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Risk Management Strategies in a Dependent Perturbed Compound Poisson Model

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Abstract This paper deals with the optimal risk management strategies for an insurer with a diffusion approximation of dependent compound Poisson process who wants to maximize the expected utility by purchasing proportional reinsurance and managing reinsurance counterparty risk with investment and he/she can invest in the financial market and in a risky asset such as stocks. It is assumed that this dependent risk model consists of the constant reinsurance premium rate, combination of the number of claims occurring by policyholders within a finite time, and perturbed by correlated standard Brownian motions, where the price of the risk-free bond is described by a stochastic differential equation. We use the alternative real measure technique to derive the optimal strategies and solution of the associated Hamilton-Jacobi-Bellman equation for the optimization problem which is formed by the expectation of combination of financial market factors and an exponential utility function. We prove the verification theorem to guarantee the optimal strategy. Finally, some numerical illustrations are presented to analyze our theoretical results and investigate the sensitivity of optimal strategies on some parameters.

Keywords Compound Poisson process, Financial market, Insurer's ambiguity aversion, Optimization problem, Proportional reinsurance.

AMS 2010 subject classifications Primary 62G32; Secondary 62F99, 62E20.

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1. Introduction

The study of optimal reinsurance and investment has been an active field over the past three decades. The researches in actuarial science and mathematical finance belongs to analysis and optimisation the investment returns of insurance companies including ruin probabilities, dividend payment, reinsurance, and investment. The level of development of insurance and the overall economy are interlinked and mutually reinforced. The role of insurance in economic development is highlighted by the fact that insurance promotes economic development while maintains social stability. In the insurance market, insurers improve the economic efficiency of the system by spreading individual risks. Additionally, the insurance companies might face the risk of making substantial claim payments in a short period. In the event of major disasters, they may not be able to cover excessive losses. Therefore, reinsurance is a powerful tool for insurance companies to share some of their losses, control their liability to some extent, and reduce the risks involved. In the reinsurance market, by paying reinsurance fees, the insured business is partially transferred to other insurers. As reinsurance is for insurance, reinsurance is a very important part of the overall insurance system. on the cash flows created by their insurance portfolio, reinsurance can also function as risk management and a financial decision. Optimal reinsurance and investment problems have been widely investigated under different criteria, especially through expected utility maximization, the mean variance criterion, and ruin probability minimization. In the literature, most optimal strategies with investment and reinsurance dynamic were studied under the assumption of independent risks. However, most of the portfoilios of insurance companies in

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practice are often dependent on each other in some way.
Currently, the main approach of dealing with model uncertainty is the robust control approach developed by [1]. The fundamental idea behind this method lies in that the decision-maker takes the reference model as a starting point, and she knows that the reference model cannot describe the real insurance or financial market correctly. Based on the above approach, [2] considered a robust optimal reinsurance and investment problem under the risk stochastic volatility model for an insurer with ambiguity aversion, the closed-form expression of the optimal strategy is obtained under the objective of maximizing the expected exponential utility. [3] investigated the optima investment and reinsurance problem with more generally multiscale stochastic volatility. When the price process of the risky asset satisfies constant elasticity of variance model, [4] derived the optimal investment and proportional reinsurance strategies. To investigate the influence of the misspecification for jump parameter on the optimal strategy of the insurer, [5] considered the robust optimal excess-of-loss reinsurance and investment strategies for the model with jumps. Under the mean-variance criteria, [6] obtained the robust optimal reinsurance and investment strategies with a benchmark. [7] derived the robust equilibrium reinsurance–investment strategies with jumps in the framework of game theory.
[8] considered the robust optimal reinsurance-investment strategy selection problem with price jumps and correlated claims for an ambiguity-averse insurer and obtained closed-form solutions for the robust optimal reinsurance–investment strategy and the corresponding value function by using the stochastic dynamic programming approach. [9] analyzed a optimal reinsurance and investment problem in a model with default risks and jumps for a general company which holds shares of an insurer and a reinsurer. Applying stochastic dynamic programming approach, they established the robust Hamilton-Jacobi-Bellman (HJB) equations for the post-default case and the pre-default case, respectively. [10] studied the optimal excess-of-loss reinsurance contract between an insurer and a reinsurer in a dynamic risk model and obtained the simultaneous equilibrium strategy in this reinsurance dynamic risk setting using the objective functions of insurer and reinsurance.
In the literature mentioned above, the optimal reinsurance or/and investment problems are investigated under the risk model with only one business for an insurer. To the best of our knowledge, there is little research on the robust optimal decision-making problem under the multiple dependent risks for an insurer. Many insurance companies have two or more claims occurring by policyholders within a finite time, and most of them are not independent of each other due to the risk of suffering from a common claim shock. In this structure of risk model, the investigation of risk management strategies is the key point. Thus, many scholars begin to investigate the optimal strategies under the multivariable dependent risks.
For example, [11] firstly converted the two-dimensional compound Poisson reserve risk process into a two-dimensional diffusion approximation process, and derived the optimal reinsurance strategy to minimize the ruin probability of the insurer. [12] obtained the optimal proportional reinsurance strategy when the surplus of insurance company is described by a two-dimensional dependent compound Poisson process and its diffusion approximation, respectively. Meanwhile, [13] extended the work of [12] to the risk model with multiple dependent classes of insurance business. [14] investigated the optimal reinsurance strategy with common shock dependence based on mean-variance criteria. Later, [15] extended the model of [14] to the case that the surplus can be invested in the financial market, and both the optimal investment and reinsurance strategies are obtained.
In this paper, we focus on the effect of uncertainty about the diffusion risk arising from risky asset and surplus process of the insurer, and consider the robust optimal investment and reinsurance problem with multiple dependent risks. We assume that the insurer adopts proportional reinsurance to disperse risk. Refer to [16] the reinsurance premium is calculated under the generalized mean-variance premium principle, which includes the expected value principle and the variance principle as special cases. The surplus of the insurance company can be invested in the financial market consisting of one risk-free asset and a risky asset or a market index. Inspired by [17] and [18] the price process of the risky asset is assumed to satisfy a square-root factor process, which can describe the randomness of volatility. We assume that the insurer is both risk and ambiguity averse. Thus, under the objective of maximizing expected exponential utility, using the method of robust optimal control, we obtain the closed-form expressions of optimal investment and reinsurance strategies and corresponding value function.
Our goal in this paper is to depart from the common assumption of independent risks occurring by policyholders which are linked by a common variation (or shock) in the parameters of each line's risk model by a counting

process within a finite time to obtain the explicit expressions for optimal strategies of dependent risk model using the alternative real measure. Motivated by the above-mentioned literatures, considering that maximizing the wealth process of insurance company with a dependent structure where the self-financing insurance company is allowed to invest a risk-free bond and a risky asset under the reinsurance premium rate, using a set of progressively measurable process is embedded in this paper and thus, it is necessary to take the management of the insurer into account.
Although research on the optimal risk management strategies problem has been rapidly increasing in recent years, none of these contributions deals with finding the optimization problem with consideration of the several dependent risks occurring by policyholders. The main novelty of this paper is to consider the optimal risk management strategies to maximize the wealth process of insurance company according to the dependent compound Poisson process whose the optimization problem is the expectation of combination of the financial market factors and an exponential utility function contains a set of progressively measurable process. To find the optimal strategies, we consider the risk model and its parameters under an alternative real measure.
The rest of this paper is organized as follows. In Section 2, we present the well-known compound Poisson process and describe the details of financial maket and dependent structure of risk model. Moreover, we exhibit the reformulation of the surplus process in investment and reinsurance dynamic strategies and give the framework of optimization problem which is formed by the combination expectation of financial market factors and an exponential utility function. Section 3 is devoted to solve the HJB equation related to our optimal control problem with exponential objective function and progressively measurable process, to find the explicit expressions for the given strategies in the dependent risk model. The verification theorem is proved in Section 4. In Section 5, we present the numerical examples and offer detailed interpretations of model parameters effects on investment and reinsurance strategies of the outcomes. Concluding remarks are provided in Section 6.

2. Model formulation and the optimization problem

The main mission of insurance companies is to market insurance contracts and provide risk protection to policyholders. However, with continuous evolution of the economy and the subsequent accumulation of wealth, maybe there are implications for insurers. In this study, we describe the modified compound Poisson process with a dependent structure and give the optimization problem faced by a value-maximizing insurance company that seeks to enhance its profitability (value) through dynamic risk management strategies. To maximize its profitability, the company dynamically adjusts its dependent risk exposure through a risky asset and one risky asset under the reinsurance premium rate. The optimization problem is to determine the optimal investment and reinsurance strategies that maximize the company profitability. In the following section, we will employ a dependent compound Poisson process and present the mathematical set-up that formalizes this optimization problem and provide a framework for analysing the optimal strategies.

