set of G. If the central set $S(G) \subseteq W$, then W is called a central local metric set, and the minimum cardinality among the central local metric sets of G is called the central local metric dimension of G, denoted by $lmd_s(G)$.

Theorem 1.1
Let G be a connected graph with order n. If W is a local metric set of G, then:

a) $max\{|S(G)|, lmd(G)\} \leq lmd_s(G) \leq min\{|V(G)|, |S(G) \cup W|\}$
b) If $diam(G) = rad(G)$ if and only if $S(G) = V(G)$
c) If $S(G) = V(G)$ then $S(G)$ is a central local metric set of G
d) $lmd_s(G) = n$ if and only if $diam(G) = rad(G)$

Theorem 1.2
Let $G \cong K_1 + H$ with $V(K_1) = \{c\}$ and $|V(H)| = m$. The central set G is $S(G) = \{c\}$ if and only if no vertex in H has degree $m - 1$.

Theorem 1.3
Let $S(G)$ and W be a central set and local basis set of $G \cong K_1 + H$, respectively. If there is no vertex in H has degree $|V(H)| - 1$, then $lmd_s(G) = \begin{cases} lmd(G) & \text{if } S(G) \subset W \\ lmd(G) + 1 & \text{if } S(G) \cap W = \emptyset \end{cases}$.

For graph $K_1 + H$, let H be a graph nP_2, then $K_1 + nP_2 \cong f_n$. Based on Theorem 1.2, graph f_n has a central set is $S(f_n) = c$. Let $U = \{x_1, x_2, x_3\} \subset V(f_n)$ be a local basis set of f_3, then $S(G) \cap U = \emptyset$. By Theorem 1.3, $lmd_s(f_3) = |S(f_3)| + lmd(f_3)$. Let $W = \{c, x_1, x_2, x_3\} \subset V(G)$, since $U \subset W$, by Lemma 1.2, W is also a local metric set of f_n. W contains a central vertex c or $S(f_3) \subset W$. So, W is a central local basis set of f_3 and $lmd_s(f_3) = 4$. Figure 1(a) illustrates the central local metric dimension of f_3 with the central local basis set $W = \{c, x_1, x_2, x_3\}$. In contrast, Figure 1(b) illustrates the local metric dimension of f_3 with local basis set $U = \{x_1, x_2, x_3\}$. It shows that the f_n has a central point that is not included in the local basis set.

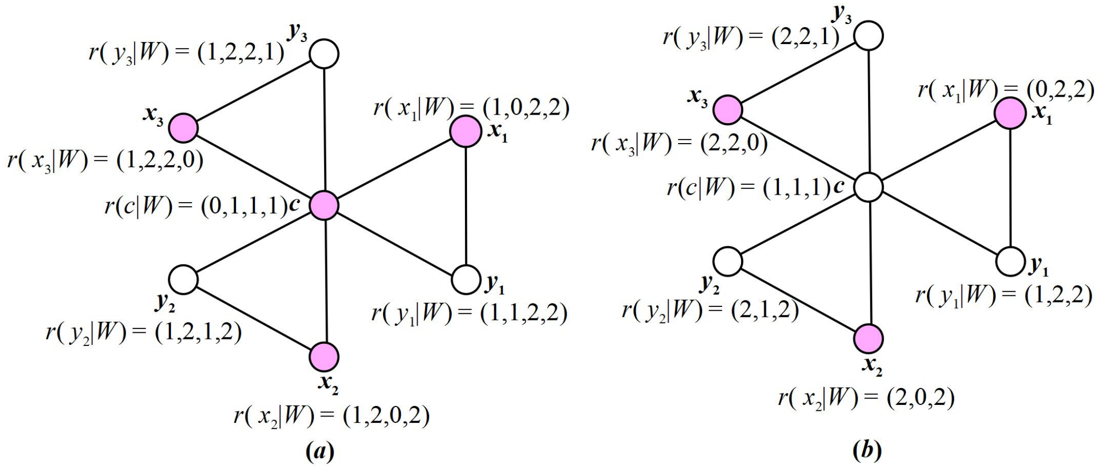

Figure 1. (a) The central local metric dimension of f_3 and (b) The local metric dimension of f_3

The central vertex and local metric set are two concepts related to distance. Thus, in real life, the concept of central local metric dimension can be used to support the government in solving transportation management problems involving vital objects, such as the placement of train stations that are easily accessible to the public and can recognize two nearby locations. Another example is the optimization of the placement of a health center, water stations, education facilities, and a disaster command post.

Susilowati et al. in [5] explored the central metric dimension of the edge coronation graph. $W \subseteq V(G)$ is a resolver set of G if every u and v in $V(G)$ has a distinct metric code with respect to W. Then, the minimal cardinality of a resolver set in G is called the central metric dimension of G, denoted by $dim_{cen}(G)$. They showed some characteristics of central metric dimension, such as $dim_{cen}(G) = 1$ if and only if $G \cong K_1$ and $dim_{cen}(G) = |V(G)| - 1$ if and only if $G \cong K_{1,n}$.

A vertex x in a graph G with order n is called a dominant vertex if $deg(x) = n - 1$ [6]. So, based on Theorem 1.2, a graph $K_1 + H$ has a single central vertex if H has only one dominant vertex. Graph friendship f_3 in Figure 1 is an example of a graph with a dominant vertex, since vertex c is adjacent to all other vertices in f_3 or $deg(c) = 6$.

In this paper, we explored the central local metric dimension of $G \odot H$. The formal definition of $G \odot H$ refers to [7]. Some properties of the local metric dimension of $G \odot H$ can be seen in [8]. Theorem 1.4 and Theorem 1.5 are the result of the local metric dimension of $G \odot H$ that also refers to [8]. This theorem will support the proof of our next theorem.

