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Repair Alert Model for One Component Systems with Discrete Lifetimes Belonging to the Power Series Family

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Abstract Repair alert models are essential tools for optimizing preventive maintenance in engineering systems. However, the development of these models for systems with discrete lifetime measurements—such as operational cycles, weekly failure reports, or counts of pages printed—has not been systematically addressed under a general class of discrete lifetime distributions. This paper specifically addresses this research gap. We propose a comprehensive framework for repair alert modeling by assuming that the discrete lifetimes of devices belong to the "power series family" of distributions. This approach encompasses a wide class of practically relevant discrete distributions. As a key component of this framework, we address the parameter estimation challenges for three significant and well-known distributions within this family. The Akaike Information Criterion is employed for optimal model selection, and approximate confidence intervals for the parameters of the chosen distribution are derived. The validity and practical utility of the proposed model are demonstrated through an insightful analysis of a real dataset.

Keywords Power Series Family, Maximum Likelihood Estimation, Random Sign Censoring, Repair Alert Model

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1. Introduction

Reliability is a fundamental concept in engineering, statistics, and system management. It is the probability that a system, product, or process will perform its intended function under specified conditions over a defined period. In simpler terms, reliability indicates a system's capability to operate effectively without failure or defects. A crucial aspect of reliability is Reliability-Centered Maintenance (RCM), which is a systematic approach focused on ensuring the reliability and functionality of equipment and systems throughout their life cycle. The primary objective of RCM is to identify the most effective maintenance strategies that can prevent failures and minimize downtime while optimizing costs. Consider a system that begins operating at time zero. It may either fail at a random time X or be replaced by a new one at a random time Z, which may occur first. Typically, it is assumed that X and Z are independent. This policy is referred to as *random age replacement maintenance*; see, for example, [11, Chapter 2]. Thus, only the pair (Y, δ) is observed, where $Y = \min(X, Z)$ and $\delta = I(Z < X)$. Here $I(A)$ denotes the indicator function for the event A, such that $I(A) = 1$ when A occurs and $I(A) = 0$ otherwise. When Z and X are independent, the marginal distribution functions (DFs) of X and Z, denoted by $F_X(x)$ and $F_Z(z)$, respectively, can be identifiable based on the joint DF (Y, δ); For example, see [5, Theorem 1]. However, in practice, engineers often receive signals from an operating device that prompt them to replace it with a new device

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to avert the costs associated with random failure, suggesting that the independence assumption may not hold. In competing risks theory, it is established that the marginal DFs are unidentifiable based solely on observations (Y, δ) without considering the independence assumption between X and Z (see [12]).This unidentifiability issue becomes more complex when the failure processes are dependent. In recent years, extensive research has been conducted on modeling dependent competing failure processes , particularly in the context of reliability and maintenance. For instance, [14] provides a systematic and critical literature review on dependent failure behavior modeling for risk and reliability analysis. Furthermore, [7] investigates reliability and maintenance modeling for systems subject to dependent competing failure processes with shifting failure thresholds. Additionally, [13] focuses on degradation modeling and the prediction of remaining useful life in the presence of such dependencies. To address the identifiability challenge mentioned earlier, Cook [6] introduced the concept of *random sign censoring* (RSC), where X and Z are dependent, yet the marginal DF of X remains identifiable.

Definition 1.1. [9, 5] Let (X, Z) be a pair of lifetimes. Then $\{(Y = \min(X, Z), \delta = I(Z < X))\}$ is called a random signs censoring of X by Z if the event $\{Z < X\}$ is (stochastically) independent of X.

Definition 1.2. The subdistribution functions (sub-DFs) of X and Z are defined by $F_X^*(x) = P(X \leq x, X < Z)$ and $F_Z^*(z) = P(Z \leq z, Z < X)$, respectively. The conditional DFs of X and Z are defined by $\tilde{F}_X(x) = P(X \leq x|X < Z)$ and $\tilde{F}_Z(z) = P(Z \leq z|Z < X)$, respectively. Similarly, the sub-probability mass functions (sub-PMFs) and the conditional PMFs of X and Z are defined.

Lindqvist [9] proposed the Repair Alert Model (RAM) for analyzing the random sign censoring data (RSCD). The formal definition of the RAM is provided.

