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The Alpha Power Modified Weibull-Geometric Distribution: A Comprehensive Mathematical Framework with Simulation, Goodness-of-fit Analysis and Informed Decision making using Real Life Data

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Abstract A new lifetime distribution by compounding the Alpha Power Modified Weibull distribution, named Alpha Power Modified Weibull-Geometric distribution is introduced and discussed.The compounding of the distribution is motivated from the failure time of the system with series structure, where only the minimum lifetime value is considered. Various Statistical properties of the proposed distribution are investigated. Maximum likelihood estimation method is used to estimate the model parameter. To assess the performance of the proposed method, Monte Carlo simulation study is conducted using various choices of effective sample size and parameter value. Finally, to illustrate the capability and flexibility of the proposed distribution, three real life data sets are considered and showed that the proposed distribution is more compatible by comparing with other competing lifetime distributions.

Keywords Compounding, Failure Times, Maximum Likelihood Estimation, Lifetime Distribution, Series Structure

AMS 2010 subject classifications 60-XX, 60E05

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1. Introduction

In the new era of statistical literature, researchers have shown an increased interest on finding and developing different ways of expanding the existing families of distributions by incorporating one or more additional parameters to the well-known distributions. Their motivation is to make the well known and existing distributions more flexible with wider characteristics so that it can be used for modeling data in various disciplines.Through this numerous distributions of modeling lifetime data has also been introduced.

[3] proposed a way to add a parameter to the lifetime distribution through compounding based on the failure time of a series or parallel system with unknown components. This approach generates a new distribution with wider characteristics which is more flexible in modeling many complex phenomenon; it also gives new distributions that extend well-known families of distributions. The flexibility of such compound distributions comes in terms of having better fit, more variation of the hazard rate function that maybe decreasing, increasing, bathtub shaped, etc. Many extension of their work has been done since then. Among such compound distributions introduced recently is the exponential-geometric distribution with decreasing hazard rate developed by compounding geometric and

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exponential distributions (EG) by [4]. In the same way, [7] and [8], respectively, introduced the exponential-Poisson (EP) and exponential logarithmic (EL) distributions, which have decreasing hazard rates, and studied their properties.

In practice, Weibull distribution has been considered as a multifaceted distribution due to having various shapes of failure rate function i.e increasing, decreasing and constant. This make the Weibull distribution capable of modeling monotone failure rate. However, In many real life phenomenon like biological, reliability studies and many more, we encounter non-monotonic failure rate such as bathtub shape, upside down bathtub shape and unimodal shape failure rates and Weibull distribution does not provide a reasonable parametric fit for modeling phenomenon. Hence many generalizations of the Weibull distribution have been attempted by various researchers. Some of the generalized Weibull distributions developed by following the idea of [1] by compounding the Weibull distribution or its generalized form with other distributions are namely, the Weibull-geometric [10], Weibull-Poisson [11], Exponential-Weibull [12], Modified Weibull geometric [13], Additive Weibull-geometric [14], Exponentiated Transmuted Weibull geometric [16],Exponentiated Inverse Weibull-geometric [15], Weilbull lindley [18] and Inverse Weibull-geometric [19]. Some other recent proposed distribution considering the samilar approach are , the Ishita Power Series [21], Inverse Gamma Power Series[22], Inverse Lindeley Power Series [24], Pareto-Poisson [25], Unit Gambertz Power Series [26] and the Zhang-Power Series[27].

[20] introduced a relatively new distribution called the Alpha power modified Weibull (APMW) distribution which is derived based on the alpha power transformation method suggested by [17] which has increasing failure rate, decreasing failure rate and unimodal probability density function (pdf) and can have increasing failure rate, decreasing failure rate, increasing-decreasing-increasing, bathtub and upside-down bathtub hazard rate function (hrf). It was also shown by analysing three real life data sets that this distribution provide better fit compared to other competing distribution. Thus, this make the APMW distribution a good member of the compounding distribution instead of other generalized distribution.

The aim of this paper is to introduced a new lifetime distribution by compounding the Alpha Power Modified Weibull distribution and the geometric distribution. The main motivations for this study are as follows:
(i) the proposed distribution have various shapes of failure rate function, indicating the great flexible nature of the distribution, which is also one of the important characteristics for modeling lifetime scenario;
(ii) The proposed distribution is capable of modeling the first failure of a system having a series structure;
(iii) The proposed distribution compares well with other competing distribution in modeling survival and failure as shown by three real data applications.

The rest of the paper is organized as follows: In Section 2, we introduce the new Alpha Power Modified Weibull Geometric (APMWG) distribution by compounding the Alpha Power Modified Weibull distribution and the geometric distribution. In Section 3, we discuss some of its statistical properties. In Section 4, the estimation of the parameters is performed by the method of maximum likelihood estimation. In Section 5, a simulation study has been performed to illustrate the behavior of the MLEs in terms of different sample size n and different sets of parameter value. Three real data sets are studied to illustrate the importance of the distribution in section 6. Finally, we conclude the paper in section 7.

2. The Alpha Power Modified Weibull-Geometric Distribution (APMWG)

In this section, we introduce the new lifetime distribution known as the Alpha Power Modified Weibull-Geometric (APMWG) distribution. The new distribution is defined as follows. Suppose that a system has n-components in series structure, assumed to be independent and identically distributed at a given time, where the lifetime of the i^{th}-components is denoted by X_i(i=1,2,3,...n). Further, suppose that the random variable N follow Geometric distribution with probability mass function (pmf) given by

$$P(N=n)=(1-p)p^{n-1},\ n=1,2,3,..., \text{and}\ \ 0<p<1 \tag{1}$$

and the lifetime X_i(i=1,2,3,..n) follows Alpha Power Modified Weibull distribution with cumulative distribution function (cdf) and probability density function (pdf) respectively given by

$$F_{APMW}(x,\theta) = F(x) = \frac{\alpha^{1-e^{-(\lambda x)^{\gamma}}} - 1}{\alpha - 1} \text{ if } \alpha > 0, \alpha \neq 0 \tag{2}$$

and

$$f_{APMW}(x,\theta) = f(x) = \frac{log\alpha}{\alpha - 1}\gamma\lambda^{\gamma}e^{\gamma-1}\alpha^{1-e^{-(\lambda x)^{\gamma}}}e^{-(\lambda x)^{\gamma}}; \text{if } \alpha > 0, \alpha \neq 0. \tag{3}$$

As the system is having a series structures thus, the failure time of the system is

$$X = min(X_1, X_2, ...X_n)$$

Then the conditional cdf of X given N=n is given as

$$F(x \mid n;\tau) = 1 - (1 - F(x))^n = 1 - \left[1 - \frac{\alpha^{1-e^{-(\lambda x)^{\gamma}}} - 1}{\alpha - 1}\right]^n. \tag{4}$$

So, the required marginal cdf of X is given by

$$F_{APMWG}(x;\theta,p) = \frac{1 - \alpha^{1-e^{-(\lambda x)^{\gamma}}}}{1 - \alpha + \alpha p - p\alpha^{1-e^{-(\lambda x)^{\gamma}}}} , \tag{5}$$

where, $x > 0$, $\theta > 0$ and $0 < p < 1$, and the corresponding pdf of X is given by

$$f_{APMWG}(x;\theta,p) = \frac{(\alpha + p - p\alpha - 1)log\alpha\gamma\lambda^{\gamma}x^{\gamma-1}\alpha^{1-e^{-(\lambda x)^{\gamma}}}e^{-(\lambda x)^{\gamma}}}{\left[1 - \alpha + \alpha p - p\alpha^{1-e^{-(\lambda x)^{\gamma}}}\right]^2}. \tag{6}$$

The survival function and the hazard rate function (hrf) of APMWG distribution are given, respectively, by

$$S_{APMWG}(x;\theta,p) = \frac{(p-1)\left(\alpha - \alpha^{1-e^{-(\lambda x)^{\gamma}}}\right)}{1 - \alpha + \alpha p - p\alpha^{1-e^{-(\lambda x)^{\gamma}}}} \tag{7}$$

and

$$h_{APMWG}(x;\theta,p) = \frac{(\alpha + p - p\alpha - 1)log\alpha\gamma\lambda^{\gamma}x^{\gamma-1}\alpha^{1-e^{-(\lambda x)^{\gamma}}}e^{-(\lambda x)^{\gamma}}}{(p-1)\left(\alpha - \alpha^{1-e^{-(\lambda x)^{\gamma}}}\right)\left(1 - \alpha + \alpha p - p\alpha^{1-e^{-(\lambda x)^{\gamma}}}\right)} , \tag{8}$$

Where, θ=(α,γ,λ) is considered to be the set of parameters.

Lemma 1: The hrf of the APMWG distribution is
i. decreasing if η'(x) $<$ 0.
ii. increasing if $\eta'(x) > 0$.
iii. bathtub if there exist x= x_o, such that $\eta'(x_o)$ = 0, η'(x) $<$ 0 for x $<$ x_0 and η'(x) $>$ 0 for x $>$ x_0.
iv. upside down bathtub if there exist x= x_o, such that $\eta'(x_o)$ = 0, η'(x) $<$ 0 for x $>$ x_0 and $\eta'(x) > 0$ for x $<$ x_0.
For η (x) = $\frac{f'(x;\theta,p)}{f(x;\theta,p)}$, $f'(x;\theta,p)$ is the first derivative of the pdf f(x;θ,p) of APMWG distribution with respect to x and η' (x) is the first derivative of η (x) with respect to x. The lemma was introduced and proven by [1], and it is widely recognized as Glaser's Theorem.

In addition to the above mathematical derivation, Figure 1 and 2 represent the graphical representation of the pdf and the hrf, showing the possible shapes for some selected values of the parameters respectively. Figure 1 and 2 shows that the APMWG Distribution has a very supple and agile trait , with the pdf having a diverse unimodal

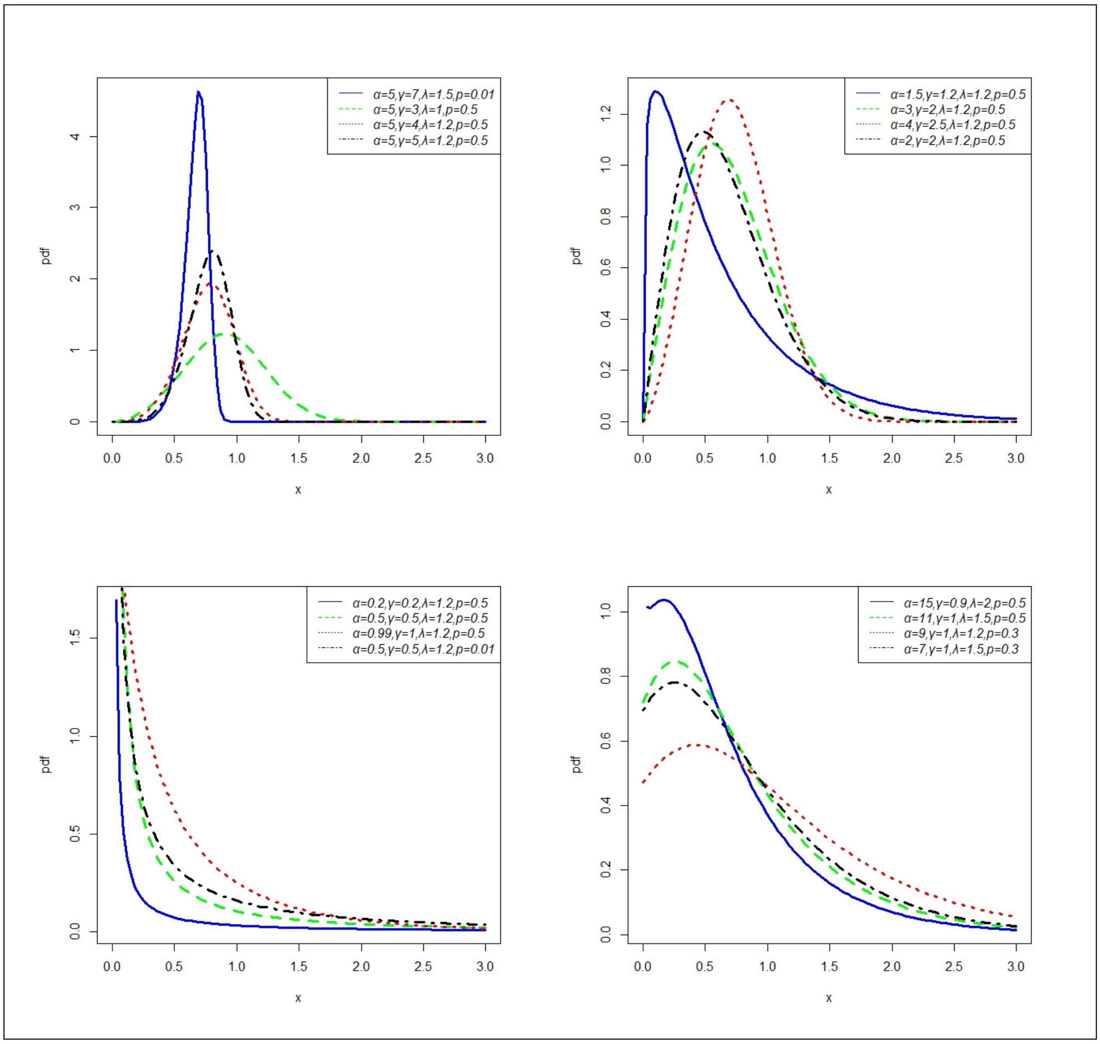

Figure 1. Plots of the pdfs of APMWG distribution for four different set of parameters values

shapes and the hrf with increasing, decreasing, bathtub, upsidedown bathtub and increasing-decreasing-increasing shapes.