Theorem 1.4
Let G be a connected graph with order n and H is non-empty set, then

a) if a vertex of K_1 is not element of basic local set for $K_1 + H$, then $lmd(G \odot H) = n(lmd(K_1 + H))$
b) if a vertex of K_1 is an element of basic local set for $K_1 + H$, then for $n \geq 2$, $lmd(G \odot H) = n(lmd(K_1 + H) - 1)$

Theorem 1.5
For any graph H with $diam(H) = 2$ and any connected graph with order $n \geq 2$, then $lmd(G \odot H) = n(lmd(H))$.

2. Main Results

Let G be a connected graph with order n and H be a graph with order m. The corona product graph $G \odot H$ is a connected graph obtained by taking one copy of graph G and n copies of graph H, where the i-th vertex in graph G is connected to all vertices in the i-th copy of graph H, for $1 \leq i \leq n$. Furthermore, the i-th copy of the graph H on $G \odot H$ is called H_i.

Let $V(G) = \{x_1, x_2, x_3, \ldots, x_n\}$ be the vertex set of G and $V(H) = \{y_1, y_2, y_3, \ldots, y_m\}$ be the vertex set of H, then $V(G \odot H)$ can be defined as $V(G \odot H) = V(G_0) \cup \{y_j^i | 1 \le i \le n \text{ and } 1 \le j \le m\}$, where $V(G_0) = \{x_{0i} \in V(G \odot H) | x_i \in V(G)\}$. Given a graph P_4, with $V(P_4) = \{x_1, x_2, x_3, x_4\}$, and a graph $\overline{K_3}$, with $V(\overline{K_3}) = \{y_1, y_2, y_3\}$, as illustrated in Figure 2(a). Then, we can see the illustration of graph $P_4 \odot \overline{K_3}$ in Figure 2(b), which the vertex set of $P_4 \odot \overline{K_3}$ is $V(P_4 \odot \overline{K_3}) = \{x_{01}, x_{02}, x_{03}, x_{04}\} \cup \{y_j^i | 1 \le i \le 4, 1 \le j \le 3\}$.

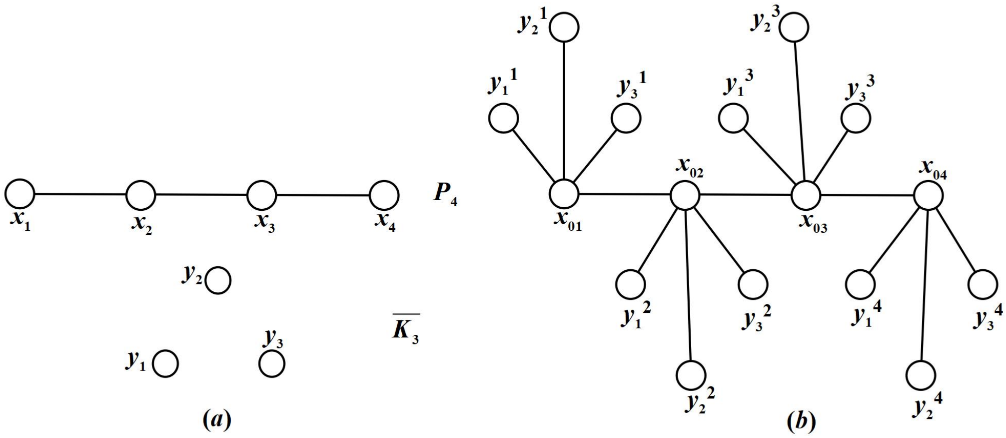

Figure 2. (a) Graph P_4 and $\overline{K_3}$ (b) Graph $P_4 \odot \overline{K_3}$

Figure 3 is another example of graph $G \odot H$, where $G = C_4$ and $H = C_5$. Let $V(C_4) = \{x_1, x_2, x_3, x_4\}$ be a vertex set of C_4 and $V(C_5) = \{y_1, y_2, y_3, y_4, y_5\}$ be a vertex set of C_5, then the vertex set of $C_4 \odot C_5$ is $V(C_4 \odot C_5) = \{x_{01}, x_{02}, x_{03}, x_{04}\} \cup \{y_j^i | 1 \le i \le 4, 1 \le j \le 5\}$.

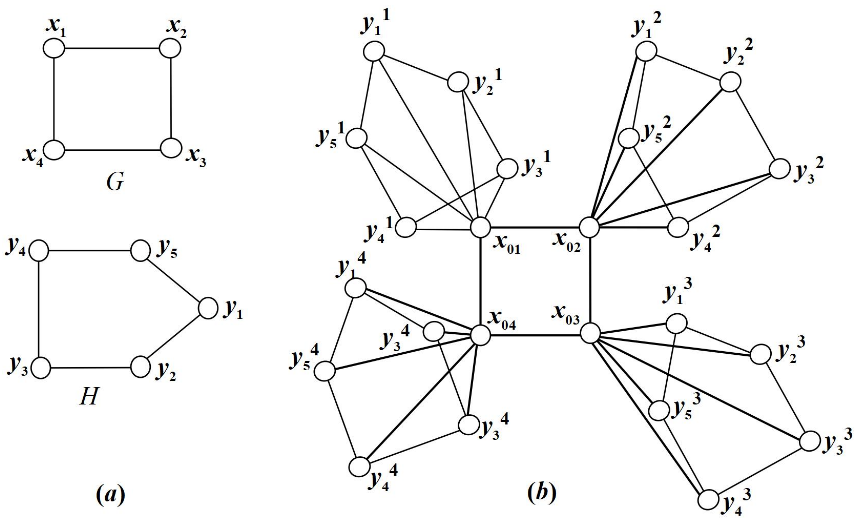

Figure 3. (a) Graph C_4 and C_5 (b) Graph $C_4 \odot C_5$

Lemma 2.1 relates to the diameter and radius of $G \odot H$. We also emphasise that in this paper, we use a connected graph G of order n, while graph H is an arbitrary graph of order m, if not otherwise stated.