Definition 1.3. [9] The pair (X, Z) satisfies the requirements of the RAM if

(i) Z is a random signs censoring of X; that is, the event $\{Z < X\}$ is stochastically independent of X;
(ii) there exists an increasing function $G : [0, +\infty) \to [0, +\infty)$ with $G(0) = 0$, such that for all $x \in (0, +\infty)$,

$$P(Z \leq z|Z < X, X = x) = \frac{G(z)}{G(x)}, \qquad 0 < z \leq x. \tag{1}$$

The function G is called the cumulative repair alert function(CRAF). Its derivative g (which we shall assume exists) is called the repair alert function(RAF). Statistical inferences based on RSCD have also been considered in the literature; for instance, see [9, 1, 2]. They assumed that the lifetime X is continuous, but lifetimes can also be discrete. Examples of discrete lifetimes include the functioning of equipment in cycles, weekly field failure reports, and the number of pages printed by a device before failure. Recently, Atlehkhani and Doostparast [3] examined the RAM in situations where the discrete lifetime distribution X belongs to a broad class known as the telescopic family. They modified Definition 1.3 for discrete lifetimes. For the sake of brevity, the supports of X and Z are assumed to be $\mathbb{N} := \{0, 1, 2, \ldots\}$, i. e. $S_x = S_z = \mathbb{N}$.

Example 1.1. [9] Let (X, Z) be a pair of life variables with joint density

$$f_{X,Z}(x, z) = (2x)^{-1}e^{-x} \quad \text{for } x > 0, \quad 0 < z < 2x.$$

The marginal distribution of X is the standard exponential distribution with density $f_X(x) = e^{-x}$, while the conditional distribution of Z given $X = x$ is the uniform distribution on $(0, 2x)$. From this we obtain $P(Z < X|X = x) = \frac{1}{2}$ for all $x > 0$. Thus the event $Z < X$ is independent of X and condition (i) of Definition 1.3 is satisfied. The following computation shows that condition (ii) holds as well. Let $0 < z < x$, Then

$$\begin{aligned} P(Z \leq z|X = x) &= \frac{P(Z \leq z, Z < X|X = x)}{P(Z < X|X = x)} \\ &= \frac{P(Z \leq z|X = x)}{(1/2)} = \frac{z}{x}, \end{aligned}$$

which implies condition (ii) of Definition 1.3 with $G(t) = t$.

Definition 1.4. [3] The pair (X, Z) is called the discrete repair alert model (DRAM) if

(i) the event $\{Z < X\}$ is stochastically independent of X;
(ii) there exists an increasing function $G : \mathbb{N} \to \mathbb{N}$ with $G(0) = 0$, such that for all $x \in \mathbb{N}$

$$P(Z \leq z | Z < X, X = x) = \frac{G(z)}{G(x)}, \qquad z = 0, 1, \ldots, x. \tag{2}$$

The next proposition provides the sub-DFs and conditional DFs under a DRAM. To do this, let

$$q = P(Z < X). \tag{3}$$

Proposition 1.2. *[3] Let the random vector (X, Z) follow the DRAM. The sub-DFs and the conditional DFs of X and Z are, respectively, given by*

$$\tilde{F}_X(x) = F_X(x), \tag{4}$$
$$F^*_X(x) = (1 - q)F_X(x), \tag{5}$$
$$\tilde{F}_Z(z) = F_X(z) + \sum_{k=z+1}^{\infty} \frac{G(z)}{G(k)} P(X = k), \tag{6}$$
$$F^*_Z(z) = q\tilde{F}_Z(z), \tag{7}$$

for all $x \in \mathbb{N}$ and $z \in \mathbb{N}$.

Note that the conditional PMF of Z in Proposition 1.2 is derived as

$$\begin{aligned} \tilde{f}_Z(z) &= P(Z = z | Z < X) \\ &= g(z) \sum_{k=z}^{\infty} \frac{P(X = k)}{G(k)}, \quad \text{for all } z \in \mathbb{N}, \end{aligned} \tag{8}$$