2.1. Dependent compound Poisson process and financial maket

Let us start with a Cramér-Lundberg model, which is a classical actuarial model used to analyse the risk of an insurance portfolio. The company experiences two opposing cash flows: incoming premiums from the policyholder and outgoing claims. Let T be a finite-time index, where $T < \infty$ is the terminal time of decision making. In the sequel, we will always work on the probability space $(\Omega, \mathcal{F}, P)$, which is endowed with the information filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ which carries all stochastic quantities and right continuous, where P is a real-world probability measure.
Let us start with a compound Poisson process, which is a classical actuarial model used to analyze and management the risks of an insurance portfolio. The insurance company actives in a continuously evolving environment, where it receives premiums continuously and faces dependent claims modeled by a compound Poisson process. The main application of this risk model is in casualty insurance, health insurance and so on. We suppose that the insurer has $k \geq 2$ dependent policyholders in its portfolio and he/she manages them simultaneously. Let X_i^m, $i = 1, 2, \ldots$, be the claim amount random variable occurring by the mth policyholder with common distribution function $F_m(x)$ and Poisson process $N_m(t)$ with intensity parameter λ_m, $m = 1, 2, \ldots, k$, and assume that the

k policyholders are linked by a common variation in the parameters of each line's risk model by the counting process $N(t)$ within the interval time $[0, t]$ as a Poisson process with intensity parameter λ. Moreover, the processes $N_1(t), N_2(t), \dots, N_k(t)$, and $N(t)$ are $k+1$ independent Poisson processes with intensity parameters $\lambda_1, \lambda_2, \dots, \lambda_k$ and λ, respectively. The first-order and second-order moments of claim random variable X_i^m, $i = 1, 2, \dots$, are denoted by $\mu_m = E(X_i^m)$ and $\nu_m = E\big((X_i^m)^2\big)$, respectively. According to the Cramér-Lundberg model (also known as compound Poisson model or classical risk model), the surplus process $dX(t)$ of a homogeneous insurance portfolio can be described by the modified risk process

$$dX(t) = cdt - \sum_{m=1}^{k} d\Big(\sum_{i=1}^{N_m(t)+N(t)} X_i^m\Big), \qquad t \geq 0, \tag{1}$$

with an initial deterministic surplus $X(0) = u$ is the positive initial reserve, where $N_m(t) + N(t)$ represents the total claim number for the mth policyholder in the insurance portfolio at time interval $[0, t]$, the positive amount c corresponds to the premium income rate, which is calculated according to the expected value premium principle with positive safety loading θ as

$$c = (1+\theta)\Big(\sum_{m=1}^{k} \mu_m(\lambda_m + \lambda)\Big), \tag{2}$$

where $S_m(t) = \sum_{i=1}^{N_m(t)+N(t)} X_i^m$ is the aggregate claim process generated from mth insured. Therefore, the risk model (1) can be rewritten as

$$dX(t) = cdt - \sum_{m=1}^{k} dS_m(t), \qquad t \geq 0. \tag{3}$$

Now, we approximate the compound Poisson risk process in terms of reinsurance contract with the standard Brownian motion and constant reinsurance premium. To spread risk in the portfolio and protect from potential large claims, it is assumed that the insurer is allowed to purchase proportional reinsurance with a constant reinsurance premium rate. More precisely, we allow the insurance company to continuously reinsure a fraction of its claim with the retention level $q_m(t) \in [0, 1]$, $t \in [0, T]$ for mth policyholder at time t. If for the mth policyholder the risk exposure of the insurance company is fixed, then the cedent pays $100q_m(t)\%$ of each claim while the rest $100(1 - q_m(t))\%$ is paid by the reinsurer. Therefore, $1 - q_m(t)$ is the proportional reinsurance to the reinsurance company associated to the mth policyholder.
Let $R^q(t)$ be the proportional reinsurance risk process associated with the strategy $q(t) = \big(q_1(t), q_2(t), \dots, q_k(t)\big)$, and for $\eta > 0$, $\eta q(t)$ be the constant reinsurance premium rate at time t, then the corresponding risk process (3) for insurer in term of dynamic proportional reinsurance becomes:

$$dX^q(t) = \big(c - \eta q(t)\big) - \sum_{m=1}^{k} q_m(t) dS_m(t), \qquad t \geq 0. \tag{4}$$

From Grandell (1991), the compound Poisson $S_m(t)$, $m = 1, 2, \dots, k$, can be approximated by the following Brownian motion:

$$dS_m(t) = a_m dt + \zeta_m dW_m(t), \qquad t \geq 0, \tag{5}$$

with $a_m = \mu_m(\lambda_m + \lambda)$, and $\zeta_m^2 = \nu_m(\lambda_m + \lambda)$, where $W_i(t)$ and $W_j(t)$, $i \neq j$, $i, j = 1, 2, \dots, k$, are standard Brownian motions with the correlation coefficient

$$\rho_{ij} = \frac{\lambda E(X_i)E(X_j)}{\sqrt{(\lambda_i + \lambda)E(X_i^2)}\sqrt{(\lambda_j + \lambda)E(X_j^2)}} = \frac{\lambda \mu_i \mu_j}{\zeta_i \zeta_j},$$

where $\rho_{ij} \in [-1, 1]$. From [12] the diffusion approximation of the surplus process is given by

$$\begin{aligned} dX^q(t) &= \big(c - \eta q(t)\big) - \sum_{m=1}^{k} a_m q_m(t)\big)dt \\ &\quad + \Big(\sum_{m=1}^{k} \zeta_m^2 q_m^2(t) + \sum_{i \neq j}^{k} \lambda \mu_i \mu_j q_i(t) q_j(t)\Big)^{\frac{1}{2}} dB_1(t), \end{aligned} \tag{6}$$

where $B_1(t)$ is a standard Brownian motion.
To manage the reinsurance risk, selection of the reinsurance premium rate in relation (6) is so important. We assume that the insurance company can choose the reinsurance premium rate policy according to [16] by the following generalized mean-variance principle

$$\eta q(t) = (1 + \delta)\Big[\sum_{m=1}^{k} a_m\big(1 - q_m(t)\big) + \beta f\big(q(t)\big)\Big], \tag{7}$$

with $\delta > 0$ and $\beta > 0$, where $f\big(q(t)\big) = \sum_{m=1}^{k} \zeta_m^2 \big(1 - q_m(t)\big)^2 + \sum_{i \neq j}^{k} \big(1 - q_i(t)\big)\big(1 - \lambda \mu_i \mu_j q_j(t)\big)$.

2.2. *Dynamics portfolio choice of financial market*

We assume that the standard assumptions of continuous-time financial models hold, that is, (1) continuous trading is allowed, (2) no transaction cost or tax is involved in trading, and (3) assets are infinitely divisible. For simplicity, we assume that there are only two assets in the financial market: a risk-free bond and a risky asset, namely stocks, in this section. The risk-free interest rate is assumed to be non-negative in the model. The price of the risk-free bond follows

$$dR(t) = rR(t)dt,$$

where $R(t)$ is the price of the risk-free bond at time t, and r is the risk-free interest rate, which is assumed to be constant $r \geq 0$. In addition, the price process of stock at time t is described by the following stochastic differential equation

$$dP(t) = P(t)\big(\kappa(t)dt + \tau(t)dB_2(t)\big), \tag{8}$$

where $\kappa(t)$ and $\tau(t)$, for $t \in [0, T]$, are the expected instantaneous rate of return of the stock rate and volatility of the stock price rate, respectively, and $B_2(t)$ is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$. As given in [17], we denote the market price of risk by the fraction

$$\theta(t) = \frac{\kappa(t) - r}{\tau(t)}, \qquad t \in [0, T], \tag{9}$$

and assume that the market price of risk process (9) is related to a stochastic factor process $\{\alpha(t)\}_{t \in [0,T]}$ as

$$\theta(t) = \gamma\sqrt{\alpha(t)}, \qquad t \in [0, T], \quad \gamma \in \mathbb{R} \setminus \{0\}, \tag{10}$$

where the stochastic factor process $\{\alpha(t)\}_{t \in [0,T]}$ given in (10) satisfies the following affine-form mean-reverting square root model

$$\begin{cases} d\alpha(t) = a\big(b - \alpha(t)\big)dt + \sqrt{\alpha(t)}\big(l_1 dB_2(t) + l_2 dB_3(t)\big), \\ \qquad\qquad \alpha(0) = \alpha_0 \geq 0, \end{cases}$$

where a, b, l_1 and l_2 are all positive constants and $B_3(t)$ is an other standard Brownian motion. Throughout this paper, we assume that the three Brownian motions $B_1(t)$, $B_2(t)$, and $B_3(t)$ are mutually independent. **Remark 1.** To provide the dynamics wealth process with an unique state variable and a tractable reformulation of the

optimization problem, we assume that in the price process of risky asset model (8), for increasing rate $\{\kappa(t)\}_{t\in[0,T]}$ and volatility rate $\{\tau(t)\}_{t\in[0,T]}$, at least one of these two factors is stochastic process and simultaneously related to th process $\{\alpha(t)\}_{t\in[0,T]}$ which satisfies the equality $\frac{\kappa(t)-r}{\tau(t)} = \gamma\sqrt{\alpha(t)}$.
Beside proportional reinsurance, the insurance companies have two types of investments at time t: risky assets (such as securities or funds) and risk-free assets (fixed-rate income investments). Here, $\pi(t)$ denotes the amount of insurance company invested in risky securities, According to [19] assumption, the investment amount can be either positive or negative, that is, $-\infty < \pi(t) < \infty$. A negative investment amount indicates a short position, while a positive amount indicates a long position. In our risk optimization problem, we find a pair of estimator for investment and reinsurance strategy $s(t) = \big(\pi(t), q(t)\big)$, $t \in [0, T]$. The dynamics wealth process associated with strategy $s(t)$ is given by

$$\begin{aligned} dX^s(t) &= \big(c - \eta q(t) - \sum_{m=1}^{k} a_m q_m(t)\big)dt + \big(\sum_{m=1}^{k} \zeta_m^2 q_m^2(t) + \sum_{i\neq j}^{k} \lambda\mu_i\mu_j q_i(t)q_j(t)\big)^{\frac{1}{2}} dB_1(t), \\ &\quad + \pi(t)\frac{dP(t)}{P(t)} + \big(X^s(t) - \pi(t)\big)\frac{dR(t)}{R(t)} \\ &= \big(c - \eta q(t) - \sum_{m=1}^{k} a_m q_m(t) + \pi(t)\big(\kappa(t) - r\big) + rU^s(t)\big)dt \\ &\quad + \big(\sum_{m=1}^{k} \zeta_m^2 q_m^2(t) + \sum_{i\neq j}^{k} \lambda\mu_i\mu_j q_i(t)q_j(t)\big)^{\frac{1}{2}} dB_1(t), \\ &\quad + \pi(t)\tau(t)dB_2(t), \end{aligned} \tag{11}$$

where $\tau(t) = \frac{\kappa(t)-r}{\gamma\sqrt{\alpha(t)}}$.
Remark 2. We find that the dynamics wealth process given in (11) depends on the factors $\kappa(t)$ and $\tau(t)$. As mentoined in Remark 1, if for any $t \in [0, T]$, the process $\kappa(t)$ is stochastic which is assumed to be a function of $\alpha(t)$, then the wealth process given in (11) has the unique state variable $\alpha(t)$.