Lemma 2.1
If G is a connected graph and H is any graph, then $rad\,(G \odot H) = rad\,(G) + 1$ and $diam\,(G \odot H) = diam\,(G) + 2$.

Proof. Given that G is a connected graph and H is any graph. To prove the Lemma, we divided it into two steps as follows:

- Take a vertex u in $V(G)$ so that u is a central vertex of G. Let $u = x_k$, since u is a central vertex of G, then the eccentricity of x_k is $e(x_k) = rad(G)$. In $G \odot H$, each vertex of H_i is connected to the vertex $x_{0i} \in V(G \odot H)$, where x_i is a vertex in $V(G)$, so that $d(x_{0i}, y_j^i) = 1, \forall y_j^i \in V(G \odot H)$. Take $i = k$, then the eccentricity of x_{0k} is $e(x_{0k}) = e(x_k) + 1$. Since $e(x_k) = rad(G)$, then $e(x_{0k}) = rad(G) + 1 = rad(G \odot H)$. Consequently, x_{0k} is a central vertex of $G \odot H$ and $rad(G \odot H) = rad(G) + 1$.
- In the same way, take vertex u and v in G, so that $d(u, v) = diam(G)$. Let $u = x_k$ and $v = x_l$, then $d(x_k, x_l) = diam(G)$. Now, take $y_j^k, y_j^l \in V(G \odot H)$, so that there are x_{0k} and x_{0l} in $V(G \odot H)$ with $d(x_{0k}, x_{0l}) = diam(G)$. It is easy to see that $d(x_{0k}, y_j^k) = 1$ and $d(x_{0l}, y_j^l) = 1$. Consequently, $d(y_j^k, y_j^l) = d(y_j^k, x_{0k}) + d(x_{0k}, x_{0l}) + d(x_{0l}, y_j^l) = 1 + d(x_{0k}, x_{0l}) + 1 = d(x_{0k}, x_{0l}) + 2 = diam(G \odot H)$. Since $d(x_{0k}, x_{0l}) = diam(G)$, then $d(y_j^k, y_j^l) = diam(G) + 2$. Thus $diam(G \odot H) = diam(G) + 2$.

It is proved that $diam(G \odot H) = diam(G) + 2$ and $rad(G \odot H) = rad(G) + 1$. □

In $G \odot H$, there is a subgraph $\langle x_i \rangle + H_i \cong K_1 + H$. Let $H = K_m$, then $K_1 + K_m \cong K_{1+m}$. It is easy to see that $diam(K_{1+m}) = rad(K_{1+m}) = 1$ and by Theorem 1.1, $lmd_s(K_1 + K_m) = |V(K_{1+m})| = 1 + m$. Lemma 2.2 depicts about the diameter of $K_1 + H$ when $H \neq K_m$.

Lemma 2.2
Let $x_0 \in V(K_1)$ and H is a connected graph with order m:

a) If $\forall y \in V(H), deg(y) \neq m - 1$, then $diam(K_1 + H) = 2$
b) If $\exists y \in V(H), deg(y) = m - 1$ then $diam(K_1 + H) = diam(H) = 2$.

Proof. Given a trivial graph K_1 with $x_0 \in V(K_1)$, and H is a connected graph of order m. Then $K_1 + H$ is defined as a graph with vertex set $V(K_1 + H) = \{x_{01} | x_0 \in V(K_1)\} \cup \{y_j^1 | y_j \in V(H), 1 \leq j \leq m\}$. So, two conditions in graph H must be considered.

a) When $\forall y \in V(H)$, $deg(y) \neq m - 1$. Take $x_{01} \in V(K_1 + H)$, then x_{01} is connected to $\forall y_j^1 \in V(K_1 + H)$, $1 \leq j \leq m$. Based on Lemma 1.1, x_{01} has a maximum degree of $K_1 + H$, which means that vertex x_{01} is connected to all other vertices in $K_1 + H$. Consequently, $d(x_{01}, y_j^1) = 1$, $\forall y_j^1 \in V(K_1 + H)$ and the eccentricity of x_{01} is $e(x_{01}) = 1$. Take any two vertices y_s and y_t in H, where $y_s \nsim y_t$. Since in graph H there is no vertex y with $deg(y) = m - 1$, the eccentricity of y_j^1 is $e(y_j^1) = 2$, for $1 \leq j \leq m$. Thus, $diam(K_1 + H) = e(y_j^1) = 2$. It is proved that $diam(K_1 + H) = 2$.
b) When $\exists y \in V(H)$, $deg(y) = m - 1$. Take $y = y_s$, so y_s is connected to all other vertices in graph H and the eccentricity of y_s is $e(y_s) = 1$. Take any two vertices y_k and y_l in H, where $y_k \nsim y_l$, then the eccentricity of y_k and y_l are $e(y_k) = 2$ and $e(y_l) = 2$. Consequently, $diam(H) = e(y_k) = e(y_l) = 2$. Further, for $x_{01} \in V(K_1 + H)$, where $x_0 \in V(K_1)$, we have the eccentricity of x_{01} is $e(x_{01}) = e(y_s^1) = 1$. While, the eccentricity of y_k^1 and y_l^1 are $e(y_k^1) = e(y_l^1) = 2$, for $1 \leq j \leq m$. Thus, $diam(K_1 + H) = 2$. It is proved that $diam(K_1 + H) = diam(H) = 2$.