The graphical statistical analysis of Figure 1 and 2, shows that the peak of the pdf of APMWG distribution sharpen gradually as we increase the value of the parameter α and γ, where as the pdf flattens and stretches with the decrease in the value of the parameter λ, giving a heavy-tailed behaviour. Similarly, We observe that the shape of the hrfs changes significantly with each of the four parameters, resulting in a wide range of dynamic hrf behaviors. Additionally, as shown in Figure 2, decreasing the value of parameter p while holding the other parameters constant leads to a noticeable flattening of the hrf.

3. Statistical Properties

In this section various statistical properties of the APMWG distribution have been studied

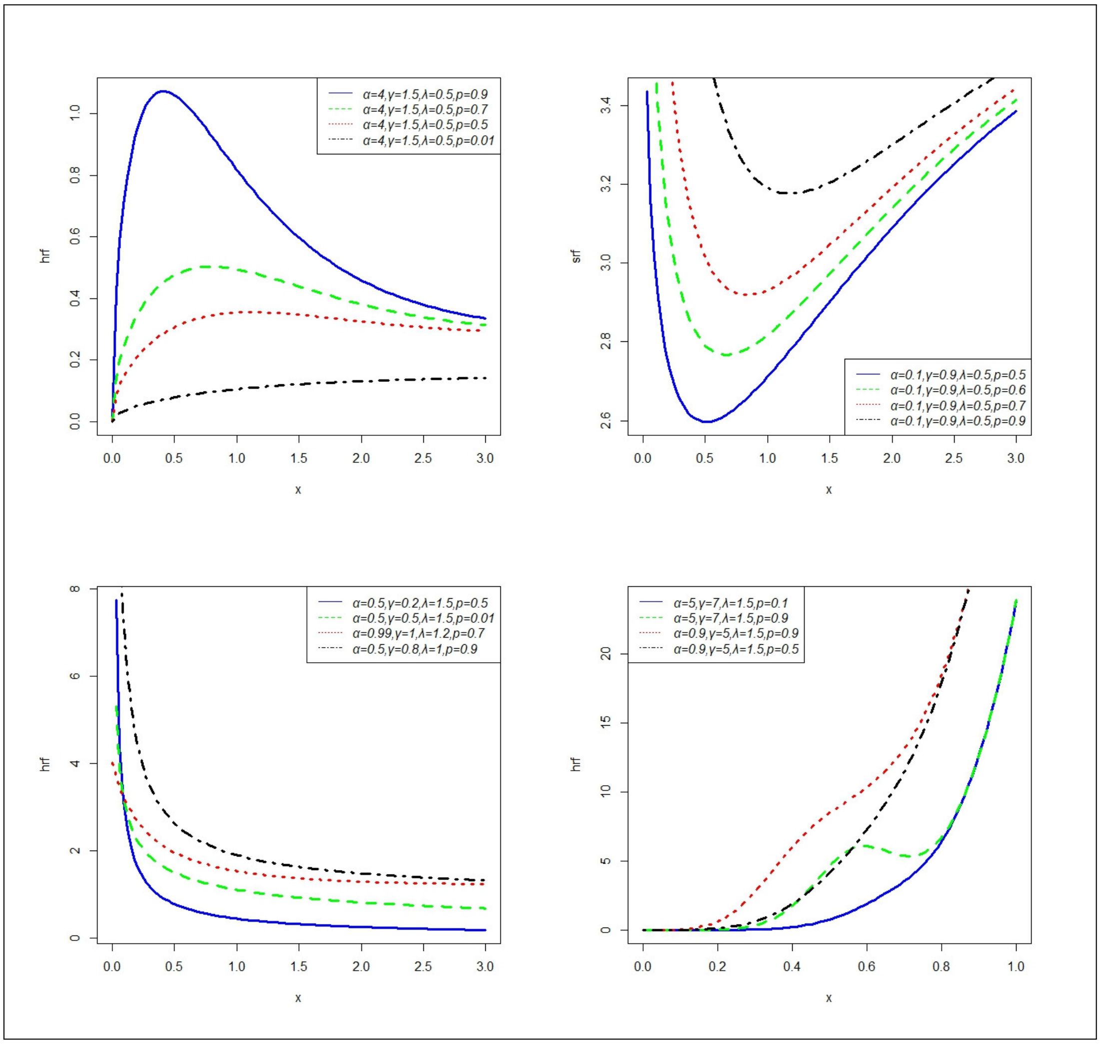

Figure 2. Plots of the hrfs of APMWG distribution for four different set of parameters values

3.1. Quantiles

The quantile function of a random variable X is the inverse of its distribution function. Thus, the quantile function denoted by x_q, for $0 < q < 1$, can be easily obtained from Eq. 5 and is obtained as

$$x_q = \left[\frac{-1}{\lambda^\gamma} log\left(1 - \frac{log\left(\frac{q-q\alpha+p\alpha q-1}{pq-1}\right)}{log\alpha}\right)\right]^{\frac{1}{\gamma}} \tag{9}$$

Hence, the first (25 percentile), second (median or 50 percentile) and the third (75 percentile) quantile of the distribution can be obtained by simply putting q= 0.25, 0.5 and 0.75 respectively in the above equation.

3.2. Moments

Let the rv X have APMWGD with pdf as given in Eq. 6. Then, for r=1,2,...., the r^{th} moment of X is given by

$$\mu_r' = \left(\int_0^\infty x^r f(x)\, dx \right) \tag{10}$$

Substituting Eq. 6 in the above equation, we get

$$\mu_r' = \left[\int_0^\infty x^r \frac{(\alpha + p - p\alpha - 1) log\alpha \gamma \lambda^\gamma x^{\gamma-1} \alpha^{1-e^{-(\lambda x)^\gamma}} e^{-(\lambda x)^\gamma}}{\left[1 - \alpha + \alpha p - p\alpha^{1-e^{-(\lambda x)^\gamma}}\right]^2} dx \right]$$

If $|z| < 1$ and k > 0, we have the series representation

$$(1-z)^{-k} = \sum_{i=0}^{\infty} \frac{\Gamma(k+i) z^i}{\Gamma(k) i!} \tag{11}$$

We also have the series representation

$$\alpha^{-w} = \sum_{k=0}^{\infty} \frac{-(-log\alpha)^k w^k}{k!} \tag{12}$$

Using the series representation in Eq. 11, the binomial expansion and the series representation in Eq.12, the r^{th} order moment of X following the APMWG distribution is obtained as,

$$\mu_r' = \frac{(\alpha + p - p\alpha - 1)\Gamma(\frac{r}{\gamma} + 1)}{\lambda(\gamma)^{\frac{r}{\gamma}}} \sum_{i=0}^{\infty} (i+1) \sum_{m=0}^{i} \binom{i}{m} (\alpha - p\alpha)^m p^{(i-m)} \alpha^{(i-m+1)} \sum_{k=0}^{\infty} \frac{(-log)^{(k+1)} (i-m+1)^k}{k!(k+1)^{\frac{r}{\gamma}+1}}. \tag{13}$$

Thus, the mean of the distribution is obtained as,

$$\mu_1' = \frac{(\alpha + p - p\alpha - 1)\Gamma(\frac{1}{\gamma} + 1)}{\lambda(\gamma)^{\frac{1}{\gamma}}} \sum_{i=0}^{\infty} (i+1) \sum_{m=0}^{i} \binom{i}{m} (\alpha - p\alpha)^m p^{(i-m)} \alpha^{(i-m+1)} \sum_{k=0}^{\infty} \frac{(-log)^{(k+1)} (i-m+1)^k}{k!(k+1)^{\frac{1}{\gamma}+1}}. \tag{14}$$

In addition, using the first four comulants denoted by C_r, the coefficient of skewness and kurtosis can also be calculated from the ordinary moments of X , where

$$C_r = \mu_r' - \sum_{i=0}^{r-1} \binom{r-1}{i-1} C_i \mu_{r-i}' \tag{15}$$

Therefore we have,
$C_1 = \mu_1'$ (mean)
$C_2 = \mu_2' - (\mu_1')^2$ (variance)
$C_3 = \mu_3' - 3\mu_2'\mu_1' + (\mu_1')^3$
$C_2 = \mu_4' - 4\mu_3'\mu_1' - 3(\mu_2')^2 + 12\mu_2'(\mu_1')^2 - 6(\mu_1')^4$
Now, the coefficients of skewness (denoted by CS) and kurtosis (denoted by CK) is respectively obtained, as
i. CS=$\frac{C_3}{C_2^{3/2}}$ and
ii. CK=$\frac{C_4}{C_2^2}$

3.3. Moment Generating Function

The Moment Generating Function of a random variable X plays a very crucial role in Probability theory and statistics, providing the basis for an alternative route of analysis, enabling us to derive moments and probability distribution with ease, as the name suggests. The MGF of X is defined as the expected value of e^{tx}, given by

$$M_X(t) = \left(\int_0^\infty e^{tx} f(x)\, dx \right) \tag{16}$$

Using the Maclaurin series expansion $e^{tx} = \sum_{l=0}^{\infty} \frac{(tx)^l}{l!}$, we have

$$M_X(t) = \left(\int_0^\infty \sum_{l=0}^{\infty} \frac{(tx)^l}{l!} f(x)\, dx \right).$$

In a way similar to the moments, using the series representation in Eq. 11, the binomial expansion, and the series representation in Eq. 12, we have

$$\begin{aligned} M_X(t) = \frac{(\alpha + p - p\alpha - 1)\Gamma(\frac{r}{\gamma} + 1)}{\lambda^{(\gamma)^{\frac{r}{\gamma}}}} \sum_{i=0}^{\infty} (i+1) \sum_{m=0}^{i} \binom{i}{m} (\alpha - p\alpha)^m p^{(i-m)} \alpha^{(i-m+1)} \\ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-log)^{(k+1)} (i-m+1)^k t^l}{k! l! (k+1)^{\frac{r}{\gamma}+1}}. \end{aligned} \tag{17}$$