where $g(z) = G(z) - G(z - 1)$. In this study, we assume that an individual has independently implemented a random age replacement policy N times. The available observations are denoted as $(Y_1, \delta_1); \ldots; (Y_N, \delta_N)$, representing the Random Sign Censoring Data (RSCD). Our focus will be on the RAM when the lifetimes are represented as discrete random variables, with the distribution of device lifetimes $F_X(t)$ belonging to a class known as the *power series family*; see Section 2. This family encompasses a diverse range of discrete lifetime distributions, including logarithmic and negative binomial distributions, etc. Furthermore, we will explore the issue of parameter estimation and illustrate our findings through an analysis of a real dataset. Therefore, the remainder of this paper is organized as follows: In Section 2, the power series family and two important subclasses, i.e., logarithmic and negative binomial distributions, are studied in detail. In Section 3, the maximum likelihood estimates (MLEs) of the parameters and approximate confidence intervals are also derived. In Section 4, a real data set on ARC-1 VHF, reported by [10], is analyzed. Section 5 provides conclusions. Table 1 displays some notations and acronyms used in this paper.

2. DRAM in the power series family

In this section, we assume that the DF of lifetime X under the DRAM belongs to the power series (PS) family. First, the formal definition of this family is given.

Definition 2.1. Suppose that the random variable X has the PMF as

$$f(x; \theta) = \frac{a(x)\theta^x}{c(\theta)}, \tag{9}$$

where $x \in \mathbb{N}$, $\theta > 0$, $a(x) > 0$, and $c(\theta) = \sum_{x=0}^{\infty} a(x)\theta^x$. Then X has the power series model and denoted by $X \sim PS(\theta)$.

Notations and acronyms	Description
X	device lifetime
Z	repair time
$F_X(x)$	$P(X \leq x)$, DF of X
$f_X(x)$	PMF of X
$F_Z(z)$	$P(Z \leq z)$, DF of Z
$f_Z(z)$	PMF of Z
$F^*_X(x)$	$P(X \leq x, X < Z)$, sub-DF of X
$f^*_X(x)$	submass function of X
$F^*_Z(z)$	$P(Z \leq z, Z < X)$, sub-DF of Z
$f^*_Z(z)$	submass function of Z
$\tilde{F}_X(x)$	$P(X \leq x \mid X < Z)$, conditional DF of X
$\tilde{f}_X(x)$	conditional PMF of X
$\tilde{F}_Z(z)$	$P(Z \leq z \mid Z < X)$, conditional Df of Z
$\tilde{f}_Z(z)$	conditional PMF of Z
RAM	repair alert model
FI	Fisher information
RSCD	random sign censoring data
LF	likelihood function
LLF	log of likelihood function
PMF	probability mass function
DRAM	discrete repair alert model
$\mathbb{R}^+$	$(0, +\infty)$, the set of positive real numbers
$\mathbb{N}$	$\{0, 1, 2, \ldots\}$, the set of natural numbers

Table 1. Notations and acronyms

Table 2 displays some members of the PS family (9). For example, if $\theta = p/(1-p)$, $c(\theta) = (1-p)^{-n}$ and $a(x) = \frac{n!}{x!(n-x)!}$ then the PMF (9) simplifies to the binomial PMF, denoted by $Bin(n, p)$.

Distribution	θ	$c(\theta)$	$a(x)$	Notation
Binomial	$\frac{p}{1-p}$	$(1-p)^{-n}$	$\frac{n!}{x!(n-x)!}$	Bin(n,p)
Negative binomial	p	$(1-p)^{-r}$	$\frac{\Gamma(x+r)}{x!\Gamma(r)}$	NB(r,p)
Poisson	λ	$exp(\lambda)$	$\frac{1}{x!}$	Poisson(λ)
Logarithmic	p	$-\ln(1-p)$	$\frac{1}{x}$	L(p)
Geometric	$(1-p)$	p^{-1}	1	Geo(p)

Table 2. Some members of the PS family

Proposition 2.1. *Under the DRAM, assume that $X \sim PS(\theta)$. Then, the sub-PMFs of X and Z are, respectively, given by*

$$f^*_X(x;\theta) = (1-q)\frac{a(x)\theta^x}{c(\theta)}, \quad \forall x \in \mathbb{N}, \tag{10}$$

and

$$f_Z^*(z;\theta) = qg(z)\sum_{k=z}^{\infty}\frac{a(k)\theta^k}{c(\theta)G(k)}, \quad \forall z \in \mathbb{N}. \tag{11}$$