□

Now, we discuss the central local metric dimension of $G \odot H$. The following are some properties related to the central set on the graph $G \odot H$ and a lower bound on the central local metric dimension on the corona graph $G \odot H$.

Lemma 2.3
Let $S(G) = \{s_1, s_2, ..., s_p\}$ be a central set of G with $p \leq n$, then a central set of $G \odot H$ be $S(G \odot H) = \{s_{0k} \in V(G \odot H) | s_k \in S(G)\}$.

Proof. Given graph $G \odot H$ with vertex set $V(G \odot H) = V(G_0) \cup \{y_j^i | 1 \leq i \leq n, 1 \leq j \leq m\}$, where $V(G_0) = \{x_{0i} \in V(G \odot H) | x_i \in V(G)\}$. Let $S(G) = \{s_1, s_2, ..., s_p\}$ be a central set of G with $p \leq n$. Take $s_k \in S(G)$, then $e(s_k) = rad(G)$, $\forall s_k \in S(G)$. Since $S(G) \subseteq V(G)$, then $s_k \in S(G)$ and $s_k \in V(G)$. Let $s_k = x_k$, then for every $x_k \in V(G)$ exist $x_{0k} \in V(G \odot H)$ so that based on Lemma 2.1, $e(x_{0k}) = rad(G) + 1 = rad(G \odot H)$. Thus, x_{0k} is a central vertex of $G \odot H$, $\forall x_k = s_k \in S(G)$. Let $x_{0k} = s_{0k} \in S(G \odot H)$, then central set of $G \odot H$ be $S(G \odot H) = \{s_{0k} \in V(G \odot H) | s_k \in S(G)\}$. □

In $G \odot H$, a vertex $x_{0i} \in V(G \odot H)$ is adjacent to every vertex in H_i. Consequently, $d(x_{0i}, y_j^i) = 1$. So, the vertex x_{0i} cannot distinguish any two adjacent vertices in H_i. Let W be the local basis set of graph $G \odot H$, then $x_{0i} \notin W$. We have lemma 2.4 inspired by this condition to facilitate the proof of the next theorem.

Lemma 2.4
Let W be a local basis set of $G \odot H$, then $S(G) \cap W = \emptyset$.

Proof. Let W be a local basis set of $G \odot H$, so $lmd(G \odot H) = |W|$. From Lemma 2.3, $S(G \odot H) = \{s_{0k} \in V(G \odot H) | s_k \in S(G)\}$. Suppose that $W \cap S(G \odot H)$ is non-empty set, that is, there exist $x_{0k} \in W$ and $x_{0k} \in S(G \odot H)$ such that for any two adjacent vertices $u, v \in V(G \odot H)$ it follows that $d(u, x_{0k}) \neq d(v, x_{0k})$. Let $u = y_j^k, v = y_l^k \in V(H_i)$, where $j \neq l$, then vertices u and v are adjacent to x_{0i}. Consequently, $d(u, x_{0i}) = d(v, x_{0i}) = 1$. It is contradiction with $d(u, x_{0i}) \neq d(v, x_{0i})$. Thus, $W \cap S(G \odot H)$ is an empty set. □

Based on Definition 1.1, let W be a local metric set of a connected graph G and $S(G) \subseteq W$, then W is a central local metric set of G. Whereas, in Lemma 2.4, a central set of $G \odot H$ is disjoint from its local basis set. Then, we obtain Corollary 2.1 and Corollary 2.2 as a consequence of Definition 1.1 and Lemma 2.4.

Corollary 2.1
$lmd_s(G \odot H) > lmd(G \odot H)$.

Corollary 2.2
The central local metric dimension of $G \odot H$ is $lmd_s(G \odot H) = |S(G \odot H)| + lmd(G \odot H)$.

Corollary 2.1 and Corollary 2.2 show the relationship between the central local metric dimension of $G \odot H$, $lmd_s(G \odot H)$, and the local metric dimension of $G \odot H$, $lmd(G \odot H)$. They show that the central local metric dimension of $G \odot H$ is greater than its local metric dimension. Based on Lemma 2.4, the central sets of the graph $G \odot H$ are disjoint from the local basis set. Thus, the central local metric dimension of the graph $G \odot H$ is the sum of its central set's cardinality and its local metric dimension.

Let H be an empty graph $\overline{K_m}$, then $G \odot \overline{K_m}$ applies $deg(y_j^i) = 1$, $\forall y_j^i \in V(H_i)$, $1 \leq i \leq n$ and $1 \leq j \leq m$. Theorem 2.1 discussed a central local metric dimension of $G \odot H$ when H is an empty graph.