3.4. Mean Deviation

The Mean Deviation about the mean and the median gives us insight into the variability of a population. Now, let us denote the mean and median of the APMWG distribution by μ and M respectively. The Mean deviation about the mean and the mean deviation about the median, denoted by $D_\mu(X)$ and $D_M(X)$ respectively can be calculated as,

$$\begin{aligned} D_\mu(X) = E(|X - \mu|) &= \int_0^\infty |X - \mu| f(x) dx \\ &= 2 \int_0^\mu (\mu - x) f(x) dx \\ &= 2\mu F(\mu) - 2 \int_0^\mu x f(x) dx \end{aligned} \tag{18}$$

$$\begin{aligned} D_M(X) = E(|X - M|) &= \int_0^\infty |X - M| f(x) dx \\ &= \mu - 2 \int_0^M x f(x) dx \end{aligned} \tag{19}$$

For short, let us set ϕ=(μ or M) and $\psi(\phi)$=$\int_0^\phi x f(x) dx$. Therefore, using the series representation given in Eq. 11, the binomial expansion and the series representation given in Eq. 12, we have

$$\psi(\phi) = \int_0^{\phi} x \frac{(\alpha + p - p\alpha - 1) log\alpha\gamma\lambda^{\gamma}x^{\gamma-1}\alpha^{1-e^{-(\lambda x)^{\gamma}}}e^{-(\lambda x)^{\gamma}}}{\left[1 - \alpha + \alpha p - p\alpha^{1-e^{-(\lambda x)^{\gamma}}}\right]^2} dx$$
$$= \frac{(\alpha + p - p\alpha - 1)}{\lambda^{(\gamma)^{\frac{1}{\gamma}}}} \sum_{i=0}^{\infty}(i+1) \sum_{m=0}^{i} \binom{i}{m} (\alpha - p\alpha)^m p^{(i-m)} \alpha^{(i-m+1)}$$
$$\sum_{k=0}^{\infty} \frac{(-log)^{(k+1)}(i-m+1)^k}{k!(k+1)^{\frac{1}{\gamma}+1}} \gamma \left[(\frac{1}{\gamma}+1), (k+1)\lambda^{\gamma}\phi^{\gamma} \right] \quad (20)$$

By substituting the above in Eq. 18 and Eq. 19, the mean deviation about the mean and about the median of APMWG distribution is repectively obtained as

$$D_{\mu}(X) = 2\mu \left[\frac{1 - \alpha^{1-e^{-(\lambda x)^{\gamma}}}}{1 - \alpha + \alpha p - p\alpha^{1-e^{-(\lambda x)^{\gamma}}}} \right] - 2\frac{(\alpha + p - p\alpha - 1)}{\lambda^{(\gamma)^{\frac{1}{\gamma}}}}$$
$$\sum_{i=0}^{\infty}(i+1) \sum_{m=0}^{i} \binom{i}{m} (\alpha - p\alpha)^m p^{(i-m)} \alpha^{(i-m+1)}$$
$$\sum_{k=0}^{\infty} \frac{(-log)^{(k+1)}(i-m+1)^k}{k!(k+1)^{\frac{1}{\gamma}+1}} \gamma \left[(\frac{1}{\gamma}+1), (k+1)\lambda^{\gamma}\mu^{\gamma} \right] \quad (21)$$

$$D_M(X) = \mu - 2\frac{(\alpha + p - p\alpha - 1)}{\lambda^{(\gamma)^{\frac{1}{\gamma}}}} \sum_{i=0}^{\infty}(i+1) \sum_{m=0}^{i} \binom{i}{m} (\alpha - p\alpha)^m p^{(i-m)} \alpha^{(i-m+1)}$$
$$\sum_{k=0}^{\infty} \frac{(-log)^{(k+1)}(i-m+1)^k}{k!(k+1)^{\frac{1}{\gamma}+1}} \gamma \left[(\frac{1}{\gamma}+1), (k+1)\lambda^{\gamma}M^{\gamma} \right] \quad (22)$$

3.5. Numerical computation of Descriptive statistics and Sensitivity analysis

In this section, we have obtained some numerical value of the mean, variance, skewness, and kurtosis of the APMWG distribution for different parameter values using R software by generating random numbers following the distribution and is presented in Table 1, 2, 3 and 4.

Some of the conclusions that can be made about the APMWG distribution from the above Table 1, 2, 3 and 4 are;

(i) The APMWG distribution is suitable for modeling both positively and negatively skewed data sets.

(ii) The APMWG distribution is suitable for modeling both platykurtic (kurtosis < 3) and leptokurtic (kurtosis > 3) data sets.

(iii) Mean and variance are an increasing function of α for fixed value of γ, λ and p in the APMWG distribution.

(iv) The mean and variance are decreasing function of the parameter λ and p in the APMWG distribution.

(v) The mean is an increasing function of the parameter γ, while the variance is decreasing function of the parameter γ in the APMWG distribution.

The Latin Hypercube Sampling (LHS) is employed to evaluate the sensitivity of key distributional characteristics—namely, skewness, kurtosis, and hazard rate trend to the parameters. Through systematic sampling , simulation and 3D plots for Visualize parameter interactions, as shown in Figure 3, 4 and 5, we derived several key insights into how these parameters influence the shape and reliability properties of the distribution.

In Figure 3 we see that skewness varies with parameters α and λ. Most parameter combinations lead to low skewness (symmetric distributions), but specific regions—particularly at moderate α and lower λ—exhibit high skewness, indicating significant asymmetry. This highlights that skewness is highly sensitive to certain parameter ranges. We also see that skewness tends to be below for a wide range of p and γ values but Skewness becomes

Table 1. Mean, variance, skewness and kurtosis of APMWG distribution as α value increases

α	γ	λ	p	mean	variance	skewness	kurtosis
0.5	2	0.05	0.01	16.4919	84.27481	0.4961771	2.575285
1.5				18.563	94.9559	0.3094247	2.369644
2.5				20.63	97.13079	0.2302769	2.344075
3.5				21.483	97.59072	0.182331	2.348406
4.5				22.108	97.97559	0.149058	2.361107

Table 2. Mean, variance, skewness and kurtosis of APMWG distribution as γ value increases

α	γ	λ	p	mean	variance	skewness	kurtosis
0.5	2	0.05	0.01	16.2394	94.7416	0.6817071	3.003957
	2.5			16.42333	65.89901	0.4041645	2.676255
	3			16.66	49.6011	0.1959899	2.596411
	3.5			16.903	39.16202	0.0308571	2.629892
	4.5			17.319	26.65246	-0.217905	2.834037

Table 3. Mean, variance, skewness and kurtosis of APMWG distribution as λ value increases

α	γ	λ	p	mean	variance	skewness	kurtosis
0.5	2	0.05	0.01	16.3404	74.33436	0.5217119	2.817425
		0.5		1.63403	0.7433436	0.5217119	2.817425
		0.9		0.90780	0.229427	0.5217119	2.817425
		1.5		0.54468	0.08259374	0.5217119	2.817425
		2.5		0.3268	0.02973374	0.5217119	2.817425

Table 4. Mean, variance, skewness and kurtosis of APMWG distribution as p value increases

α	γ	λ	p	mean	variance	skewness	kurtosis
0.5	2	0.05	0.0001	1.7022	0.8419467	0.5576899	2.758013
			0.009	1.6977	0.8403312	0.5610282	2.762775
			0.01	1.6972	0.8401484	0.5614055	2.763316
			0.5	1.36884	0.701565	0.8299848	3.26549
			0.9	0.75400	0.3606913	1.58557	5.84756