Proof
From Equation (9), the sub-PMF of X is derived as

$$\begin{aligned} f_X^*(x;\theta) &= P(X=x, X<Z) \\ &= P(X=x)P(X<Z) \\ &= (1-q)P(X=x) \\ &= (1-q)\frac{a(x)\theta^x}{c(\theta)}. \end{aligned}$$

The second equality follows by the fact that X and the event $\{X < Z\}$ are independent. From Equation (8), we also obtain the sub-PMF of Z as

$$\begin{aligned} f_Z^*(z;\theta) &= P(Z=z, Z<X) \\ &= P(Z<X)P(Z=z|Z<X) \\ &= q\tilde{f}_Z(z) = qg(z)\sum_{k=z}^{\infty}\frac{P(X=k)}{G(k)} \\ &= qg(z)\sum_{k=z}^{\infty}\frac{a(k)\theta^k}{c(\theta)G(k)}, \end{aligned}$$

and the proof completes. □

Remark 2.2. In Proposition 2.1, the conditional PMFs of X and Z are easily derived by dividing the corresponding sub-PMFs by the quantity q in Equation (3).

Now, we simplified the DRAM for three important members of the PS family.

2.1. Logarithmic distribution

In the PS distribution (9) if $\theta = p$, $c(\theta) = -\ln(1-p)$, and $a(x) = x^{-1}$ that is $f_X(x;p) = \frac{-p^x}{x\ln(1-p)}$ then, the logarithmic distribution is obtained. The logarithmic distribution is a significant statistical model utilized to analyze income distribution, where a small number of individuals hold a disproportionately large share of wealth. This distribution is also applied to model species abundance and biodiversity, particularly in understanding the distribution of various species within a given area. In communication systems, the logarithmic distribution is employed to analyze the distribution of information content in datasets, providing insights into data organization and transmission efficiency. In reliability engineering, it is used to model the time until failure of systems or components, especially in scenarios where failures are rare and have critical implications. In the social sciences, the logarithmic distribution is implemented to model the distribution of social phenomena, such as the number of social interactions or connections among individuals, thus enhancing our understanding of social networks. Within telecommunications, this distribution is utilized to model the distribution of packet arrival times in network traffic, which is essential for optimizing network performance and reliability. In marketing, the logarithmic distribution is applied to analyze consumer behavior and the distribution of product usage among different consumer segments, enabling businesses to tailor their strategies effectively. Additionally, environmental studies, serve to model the distribution of pollutants or species in ecological contexts, contributing to the assessment of environmental health and biodiversity. Overall, the logarithmic distribution proves to be a versatile tool across various disciplines, facilitating the analysis of complex phenomena and enhancing our understanding of diverse systems.

Under the DRAM, assume that $G(t) = t^p$. Then $g(t) = G(t) - G(t-1) = t^p - (t-1)^p$. So, Equation (11) yields

$$f_X^*(x;p) = (1-q)\frac{-p^x}{x\ln(1-p)}, \quad \forall x \in \mathbb{N}, \tag{12}$$

Also,

$$\begin{aligned} f_Z^*(z;p) &= qg(z)\sum_{k=z}^{\infty}\frac{a(k)\theta^k}{c(\theta)G(k)} \\ &= q\big(z^p-(z-1)^p\big)\sum_{k=z}^{\infty}\frac{-p^k}{k\ln(1-p)k^p}, \quad \forall z \in \mathbb{N}. \end{aligned} \tag{13}$$

2.2. Negative binomial distribution

In the PS distribution (9) if $\theta = p$, $c(\theta) = (1-p)^{-r}$ and $a(x) = \Gamma(x+r)(x!\Gamma(r))^{-1}$ that is $f_X(x;r,p) = \Gamma(x+r)(x!\Gamma(r))^{-1}p^x(1-p)^r$ then the negative binomial distribution is derived. The negative binomial distribution is a powerful statistical tool used to model the number of trials until a specified number of successes occurs. For instance, it can be employed to determine the number of treatments required until a patient achieves a specific health outcome. In manufacturing, this distribution is applied to ascertain the number of defective items produced until a predetermined number of acceptable items are found. Additionally, the negative binomial distribution is useful for modeling the number of species observed in a given area until a certain number of species are recorded, providing valuable insights into ecological studies. In the insurance industry, this distribution models the number of claims made by policyholders until a certain threshold of claims is reached, aiding in risk management and financial planning. In the realm of sports analytics, it is utilized to model the number of games played until a team wins a specified number of matches, offering insights into team performance and strategy. The negative binomial distribution also finds application in telecommunications, where it can model the number of calls made until a fixed number of successful connections are established. In epidemiology, the negative binomial distribution is used to model the number of infections that occur until a certain number of recoveries is achieved within a population, contributing to public health assessments and strategies. In marketing, it is instrumental in analyzing customer behavior, particularly in determining the number of purchases made until a customer executes a repeat purchase. Furthermore, in finance, the number of trades required to achieve a specific profit target can be effectively modeled using the negative binomial distribution, helping investors in strategy formulation. In the context of corporate liabilities, the distribution is applied to model the number of attempts a student makes until they pass a certain number of exams, providing insights into educational outcomes and student performance. Overall, the negative binomial distribution serves as a versatile mechanism for analyzing various phenomena across multiple disciplines. In the DRAM, the sub-PMF of the lifetime X is