2.3. Optimization problem using the alternative real measure

The insurance company tries to increase the amount of its portfolio in order to prevent losses and ruin at the terminal time T. In the next section, we will derive the optimal reinsurance and investment strategies for insurer by solving the HJB equation related to our optimal control problem with the exponential utility function $U(x) = -\frac{1}{y}\exp(-yx)$ at point x, where $y > 0$ a positive constant absolute risk aversion coefficient. In our risk model, the insurer desires to use the following objective function:

$$\sup_{s\in\mathcal{S}} E^P\big[U\big(X^s(T)\big)\big] = \sup_{s\in\mathcal{S}} E^P\big[-\frac{1}{y}\exp(-yX^s(T))\big],$$

where $\mathcal{S}$ is the set of all admissible strategy s in the financial market, and E^P is the expectaion under the probability measure P.
A risk model which is created by statistical methods under the real-world data and probability measurer P can only be considered as a reference model for the real. Therefore, the parameters are called reference model parameters. Due to the estimation error, the insurer is uncertain about the reference model. He/she is aware that he/she does not know the true model, but only looks at the reference model as an approximation of the true model. Since the decision about the model depends on the value of the parameters of that model, the uncertainty of the parameter estimates directly affects the validity of the strategy. To find the optimal strategies in the risk model, one proposed approach is to reconsider the model and parameters under an alternative measure that is equivalent to the real-world measure. This method is an effective measurement of the risk associated with an insurance portfolio.

Let $\mathcal{P}$ denotes such a candidate set of probability measures. Then we have $\mathcal{P} = \{Q : Q \sim P\}$. This assumption allows us to use Girsanov's theorem for changing measures, further restricting the alternative models so that they only differ in terms of their drift coefficients. According to Girsanov's theorem, for any $Q \in \mathcal{P}$, there exists a progressively measurable proces $\omega(t) = \big(\omega_1(t), \omega_2(t), \omega_3(t)\big)' \in R^3$ and a measurable real-valued process $a(t)$, such that

$$\frac{dQ}{dP} = a(t),$$

where

$$a(t) = \exp\big(-\int_0^t \omega'(s)dB(s) - \frac{1}{2}\int_0^t ||\omega(s)||^2 ds\big), \tag{12}$$

is a P martingale with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ and the process $\omega(t)$ satisfies Novikov's condition

$$E^P\big[\exp\big(\frac{1}{2}\int_0^t ||\omega(s)||^2 ds\big)\big] < \infty,$$

where $B(t) = \big(B_1(t), B_2(t), B_3(t)\big)'$ and $||\omega(t)||^2 = \omega_1^2(t) + \omega_2^2(t) + \omega_3^2(t)$. Moreover, by Girsanov's theorem for $Q \in \mathcal{P}$, we can define the following Brownian motions:

$$\begin{aligned} dB_1^Q(t) &= \omega_1(t)dt + dB_1(t), \qquad dB_2^Q(t) = \omega_2(t)dt + dB_2(t), \\ dB_3^Q(t) &= \omega_3(t)dt + dB_3(t). \end{aligned}$$

Therefore, the dynamic price process $P(t)$ of risky asset, the process $\alpha(t)$ and the wealth process $X^s(t)$ under the alternative real measure Q can be rewritten as

$$dP(t) = P(t)\big(\kappa(t) - \omega_1(t)\tau(t) + \tau(t)dB_1^Q(t)\big),$$

$$\begin{aligned} d\alpha(t) &= \Big(a\big(b - \alpha(t)\big) - \sqrt{\alpha(t)}\big(l_1\omega_2(t) + l_2\omega_3(t)\big)\Big)dt \\ &\quad + \sqrt{\alpha(t)}\big(l_1 dB_2^Q(t) + l_2 dB_3^Q(t)\big), \end{aligned}$$

and

$$\begin{aligned} dX^s(t) &= \Big\{c - \eta q(t) - \sum_{m=1}^k a_m q_m(t) - \zeta_1(t)\big(\sum_{m=1}^k \zeta_m^2 q_m^2(t) + \sum_{i\neq j}^k \lambda\mu_i\mu_j q_i(t)q_j(t)\big)^{\frac{1}{2}} \\ &\quad + \pi(t)\big(\kappa(t) - r - \zeta_2(t)\tau(t)\big) + rX^s(t)\Big\}dt \\ &\quad + \big(\sum_{m=1}^k \zeta_m^2 q_m^2(t) + \sum_{i\neq j}^k \lambda\mu_i\mu_j q_i(t)q_j(t)\big)^{\frac{1}{2}} dB_1^Q(t), \\ &\quad + \pi(t)\tau(t)dB_2^Q(t), \end{aligned} \tag{13}$$

respectively.

Definition 1. The strategy $s(t) = \big(\pi(t), q(t)\big)$ for any $t \in [0,T]$, is said to be admissible if it satisfies the following conditions:

(i) For any $t \in [0,T]$, the strategy $s(t)$ is $\{\mathcal{F}_t\}$-progressively measurable process with $-\infty < \pi(t) < \infty$, and $E^Q_{t,x,\alpha}\big(\int_0^t \big|s(t)\big|^4 dt\big) < \infty$.

(ii) The stochastic equation (13) has an unique solution $X^s(t)$ on the interval $[0,T]$, with $E^Q_{t,x,\alpha}\big|U(X^s(T)\big| < \infty$, where $E^Q_{t,x,\alpha}(.) = E^Q\big(\ .\ \big|X^s(t) = x, \alpha(t) = \alpha\big)$.

The optimization problem is to determine the risk management strategies, which in our paper is the optimal investment and reinsurance strategies that maximize the insurance company profitability. In the present paper, to obtain the risk management strategies, we assume that the optimization problem function following the form given in [20] as

$$\sup_{s\in\mathcal{S}} \inf_{Q\in\mathcal{P}} E^Q\big[U\big(X^s(T)\big) + \int_0^T K(t)dt\big], \tag{14}$$

where the wealth process $X^s(t)$ given in (11) and $K(t)$ is a combination of progressively measurable process as $K(t) = \frac{1}{2}\big(\frac{\omega_1^2(t)}{\Delta_1(t)} + \frac{\omega_2^2(t)}{\Delta_2(t)} + \frac{\omega_3^2(t)}{\Delta_3(t)}\big)$, and by [21] the functions $\Delta_1(t)$, $\Delta_2(t)$ and $\Delta_3(t)$ are defined as follow

$$\Delta_1(t) = -\frac{\Upsilon_1}{yO(t,x,\alpha)}, \qquad \Delta_2(t) = -\frac{\Upsilon_2}{yO(t,x,\alpha)}, \qquad \Delta_2(t) = -\frac{\Upsilon_3}{yO(t,x,\alpha)}, \tag{15}$$

where the positive parameters Υ_1, Υ_2 and Υ_3 are three main financial market factors representing the insurer's ambiguity aversion to the risk model from the insurance market, risky asset and the stochastic factor process, respectively and the objective function $O(t,x,\alpha)$ is given by

$$O(t,x,\alpha) = \sup_{s\in\mathcal{S}} \inf_{Q\in\mathcal{P}} E^Q_{t,x,\alpha}\big[U\big(X^s(T)\big) + \int_t^T K(s)ds\big].$$

In section 4, the value of objective function $O(t,x,\alpha)$ will be computed based on the optimal proportional reinsurance strategy.

3. Optimal risk management strategies

Now we are ready to state our main result regarding the optimal strategies for the perturbed continuous compound Poisson risk model. We study the optimal stochastic control problem (14) to get the explicit expressions for the optimal strategies in the reinsurance and investment treaty aiming to minimise the risk of insurance company. This leads to novel concepts of optimality which require development of new methodologies for solving the problems.