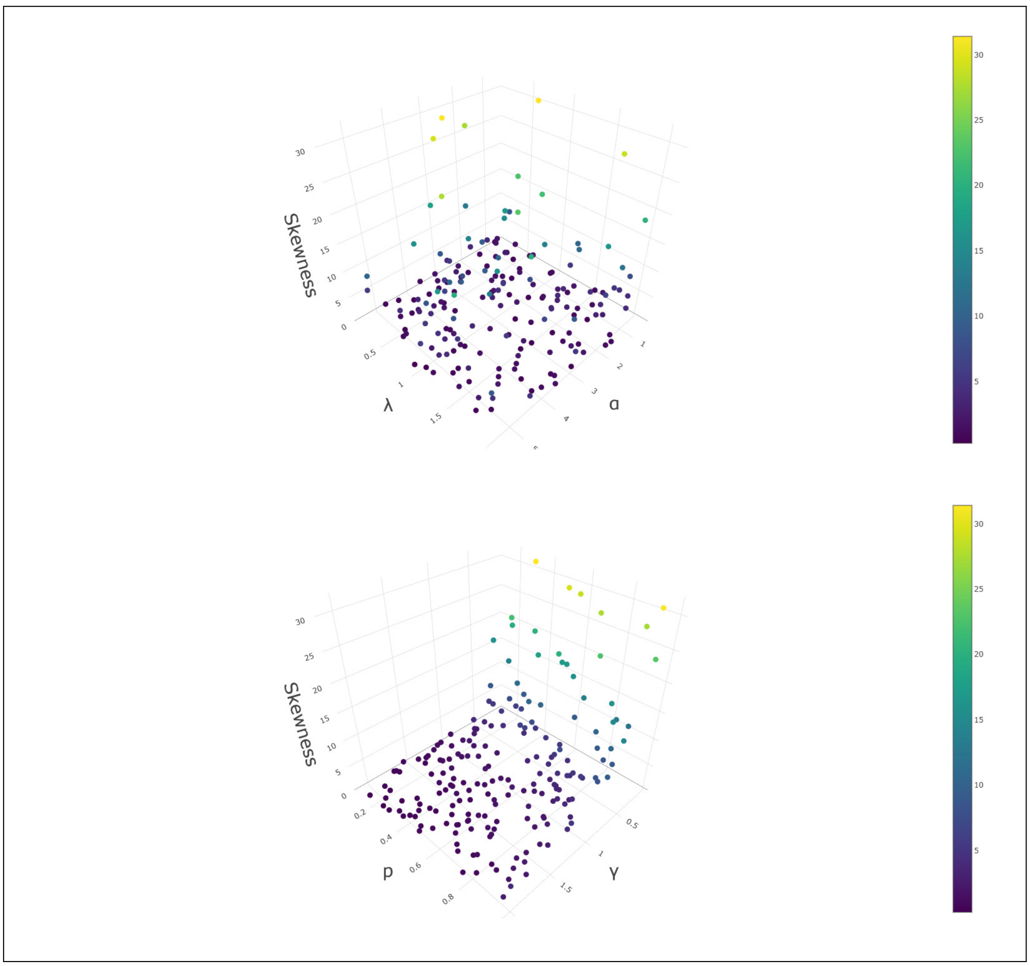

Figure 3. 3D-Plots of the Skewness as the function of the parameters

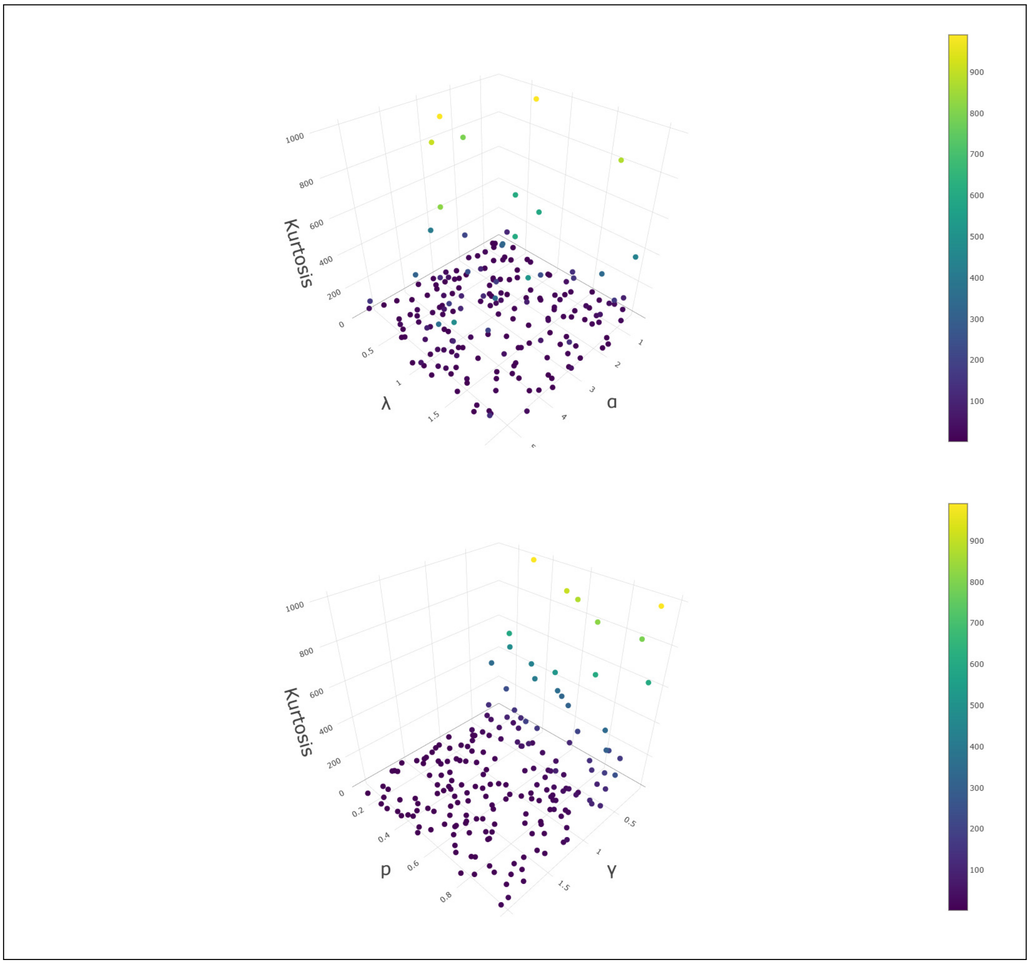

Figure 4. 3D-Plots of the Kurtosis as the function of the parameters

higher in certain regions, suggesting that the underlying distribution becomes more asymmetric depending on specific combinations of p and γ. Figure 4 shows that most combinations yield low kurtosis (thin tails), but some lead to extremely heavy tails. Low values of α, λ and γ tend to be associated with high kurtosis, suggesting these parameters drive tail risk.This suggests that the model is capable of capturing heavy-tailed behavior, a desirable property in contexts such as finance, reliability, or survival analysis where extreme values play a critical role.

Analysis of the hazard rate trend revealed predominantly positive values, indicating that the hazard function often increases with time. This characteristic is consistent with systems exhibiting wear-out failures or aging-related degradation. From Figure 5, we see that Hazard rate is highly sensitive to all the parameters. Hazard rate tends to decrease with increasing of the parameters α and γ, but the influence of the other parameters modulates this effect. The flexibility in hazard rate behavior further validates the distribution for a wide range of applications involving lifetime or failure-time data.

3.6. Incomplete Moments

Incomplete moments are easily interpreted and form natural building blocks from which measure of inequality may be constructed.

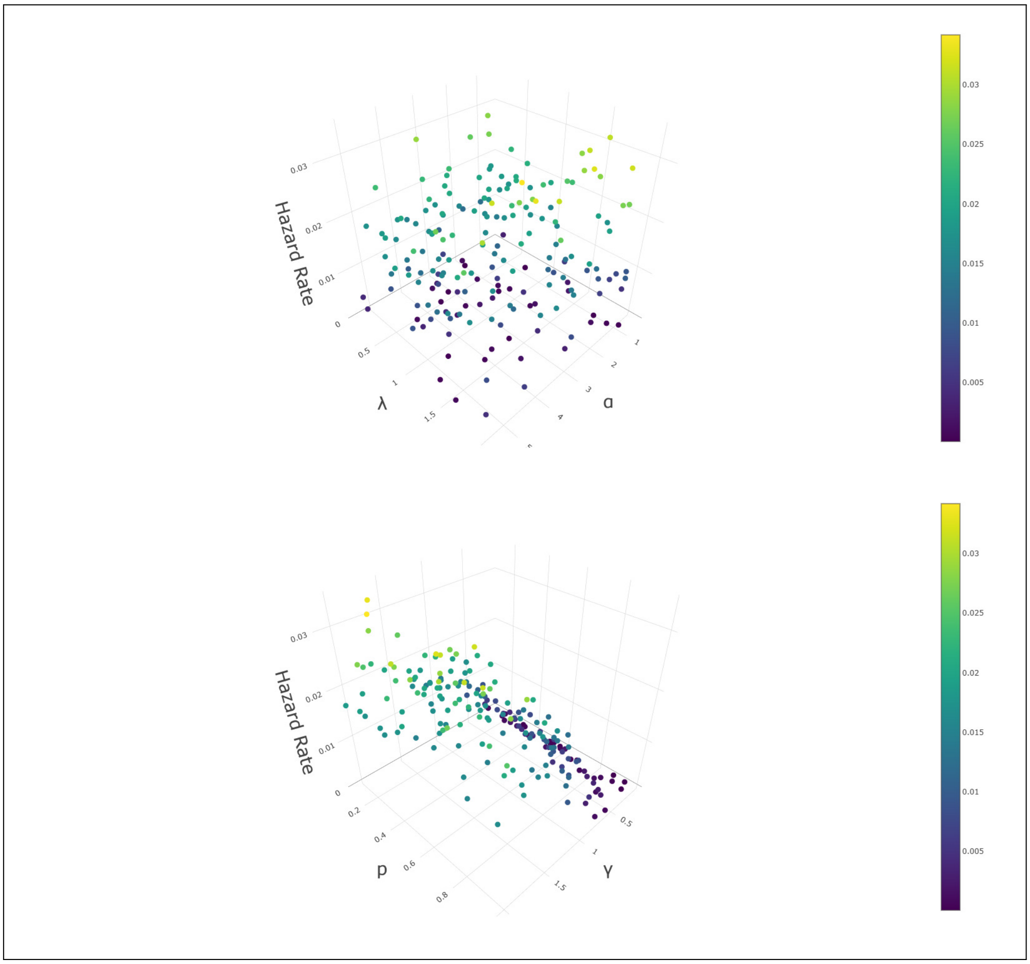

Figure 5. 3D-Plots of the Hazard Rate as the function of the parameters

The h^{th} incomplete moment of the APMWG distribution is define by

$$I(t;h) = \int_0^t x^h f(x)\, dx \tag{23}$$

Similarly using the series representation as given in Eq. 11 and Eq. 12 and the binomial expansion, the h^{th} incomplete momemt of X following APMWG distribution is obtained as

$$I(t;h) = \frac{(\alpha + p - p\alpha - 1)}{\lambda(\gamma)^{\frac{h}{\gamma}}} \sum_{i=0}^{\infty}(i+1) \sum_{m=0}^{i} \binom{i}{m} (\alpha - p\alpha)^m p^{(i-m)} \alpha^{(i-m+1)}$$
$$\sum_{k=0}^{\infty} \frac{(-log)^{(k+1)}(i-m+1)^k}{k!(k+1)^{\frac{h}{\gamma}+1}} \gamma \left[(\frac{h}{\gamma}+1), (k+1)\lambda^\gamma t^\gamma \right], \tag{24}$$

where, γ(a,x) is the lower incomplete gamma function.

The first two incomplete moments, where we set h=0 and h=1, provide especially useful information in many application about the shape of the distribution. The first incomplete moment is also known to be related to Bonferroni and Lorenz curve, mean residual and mean waiting time.

3.7. Conditional Moments

The conditional moment of random variable X is defined by

$$E\left[X^n/X>n\right]=\frac{1}{S(x)}\int_t^{\infty}x^n f(x)\,dx \tag{25}$$

Now substituting from Eq. 6 and Eq. 7, the conditional moment of the APMWG distribution is obtained as

$$E\left[X^n/X>n\right]=\frac{(\alpha+p-p\alpha-1)}{\lambda^{(\gamma)^{\frac{n}{\gamma}}}}\sum_{i=0}^{\infty}(i+1)\sum_{m=0}^{i}\binom{i}{m}(\alpha-p\alpha)^m p^{(i-m)}\alpha^{(i-m+1)}$$
$$\sum_{k=0}^{\infty}\frac{(-log)^{(k+1)}(i-m+1)^k}{k!(k+1)^{\frac{n}{\gamma}+1}}\Gamma\left[(\frac{n}{\gamma}+1),(k+1)\lambda^{\gamma}t^{\gamma}\right], \tag{26}$$

where, Γ(a,x) is the upper incomplete gamma function.

3.8. Mean Residual Life

In the context of life testing situation or reliability studies, the expected additional lifetime that a component has survived until time "t" is a function t, called the mean residual life $\mu_R(t)$ and is defined as

$$\mu_R(t)=\frac{1}{S(t)}\left(\int_t^{\infty}xf(x)dx-t\right) \tag{27}$$

$$\mu_R(t)=\frac{1}{S(t)}\Bigg(\mu_1'-t-\frac{(\alpha+p-p\alpha-1)}{\lambda^{(\gamma)^{\frac{1}{\gamma}}}}\sum_{i=0}^{\infty}(i+1)$$
$$\sum_{m=0}^{i}\binom{i}{m}(\alpha-p\alpha)^m p^{(i-m)}\alpha^{(i-m+1)}$$
$$\sum_{k=0}^{\infty}\frac{(-log)^{(k+1)}(i-m+1)^k}{k!(k+1)^{\frac{1}{\gamma}+1}}\gamma\left[(\frac{1}{\gamma}+1),(k+1)\lambda^{\gamma}t^{\gamma}\right]\Bigg), \tag{28}$$

where, S(t) is the survival function defined in Eq. 7, μ_1' is the mean define in Eq. 14 and γ(a,x) is the lower incomplete gamma function.

3.9. Measures of Inequality and uncertainty

In this subsection, we will obtained the Lorenz and Bonferroni curve as the measures of inequality. Also, Renyi Entropy will be obtained as an important measures of uncertainty.

3.9.1. Lorenz and Bonferroni curve The Lorenz and Bonferroni curve for a random variable X is obtained as,

$$L(x)=\frac{I(t;1)}{E(x)}\quad\text{and}\quad B(x)=\frac{L(x)}{E(x)}, \tag{29}$$

Where, I(t;1) is the first incomplete moment. Now, using Eq. 12 and Eq. 14, the Lorenz curve expression is obtained as,

$$L(x) = \frac{1}{\Gamma(\frac{1}{\gamma}+1)} \sum_{i=0}^{\infty}(i+1) \sum_{m=0}^{i} \binom{i}{m} (\alpha - p\alpha)^m p^{(i-m)} \alpha^{(i-m+1)}$$
$$\sum_{k=0}^{\infty} \frac{(-log)^{(k+1)}(i-m+1)^k}{k!(k+1)^{\frac{1}{\gamma}+1}} \gamma \left[(\frac{1}{\gamma}+1), (k+1)\lambda^\gamma t^\gamma \right]$$
$$\left[\sum_{i=0}^{\infty}(i+1) \sum_{m=0}^{i} \binom{i}{m} (\alpha - p\alpha)^m p^{(i-m)} \alpha^{(i-m+1)} \right.$$
$$\left. \sum_{k=0}^{\infty} \frac{(-log)^{(k+1)}(i-m+1)^k}{k!(k+1)^{\frac{1}{\gamma}+1}} \right]^{-1}, \quad (30)$$

where, γ(a,x) is the lower incomplete gamma function.

Thus, the Bonferroni curve can be obtained by substituting the value from the above equation and the mean value in the given formula.