Theorem 2.1
Let G be a connected graph with order n. If $H = \overline{K_m}$, then $lmd_s(G \odot H) = lmd_s(G)$

Proof. Let W_1 be a central local basis set of G. By Lemma 2.3, we have $S(G \odot H) = \{s_{0k} \in V(G \odot H) | s_k \in S(G)\}$, then $\forall s_{0k} \in S(G \odot H)$, exist $s_k \in S(G)$, so that $s_k \in W_1$. It is easy to see that $|S(G \odot H| = |S(G)|$. Let W_2 be a central local basis set of $G \odot H$, then $\forall s_{0k} \in S(G \odot H), s_{0k} \in W_2$. Thus, $S(G \odot H) \subseteq W_2$. Take $W_2 = \{w_{0l} \in V(G \odot H) | w_l \in W_1\}$, so $|W_2| = |W_1| = lmd_s(G)$. Take any two vertices $y_j^i, y_{j+1}^i \in V(H_i)$ and $x_{0i} = w_{0i} \in W_2$, then $d(y_j^i, w_{0i}) = d(y_{j+1}^i, w_{0i}) = 1$. Similarly, if the vertex $x_{0l} = w_{0l} \in W_2$ is taken, then $d(y_j^i, w_{0l}) = 1 + d(w_{0i}, w_{0l}) = d(y_{j+1}^i, w_{0l})$. Consequently, $r(y_j^i | W) = r(y_{j+1}^i | W)$. Since $H = \overline{K_m}$ is an empty graph, then y_j^i and y_{j+1}^i are not adjacent, $\forall y_i^i, y_{j+1}^i \in V(H_i)$, $1 \leq i \leq n$. Thus, W_2 is a central local basis set of $G \odot H$. Next, we prove that W_2 is the central local basis set. Take U where $S(G \odot H) \subseteq U \subseteq V(G \odot H)$ with $|U| < |W_2|$. Let $|U| = |W_2| - 1$, then there are two adjacent vertices x_{0s} and x_{0t} where $x_s \vee x_t \notin W_1$ such that $r(x_{os} | W_2) = r(x_{ot} | W_2)$. Consequently, U is not the central local metric set of $G \odot H$. So W_2 is the central local metric set on $G \odot H$ with minimal cardinality. In other words, W_2 is the central local basis set of $G \odot H$. Hence $lmd_s(G \odot H) = |W_2| = |W_1| = lmd_s(G)$. □

By Theorem 2.1, for $G = C_n$, we get $lmd_s(C_n \odot \overline{K_m}) = lmd_s(C_n) = n$, for $n \geq 3$ and $m \geq 2$ as in [9]. Figure 4 is an illustration of the central local metric dimension of $P_4 \odot \overline{K_3}$. Graph P_4 is a graph with $S(P_4) = \{x_2, x_3\}$, which for every two adjacent vertices u, v in P_4 applies $r(u | S(P_4)) \neq r(v | S(P_4))$, so $lmd_s(P_4) = 2$. Based on Lemma 2.4 we have $S(P_4 \odot \overline{K_3}) = \{x_{02}, x_{03}\}$ and based on Theorem 2.1, $lmd_s(P_4 \odot \overline{K_3}) = lmd_s(P_4) = 2$.

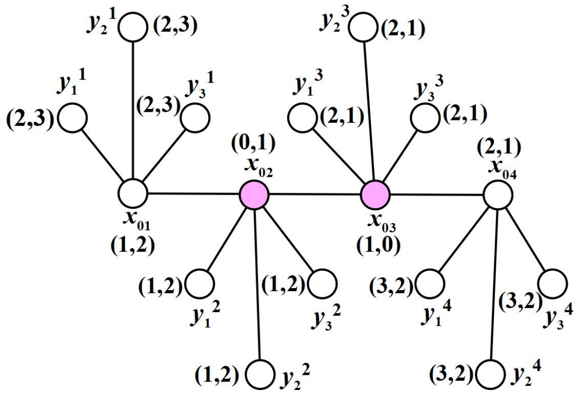

Figure 4. The illustration of the central local metric dimension of $P_4 \odot \overline{K_3}$

Next, Theorem 2.2 and Theorem 2.3 discussed the central local metric dimension of $G \odot H$ when H is not an empty graph.

Theorem 2.2
Let G be a connected graph with order n and H be a graph with no dominant vertex, then

$$lmd_s(G \odot H) = |S(G \odot H)| + n(lmd(K_1 + H) - \alpha_i)$$

for $\alpha_i = 1$ when x_{0i} is an element of a local basis set of $K_1 + H_i$, or $\alpha_i = 0$ when x_{0i} is not an element of a local basis set of $K_1 + H_i$.

Proof. Graph H is a graph without a dominant vertex, it is means that $\forall y \in V(H)$, $deg \neq m - 1$. So, there is a subgraph $\langle x_{0i} \rangle + H_i \cong K_1 + H$ in $G \odot H$ and a vertex x_{0i} may be an element of the local basis set of $\langle x_{0i} \rangle + H_i$. We know that every x_{0i} in $G \odot H$ can not distinguish each vertex in H_i. By Theorem 1.4 and Corollary 2.2, we have $lmd_s(G \odot H) = |S(G \odot H)| + n(lmd(K_1 + H) - 1)$ when x_{0i} is an element of $\langle x_i \rangle + H_i$ or $lmd_s(G \odot H) = |S(G \odot H)| + n(lmd(K_1 + H))$ when x_{0i} is not an element of $\langle x_i \rangle + H_i$. Consequently, $lmd_s(G \odot H) = |S(G \odot H)| + n(lmd(K_1 + H) - \alpha_i)$ for $\alpha_i = 1$ when x_{0i} is an element of a local basis set of $K_1 + H_i$, or $\alpha_i = 0$ when x_{0i} is not an element of a local basis set of $K_1 + H_i$. □

Theorem 2.3
Let G and H be connected graphs with orders n and m, respectively. If $\exists y \in V(H)$, $deg(y) = m - 1$, then

$$lmd_s(G \odot H) = |S(G \odot H)| + n(lmd(H))$$

Proof. Since $\exists y \in V(H)$, $deg(y) = m - 1$, based on Lemma 2.2, $diam(H) = 2$. Then, from Corollary 2.2 and Theorem 1.5, we have $lmd_s(G \odot H) = |S(G \odot H)| + n(lmd(G \odot H))$. It is proved that $lmd_s(G \odot H) = |S(G \odot H)| + n(lmd(H))$. □

In Theorem 1.1, it is stated that for any connected graph G with order n, the central set $S(G) = V(G)$ if and only if $diam(G) = rad(G)$. That is, all vertices x_i in the graph G are central vertices. Then, from Theorem 1.2, it is stated that for a graph H with order m where $\forall y \in V(H)$, $deg(y) \neq m - 1$, the central vertex of the graph $K_1 + H$ is a single vertex in K_1. From these two conditions, Theorem 2.4 is as follows.