Theorem 3.1
Consider the optimal stochastic control problem (14). Then for $y > 0$, $(t,x,\alpha) \in [0,T] \times R \times R^+$ with the condition $O(T,x,\alpha) = -\frac{1}{y}\exp(-yx)$, the optimal investment proportional strategt is given by

$$\begin{aligned}\pi^*(t,\alpha) &= \frac{\alpha\gamma^2}{\kappa(t)-r}e^{-r(T-t)}\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}H_2(t)\big)\\ &= \frac{\gamma\sqrt{\alpha}}{\tau(t)}e^{-r(T-t)}\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}H_2(t)\big),\end{aligned} \tag{16}$$

where

$$H_2(t) = \frac{d_3\big(1-e^{d_1(T-t)}\big)}{2d_1 + (d_1+d_2)\big(e^{d_1(T-t)}-1\big)}, \tag{17}$$

with $d_1 = \big((a+\gamma l_1)^2 + (l_2^2\gamma^2)\frac{y+\Upsilon_2}{y+\Upsilon_1}\big)^{\frac{1}{2}}$, $d_2 = a+\gamma l_1$, $d_3 = \frac{y\gamma^2}{y+\Upsilon_1}$ and the condition $H_2(T) = 0$. Moreover, the optimal proportional reinsurance is given by

$$\hat{q}(t) = \frac{2\beta(1+\delta)\mathbf{1} + \delta\mathbf{M}^{-1}.\mathbf{a}}{2\beta(1+\delta) + (y+\Upsilon_1)e^{r(T-t)}}, \tag{18}$$

where the positive matrix $\mathbf{M}$ is defined by

$$\mathbf{M} = \begin{pmatrix} \zeta_1^2 & \lambda\mu_1\mu_2 & \lambda\mu_1\mu_3 & \dots & \lambda\mu_1\mu_k \\ \lambda\mu_2\mu_1 & \zeta_2^2 & \lambda\mu_2\mu_3 & \dots & \lambda\mu_2\mu_k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda\mu_k\mu_1 & \lambda\mu_k\mu_2 & \lambda\mu_k\mu_3 & \dots & \zeta_k^2 \end{pmatrix}. \tag{19}$$

Proof
Using the principle of dynamic programming, for $y > 0,\ (t, x, \alpha) \in [0, T] \times R \times R^+$ with the condition $O(T, x, \alpha) = -\frac{1}{y}\exp(-yx)$, we have the HJB equation for the optimal stochastic control problem (14) as

$$\begin{aligned}
&\sup_{s\in\mathcal{S}} \inf_{\zeta(t)=\big(\zeta_1(t),\zeta_2(t),\zeta_3(t)\big)\in R^3} \Big\{\frac{\partial O(t,x,\alpha)}{\partial t} + \frac{\partial O(t,x,\alpha)}{\partial x}\big(c - \eta q(t) \\
&-\sum_{m=1}^{k} a_m q_m(t) - \zeta_1(t)\big(\sum_{m=1}^{k}\zeta_m^2 q_m^2(t) + \sum_{i\neq j}^{k}\lambda\mu_i\mu_j q_i(t)q_j(t)\big)^{\frac{1}{2}} \\
&+\pi(t)\big(\kappa(t) - r - \omega_2(t)\tau(t)\big) + rx\big) \\
&+\frac{\partial O(t,x,\alpha)}{\partial \alpha}\big(a(b-\alpha) - \sqrt{\alpha}\big(l_1\omega_2(t) + l_2\omega_3(t)\big)\big) \\
&+\frac{1}{2}\frac{\partial^2 O(t,x,\alpha)}{\partial x^2}\big(\sum_{m=1}^{k}\zeta_m^2 q_m^2(t) + \sum_{i\neq j}^{k}\lambda\mu_i\mu_j q_i(t)q_j(t) + \frac{\big(\pi(t)(\kappa(t)-r)\big)^2}{\alpha\gamma^2}\big) \\
&+\frac{1}{2}\alpha(l_1^2 + l_2^2)\frac{\partial^2 O(t,x,\alpha)}{\partial \alpha^2} + \frac{l_1\pi(t)(\kappa(t)-r)}{\gamma}\frac{\partial^2 O(t,x,\alpha)}{\partial x\partial\alpha} \\
&+\frac{1}{2}\big(\frac{\omega_1^2(t)}{\Delta_1(t)} + \frac{\omega_2^2(t)}{\Delta_2(t)} + \frac{\omega_3^2(t)}{\Delta_3(t)}\big)\Big\} = 0,
\end{aligned} \tag{20}$$

Now, we try to solve the euation (20) to obtain the optimal values for $\pi(t)$ and $q(t)$. We guess that the solution to euation (20) is specified by the following form

$$G(t,x,\alpha) = -\frac{1}{y}\exp\big\{-yxe^{r(T-t)} + H_1(t) + \alpha H_2(t)\big\}, \tag{21}$$

where $H_2(t)$ is given in (17) and $H_1(t)$ is a function of time t, which will need to be determined later with the condition $H_1(T) = 0$. From (17) the corresponding partial derivatives of the function $G(t, x, \alpha)$ are given by

$$\frac{\partial G(t,x,\alpha)}{\partial t} = \big(yxre^{r(T-t)} + \frac{\partial H_1(t)}{\partial t} + \alpha\frac{\partial H_2(t)}{\partial t}\big)G(t,x,\alpha),$$

$$\frac{\partial G(t,x,\alpha)}{\partial x} = -ye^{r(T-t)}G(t,x,\alpha), \qquad \frac{\partial^2 G(t,x,\alpha)}{\partial x^2} = y^2e^{2r(T-t)}G(t,x,\alpha),$$

$$\frac{\partial G(t,x,\alpha)}{\partial \alpha} = H_2(t)G(t,x,\alpha), \qquad \frac{\partial^2 G(t,x,\alpha)}{\partial \alpha^2} = H_2^2(t)G(t,x,\alpha),$$

$$\frac{\partial^2 G(t,x,\alpha)}{\partial x\partial\alpha} = -ye^{r(T-t)}H_2(t)G(t,x,\alpha).$$

We first fix s and take the derivative of the equation in euation (16) with respect to ω_1, ω_2 and ω_3, respectively. Based on the first-order necessary condition, we obtain the minimum value $\omega^*(t,\alpha) = \big(\omega_1^*(t), \omega_2^*(t,\alpha), \omega_3^*(t,\alpha)\big)$ which minimizes the equation in (16) as

$$\begin{aligned}
\omega_1^*(t) &= \Upsilon_1 e^{r(T-t)}\big(\sum_{m=1}^{k}\zeta_m^2 q_m^2(t) + \sum_{i\neq j}^{k}\lambda\mu_i\mu_j q_i(t)q_j(t)\big)^{\frac{1}{2}}, \\
\omega_2^*(t,\alpha) &= \Upsilon_2 e^{r(T-t)}\pi(t)\tau(t) - \frac{\Upsilon_2}{y}l_1\sqrt{\alpha(t)}H_2(t), \\
\omega_3^*(t,\alpha) &= -\frac{\Upsilon_3}{y}l_2\sqrt{\alpha(t)}H_2(t).
\end{aligned} \tag{22}$$

Substituting the equation (18) and partial derivatives of the function $G(t,x,\alpha)$ in equation (16), we obtain

$$\frac{\partial H_1(t)}{\partial t} + \alpha\frac{\partial H_2(t)}{\partial t} - cye^{r(T-t)} + a(b-\alpha)H_2(t) + \frac{1}{2}\alpha\big(l_1^2\big(\frac{\Upsilon_2}{y}+1\big) + l_2^2\big(\frac{\Upsilon_3}{y}+1\big)\big)H_2^2(t)$$
$$+ \inf_{\pi\in R}\big\{\frac{1}{2}y(y+\Upsilon_2)e^{2r(T-t)}\frac{\pi^2(t)(\kappa(t)-r)}{\alpha\gamma^2} - \frac{l_1(y+\Upsilon_2)}{\gamma}(\kappa(t)-r)\pi(t)H_2(t)e^{r(T-t)}$$
$$-y\pi(t)(\kappa(t)-r)e^{r(T-t)}\big\} + \inf_{q(t)\in[0,1]} L\big(q(t)\big) = 0, \tag{23}$$

where the function $L\big(q(t)\big)$ is defined by

$$\begin{aligned} L\big(q(t)\big) &= ye^{r(T-t)}\big(\eta q(t) + \sum_{m=1}^{k} a_m q_m(t)\big) \\ &\quad + \frac{1}{2}y(y+\Upsilon_1)e^{2r(T-t)}\big(\sum_{m=1}^{k}\zeta_m^2 q_m^2(t) + \sum_{i\neq j}^{k}\lambda\mu_i\mu_j q_i(t)q_j(t)\big). \end{aligned}$$

Taking the derivative of the equation in equation (19) with respect to $\pi(t)$, we obtain the minimum value $\pi^*(t,\alpha)$ as given in (16).

Now, by taking the first and second derivatives of the function $L\big(q(t)\big)$ with respect to $q(t)$ for any $t\in[0,T]$, we get

$$\begin{aligned} \frac{\partial L\big(q(t)\big)}{\partial q_m(t)} &= ye^{r(T-t)}\big(-\delta a_m q_m(t) - 2\beta(1+\delta)(1-q_m(t))\zeta_m^2 \\ &\quad + \sum_{j=1,j\neq m}^{k}\lambda\mu_m\mu_j(1-q_j(t))\big) \\ &\quad + y(y+\Upsilon_1)e^{2r(T-t)}\big(\zeta_m^2 q_m(t) + \sum_{j=1,j\neq m}^{k}\lambda\mu_m\mu_j q_j(t)\big), \end{aligned}$$

$$\frac{\partial^2 L\big(q(t)\big)}{\partial q_m^2(t)} = ye^{r(T-t)}\zeta_m^2\big(2\beta(1+\delta) + (y+\Upsilon_1)e^{r(T-t)}\big),$$

and for $j\neq m$, $j=1,2,\ldots,k$, we have

$$\frac{\partial^2 L\big(q(t)\big)}{\partial q_m(t)\partial q_j(t)} = \frac{\partial^2 L\big(q(t)\big)}{\partial q_j(t)\partial q_m(t)} = ye^{r(T-t)}\lambda\mu_m\mu_j\big(2\beta(1+\delta) + (y+\Upsilon_1)e^{r(T-t)}\big).$$