$$\begin{aligned} f_X^*(x;p) &= P(X=x, X<Z) \\ &= (1-q)P(X=x) \\ &= (1-q)\frac{\Gamma(x+r)}{x!\Gamma(r)}p^x(1-p)^r, \quad \forall x \in \mathbb{N}. \end{aligned} \tag{14}$$

If $G(t) = t^p$, then $g(t) = G(t) - G(t-1) = t^p - (t-1)^p$ and from Equation (11), we obtain

$$\begin{aligned} f_Z^*(z;p) &= qg(z)\sum_{k=z}^{\infty}\frac{P(X=k)}{G(k)} \\ &= q\big(z^p-(z-1)^p\big)\sum_{k=z}^{\infty}\frac{\Gamma(k+r)}{k!\Gamma(r)k^p}p^k(1-p)^r, \quad \forall z \in \mathbb{N}. \end{aligned} \tag{15}$$

2.3. Poisson distribution

In the PS distribution (9) if $\theta = \lambda$, $c(\theta) = \exp \lambda$ and $a(x) = x!^{-1}$ that is $f_X(x;\lambda) = \lambda^x e^{-\lambda} x!^{-1}$ then the Poisson distribution is derived. The Poisson distribution is a fundamental tool in statistical modeling, with diverse applications across various fields. In call center management, it is employed to model the number of incoming calls to a call center within a specified period, aiding in resource allocation and staffing decisions to optimize service levels. In traffic flow analysis, the distribution is utilized to estimate the number of vehicles passing through a checkpoint during a defined time frame, assisting urban planners in traffic management and infrastructure development. In the realm of epidemiology, the Poisson distribution plays a crucial role in modeling the incidence of diseases within a fixed population over a specified time interval, which is essential for public health officials in understanding disease dynamics and implementing control measures. Quality control benefits from this distribution as it helps determine the number of defects in a batch of products, thus maintaining product quality and minimizing waste. In natural disaster management, the Poisson distribution models the frequency of earthquakes occurring in a particular region over a fixed period, aiding in risk assessment and disaster preparedness planning. Similarly, in website traffic analysis, the Poisson distribution is used to predict the number of hits on a website within a given time interval, supporting businesses in understanding user engagement and optimizing their online presence. Additionally, in insurance claims analysis, the number of claims received by an insurance company in a specific time frame can be effectively modeled using the Poisson distribution, which assists insurers in risk assessment and financial forecasting. In biological research, this distribution is applied to model the number of mutations occurring in a specified stretch of DNA over time, contributing to our understanding of genetic variation and evolution. Furthermore, in sports analytics, the distribution is utilized to model the number of goals scored in a soccer match, providing insights into team performance and game dynamics. Finally, in manufacturing settings, the Poisson distribution assists in predicting the number of machine failures over a certain period, which is crucial for maintenance scheduling and operational efficiency. Overall, the Poisson distribution serves as a versatile statistical tool, facilitating the analysis and modeling of count-based phenomena across various disciplines. In the DRAM, the sub-PMF of the lifetime X is

$$\begin{aligned} f_X^*(x;\lambda) &= P(X = x, X < Z) \\ &= (1-q)P(X = x) \\ &= (1-q)\frac{\lambda^x e^{-\lambda}}{x!}, \quad \forall x \in \mathbb{N}. \end{aligned} \tag{16}$$