Theorem 2.4
Let G be a connected graph with order n and $diam(G) = rad(G)$. If there is H with $|V(H)| = m$ and $deg(y) \neq m - 1, \forall y \in V(H)$, then $lmd_s(G \odot H) = n(lmd_s(K_1 + H))$.

Proof. Let H_i be the $i - th$ copies of H in $G \odot H$. For every $\langle x_{0i} \rangle + H_i$, take $W_{0i} = \{w_{or} \in V(G \odot H) | w_r \in W_i\}$ as a central local basis set of $\langle x_{0i} \rangle + H_i$, where W_i is a central local basis set of $K_1 + H$. Since $\forall y \in V(H)$,

$deg(y) \neq m-1$, then based on Theorem 1.2, a central vertex of $K_1 + H$ is a single vertex in $V(K_1)$. For $\langle x_{0i} \rangle + H_i \cong K_1 + H$, take x_{0i} as a single central vertex in $\langle x_{0i} \rangle + H_i$, then $\forall x_{0i} \in V(G \odot H), x_{0i} \in W_{0i}$. Since $x_i \in W_i$, where $diam(G) = rad(G)$, by Theorem 1.1 we have $S(G) = V(G)$, so $\forall x_i \in V(G), x_i \in W_i, 1 \leq i \leq n$. Whereas in Lemma 2.3, a central set of $S(G \odot H) = \{s_{0k} \in V(G \odot H) | s_k \in S(G)\}$. Take $s_k = x_i$, it is applies $\forall x_i \in W_i$, there is $x_{0i} \in W_{0i}$, so that $x_{0i} \in S(G \odot H)$, $\forall x_{0i} \in V(G \odot H)$. Take $X = \bigcup_{i=1}^{n} W_{0i}$, since $x_i \in W_i$, for $1 \leq i \leq n$, then $S(G) \subset W_i$ and $S(G \odot H) \subset W_{0i}$. There are several possibilities for any two adjacent vertices u, v and v on the graph $G \odot H$ with respect to X as follows.

- $u, v \in V(H_i)$. Since each of u and v is adjacent to x_{0i} and W_{0i} is central local basis set of $\langle x_{0i} \rangle + H_i$, then there is $w_{0i} \in W_{0i} \backslash \{x_{0i}\}$, so that $d(u, w_{0i}) \neq d(v, w_{0i})$. Consequently, $r(u|X) \neq r(v|X)$.
- $u, v \in V(G_0)$. Since $x_{0k} \in W_{0i}$ and $x_{0l} \in W_{0i}$, then $x_{0k}, x_{0l} \in X$ and by Lemma 1.3, $r(u|X) \neq r(v|X)$.
- $u \in V(H_i)$ and $v \in V(G_0)$. Since for every $v = x_{0i} \in V(G_0)$ there is $x_{0i} \in W_{0i}$ so that $x_{0i} \in X$, then by Lemma 1.3, $r(u|X) \neq r(v|X)$.

Thus, X is the central local metric set of $G \odot H$. Further, it is shown that X is the minimal central local metric set. Take any U where $S(G \odot H) \subset W$ with $|U| < |X|$. Since $X = \bigcup_{i=n}^{n} W_{0i}$, where W_{0i} is a central local basis set of $\langle x_{0i} \rangle + H_i$, then there exists i such that at most as many as $|W_{0i}| - 1$ vertices in $\langle x_{0i} \rangle + H_i$ are element of U. Since W_{0i} is the central local basis of the graph $\langle x_{0i} \rangle + H_i$, there exist two adjacent vertices in $K_1 + H_i$ that have the same representation in U. Thus, U is not a central local metric set of $G \odot H$. Consequently, $X = \bigcup_{i=1}^{n} W_{0i}$ is the central local basis of $G \odot H$. Thus, it is proved that $lmd_s(G \odot H) = |X| = \sum_{i=1}^{n} (lmd_s(K_1 + H)) = n(lmd_s(K_1 + H))$. □

Let $\mathcal{H}$ be a sequence of n connected graph $H_1, H_2, ..., H_n$, where the order of H_i is m_i, $1 \leq i \leq n$, $m_i \geq 2$. Let $V(G_0) = \{x_{0i} \in V(G \odot \mathcal{H}) | x_i \in V(G)\}$ and $V(H_i) = \{y_j | 1 \leq j \leq m_i\}$, then the vertex set of $G \odot \mathcal{H}$ is $V(G \odot \mathcal{H}) = V(G_0) \cup \{y_j^i | 1 \leq i \leq n, 1 \leq j \leq m_i\}$. Hence, we obtain Lemma 2.5 about the central set in the graph $G \odot \mathcal{H}$.