For simplicity and better understanding, we can give these derivatives in the matrix form as

$$\begin{pmatrix} \frac{\partial^2 L(q(t))}{\partial q_1^2(t)} & \frac{\partial^2 L(q(t))}{\partial q_1(t)\partial q_2(t)} & \cdots & \frac{\partial^2 L(q(t))}{\partial q_1(t)\partial q_k(t)} \\ \frac{\partial^2 L(q(t))}{\partial q_2(t)\partial q_1(t)} & \frac{\partial^2 L(q(t))}{\partial q_2^2(t)} & \cdots & \frac{\partial^2 L(q(t))}{\partial q_2(t)\partial q_k(t)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 L(q(t))}{\partial q_k(t)\partial q_1(t)} & \frac{\partial^2 L(q(t))}{\partial q_k(t)\partial q_2(t)} & \cdots & \frac{\partial^2 L(q(t))}{\partial q_k^2(t)} \end{pmatrix} = ye^{r(T-t)}\big(2\beta(1+\delta) + (y+\Upsilon_1)e^{r(T-t)}\big)\mathbf{M},$$

where the matrix $\mathbf{M}$ given in (19). By the Lemma 1 of [22], $\mathbf{M}$ is a positive definite matrix, therefore, $L\big(q(t)\big)$ is a convex function with respect to the points $q_1(t), q_2(t), \ldots, q_m(t)$. Therefore, applying the first-order optimization condition, the vactor $\hat{q}(t) = \big(\hat{q}_1(t), \hat{q}_2(t), \dots, \hat{q}_m(t)\big)'$ which minimizes the function $L\big(q(t)\big)$ satisfies the equation

$$\big(2\beta(1+\delta) + (y+\Upsilon_1)e^{r(T-t)}\big)\mathbf{M}.\hat{q}(t) = 2\beta(1+\delta)\mathbf{M}.\mathbf{1} + \delta.\mathbf{a},$$

where $\mathbf{1}$ and $\mathbf{a}$ are defined by $\mathbf{1} = \big(1, 1, \dots, 1\big)'$ and $\mathbf{a} = \big(a_1, a_2, \dots, a_k\big)'$, respectively. Since $\mathbf{M}$ is a positive definite matrix, the invertibility of this matrix exists, therefore, the minimum value $\hat{q}(t)$ as given in (18), and this completes the proof. □

We note that in the equation (18) if $\delta = 0$, then the strategy for proportional reinsurance simplifies as $\hat{q}_m(t) = \frac{2\beta}{2\beta+(y+\Upsilon_1)e^{r(T-t)}}$, $m = 1, 2, \dots, k$, which satisfies the condition $\hat{q}_m(t) \in [0, 1]$. In addition, when $\delta \neq 0$, to make sure that the strategy given in (18) is a general optimal strategy for proportional reinsurance risk process, we need to investigate this strategy for different cases of the value of parameters. In the following subsection, we only consider two dependent policyholders, i.e., $k = 2$, and sudy the optimal strategies for proportional reinsurance. The following methods are useful for deriving the results on optimal strategies for more than dependent policyholders. If the value of k takes larger, then the parameters and variables of the risk model increase and the analysis of model increases geometrically.

$$\begin{cases} \frac{24y^2\gamma^2}{(y+\Upsilon)^2} - \frac{56y\gamma l_1 d_3}{(y+\Upsilon)(d_1+d_2)} + \frac{32l_1^2 d_3^2}{(d_1+d_2)^2} \leq \frac{a^2}{2(l_1^2+l_2^2)^2}, \\ \\ \frac{24y^2\gamma^2}{(y+\Upsilon)^2} \leq \frac{a^2}{2(l_1^2+l_2^2)^2}. \end{cases}$$

3.1. Optimal proportional reinsurance for two dependent policyholders

To obtain the optimal proportional reinsurance of risk process for two dependent policyholders, i.e., $k = 2$, define the parameters

$$\begin{aligned} &D_1 = a_1\zeta_2^2 - a_2\lambda\mu_1\mu_2, \qquad D_2 = a_2\zeta_1^2 - a_1\lambda\mu_1\mu_2, \\ &D_3 = \zeta_1^2\zeta_2^2 - \lambda^2\mu_1^2\mu_2^2. \end{aligned} \tag{24}$$

Using the parameters given in (23), for $k = 2$, the equation (21) becomes

$$\begin{aligned} \hat{q}_1(t) &= \frac{\delta D_1 + 2\beta(1+\delta)D_3}{\big(2\beta(1+\delta) + (y+\Upsilon_1)e^{r(T-t)}\big)D_3}, \\ \hat{q}_2(t) &= \frac{\delta D_2 + 2\beta(1+\delta)D_3}{\big(2\beta(1+\delta) + (y+\Upsilon_1)e^{r(T-t)}\big)D_3}. \end{aligned} \tag{25}$$

For this case, the following theorem gives the optimal proportional reinsurance and the value of objective function $O(t, x, \alpha)$.

Theorem 3.2
Suppose that the conditions in Theorem 3.1 hold for $k = 2$. When $D_1 \leq D_2$, the optimal proportional reinsurance is given by

$$q^*(t) = \begin{cases} \big(q_1^*(t), q_2^*(t)\big), & \text{if} \quad 0 \leq t \leq t_0' \,, \\ \big(\tilde{q}_1(t), 1\big), & \text{if} \quad t_0' \leq t \leq t_1 \,, \\ (1, 1), & \text{if} \quad t_1 \leq t \leq T, \end{cases} \tag{26}$$

with the objective function

$$O(t, x, \alpha) = G(t, x, \alpha) = \begin{cases} -\frac{1}{y}\exp\big(yxe^{r(T-t)} + H_{11}(t) + \alpha H_2(t)\big), & \text{if} \quad 0 \leq t \leq t_0' \,, \\ -\frac{1}{y}\exp\big(yxe^{r(T-t)} + H_{12}(t) + \alpha H_2(t)\big), & \text{if} \quad t_0' \leq t \leq t_1 \,, \\ -\frac{1}{y}\exp\big(yxe^{r(T-t)} + H_{13}(t) + \alpha H_2(t)\big), & \text{if} \quad t_1 \leq t \leq T, \end{cases} \tag{27}$$

and when $D_1 > D_2$, the optimal proportional reinsurance is given by

$$q^*(t) = \begin{cases} \big(q_1^*(t), q_2^*(t)\big), & \text{if} \quad 0 \le t \le t_0 \ , \\ \big(1, \tilde{q}_2(t)\big), & \text{if} \quad t_0 \le t \le t_1' \ , \\ (1,1), & \text{if} \quad t_1' \le t \le T \ , \end{cases} \tag{28}$$

with the objective function

$$O(t,x,\alpha) = G(t,x,\alpha) = \begin{cases} -\frac{1}{y}\exp\big(yxe^{r(T-t)} + H_{14}(t) + \alpha H_2(t)\big), & \text{if} \quad 0 \le t \le t_0 \ , \\ -\frac{1}{y}\exp\big(yxe^{r(T-t)} + H_{15}(t) + \alpha H_2(t)\big), & \text{if} \quad t_0 \le t \le t_1' \ , \\ -\frac{1}{y}\exp\big(yxe^{r(T-t)} + H_{13}(t) + \alpha H_2(t)\big), & \text{if} \quad t_1' \le t \le T \ , \end{cases} \tag{29}$$

where

$$\tilde{q}_1(t) = \frac{\delta a_1 + 2\beta(1+\delta)\zeta_1^2 - \lambda\mu_1\mu_2(y+\Upsilon_1)e^{2r(T-t)}}{\zeta_1^2\big(2\beta(1+\delta) + (y+\Upsilon_1)e^{2r(T-t)}\big)}, \tag{30}$$

$$\tilde{q}_2(t) = \frac{\delta a_2 + 2\beta(1+\delta)\zeta_2^2 - \lambda\mu_1\mu_2(y+\Upsilon_1)e^{2r(T-t)}}{\zeta_2^2\big(2\beta(1+\delta) + (y+\Upsilon_1)e^{2r(T-t)}\big)}, \tag{31}$$

$$t_1 = T - \frac{1}{r}\ln\big(\frac{\delta a_1}{(y+\Upsilon_1)(\zeta_1^2 + \lambda\mu_1\mu_2)}\big), \tag{32}$$

$$t_1' = T - \frac{1}{r}\ln\big(\frac{\delta a_2}{(y+\Upsilon_1)(\zeta_2^2 + \lambda\mu_1\mu_2)}\big), \tag{33}$$

$$t_0 = \begin{cases} T, & \text{if } \delta D_1 \le (y+\Upsilon_1)D_3 \ , \\ T - \frac{1}{r}\ln\big(\frac{\delta D_1}{(y+\Upsilon_1)D_3}\big), & \text{if } (y+\Upsilon_1)D_3 < \delta D_1 < (y+\Upsilon_1)D_3e^{rT}, \\ 0, & \text{if } \delta D_1 \ge (y+\Upsilon_1)D_3e^{rT}, \end{cases} \tag{34}$$

$$t_0' = \begin{cases} T, & \text{if } \delta D_2 \le (y+\Upsilon_1)D_3 \ , \\ T - \frac{1}{r}\ln\big(\frac{\delta D_2}{(y+\Upsilon_1)D_3}\big), & \text{if } (y+\Upsilon_1)D_3 < \delta D_2 < (y+\Upsilon_1)D_3e^{rT}, \\ 0, & \text{if } \delta D_2 \ge (y+\Upsilon_1)D_3e^{rT}, \end{cases} \tag{35}$$

and the functions H_{11}, H_{12}, H_{13}, H_{14} and H_{15} will be given in equations (32) and (33).