If $G(t) = t^\lambda$, then $g(t) = G(t) - G(t-1) = t^\lambda - (t-1)^\lambda$ and from Equation (11), we obtain

$$\begin{aligned} f_Z^*(z;\lambda) &= qg(z)\sum_{k=z}^{\infty} \frac{P(X = k)}{G(k)} \\ &= q\big(z^\lambda - (z-1)^\lambda\big)\sum_{k=z}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!k^\lambda}, \quad \forall z \in \mathbb{N}. \end{aligned} \tag{17}$$

3. Parameter estimation

In this section, the estimates of parameters in a given DRAM are derived. Various approaches may be used to do this. In this paper, the ML method is considered. To end this, the likelihood function (LF) of the available RSCD $(\mathbf{x}, \mathbf{z})$ under the DRAM is

$$L(\theta, q; \mathbf{x}, \mathbf{z}) = \left(\prod_{i=1}^{m} f_X^*(x_i;\theta)\right)\left(\prod_{j=1}^{n} f_Z^*(z_j;\theta)\right). \tag{18}$$

The ML estimate of the parameter vector (θ, q) is then $L(\hat{\theta}, \hat{q}) = \arg\max_{\theta,q} L(\theta, q; \mathbf{x}, \mathbf{z})$. For the PS family (9), the LF of the RSCD $(\mathbf{x}, \mathbf{z})$ is obtained by substituting (10) and (11) into (18); that is,

$$\begin{aligned} L(\theta, q; \mathbf{x}, \mathbf{z}) &= \left(\prod_{i=1}^{m}(1-q)\frac{a(x_i)\theta^{x_i}}{c(\theta)}\right)\left(\prod_{j=1}^{n} qg(z_j)\sum_{k=z_j}^{\infty}\frac{a(k)\theta^k}{c(\theta)G(k)}\right) \\ &= (1-q)^m q^n \frac{\theta^{\sum_{i=1}^m x_i}}{(c(\theta))^m}\left(\prod_{i=1}^{m} a(x_i)\right)\left(\prod_{j=1}^{n} g(z_j)\sum_{k=z_j}^{\infty}\frac{a(k)\theta^k}{c(\theta)G(k)}\right). \end{aligned} \tag{19}$$

The ML estimate for q is readily derived from Equation (19) as

$$\hat{q} = \frac{n}{m+n}. \tag{20}$$

So, the profile LF of the RSCD $(\mathbf{x}, \mathbf{z})$, denoted by L_P, is then obtained by substituting $\hat{q}$ in Equation (20) into Equation (19); that is,

$$\begin{aligned} L_P(\theta; \mathbf{x}, \mathbf{z}) &= L(\theta, \hat{q}; \mathbf{x}, \mathbf{z}) \\ &= \left(\frac{m}{m+n}\right)^m \left(\frac{n}{m+n}\right)^n \frac{\theta^{\sum_{i=1}^m x_i}}{(c(\theta))^m}\left(\prod_{i=1}^{m} a(x_i)\right)\left(\prod_{j=1}^{n} g(z_j)\sum_{k=z_j}^{\infty}\frac{a(k)\theta^k}{c(\theta)G(k)}\right). \end{aligned} \tag{21}$$

The profile ML estimate of θ is then derived by maximizing (21); that is,

$$\begin{aligned} \hat{\theta} &= \arg\max_{\theta} L_P(\theta; \mathbf{x}, \mathbf{z}) \\ &= \arg\max_{\theta} \left[\frac{\theta^{\sum_{i=1}^m x_i}}{(c(\theta))^m}\prod_{i=1}^{m} a(x_i)\prod_{j=1}^{n} g(z_j)\sum_{k=z_j}^{\infty}\frac{a(k)\theta^k}{c(\theta)G(k)}\right]. \end{aligned} \tag{22}$$

Notice that the ML estimate $\hat{q}$ in Equation (20) does not depend on the ML estimate $\hat{\theta}$ in Equation (22). The ML estimate of the parameters vector θ in the DRAM for special cases, mentioned in Section 2, are obtained in sequel.