3.9.2. Renyi Entropy The entropy of a random variable X with density function f(x) is a measure of uncertainty or randomness of a system. One of the entropy measure that is often used is Renyi entropy which is defined by

$$I_R(\beta) = \frac{1}{1-\beta} log \left(\int_0^\infty (f(x))^\beta dx \right) \quad (31)$$

Substituting Eq. 6 in the above equation, we have

$$I_R(\beta) = \frac{1}{1-\beta} log((\alpha + p - p\alpha - 1)^\beta (log\alpha\gamma\lambda^\gamma)^\beta$$
$$\int_0^\infty \left(\frac{x^{\gamma-1}\alpha^{1-e^{-(\lambda x)^\gamma}} e^{-(\lambda x)^\gamma}}{\left[1 - \alpha + \alpha p - p\alpha^{1-e^{-(\lambda x)^\gamma}}\right]^2} \right)^\beta dx) \quad (32)$$

Using the binomial expansion and the series expansion as given in Eq. 12 we have,

$$I_R(\beta) = \frac{\beta}{1-\beta} \left(log(\alpha + p - p\alpha - 1) + log(log\alpha\gamma\lambda^\gamma) + log\alpha \right)$$
$$+ \frac{1}{1-\beta} log \left[\sum_{m=0}^{2\beta} \sum_{k=0}^{\infty} \frac{-(-log\alpha)^k \Gamma\left(\frac{\beta(\gamma-1)}{\gamma}+1\right)}{\binom{m}{2\beta}(1-\alpha+p\alpha)^m (-p)^m \gamma \left(\lambda^\gamma (k(\beta - m) - \beta)\right)^{\frac{\beta(\gamma-1)}{\gamma}+1}} \right]. \quad (33)$$

3.10. Order Statistics

Let $X_1, X_2, ..., X_r$ be a random sample of size n following APMWG distribution and let $X_{1:r}, X_{2:r}, ..., X_{r:r}$ be their corresponding order statistics. The pdf of the i^{th} order statistic, $X_{i:r}$ denoted by $f_{i:r}(x)$ is given by

$$f_{i:r}(x) = \frac{1}{B(i, r-i+1)} f(x) F(x)^{i-1} (1 - F(x))^{r-i}$$

By using the binomial expansion in the above equation we have

$$f_{i:r}(x) = \frac{1}{B(i, r-i+1)} \sum_{l=0}^{k-i} C_l^{k-i} (-1)^l f(x) F(x)^{l+i-1} \quad (34)$$

Substituting the value from Eq. 5 and Eq. 6 in the above equation and using the binomial expansion, the pdf of the i^{th} order statistics is obtained as

$$f_{i:r}(x) = \frac{1}{B(i, r-i+1)} \sum_{l=0}^{k-i} \sum_{m=0}^{l+i-1} C_l^{k-i} C_m^{l+i-1} (-1)^{l+m} (\alpha + p - p\alpha - 1) \frac{\alpha^{m(1-e^{-(\lambda x)^\gamma})} + (1 - e^{-(\lambda x)^\gamma})}{(1 - \alpha + \alpha p - p\alpha^{1-e^{-(\lambda x)^\gamma}})^{l+i-1}} log\alpha\gamma\lambda^\gamma x^{\gamma-1} e^{-(\lambda x)^\gamma}. \tag{35}$$

The associate cdf of the i^{th} order statistic, $X_{i:r}$ denoted by$F_{i:r}(x)$ is given by

$$F_{i:r}(x) = \sum_{i=0}^{k} C_l^k F(x)^i (1 - F(x))^{k-i} = \sum_{i=0}^{k} \sum_{l=0}^{k-i} (-1)^l C_l^k C_l^{k-i} F(x)^{(i+l)} = \sum_{i=0}^{k} \sum_{l=0}^{k-i} (-1)^l C_l^k C_l^{k-i} \left[\frac{1 - \alpha^{1-e^{-(\lambda x)^\gamma}}}{1 - \alpha + \alpha p - p\alpha^{1-e^{-(\lambda x)^\gamma}}} \right]^{(i+l)} \tag{36}$$

The pdf and cdf of the smallest $X_{1:r}$ and largest $X_{r:r}$ order statistics can be obtain by simply setting i=1 and i=r respectively Eq. 35 and Eq. 36 respectively.

4. Parameter Estimation

In this section, we estimate the parameter of APMWG distribution by using the maximum lilelihood estimation (MLE) method.

4.1. Maximum Likelihood Estimation method

Suppose that X_1,X_2,....,X_n is a random sample of size n from the APMWG distribution with the density function as given in Eq. 6.

Then the log-likelihood function is obtained as

$$l(\theta, p) = nlog(\alpha + p - p\alpha - 1) + nloglog\alpha + nlog\left(\lambda^\gamma \gamma \alpha\right) - log\alpha \sum_{i=1}^{n} e^{-(\lambda x_i)^\gamma} - \lambda^\gamma \sum_{i=1}^{n} x_i^\gamma + (\gamma - 1) \sum_{i=1}^{n} logx_i - 2 \sum_{i=1}^{n} log\left(1 - \alpha + p\alpha - p\alpha^{1-e^{-(\lambda x_i)^\gamma}}\right). \tag{37}$$

Now differentiating Eq. (19) w.r.t α,λ, γ and p respectively and equating them to zero, we obtain the maximum likelihood estimates (MLEs) of α, λ, γ and p.

$$\frac{\delta l(\theta, p)}{\delta \alpha} = \frac{n(1-p)}{(\alpha + p - p\alpha - 1)} + \frac{n}{\alpha log\alpha} + \frac{n}{\alpha} - \frac{1}{\alpha} \sum_{i=1}^{n} e^{-(\lambda x_i)^\gamma} - 2 \sum_{i=1}^{n} \frac{p - 1 + \left(p\alpha^{-e^{-(\lambda x_i)^\gamma}} (e^{-(\lambda x_i)^\gamma})\right)}{\left(1 - \alpha + p\alpha - p\alpha^{1-e^{-(\lambda x_i)^\gamma}}\right)} = 0 \tag{38}$$

$$\frac{\delta l(\theta,p)}{\delta\lambda} = \frac{n\gamma\lambda^{(\gamma-1)}}{\lambda^{\gamma}} + log\alpha\gamma\lambda^{(\gamma-1)}\sum_{i=1}^{n} e^{-(\lambda x_i)^{\gamma}} x_i^{\gamma} - \gamma\lambda^{(\gamma-1)}\sum_{i=1}^{n} x_i^{\gamma}$$
$$+2plog\alpha\gamma\lambda^{(\gamma-1)}\sum_{i=1}^{n}\frac{\alpha^{1-e^{-(\lambda x_i)^{\gamma}}} x_i^{\gamma}}{\left(1-\alpha+p\alpha-p\alpha^{1-e^{-(\lambda x_i)^{\gamma}}}\right)} = 0 \tag{39}$$

$$\frac{\delta l(\theta,p)}{\delta\gamma} = \frac{n}{\gamma} + \frac{nlog\lambda}{\lambda^{\gamma}} + log\alpha\lambda^{\gamma}\sum_{i=1}^{n} e^{-(\lambda x_i)^{\gamma}} x_i^{\gamma} log(\lambda x_i) - \sum_{i=1}^{n}(\lambda x_i)^{\gamma} log(\lambda x_i)$$
$$+\sum_{i=1}^{n} logx_i + 2p\sum_{i=1}^{n}\frac{log\alpha\alpha^{-e^{-(\lambda x_i)^{\gamma}}} e^{-(\lambda x_i)^{\gamma}}(\lambda x_i)^{\gamma} log(\lambda x_i)}{\left(1-\alpha+p\alpha-p\alpha^{1-e^{-(\lambda x_i)^{\gamma}}}\right)} = 0 \tag{40}$$

$$\frac{\delta l(\theta,p)}{\delta p} = \frac{n(1-\alpha)}{(\alpha+p-p\alpha-1)} - 2\sum_{i=1}^{n}\frac{\alpha-\alpha^{1-e^{-(\lambda x_i)^{\gamma}}}}{\left(1-\alpha+p\alpha-p\alpha^{1-e^{-(\lambda x_i)^{\gamma}}}\right)} = 0 \tag{41}$$

The maximum likelihood estimates (MLEs) of (α, λ, γ,p) are the simultaneous solutions of the Eq. 38, Eq. 39, Eq. 40 and Eq. 41. The above mentioned equations are not in closed form, thus numerical technique like the Newton-Raphson method can be used to obtained the MLEs. Also, there are many well-established packages in R language that can be used to obtained the maximization of Eq. 37.

Under certain regularity conditions, the asymptotic distribution of the MLEs, i.e. $\sqrt{n}(\hat{\phi}-\phi)$ follows multivariate normal distribution with mean vector zero and variance covariance matrix ($I(\hat{\phi})^{-1}$), where n is the sample size, $\hat{\phi}$ is the MLE of ϕ. Using this asymptotic properties of MLE we can obtain the standard Error of the estimates and also construct the asymptotic confidence interval (ACI) of the parameters.

Therefore, for any arbitrary $0<\tau<1$, the $100(1-\tau)\%$ ACI of the unknown parameters can be determined as follows:

$$(\hat{\alpha}) \pm z_{\frac{\tau}{2}}\sqrt{var(\hat{\alpha})}, (\hat{\lambda}) \pm z_{\frac{\tau}{2}}\sqrt{var(\hat{\lambda})}, (\hat{\gamma}) \pm z_{\frac{\tau}{2}}\sqrt{var(\hat{\gamma})}, (\hat{p}) \pm z_{\frac{\tau}{2}}\sqrt{var(\hat{p})}, \tag{42}$$

Where, $z_{\frac{\tau}{2}}$ is the upper $(\frac{\tau}{2})^{th}$ percentile point of the standard normal distribution.

4.2. Maximum Likelihood estimation for progressive type II censored samples

Researchers often faced problem with incomplete or censored data, as censored samples are always present in life testing experiment when failures times of all units place on the life test is not observe.This may happen intentionally based on the requirements or unintentionally.

Progressive censoring can be describe as a censoring method where units are removed from the life test at some prefixed or random inspection times.Although various models of progressively censored data have been discussed in the literature, Progressive type II can be considered as the most popular model. Under this scheme of censoring, the removal are carried out at observed failure times. The prefixed number of units is immediately withdrawn from the surviving units open observing a failure. Therefore, the number of observations is fixed in advance, while the duration of the experiment is random[9].

Under this scheme, from a total of n units placed simultaneously under life test, only m($<$ n)are completely observed until failure. Then given a censoring plan R=($R_1, R_2, ..., R_m$), at the time $x_{1:m:n}$ of the first failure, R_1 units are randomly censored from the (n-1) surviving units.At the time $x_{2:m:n}$ of the second failure, R_2 units are randomly censored from the (n-2-R_1) surviving units and its continue until at the time $x_{m:m:n}$ of the m^{th} failure, all the remaining $R_m = n-m-R_1-...-R_{m-1}$ surviving units are censored. The ordered failure times $X_{1:m:n;R_1} \leq X_{2:m:n;R_2} \leq ... \leq X_{m:m:n;R_m}$ are the progressive type II censored samples.