Proof
(a) If $D_1 \le D_2$, then $t_0 \ge t_0'$, therefore, in this case we have the following different cases:
(a_1) When $0 \le t < t_0'$, then the optimal proportional reinsurance strategy is $q^*(t) = \big(q_1^*(t), q_2^*(t)\big)$, where $q_1^*(t)$ and $q_2^*(t)$ are equal to $\hat{q}_1(t)$ and $\hat{q}_2(t)$ given in (24), respectively.
(a_2) When $t \ge t_0'$, then $\hat{q}_2(t) \ge 1$, therefore, we let $\hat{q}_2^*(t) = 1$. Putting $\hat{q}_2^*(t) = 1$ into the last part of equation (19), we obtain the optimization problem

$$\begin{aligned} \inf_{q_1(t)} L\big(q(t)\big) \quad &= \quad \inf_{q_1(t)} \big\{ye^{r(T-t)}\big((1+\delta)\big((1-q_1(t))a_1 + \beta(1-q_1(t))^2\zeta_1^2\big) + a_1q_1(t) + a_2\big) \\ &\quad + \frac{1}{2}y(y+\Upsilon_1)e^{2r(T-t)}\big(\zeta_1^2q_1^2(t) + \zeta_2^2 + 2\lambda\mu_1\mu_2q_1(t)\big\}. \end{aligned} \tag{36}$$

For $t \le T$, it can be shown that the minimum value of $q_1(t)$ in the problem (25) is $\tilde{q}_1(t)$ as given in (28), therefore, for $t_0' \le t \le t_1$, the optimal proportional reinsurance strategy is $q^*(t) = \big(q_1^*(t), q_2^*(t)\big) = \big(\tilde{q}_1(t), 1\big)$.

(a_3) When $t'_0 \leq t \leq T$, it is easy to see that the optimal proportional reinsurance strategy is $q^*(t) = \big(q_1^*(t), q_2^*(t)\big) = \big(1, 1\big)$.
(b) If $D_1 > D_2$, then $t_0 < t'_0$, and as the similar method presented above, we have the following different cases:
(b_1) When $0 \leq t < t_0$, then the optimal proportional reinsurance strategy is $q^*(t) = \big(q_1^*(t), q_2^*(t)\big)$, where $q_1^*(t)$ and $q_2^*(t)$ are equal to $\hat{q}_1(t)$ and $\hat{q}_2(t)$ given in (24), respectively.
(b_2) When $t \geq t_0$, then $\hat{q}_1(t) > 1$, therefore, we let $\hat{q}_1^*(t) = 1$. Putting $\hat{q}_1^*(t) = 1$ into the last part of equation (19), we obtain the optimization problem

$$\begin{aligned} \inf_{q_2(t)} L\big(q(t)\big) &= \inf_{q_2(t)} \big\{ y e^{r(T-t)} \big((1+\delta)\big((1-q_2(t))a_2 + \beta(1-q_2(t))^2\zeta_2^2\big) + a_2 q_2(t) + a_1\big) \\ &\quad + \frac{1}{2} y(y+\Upsilon_1) e^{2r(T-t)} \big(\zeta_2^2 q_2^2(t) + \zeta_1^2 + 2\lambda\mu_1\mu_2 q_2(t)\big\}. \end{aligned} \tag{37}$$

For $t \leq T$, it can be shown that the minimum value of $q_1(t)$ in the problem (27) is $\tilde{\tilde{q}}_1(t)$ as given in (29), therefore, for $t_0 \leq t < t'_1$, the optimal proportional reinsurance strategy is $q^*(t) = \big(q_1^*(t), q_2^*(t)\big) = \big(1, \tilde{\tilde{q}}_1(t)\big)$.
(b_3) When $t'_1 \leq t \leq T$, it is easy to see that the optimal proportional reinsurance strategy is $q^*(t) = \big(q_1^*(t), q_2^*(t)\big) = \big(1, 1\big)$.
Now substituting the optimal reinsurance strategy (16) and $\big(q_1^*(t), q_2^*(t)\big)$ into (24), and solve the equation, we have

$$\frac{\partial H_2(t)}{\partial t} - (a + \gamma l_1) H_2(t) + \frac{l_2^2 (y + \Upsilon_3)}{2y} H_2^2(t) - \frac{y\gamma^2}{2(y + \Upsilon_2)} = 0, \tag{38}$$

and

$$\frac{\partial H_1(t)}{\partial t} + abH_2(t) + L\big(q_1^*(t), q_2^*(t)\big) - yce^{r(T-t)} = 0. \tag{39}$$

Solving equation (30) leads to $H_2(t)$ given in (17). On the other hand, since the equation (31) depends on $\big(q_1^*(t), q_2^*(t)\big)$, therefore, to solve it we consider $D_1 \leq D_2$ and $D_1 < D_2$ for the following different cases.
(a) If $D_1 \leq D_2$, we have the following different cases:
(a_1) When $0 \leq t < t'_0$, use the strategies in (24), from (31) we get

$$H_1(t) = H_{11}(t) = \int_t^T \big(abH_2(s) + L\big(\hat{q}_1(s), \hat{q}_2(s)\big) - yce^{r(T-s)}\big) ds + c_1, \tag{40}$$

where c_1 is a constant that will be determined later.
(a_2) When $t'_0 \leq t \leq t_1$, we use the strategy $q^*(t) = \big(q_1^*(t), q_2^*(t)\big) = \big(\tilde{q}_1(t), 1\big)$ into (31), we get

$$H_1(t) = H_{12}(t) = \int_t^T \big(abH_2(s) + L\big(\tilde{q}_1(t), 1\big) - yce^{r(T-s)}\big) ds + c_2, \tag{41}$$

where c_2 is a constant that will be determined later.
(a_3) When $t_1 \leq t \leq T$, we use the strategy $q^*(t) = \big(q_1^*(t), q_2^*(t)\big) = (1, 1)$ into (31), we get

$$H_1(t) = H_{13}(t) = \int_t^T \big(abH_2(s) + L(1,1) - yce^{r(T-s)}\big) ds. \tag{42}$$

On the other hand, since for any $(t, x, \alpha) \in [0, T] \times R \times R^+$, $G(t, x, \alpha)$ is a solution of HJB equation (20), this means that the equalities

$$\begin{aligned} &H_{11}(t'_0) = H_{12}(t'_0), && H_{12}(t_1) = H_{13}(t_1), \\ &\frac{\partial H_{11}(t'_0)}{\partial t'_0} = \frac{\partial H_{12}(t'_0)}{\partial t'_0}, && \frac{\partial H_{12}(t_1)}{\partial t_1} = \frac{\partial H_{13}(t_1)}{\partial t_1}, \end{aligned}$$

must be hold. By solving these equalities, we derive the constants c_1 and c_2 as

$$c_2 = \int_{t_1}^{T} \big(L(1,1) - L\big(\tilde{q}_1(s),1\big)\big)ds, \qquad c_1 = \int_{t_0'}^{T} \big(L\big(\tilde{q}_1(s),1\big) - L\big(\hat{q}_1(s),\hat{q}_2(s)\big)\big)ds + c_2.$$

(b) If $D_1 > D_2$, we have the following different cases:
(b_1) When $0 \le t < t_0$, use the strategies in (28) and (29), from (31) we get

$$H_1(t) = H_{14}(t) = \int_t^T \big(abH_2(s) + L\big(\hat{q}_1(s),\hat{q}_2(s)\big) - yce^{r(T-s)}\big)ds + c_3, \tag{43}$$

where c_3 is a constant that will be determined later.
(b_2) When $t_0 \le t < t_1'$, we use the strategy $q^*(t) = \big(q_1^*(t), q_2^*(t)\big) = \big(1, \tilde{q}_2(t)\big)$ into (31), we get

$$H_1(t) = H_{15}(t) = \int_t^T \big(abH_2(s) + L\big(1,\tilde{q}_2(t)\big) - yce^{r(T-s)}\big)ds + c_4, \tag{44}$$

where c_4 is a constant that will be determined later.
(b_3) When $t_1' \le t \le T$, we use the strategy $q^*(t) = \big(q_1^*(t), q_2^*(t)\big) = (1,1)$ into (31), we get

$$H_1(t) = H_{13}(t) = \int_t^T \big(abH_2(s) + L(1,1) - yce^{r(T-s)}\big)ds. \tag{45}$$

As the same before, since for any $(t,x,\alpha) \in [0,T] \times R \times R^+$, $G(t,x,\alpha)$ is a solution of HJB equation (20), this means that the equalities

$$\begin{aligned} &H_{14}(t_0) = H_{15}(t_0), && H_{13}(t_1') = H_{15}(t_1'),\\ &\frac{\partial H_{14}(t_0)}{\partial t_0} = \frac{\partial H_{15}(t_0)}{\partial t_0}, && \frac{\partial H_{13}(t_1')}{\partial t_1'} = \frac{\partial H_{15}(t_1')}{\partial t_1'}, \end{aligned}$$

must be hold. By solving these equalities, we derive the constants c_3 and c_4 as

$$c_4 = \int_{t_1'}^{T} \big(L(1,1) - L\big(1,\tilde{q}_2(s)\big)\big)ds, \qquad c_3 = \int_{t_0}^{T} \big(L\big(1,\tilde{q}_2(s)\big) - L\big(\hat{q}_1(s),\hat{q}_2(s)\big)\big)ds + c_4,$$

and this completes the proof. □

We note that from the relation (22), for $k = 2$, the minimum value $\omega^*(t,\alpha(t)) = \big(\omega_1^*(t), \omega_2^*(t,\alpha(t)), \omega_3^*(t,\alpha(t))\big)$ is given by

$$\begin{aligned} &\omega_1^*(t) = \Upsilon_1 e^{r(T-t)}\big(\zeta_1^2 q_1^{*2}(t) + \zeta_2^2 q_2^{*2}(t) + 2\lambda\mu_1\mu_2 q_1^{*2}(t) q_2^{*2}(t)\big)^{\frac{1}{2}},\\ &\omega_2^*(t,\alpha(t)\frac{\gamma\Upsilon_2}{y+\Upsilon_2}\sqrt{\alpha(t)},\\ &\omega_3^*(t,\alpha(t)) = -\frac{\Upsilon_3}{y} l_2 H_2(t)\sqrt{\alpha(t)}. \end{aligned} \tag{46}$$

4. Verification theorem

In this section, we prove a verification theorem to show the optimal strategy $s^*(t) = \big(\pi^*(t), q^*(t)\big)$ given in Theorem 3.2. This verification theorem is based on the Corollary 1.2 in [23]. To do it, we use some mathematical expressions and give the admissibility and some properties of optimal strategy $s(t) = \big(\pi(t), q(t)\big)$ in Theorem 3.2.