Lemma 2.5
Let $S(G) = \{s_1, s_2, ..., s_p\}$ be a central set of G with $p \leq |V(G)|$, then a central set of $G \odot \mathcal{H}$ be $S(G \odot \mathcal{H}) = \{s_{0k} \in V(G \odot \mathcal{H}) | s_k \in S(G)\}$

Proof. Given graph $G \odot \mathcal{H}$ with vertex set $V(G \odot \mathcal{H}) = V(G_0) \cup \{y_j^i | 1 \leq i \leq n, 1 \leq j \leq m_i\}$, where $V(G_0) = \{x_{0i} \in V(G \odot \mathcal{H}) | x_i \in V(G)\}$. Let $S(G) = \{s_1, s_2, ..., s_p\}$ be a central set of G with $p \leq n$. Take $s_k \in S(G)$ then $e(s_k) = rad(G)$, $\forall s_k \in S(G)$. Since $S(G) \subseteq V(G)$, then $s_k \in S(G)$ and $s_k \in V(G)$. Let $s_k = x_k$, then for every $x_k \in V(G)$ exist $x_{0k} \in V(G \odot \mathcal{H})$ so that $e(x_{0k}) = e(x_k) + 1 = rad(G) + 1 = rad(G \odot \mathcal{H})$. Thus, x_{0k} is a central vertex of $G \odot \mathcal{H}$, $\forall x_k = s_k \in S(G)$. Let $x_{0k} = s_{0k} \in S(G \odot \mathcal{H})$, then the central set of $G \odot \mathcal{H}$ be $S(G \odot \mathcal{H}) = \{s_{0k} \in V(G \odot \mathcal{H}) | s_k \in S(G)\}$. □

Furthermore, the dimension of the central local metric on the graph $G \odot \mathcal{H}$ is discussed in Theorem 2.5. In this theorem, the term dominant vertex $v \in V(H)$ denotes a vertex adjacent to every other vertex in the graph H. Let H be a graph with $|V(H)| = m$, v be a dominant vertex in H if $deg(v) = m - 1$.

Theorem 2.5
Let G be a connected graph with order $n \geq 2$ and $\mathcal{H}$ be a sequence of connected graph $H_1, H_2, ..., H_n$, where the order of H_i is m_i, $1 \leq i \leq n$, $m_i \geq 2$. If there are k graph H_i contains the dominant vertex, then

$$lmd_s(G \odot \mathcal{H}) = |S(G \odot \mathcal{H})| + \sum_{i=1}^{k} (lmd(H_i)) + \sum_{i=n-k}^{n} (lmd(K_1 + H_i) - \alpha_i)$$

for $\alpha_i = 1$ when x_{0i} is an element of a local basis set of $K_1 + H_i$, or $\alpha_i = 0$ when x_{0i} is not an element of a local basis set of $K_1 + H_i$.

Proof. Let G be a connected graph with order $n \geq 2$ with the vertex set is $V(G) = \{x_i | 1 \leq i \leq n\}$ and $\mathcal{H}$ be the sequence of n connected graphs of $H_1, H_2, ..., H_n$. Graph $\langle x_i \rangle + H_i \cong K_1 + H_i$ for each $1 \leq i \leq n$, then

$lmd(\langle x_i \rangle + H_i) = lmd(K_1 + H_i)$. Since there are graphs H_i that contain the dominant vertex and there are $n-k$ graphs H_i without the dominant vertex, then there are two different local basis sets for $\langle x_i \rangle + H_i$. Let B_{0i} be the local basis set of $\langle x_i \rangle + H_i$ for H_i that contains the dominant vertex, and B'_{0i} be the local basis set of the graph $\langle x_i \rangle + H_i$ for H_i without the dominant vertex. Take $B_{0i} = \{b_{0r} \in V(G \odot \mathcal{H}) | b_r \in B_i\}$, where B_i is a local basis set of $K_1 + H$ with H contain the dominant vertex, and take $B'_{0i} = \{b'_{0r} \in V(G \odot \mathcal{H}) | b'_r \in B'_i\}$, where B'_i is a local basis set of $K_1 + H$ with H without the dominant vertex. It is important to note that when there is no dominant vertex in H_i, a vertex x_{0i} in $\langle x_{0i} \rangle + H_i$ is possible to be an element of B'_i. While in $G \odot \mathcal{H}$, vertex x_{0i} cannot distinguish any two adjacent vertices in H_i. Take $W = \bigcup_{i=1}^{k} B_{0i} \cup \bigcup_{i=n-k}^{n} (B'_{0i} - \{x_{0i}\})$. Then, there are several possibilities for any two adjacent vertices u and v on the graph $G \odot \mathcal{H}$ such that they have different representations in W.