Now let x = $(x_{1:m:n}, x_{2:m:n;}...x_{m:m:n;})$ with $x_{1:m:n} \leq x_{2:m:n} \leq ... \leq x_{m:m:n}$ be the m observations under progressive type ii censoring from a sample of size n drawn from the APMWG distribution with cdf and pdf given in Eq. 5 and Eq. 6 respectively. The likelihood function based on the progressive type ii censored sample x is given by

$$L_{PC}(\theta, p; x_{i:m:n}) = A\prod_{i=1}^{m} f(x_{i:m:n})[1-f(x_{i:m:n})]^{R_i} \tag{43}$$

Where $A = n(n-1-R_1)(n-2-R_1-R_2)...(n-m+1-R_1-...R_{m-1})$ **[?]**
Using Eq. 5 and Eq.6 and taking log on both side we obtained the log likelihood function as

$$\begin{aligned} l_{PC}(\theta, p; x_{i:m:n}) = logA + mlog(\alpha+p-p\alpha-1) + mloglog\alpha + (n-m)log(p-1) \\ +mlog\left(\lambda^{\gamma}\gamma\alpha\right) + (n-m)log\alpha - log\alpha\sum_{i=1}^{m} e^{-(\lambda x_{i:m:n})^{\gamma}} \\ -\lambda^{\gamma}\sum_{i=1}^{m} x_{i:m:n}^{\gamma} + (\gamma-1)\sum_{i=1}^{m} logx_{i:m:n} + \sum_{i=1}^{m} R_i log\left(1-\alpha^{-e^{-(\lambda x_{i:m:n})^{\gamma}}}\right) \\ -\sum_{i=1}^{m}(2+R_i)log\left(1-\alpha+p\alpha-p\alpha^{1-e^{-(\lambda x_{i:m:n})^{\gamma}}}\right). \end{aligned} \tag{44}$$

Now differentiating Eq. w.r.t α,λ, γ and p respectively and equating them to zero, we obtain the MLEs of α, λ, γ and p.

$$\begin{aligned} \frac{\delta l_{PC}(\theta,p)}{\delta\alpha} = \frac{m(1-p)}{(\alpha+p-p\alpha-1)} + \frac{m}{\alpha log\alpha} + \frac{m}{\alpha} - \frac{1}{\alpha}\sum_{i=1}^{m} e^{-(\lambda x_{1:m:n})^{\gamma}} + \frac{(n-m)}{\alpha} \\ +\sum_{i=1}^{m}\frac{R_i\left(\alpha^{1-e^{-(\lambda x_{1:m:n})^{\gamma}}}\right)\left(e^{-(\lambda x_{1:m:n})^{\gamma}}\right)}{\left(1-\alpha^{-e^{-(\lambda x_{i:m:n})^{\gamma}}}\right)} - \sum_{i=1}^{m}(2+R_i)\frac{p-1+\left(p\alpha^{-e^{-(\lambda x_{1:m:n})^{\gamma}}}\left(e^{-(\lambda x_{1:m:n})^{\gamma}}\right)\right)}{\left(1-\alpha+p\alpha-p\alpha^{1-e^{-(\lambda x_{1:m:n})^{\gamma}}}\right)} = 0 \end{aligned} \tag{45}$$

$$\begin{aligned} \frac{\delta l_{PC}(\theta,p)}{\delta\lambda} = \frac{m\gamma\lambda^{(\gamma-1)}}{\lambda^{\gamma}} + log\alpha\gamma\lambda^{(\gamma-1)}\sum_{i=1}^{m} e^{-(\lambda x_{1:m:n})^{\gamma}} x_{1:m:n}^{\gamma} - \gamma\lambda^{(\gamma-1)}\sum_{i=1}^{m} x_{1:m:n}^{\gamma} \\ +log\alpha\gamma\lambda^{(\gamma-1)}\sum_{i=1}^{m}\frac{R_i x_{1:m:n}^{\gamma}\left(\alpha^{1-e^{-(\lambda x_{1:m:n})^{\gamma}}}\right)\left(e^{-(\lambda x_{1:m:n})^{\gamma}}\right)}{\left(1-\alpha^{-e^{-(\lambda x_{i:m:n})^{\gamma}}}\right)} \\ +plog\alpha\gamma\lambda^{(\gamma-1)}\sum_{i=1}^{m}(2+R_i)\frac{\alpha^{1-e^{-(\lambda x_{1:m:n})^{\gamma}}} x_{1:m:n}^{\gamma}}{\left(1-\alpha+p\alpha-p\alpha^{1-e^{-(\lambda x_{1:m:n})^{\gamma}}}\right)} = 0 \end{aligned} \tag{46}$$

$$\begin{aligned} \frac{\delta l_{PC}(\theta,p)}{\delta\gamma} = \frac{m}{\gamma} + \frac{mlog\lambda}{\lambda^{\gamma}} + log\alpha\lambda^{\gamma}\sum_{i=1}^{m} e^{-(\lambda x_{1:m:n})^{\gamma}} x_{1:m:n}^{\gamma}log(\lambda x_{1:m:n}) - \sum_{i=1}^{m}(\lambda x_{1:m:n})^{\gamma}log(\lambda x_{1:m:n}) \\ +\sum_{i=1}^{n} logx_{1:m:n} + p\sum_{i=1}^{m}\frac{(2+R_i)log\alpha\alpha^{-e^{-(\lambda x_{1:m:n})^{\gamma}}} e^{-(\lambda x_{1:m:n})^{\gamma}}(\lambda x_{1:m:n})^{\gamma}log(\lambda x_{1:m:n})}{\left(1-\alpha+p\alpha-p\alpha^{1-e^{-(\lambda x_{1:m:n})^{\gamma}}}\right)} \\ +log\alpha\lambda^{\gamma}\sum_{i=1}^{m}\frac{R_i x_{1:m:n}^{\gamma}log(\lambda x_{1:m:n})\left(\alpha^{1-e^{-(\lambda x_{1:m:n})^{\gamma}}}\right)\left(e^{-(\lambda x_{1:m:n})^{\gamma}}\right)}{\left(1-\alpha^{-e^{-(\lambda x_{i:m:n})^{\gamma}}}\right)} = 0 \end{aligned} \tag{47}$$

$$\frac{\delta l_{PC}(\theta,p)}{\delta p} = \frac{m(1-\alpha)}{(\alpha+p-p\alpha-1)} + \frac{(n-m)}{log(p-1)} - \sum_{i=1}^{m} \frac{(2+R_i)\alpha - \alpha^{1-e^{-(\lambda x_{1:m:n})^{\gamma}}}}{\left(1-\alpha+p\alpha-p\alpha^{1-e^{-(\lambda x_{1:m:n})^{\gamma}}}\right)} = 0 \quad (48)$$

The maximum likelihood estimates (MLEs) of (α, λ, γ,p) for the progressive type ii censored samples are the simultaneous solutions of the Eq. 45, Eq. 46, Eq. 47 and Eq. 48. The above mentioned equations are not in closed form, thus numerical technique like the Newton-Raphson method can be used to obtained the MLEs. Also, there are many well-established packages in R language that can be used to obtained the maximization of Eq. 44.

The Standard Error and the ACI can be obtained in the similar way as the uncencored samples. Hence, for any arbitrary $0 < \tau < 1$, the $100(1-\tau)\%$ ACI of the unknown parameters can be determined as follows:

$$(\hat{\alpha}) \pm z_{\frac{\tau}{2}}\sqrt{var(\hat{\alpha})}, (\hat{\lambda}) \pm z_{\frac{\tau}{2}}\sqrt{var(\hat{\lambda})}, (\hat{\gamma}) \pm z_{\frac{\tau}{2}}\sqrt{var(\hat{\gamma})}, (\hat{p}) \pm z_{\frac{\tau}{2}}\sqrt{var(\hat{p})}, \quad (49)$$

Where, $z_{\frac{\tau}{2}}$ is the upper $(\frac{\tau}{2})^{th}$ percentile point of the standard normal distribution.

5. Simulation Analysis

In this section, we will be using the inversion method for simulating random samples from the APMWG distribution, that is, simulation is done using Eq. 5.Also a simulation study will be performed to assess the behaviour of the MLEs.

5.1. Generation method

The inversion method relies on the principle that continuous cdf range uniformly over the open interval (0,1). If u is a uniform random number on (0,1), then x=$F^{-1}(u)$ generates a random number x from a continuous distribution with the specified cdf F.

Thus,for APMWG distribution we have,

$$x = \left[\frac{-1}{\lambda^{\gamma}} log\left(1 - \frac{log\left(\frac{u-u\alpha+p\alpha u-1}{pu-1}\right)}{log\alpha}\right)\right]^{\frac{1}{\gamma}} \quad (50)$$

5.2. Simulation

In this subsection, we have done a simulation study to assess the behaviour of the MLEs for uncencored data in terms of sample size n. We have generated random numbers of sample size n=50, 100 and 150 following APMWG distribution by using Eq. 50 by initially setting the parameters values. Then, by using the DEoptim function over 1000 replication in r, the average MLE (MLE), average bias (bias) and average standard error (MSE) values of the parameters is obtained for different sets of parameters as follows

Set-1: α=0.9, γ=2.0, λ=0.5, p=0.9
Set-2: α=0.9, γ=2.5, λ=1.5, p=0.9
Set-3: α=0.9, γ=2.0 λ=2.5, p=0.5
Set-4: α=1.5, γ=4.5, λ=0.9, p=0.5
Set-5: α=0.5, γ=4.5, λ=1.5, p=0.9

The average MLE, bias and the Standard Error for five sets of parameter value are reported in Table 5 respectively and the conclusion that can be drawn from the result in Table 5 are as follow
1. The average MLE values of all the four parameter are very precise to the actual parameter value considered in

Table 5. MLEs and MSE for 5 different sets of parameter value

		MLE				bias				MSE			
sets	n	α	γ	λ	p	α	γ	λ	p	α	γ	λ	p
set 1	50	1.07073	1.94911	0.63598	0.71971	0.17073	-0.05089	0.13598	-0.180289	0.50687	0.11663	0.10555	0.12079
	100	0.97317	1.97086	0.54989	0.78179	0.07318	-0.02913	0.04989	-0.11820	0.48377	0.06284	0.03648	0.07396
	150	0.88851	1.97051	0.53278	0.78912	-0.01149	-0.02949	0.03278	-0.11088	0.438761	0.04476	0.02132	0.06345
set 2	50	1.03432	2.45528	1.76486	0.73459	0.134328	-0.04471	0.26486	-0.16540	0.58958	0.16812	0.45298	0.11162
	100	0.92911	2.46716	1.61486	0.77530	0.02911	-0.03283	0.11485	-0.12469	0.43338	0.08978	0.19944	0.07442
	150	0.87085	2.45845	1.58275	0.79402	-0.02914	-0.04154	0.08275	-0.10598	0.39516	0.06072	0.12507	0.05658
set 3	50	1.06434	2.09545	2.54903	0.40401	0.16434	0.09545	0.04903	-0.09598	0.43342	0.16647	0.49666	0.12896
	100	1.05074	2.01356	2.57155	0.37752	0.15074	0.01356	0.07155	-0.12247	0.37809	0.093011	0.39223	0.10744
	150	1.02251	2.01199	2.53619	0.39535	0.12251	0.01199	0.03618	-0.10464	0.33508	0.06609	0.32218	0.10002
set 4	50	1.50354	4.68743	0.89873	0.36658	0.00353	0.18743	-0.00127	-0.13341	0.90163	0.86385	0.01227	0.13127
	100	1.43813	4.62509	0.89918	0.35946	-0.06186	0.12509	-0.00082	-0.14053	0.87416	0.51208	0.00864	0.11138
	150	1.49655	4.56308	0.89906	0.37660	-0.00345	0.06307	-0.00093	-0.12340	0.76620	0.36465	0.006838	0.09779
set 5	50	1.022818	4.423928	1.63996	0.76981	0.52281	-0.07607	0.13996	-0.13018	0.64239	0.54575	0.13306	0.09248
	100	0.91524	4.42542	1.56811	0.80734	0.41524	-0.07457	0.06810	-0.09265	0.56244	0.29496	0.06337	0.06126
	150	0.83381	4.44147	1.53334	0.82623	0.33380	-0.05853	0.03334	-0.07376	0.47763	0.19670	0.03890	0.04556

the study.
2. The average bias move towards zero as the sample size n increases.
3. Also, the Standard Error of all the four parameters decreases as we increase the sample size n.

All the mention conclusion drawn shows the unbiasedness and consistency of the MLEs.

6. Applications to real data set

In this section, four different real data sets is consider to observe the efficiency of the proposed distribution. Also, a comparative study with Alpha Power Modified Weibull (APMW), Exponentiated Inverse Flexible Weibull (EIFW),Exponentoated Inverse Weibull-Geometric (EIWG), Additive Weibull- Geometric (AWG), Modified Weibull (MW) and Weibull (W) distributions has been made to show the efficacy of the proposed distribution.

We calculate the analytical measures such as the negative log likelihood (-l), Akaike information criterion (AIC), Consistent Akaike Information criterion (AICC), Bayesian Information Criterion (BIC) and Hannan-Quinn Information criterion (HQIC) for the comparative study and the model having the lowest analytical measures value will be considered to be the best model.

6.1. Application I

For the first application, we have considered a real-life data set which represents the failure times of the light-emitting diode (LED) in a life test under normal use condition. The data was also considered by [23] The data set 1 is provided in Table 6.