Theorem 4.1
Suppose that the conditions in Theorem 3.2 hold and $s^*(t) = \big(\pi^*(t), q^*(t)\big)$ candidates for the optimal strategy $s(t) = \big(\pi(t), q(t)\big)$ given in that theorem, then we have the following properties for the optimal strategy $s^*(t)$ and objective function $G(t, x, \alpha)$:
1) $s^*(t)$ is an admissible strategy.
2) For $Q \in \mathcal{P}$, $E^Q\big(\sup_{t\in[0,T]} \big|G(t, X^{s^*}(t), \alpha(t))\big|^4\big) < \infty$, where Q is defined by $a(t)$ in equation (12) with $\omega(t, \alpha(t))$.
3) $E^Q\big(\sup_{t\in[0,T]} \big|\frac{1}{2}\big(\frac{(\omega_1^*(t))^2}{\Delta_1(t)} + \frac{(\omega_2^*(t,\alpha(t)))^2}{\Delta_2(t)} + \frac{(\omega_3^*(t,\alpha(t)))^2}{\Delta_3(t)}\big)\big|^2\big) < \infty$.

Proof
We will prove the above properties one by one. But to make the proof process to be understood easily, we firstly prove the property (2) and then prove properties (1) and (3), respectively.
Proof of property (2). Substituting the strategy $\pi^*(t, \alpha(t))$ and $q^*(t) = \big(q_1^*(t), q_2^*(t)\big)$ into the equation (13), then we have

$$
\begin{aligned}
dX^{s^*}(t) &= ue^{rt} + \int_0^t e^{r(t-s)}\big\{A_1(q^*(s)) - \Upsilon_1 e^{r(T-s)} A_2(q^*(s)) \\
&\quad + \frac{y}{y+\Upsilon_2}\gamma^2\alpha(s)e^{r(T-s)}\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)\big\} \\
&\quad + \int_0^t e^{r(t-s)}\big(A_2(q^*(s))\big)^{\frac{1}{2}} dB_0^Q(s) \\
&\quad + \int_0^t e^{r(T-s)}\gamma\big(\alpha(s)\big)^{\frac{1}{2}}\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big) dB_1^Q(s),
\end{aligned}
\tag{47}
$$

where

$$
A_1(q^*(s)) = c - \eta(q^*(s)) - \sum_{m=1}^{2} a_m q_m^*(s)
$$

$$
A_2(q^*(s)) = \sum_{m=1}^{k} \zeta_m^2 q_m^{*2}(s) + 2\lambda\mu_1\mu_2 q_1^{*2}(s) q_2^{*2}(s)
$$

Then using the wealth process (47) for the given objective function, we get

$$
\begin{aligned}
\big|G(t, X^{s^*}(t), \alpha(t))^4\big| &= \frac{1}{y^4}\exp\big\{-4yX^{s^*}(t)e^{r(T-t)} + 4H_1(t) + 4H_2(t)\alpha(t)\big\} \\
&\le M_1 \exp\big\{-4yX^{s^*}(t)e^{r(T-t)}\big\} \\
&\le M_2 \exp\big\{-4y\int_0^t \frac{y}{y+\Upsilon_2}\gamma^2\alpha(s)\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)ds \\
&\quad -4y\int_0^t e^{r(T-s)}\big(A_2(q^*(s))\big)^{\frac{1}{2}} dB_0^Q(s) \\
&\quad -4y\int_0^t \gamma\big(\alpha(s)\big)^{\frac{1}{2}}\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big) dB_1^Q(s)\big\},
\end{aligned}
\tag{48}
$$

where M_1 and M_2 are two constants which satisfy the following inequalities:

$$
M_1 \ge \frac{1}{y^4}\exp\big(4H_1(t) + 4H_2(t)\alpha(t)\big),
$$

and

$$M_2 \geq M_1 \exp\big(-4y\big(ue^{rT} + \int_0^t e^{r(T-s)}\big(A_1(q^*(s)) - \omega_1 e^{r(T-s)} A_2(q^*(s))\big)\big),$$

for any $t \in [0,T]$. Moreover, since $\big(A_2(q^*(s))\big)^{\frac{1}{2}} e^{r(T-s)}$ is bounded for any $s \in [0,T]$, then we have the following exponential integral:

$$\begin{aligned}\exp\big(\int_0^t -4ye^{r(T-s)}\big(A_2(q^*(s))\big)^{\frac{1}{2}} dB_0^Q(s)\big) \quad &= \quad \exp\big(\int_0^t 8y^2 e^{2r(T-s)} A_2(q^*(s))ds\big) \\ &\times \exp\big(-\int_0^t 8y^2 e^{2r(T-s)} A_2(q^*(s))ds \\ &+ \int_0^t -4ye^{r(T-s)}\big(A_2(q^*(s))\big)^{\frac{1}{2}} dB_0^Q(s)\big).\end{aligned}$$

Therefore, it follows that

$$E^Q\big\{\exp\big(\int_0^t -4ye^{r(T-s)}\big(A_2(q^*(s))\big)^{\frac{1}{2}} dB_0^Q(s)\big)\big\} < \infty. \tag{49}$$

Now, we try to find an estimator for the expression

$$\begin{aligned}&\exp\big\{-4y\int_0^t \frac{y}{y+\Upsilon_2}\gamma^2\alpha(s)\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)ds \\ &-4y\int_0^t \gamma\big(\alpha(s)\big)^{\frac{1}{2}}\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)dB_1^Q(s)\big\}.\end{aligned}$$

First, consider the following equality

$$\begin{aligned}&\exp\big\{-4y\int_0^t \frac{y}{y+\Upsilon_2}\gamma^2\alpha(s)\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)ds - 4y\int_0^t \gamma\big(\alpha(s)\big)^{\frac{1}{2}}\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)dB_1^Q(s)\big\} \\ &= \exp\big\{-16y^2\int_0^t \theta^2\alpha(s)\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)^2 ds \\ &-4y\int_0^t \gamma\big(\alpha(s)\big)^{\frac{1}{2}}\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)dB_1^Q(s)\big\} \\ &\times \exp\big\{\int_0^t \big[16y^2\gamma^2\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)^2 ds \\ &-4y^2\frac{1}{y+\Upsilon_2}\gamma^2\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)\big]\alpha(s)ds\big\} \\ &= C_1 + C_2,\end{aligned}$$

where

$$C_1 = \exp\big\{-16y^2\int_0^t \gamma^2\alpha(s)\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)^2 ds - 4y\int_0^t \gamma\big(\alpha(s)\big)^{\frac{1}{2}}\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)dB_1^Q(s)\big\},$$

and

$$C_2 = \exp\big\{\int_0^t \big[16y^2\gamma^2\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)^2 ds - 4y^2\frac{1}{y+\Upsilon_2}\gamma^2\big(\frac{1}{y+\Upsilon_2} + \frac{l_1}{y\gamma}N(s)\big)\big]\alpha(s)ds\big\}.$$

For the term C_1, we have

$$\begin{aligned} E^Q(C_1^2) &= E^Q\Big(\exp\Big\{-32y^2\int_0^t \gamma^2\alpha(s)\big(\frac{1}{y+\Upsilon_2}+\frac{l_1}{y\gamma}N(s)\big)^2 ds \\ &\quad -8y\int_0^t \gamma\big(\alpha(s)\big)^{\frac{1}{2}}\big(\frac{1}{y+\Upsilon_2}+\frac{l_1}{y\gamma}N(s)\big)dB_1^Q(s)\Big\}\Big) < \infty, \end{aligned} \tag{50}$$

since C_1^2 is a supermartingale. On the other hand, $-8y\gamma\big(\frac{1}{y+\Upsilon_2}+\frac{l_1}{y\gamma}N(s)\big)$ is bounded on the interval $[0,T]$, therefore, according to Lemma 4.3 in [24], C_1^2 is a martingale.
For the term C_2, we have

$$\begin{aligned} E^Q(C_2^2) &= E^Q\Big(\exp\Big\{32y^2\int_0^t \gamma^2\alpha(s)\big(\frac{1}{y+\Upsilon_2}+\frac{l_1}{y\gamma}N(s)\big)^2 ds \\ &\quad -8y^2\frac{\gamma^2}{y+\Upsilon_2}\alpha(s)\big(\frac{1}{y+\Upsilon_2}+\frac{l_1}{y\gamma}N(s)\big)ds\Big\}\Big). \end{aligned}$$