- $u, v \in V(H_i)$. Vertex u and v, adjacent to x_{0i}, so $d(u, x_{0i}) = d(v, x_{0i}) = 1$. There are two conditions for this possibility. First, when u and v are two adjacent vertices in graph H_i that contains the dominant vertex, then there is $s \in B_i \backslash \{x_{0i}\}$ so that $d(u, s) \neq d(v, s)$. Consequently, $r(u|W) \neq r(v|W)$. Second, when u and v are two adjacent vertices in graph H_i without the dominant vertex, then there is $s \in B'_i \backslash \{x_{0i}\}$ so that $d(u, s) \neq d(v, s)$. Consequently, $r(u|W) \neq r(v|W)$.
- $u, v \in V(G_0)$. Take $u = x_{0p}$ and $v = x_{0r}$, then there are H_p and H_r in $G \odot \mathcal{H}$. Let H_p be a graph that contains the dominant vertex, and let H_r be a graph without the dominant vertex. Then for every $s \in B'_r \backslash \{x_{0r}\}$, $d(x_{0p}, s) = 1 + d(x_{0p}, x_{0r}) > d(x_{0r}, s)$. Similarly, $\forall s \in B_p \backslash \{x_{0p}\}$, $d(x_{0r}, s) = 1 + d(x_{0r}, x_{0p}) > d(x_{0p}, s)$. Consequently, $r(x_{op}|W) \neq r(x_{or}|W)$.
- $u \in V(H_i)$ and $v \in V(G_0)$. Take $v = x_{0i}$, so there are two conditions in this possibility. First, when $u = y_j^i$ is a vertex in H_i that contains the dominant vertex, then there is $s \in B_k \backslash \{x_{0k}\}$, $k \neq i$, so that $d(y_j^i, s) = 2 + d(x_{0i}, x_{0k}) > d(x_{0i}, s) = 1 + d(x_{0i}, x_{0k})$. Consequently, $r(y_j^i|W) \neq r(x_{0i}|W)$. Second, when $u = y_j^i$ is a vertex in H_i without the dominant vertex, then there is $s \in B'_k \backslash \{x_{0k}\}$, $k \neq i$, so that $d(y_j^i, s) = 2 + d(x_{0i}, x_{0k}) > d(x_{0i}, s) = 1 + d(x_{0i}, x_{0k})$. Consequently, $r(y_j^i|W) \neq r(x_{0i}|W)$.

Thus, $W = \bigcup_{i=1}^{k} B_i \cup \bigcup_{i=n-k}^{n} (B'_i - \{x_{0i}\})$ is a local metric set of $G \odot \mathcal{H}$. Since $d(x_{oi}, y_j^i) = 1$, for $1 \leq i \leq n$ and $1 \leq j \leq m_i$, then x_{0i} is not element of local metric set W. While, in Lemma 2.5 we have the central set of $G \odot \mathcal{H}$ is $S(G \odot \mathcal{H}) = \{s_{0k} \in V(G \odot \mathcal{H}) | s_k \in S(G)\}$. So, $S(G \odot \mathcal{H}) \cap W = \emptyset$. Take $X = S(G \odot \mathcal{H}) \cup W$, then based on Lemma 1.3, X is also a local metric set of $G \odot \mathcal{H}$ and $S(G \odot \mathcal{H}) \subset X$, so X is a central local metric set of $G \odot \mathcal{H}$. Next, it is shown that X is the minimal central local metric set. Take $U \subset V(G \odot \mathcal{H})$ with $S(G \odot \mathcal{H}) \subset U$ and $|U| < |X|$. Since $X = S(G \odot \mathcal{H}) \cup W$, then there exists i such that at most as many as $|B_i| + |B'_i - \{x_{0i}\}| - 1$ vertices in $K_1 + H_i$ are members of U. Since B_i and B'_i are the local basis of the graph $K_1 + H_i$, there exist two adjacent vertices in $K_1 + H_i$ that have the same representation in U. So, U is not a central local metric set of $G \odot \mathcal{H}$. Consequently, $X = S(G) \cup W$ is the central local basis set of $G \odot \mathcal{H}$. So, it is proved that $lmd_s(G \odot \mathcal{H}) = |S(G \odot \mathcal{H})| + |W| = |S(G \odot \mathcal{H})| + \sum_{i=1}^{k} B_i + \sum_{i=n-k}^{n} (B'_i - \{x_{0i}\}) = |S(G \odot \mathcal{H})| + \sum_{i=1}^{k} (lmd(K_1 + H_i)) + \sum_{i=n-k}^{n} ((lmd(K_1 + H_i) - 1)$. Further, for graph $K_1 + H_i$ when H_i is contains the dominant vertex, $diam(H_i) = 2$. So that, by Theorem 1.5, $lmd_s(G \odot \mathcal{H}) = |S(G \odot \mathcal{H})| + \sum_{i=1}^{k} (lmd(H_i)) + \sum_{i=n-k}^{n} ((lmd(K_1 + H_i) - 1)$. Since when H does not contain a dominant vertex, there is a possibility that a vertex in K_1 is an element of the local metric set of $K_1 + H$, then we used α_i to facilitate this condition. So, the central local metric dimension of $G \odot \mathcal{H}$ is $lmd_s(G \odot \mathcal{H}) = |S(G \odot \mathcal{H})| + \sum_{i=1}^{k} (lmd(H_i)) + \sum_{i=n-k}^{n} ((lmd(K_1 + H_i) - \alpha_i)$ for $\alpha_i = 1$ when x_{0i} is an element of a local basis set of $K_1 + H_i$, or $\alpha_i = 0$ when x_{0i} is not an element of a local basis set of $K_1 + H_i$. □

3. Conclusion

In this paper, we explore the central local metric dimension of the corona product graphs $G \odot H$ and $G \odot \mathcal{H}$, where $\mathcal{H}$ is a sequence of connected graph $H_1, H_2, H_3, ..., H_n$. The result shows that the main properties of the central set in graph $G \odot H$ is $S(G \odot H) = \{s_{0k} \in V(G \odot H) | s_k \in S(G)\}$, meaning that the corona operation on graph

$G \odot H$ can preserve the central set in graph G. Since every i-th vertex in graph G is adjacent to every vertex in the i-th copy of graph H, the central set in graph $S(G \odot H)$ is mutually disjoint to its local difference set. Thus, the central local metric dimension of the graph $S(G \odot H)$ is the sum of the cardinality of the central set $S(G)$ and the cardinality of the central local basis set of $G \odot H$.

The central local metric dimension of graph G remains open for further research. Some topics that can be explored include the application of the central local metric dimension concept to other types of graphs and network structures. The application of the central local metric dimension concept to the management of transportation systems or other vital objects is still available to explore.

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