Table 6. Data set 1

0.18, 0.19, 0.19, 0.34, 0.36, 0.40, 0.44, 0.44, 0.45, 0.46, 0.47, 0.53, 0.57, 0.57, 0.63, 0.65, 0.70, 0.71,
0.71, 0.75, 0.76, 0.76, 0.79, 0.80, 0.85, 0.98, 1.01, 1.07, 1.12, 1.14, 1.15, 1.17, 1.20, 1.23, 1.24, 1.25,
1.26, 1.32, 1.33, 1.33, 1.39, 1.42, 1.50, 1.55, 1.58, 1.59, 1.62, 1.68, 1.70, 1.79, 2.00, 2.01, 2.04, 2.54,
3.61, 3.76, 4.65, 8.97

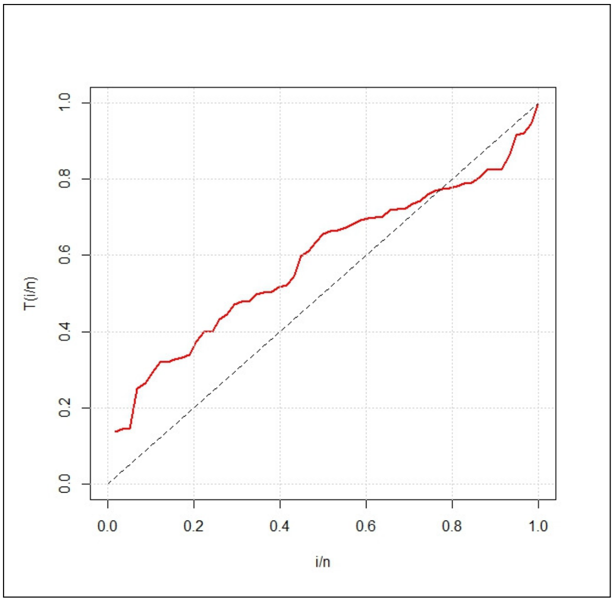

Figure 6. TTT plot for data set 1

The Total Time on Test (TTT) plot, [2] for the data is shown in Figure 6, which represent an upside-down bathtub shaped hazard rate function. Thus, the data can be consider to be suitable to model using the APMWG distribution. The maximum likelihood estimates (MLEs) along with their standard error (SE), the analytical measures i.e the -l, AIC, AICC, BIC, HQIC, K-S test statistics and its p-value of the APMWG distribution and all the others competitive ditribution is presented in Table 6 and 7 respectively.

From Table 8, we observe that the APMWG distribution has the lowest value of -l, AIC, AICC,BIC,HQIC and K-S (p-vale $>$0.05), which is the analytical measures as compared to all the other competing distribution considered with . Hence, The APMWG distribution can be considered to be more adequate for explaining the data set 1 compared to all the other competing distribution. Figure 7 (a), (b) and (c) display the plot of the estimated HRFs, estimated densities and estimated CDF respectively fitted to data set 1 for all the competing distributions. In Figure 8 we also have the QQ-plot and PP-plot of the proposed distribution to the considered data set, showing that the distribution is a good choice for the data set.

6.2. Application 2

For the second Application, we have considered the survival times (in years) of 45 patients who were randomized to chemotherapy plus Radiotherapy for 8 years as reported in [6], and the survival time (data set 2) is provided in Table 9. This data set was also recently considered by [24].

Table 7. MLEs (SE) for all the distributions for data set 1

Distribution	α	γ	λ	β	p
APMWG	0.6858(0.0263)	2.4081(0.2580)	0.1048 (0.0262)	-	0.9948(0.0019)
APMW	0.0049(0.0066)	1.6242 (0.1511)	0.2856 (0.0556)	-	-
EIFW	0.1892(0.0715)	1.9745(0.5993)	1.1654(0.1986)	-	
EIWG	0.3259(0.2749)	7.0339(0.0261)	-	-	0.9992(71.5050)
AWG	2.2870 (0.2678)	0.0025 (0.3670)	2.3890 (0.2990)	0.0021 (0.3240)	0.9955 (0.4360)
MW	0.0100 (0.6816)	0.6237(0.6070)	1.2557(0.1451)	-	-
W	1.2544 (0.1119)	0.6941 (0.0771)	-	-	-

Table 8. Analytical measures for data set 1 for all the competing distributions

Distribution	-l	AIC	AICC	BIC	HQIC	K-S	p-value
APMWG	63.9650	135.9300	136.6847	144.1718	139.1404	0.08083	0.7894
APMW	66.5174	139.0347	139.4792	145.2161	141.4425	0.08915	0.7050
EIFW	67.3977	140.7954	141.2398	146.9767	143.2031	0.1072	0.5157
EIWG	65.9582	137.9165	138.3609	144.0978	140.3243	0.0976	0.8016
AWG	63.9915	137.9830	139.1368	148.2852	141.9959	0.0889	0.7481
MW	71.6092	149.2185	149.6629	155.3998	151.6262	0.1369	0.2269
W	71.6092	147.2185	147.4367	151.3394	148.8236	0.3601	0.0583

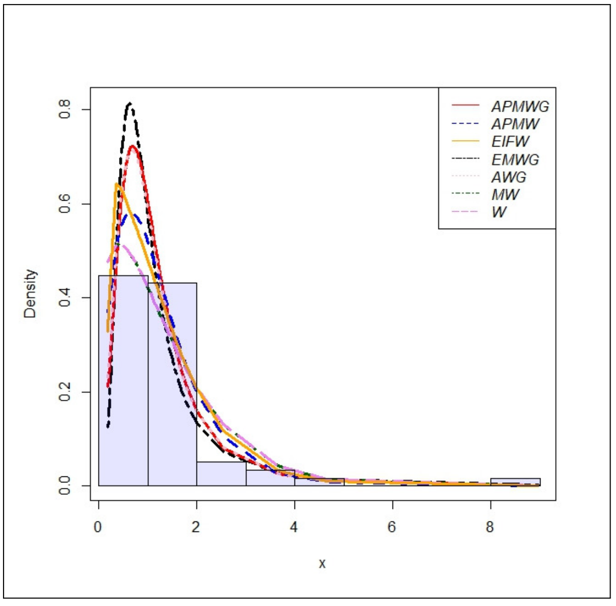

Figure 7. estimated densities with histogram for data set 1.

The TTT plot for this data reveals an upside down bathtub shaped HRF as shown in Figure 9 making APMWG distribution a suitable model for modeling the data.In Table 10 and 11 , the MLEs along with their SE and the analytical measure is presented respectively.

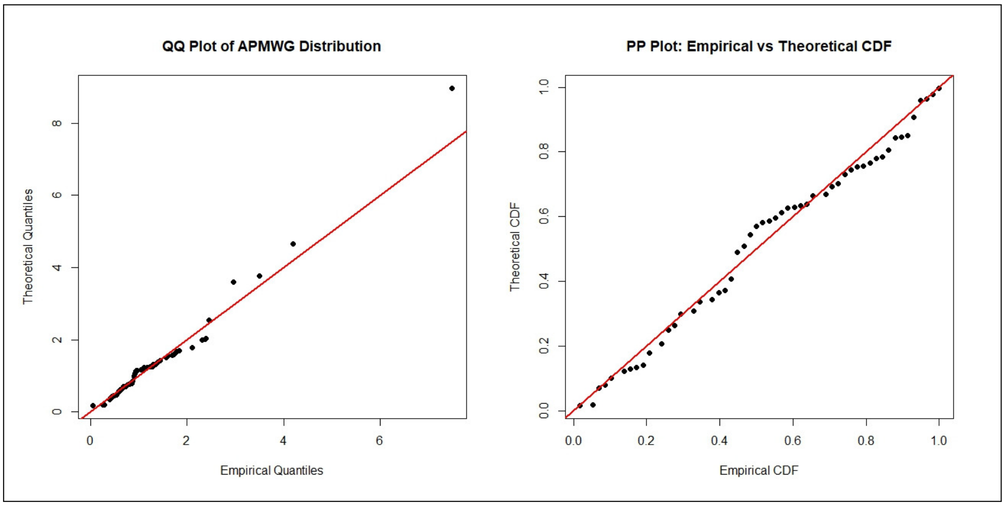

Figure 8. (a) QQ-plot; (b) PP-plot of APMWG ditribution for data set 1.

Table 9. Data set 2

17, 42, 44, 48, 60, 72, 74, 95, 103, 108, 122,144, 167, 170, 183, 185, 193,195, 197,
208, 234, 235, 254, 307, 315, 401, 445, 464, 484, 528, 542, 547, 577,580,
795, 855, 1366, 1577, 2060, 2412, 2486, 2796, 2802, 2934, 2988

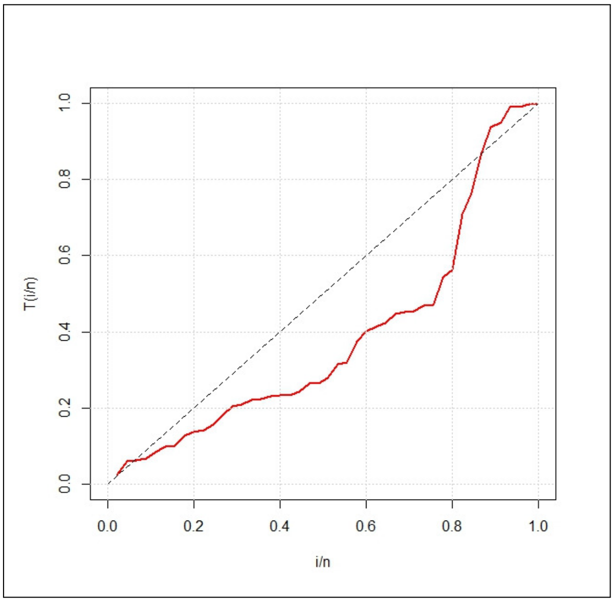

Figure 9. TTT plot for data set 2

From Table 11, we observe that the APMWG distribution has the lowest value of -l, AIC, AICC,BIC, HQIC and K-S (p-vale >0.05) , which is the analytical measures as compared to all the other competing distribution

Table 10. MLEs(SE) for all the distributions for data set 2

Distribution	α	γ	λ	β	p
APMWG	1.0215(0.7071)	1.2349 (0.1846)	0.0004 (0.0027)	-	0.9127 (7.79E-09)
APMW	0.11764(0.1653)	0.9550(0.1120)	0.0008 (0.0004)	-	-
EIFW	10(4.7249)	1.6681 (0.2968)	0.0017 (0.0003)	-	-
EIWG	0.4838 (0.1244)	10 (0.6387)	-	-	0.0050(1.2053)
AWG	0.8559 (0.0538	0.0006 (0.0025)	0.11037(0.0953)	0.0036 (0.0019)	0.3152(0.1420)
MW	0.00046 (0.0011)	0.0061 (0.0049)	0.7424 (0.1801)	-	-
W	0.8095 (0.0906)	0.0016(0.00032)	-	-	-

Table 11. Analytical measures for data set 2 for all the competing distributions

Distribution	-l	AIC	AICC	BIC	HQIC	K-S	p-value
APMWG	335.1269	678.2538	679.2538	685.4804	680.9478	0.1031	0.6867
APMW	338.4422	682.8845	683.4698	688.3045	684.9050	0.1233	0.4646
EIFW	345.7424	697.4849	698.0703	702.9049	699.5054	0.2246	0.01799
EIWG	347.3998	700.7997	701.3850	706.2197	702.8202	0.1823	0.5429
AWG	336.0936	682.1873	683.7257	691.2206	685.5548	0.1878	0.07301
MW	339.4885	684.9770	685.5624	690.3970	686.9975	0.14087	0.3044
W	337.6904	679.3807	679.6664	682.9941	680.7277	0.6339	0.0500

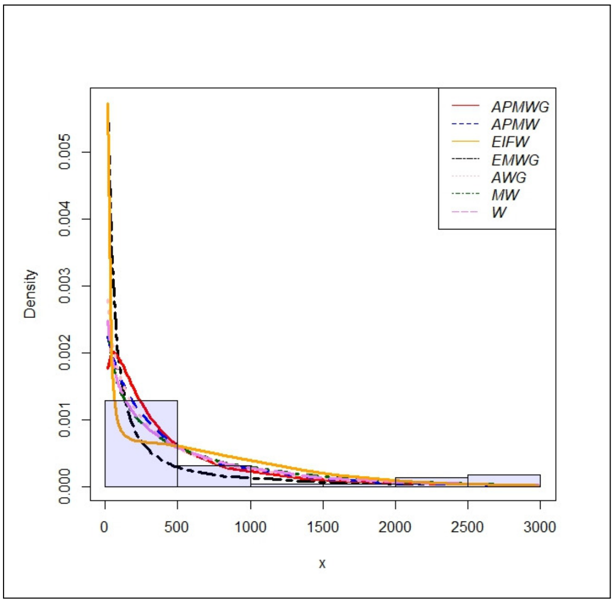

Figure 10. estimated densities with histogram for data set 2.

considered. Hence, The APMWG distribution can be considered to be the most adequate model for explaining the data set 2 compared to all the other competing distribution. Figure 10 (a), (b) and (c) display the plot of the estimated HRFs, estimated densities and estimated CDF respectively fitted to data set 1 for all the competing distributions. The QQ-plot and PP-plot of APMWG for the data set is shown in Figure 11.