On the other hand, by Theorem 5.1 in [24], we have the inequality

$$32y^2\gamma^2\big(\frac{1}{y+\Upsilon_2}+\frac{l_1}{y\gamma}N(s)\big)^2 - 8y^2\frac{\gamma^2}{y+\Upsilon_2}\big(\frac{1}{y+\Upsilon_2}+\frac{l_1}{y\gamma}N(s)\big) \le \frac{a^2}{2(l_1+l_2)},$$

which is sufficient condition for

$$E^Q(C_2^2) < \infty. \tag{51}$$

Moreover, applying the relations (48)-(51), and Cauchy-Schwartz inequality, for any $t \in [0,T]$, we have

$$\begin{aligned} E^Q\big(\big|G(t,X^{s^*}(t),\alpha(t))\big|^4\big) &\le M_2E^Q\Big\{\exp\Big(\int_0^t -4ye^{r(T-s)}\big(A_2(q^*(s))\big)^{\frac{1}{2}}dB_0^Q(s)\Big)\Big\}E^Q\big(C_1C_2\big) \\ &\le M_2E^Q\Big\{\exp\Big(\int_0^t -4ye^{r(T-s)}\big(A_2(q^*(s))\big)^{\frac{1}{2}}dB_0^Q(s)\Big)\Big\} \\ &\quad \times\big(E^Q(C_1^2)E^Q(C_2^2)\big)^{\frac{1}{2}} < \infty, \end{aligned} \tag{52}$$

and this completes the proof of part (2).
Proof of property (1). From the process of solving HJB equation, we know the optimal strategy $s^*(t)$ is progressively measurable. From the equations (16), (25) and (26), the optimal proportional reinsurances $q_1^*(t)$ and $q_2^*(t)$ are deterministic and state independent. On the other hand, the optimal strategy $s^*(t,\alpha(t))$ is a mean-reverting square root process, although it is generally an unbounded random variable for any fixed given time, its first and second order moments are bounded (the detailed proof for the boundedness of its moments can be found in ([25], p. 308), thus condition (1) in Definition 1 holds. For the condition (2) of Definition 1, by the proof of property (2), we have deduced that the solution of equation (13) has the form of equation (47). Moreover, in property (2), we proved that $E^Q\big(\big|G(t,X^{s^*}(t),\alpha(t))\big|^4\big) < \infty$ for $t \in [0,T]$, which by the similar method we can prove that $E^Q\big(\big|G(T,X^{s^*}(T),\alpha(T))\big|^4\big) < \infty$. Therefore, $s^*(t)$ is an admissible strategy.
Proof of property (3). Let $Z(t) = \frac{1}{2}\big(\frac{y(\omega_1^*(t))^2}{\Upsilon_1}+\frac{y(\omega_2^*(t,\alpha(t)))^2}{\Upsilon_2}+\frac{y(\omega_3^*(t,\alpha(t)))^2}{\Upsilon_3}\big)$. Since for any $l > 0$, $E^Q\big(\alpha(t)\big)^l < \infty$, then by (34) we result that

$$E^Q\big(Z(t)\big)^4 < \infty. \tag{53}$$

From equation (15), we have $O(t,x,\alpha) = -\frac{\Upsilon_1}{y\Delta_1} = -\frac{\Upsilon_2}{y\Delta_2} = -\frac{\Upsilon_3}{y\Delta_3}$. Therefore, applying the Cauchy-Schwarz inequality and the inequalities (52) and (53), we have

$$
\begin{aligned}
&E^Q\Big(\sup_{t\in[0,T]}\Big|\frac{1}{2}\Big(\frac{(\omega_1^*(t))^2}{\Delta_1(t)} + \frac{(\omega_2^*(t,\alpha(t)))^2}{\Delta_2(t)} + \frac{(\omega_3^*(t,\alpha(t)))^2}{\Delta_3(t)}\Big)\Big|^2\Big) \\
&= E^Q\Big(\sup_{t\in[0,T]}\big|Z(t)G(t,X^{s^*}(t),\alpha(t))\big|^2\Big) \\
&= E^Q\Big(\sup_{t\in[0,T]}\big|Z(t)\big|^2.\big|G(t,X^{s^*}(t),\alpha(t))\big|^2\Big) \\
&\leq \Big(E^Q\Big(\sup_{t\in[0,T]}\big|Z(t)\big|^4\Big)^{\frac{1}{2}} \times \Big(E^Q\Big(\sup_{t\in[0,T]}\big|G(t,X^{s^*}(t),\alpha(t))\big|^4\Big)^{\frac{1}{2}} < \infty,
\end{aligned}
$$

and this completes the proof. □

5. Numerical examples

In this section, we present some numerical examples to analyze the theoretical results and investigate the sensitivity of optimal strategies on some parameters. To investigate the sensitivity of optimal strategies, we consider two dependent policyholders with claim amounts random X_i^1 and X_i^2 follow exponential distribution with parameters $\alpha_1 = 1$ and $\alpha_2 = 2$, respectively. The Poisson processes $N_1(t)$, $N_2(t)$ and $N(t)$ within the interval time $[0,T]$, with $T = 10$, have the intensity parameters $\lambda_1 = 2$, $\lambda_2 = 4$, and $\lambda = 2.5$, respectively. We suppose that the volatility of the stock price rate has the form

$$\tau(t) = rt^{\frac{1}{2}}.$$

Moreover, to compute the optimal reinsurance strategy we use the standard Brownian motions as given in (6.1) of [26] and the basic parameters of risk model are given in Table 1.

Table 1. The basic parameters of risk model

a	b	l_1	l_2	r	δ	γ	y	Υ_1	Υ_2
3	4	0.5	0.7	0.08	0.4	0.05	0.5	0.5	0.6

With some computations, we have obtained the values of parameters. These values are reported in Table 2. From

Table 2. The computed parameters of risk model

μ_1	μ_2	a_1	a_2	ν_1	ν_2	ζ_1^2	ζ_2^2	D_1	D_2	D_3
1	0.5	4.5	3.25	2	0.5	9	3.25	10.56	23.625	27.69

this table, we see that $D_1 < D_2$ and $\delta D_2 < (y+\Upsilon_1)D_3$. Thus, we set $t_0' = 10$ and the optimal proportional reinsurance strategy is $\big(q_1^*(t), q_2^*(t)\big) = \big(\hat{q}_1(t), \hat{q}_2(t)\big)$. In Figures 1 and 2, we investigate the effect of intensity parameter λ and insurer's ambiguity aversion Υ_1 to the risk model from the reinsurance market on the optimal reinsurance proportion. Figure 1 shows that the reinsurance proportions in both two lines of business of the insurance company decrease with the λ increasing, which means that the higher dependence of the two lines of business in an insurance company, the lower reinsurance retention proportion is arranged by the insurer. From Figure 2, we find that when the insurer has higher ambiguity-averse level Υ_1 on the parameters of insurance market, he/she will arrange lower reinsurance retention proportion.

Figure 3 shows the effect of absolute risk aversion coefficient y on the optimal investment strategy. We find that no matter under the risk model, the more risk averse the insurer is, the less proportion of the insurance surplus is invested in the risky asset. Figure 4 shows the effect of ambiguity-averse level Υ_2 on the optimal investment

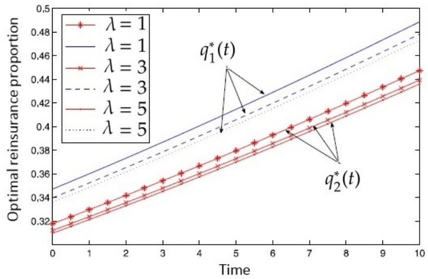

Figure 1. Effect of intensity parameter λ on the optimal reinsurance proportion

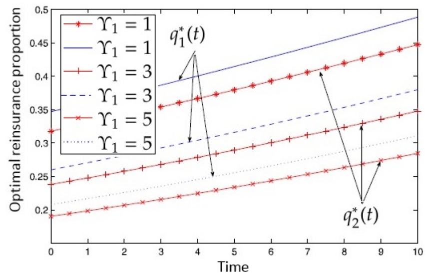

Figure 2. Effect of insurer's ambiguity aversion Υ_1 on the optimal reinsurance proportion

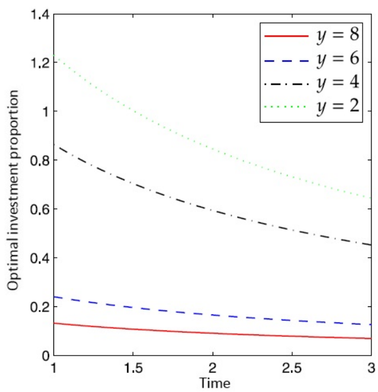

Figure 3. Effect of intensity parameter y on the optimal investment proportion

strategy. We find that when the insurer has higher ambiguity-averse level Υ_2 on the parameters of insurance market, he/she will arrange lower investment strategy. Figure 5 shows the effect of parameter γ on the optimal investment strategy. It shows that with the increase of parameter γ, the corresponding investment proportion increases too.

6. Concluding remarks

In this paper, we investigated the optimal risk management strategies for an insurer with a diffusion approximation in a dependent compound Poisson process. We assumed that the dependent risk model consists of the constant reinsurance premium rate, combination of the number of claims occurring by policyholders within a finite time, and perturbed by correlated standard Brownian motions, where the price of the risk-free bond is described by a stochastic differential equation. We derived the optimal strategies and solution of the associated HJB equation for the optimization problem. A verification theorem using some mathematical expressions were given to guarantee the

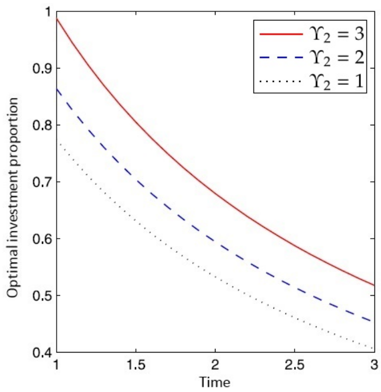

Figure 4. Effect of insurer's ambiguity aversion Υ_2 on the optimal investment proportion

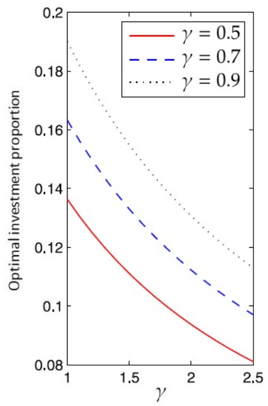

Figure 5. Effect of parameter γ on the optimal investment proportion

optimal strategy and studied the admissibility and some properties of optimal strategy. Furthermore, in the given numerical examples the effects of intensity parameter λ and insurer's ambiguity aversion Υ_1 to the risk model from the reinsurance market on the optimal reinsurance proportion and the effects of risk aversion coefficient y, insurer's ambiguity aversion Υ_2 and parameter γ on the optimal investment strategy are investigated.

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