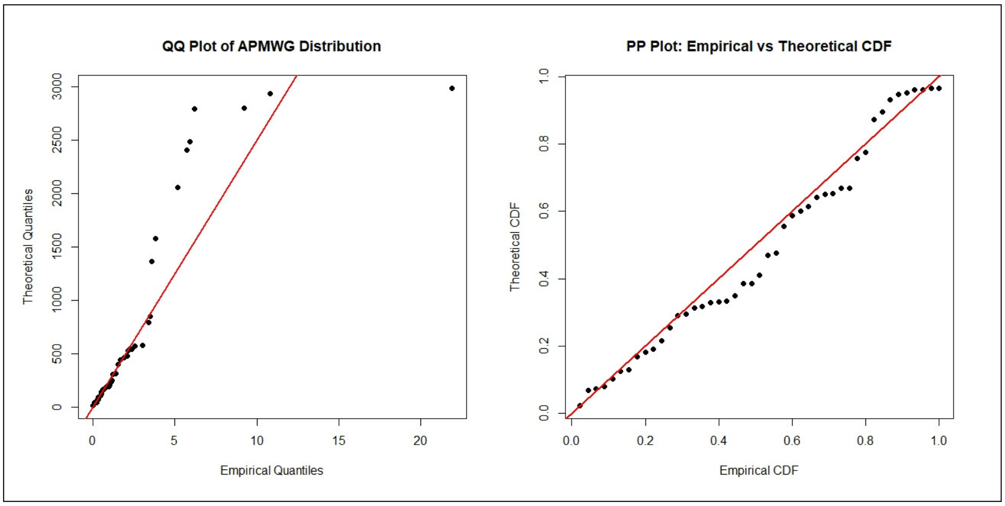

Figure 11. (a) QQ-plot; (b) PP-plot of APMWG ditribution for data set 2.

Table 12. Data set 3

0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0,
2.2, 2.5, 2.7,3.0, 3.0, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.2, 22.0, 24.5

Table 13. MLEs(SE) for all the distributions for data set 3

Distribution	α	γ	λ	β	p
APMWG	0.9043 (0.0125)	1.6478 (0.2316)	0.0448 (0.0398)	-	0.9758 (0.0538)
APMW	0.0281 (0.0532)	1.1867 (0.1376)	0.1060 (0.0458)	-	-
EIFW	0.4543 (0.5349)	1.7682 (1.9096)	0.3169 (0.1869)	-	-
EIWG	1.1136 (0.1610)	1.0214 (0.2132)	-	-	0.0518 (0.2793)
AWG	1.5979 (0.5792)	0.0024 (0.2155)	1.6826 (0.2901)	0.0040 (0.0507)	0.9762 (0.7412)
MW	0.0037 (1.3449)	0.2650 (1.4002)	0.9601 (0.1511)	-	-
W	0.9604 (0.1089)	0.2546 (0.0446)	-	-	-

6.3. Application 3

For the third application, we have considered a data sets which consists of 40 records of active repair times (in hours) for airborne communication transceiver (data set 3). The data was recently studied by [25] and the data set is given in Table 12.

As shown in Figure 12, the TTT plot of the third considered data set is an upside down bathtub shape which makes the APMWG distribution suitable for modeling the data. In Table 13 and 14, the MLEs along with their SE and the analytical measure is presented respectively.

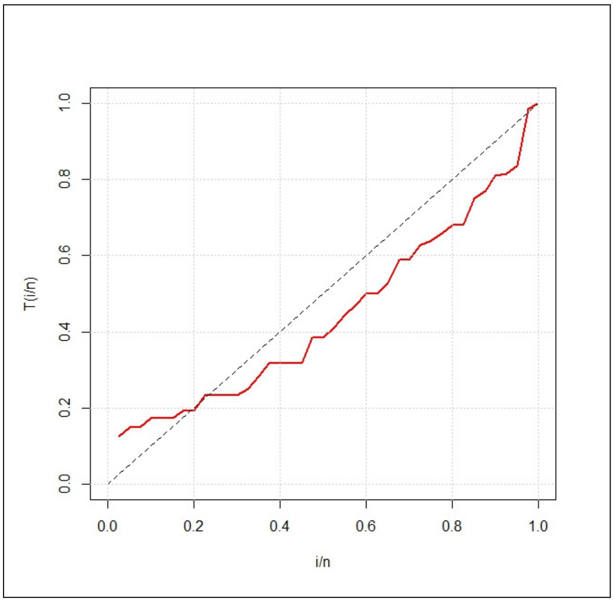

Figure 12. TTT plot for data set 3

Table 14. Analytical measures for data set 3 for all the competing distributions

Distribution	-l	AIC	AICC	BIC	HQIC	K-S	p-value
APMWG	91.6167	191.2335	192.3763	197.9890	193.6760	0.1196	0.6977
APMW	93.4721	192.9441	193.6108	198.0108	194.7761	0.1240	0.5699
EIFW	93.4986	192.9972	193.6639	198.0638	194.8291	0.2495	0.03731
EIWG	93.2363	192.4726	193.1393	197.5393	194.3046	0.2109	0.0912
AWG	91.6291	193.2582	195.0229	201.7026	196.3115	0.1178	0.7477
MW	95.5119	197.0237	197.6904	202.0904	198.8557	0.1289	0.5190
W	95.5114	195.0227	195.3470	198.4005	196.2440	0.6910	0.0014

From Table 14, we observe that the APMWG distribution has the lowest value of -l, AIC, AICC,BIC, HQIC and K-S (p-vale >0.05), which is the analytical measures as compared to all the other competing distribution considered. Hence, The APMWG distribution can be considered to be the most adequate model for explaining the data set 2 compared to all the other competing distribution. Figure 13 (a), (b) and (c) display the plot of the estimated HRFs, estimated densities and estimated CDF respectively fitted to data set 3 for all the competing distributions. The QQ-plot and PP-plot of APMWG distribution for the this data set is also shown in Figure 14.

7. Conclusion

In this work, we proposed a new four parameter lifetime distribution defined as Alpha Power Modified Weibull geometric distribution. We obtain some of its statistical properties including the moments, moment generating function, mean residual life, measure of Inequality and uncertainty and order statistics. The method of maximum likelihood is used to obtain the model parameter and Monte Carlo Simulation technique is used to assess the performance of the estimation method considered. we have analyzed three real life data to demonstrate the capability and adequacy of the proposed distribution. The APMWG distribution is model fit for modeling the considered data sets as shown by the TTT plot in Figure 6, 9 and 12. Hence we have shown and justify that the APMWG distribution provides a better fit for all the three considered data sets showing its supremacy over the base

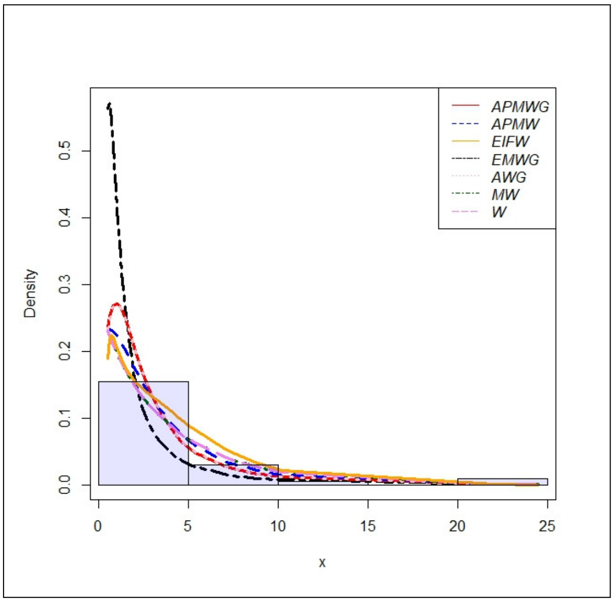

Figure 13. estimated densities with histogram for data set 3.

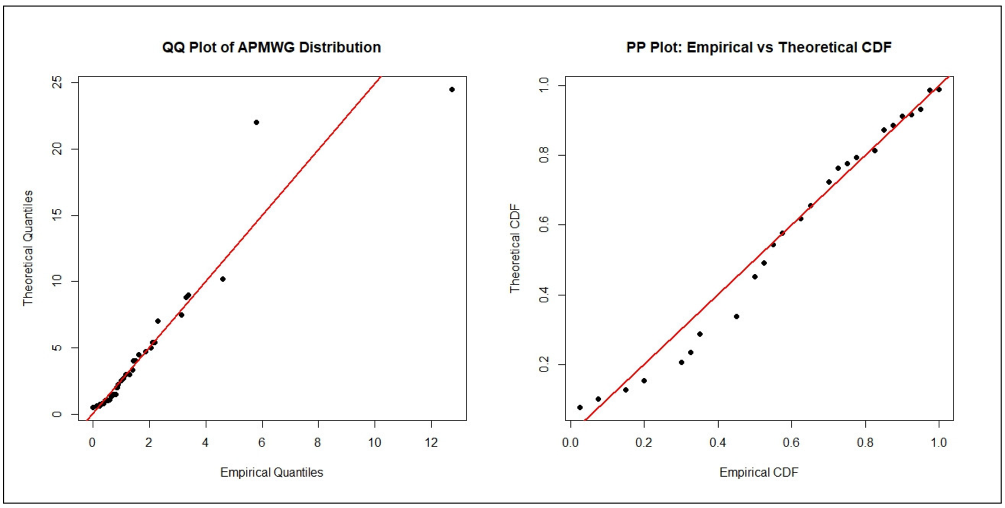

Figure 14. (a)QQ-plot; (b) PP-plot of APMWG ditribution for data set 3.

APMW distribution and the other five competing distribution viz. Exponentiated Inverse Flexible Weibull (EIFW), Exponentiated Inverse Weibull geometric (EIWG), Additive Weibull Geometric (AWG), Modified Weibull (MD) and Weibull(W) distribution for the reference data sets.

The APMWG distribution can be ethically applied to support fair, data-driven decisions in fields such as engineering, healthcare, reliability, and various biological or physical studies. However, its high flexibility poses a risk of overfitting, particularly when applied to small or limited datasets. Additionally, the large number of parameters can reduce interpretability, make communication more difficult, and increase complexity in both parameter estimation and computational tasks—these are key limitations of the flexible APMWG distribution.

This paper can be extended in several ways, which can also be consider for future research. One can develop the bi-variate and multivariate extension of the proposed distribution. Additional different estimation method can also be study alongside. Simulataneously, incorporating covariates for regression analysis, developing Bayesian estimation methods, or exploring multivariate versions can be considered. As we know, system can have different structural arrangement, with different failure time, hence compounding based on all this different failures time will give us different characteristic for the new develop distribution.Thus, work on compounding on different failure time of the system is appreciated.

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