3.1. The logarithmic distribution

For the logarithmic distribution, from Equations (12), (13) and (18), we have

$$\begin{aligned} L(p, q; \mathbf{x}, \mathbf{z}) &= \prod_{i=1}^{m}\left((1-q)\frac{-p^{x_i}}{x_i \ln(1-p)}\right)\prod_{j=1}^{n}\left(q(z_j^p - (z_j-1)^p)\sum_{k=z_j}^{\infty}(\frac{-p^k}{k\ln(1-p)k^p})\right) \\ &= \frac{(1-q)^m q^n p^{\sum_{i=1}^m x_i}}{(-\ln(1-p))^{m+n}\prod_{i=1}^m x_i}\prod_{j=1}^{n}\left((z_j^p - (z_j-1)^p)\sum_{k=z_j}^{\infty}\frac{p^k}{k^{p+1}}\right). \end{aligned} \tag{23}$$

The logarithm of the likelihood function (LLF) is then

$$\begin{aligned} l(p, q; \mathbf{x}, \mathbf{z}) = n\ln(q) + m\ln(1-q) + \Big(\sum_{i=1}^{m} x_i\Big)\ln p - (m+n)\ln(-\ln(1-p)) \\ -\sum_{i=1}^{m}\ln x_i + \sum_{j=1}^{n}\ln((z_j^p - (z_j-1)^p) + \sum_{j=1}^{n}\ln\left(\sum_{k=z_j}^{\infty}\frac{p^k}{k^{p+1}}\right) \end{aligned} \tag{24}$$

The logarithm of the profile LF, denoted by l_P, is obtained from Equations (20) and (24) as follows:

$$\begin{aligned} l_P(p,r;\mathbf{x},\mathbf{z}) &= \ln L(p,r,\hat{q};\mathbf{x},\mathbf{z}) \\ &= n\ln\left(\frac{n}{m+n}\right)+m\ln\left(\frac{m}{m+n}\right)+\big(\sum_{i=1}^{m}x_i\big)\ln p-(m+n)\ln(-\ln(1-p)) \\ &\quad -\sum_{i=1}^{m}\ln x_i+\sum_{j=1}^{n}\ln((z_j^p-(z_j-1)^p)+\sum_{j=1}^{n}\ln\left(\sum_{k=z_j}^{\infty}\frac{p^k}{k^{p+1}}\right) \end{aligned} \tag{25}$$

The ML estimate of p is obtained by maximizing (24). For this purpose, one may use numerical methods; see Section 4.

3.2. *The negative binomial distribution*

For the negative binomial distribution, from Equations (14), (15) and (18), the LF (19) is simplified to

$$\begin{aligned} L(p,r,q;\mathbf{x},\mathbf{z}) &= \prod_{i=1}^{m}\left((1-q)\frac{\Gamma(x_i+r)}{x_i!\Gamma(r)}p^{x_i}(1-p)^r\right)\prod_{j=1}^{n}\left(q\big(z_j^p-(z_j-1)^p\big)\sum_{k=z_j}^{\infty}\frac{\Gamma(k+r)}{k!\Gamma(r)k^p}p^k(1-p)^r\right) \quad (26) \\ &= p^{\sum_{i=1}^{m}x_i}(1-q)^m q^n\frac{(1-p)^{r(m+n)}}{(\Gamma(r))^{m+n}}\prod_{i=1}^{m}\left(\frac{\Gamma(x_i+r)}{x_i!}\right)\prod_{j=1}^{n}\left(\big(z_j^p-(z_j-1)^p\big)\sum_{k=z_j}^{\infty}\frac{\Gamma(k+r)}{k!k^p}p^k\right). \end{aligned}$$

The LLF is then

$$\begin{aligned} l(p,r,q;\mathbf{x},\mathbf{z}) &= \ln L(p,r,q;\mathbf{x},\mathbf{z}) \\ &= n\ln q+m\ln(1-q)+\left(\sum_{i=1}^{m}x_i\right)\ln p+r(m+n)\ln(1-p)-(m+n)\ln\Gamma(r) \\ &\quad +\sum_{i=1}^{m}\big(\ln\Gamma(x_i+r)-\ln x_i!\big)+\sum_{j=1}^{n}\ln\big(z_j^p-(z_j-1)^p\big)+\sum_{j=1}^{n}\ln\left(\sum_{k=z_j}^{\infty}\frac{\Gamma(k+r)}{k!k^p}p^k\right). \end{aligned} \tag{27}$$

The logarithm of the profile LF, denoted by l_P, is obtained from Equations (20) and (27) as follows:

$$\begin{aligned} l_P(p,r;\mathbf{x},\mathbf{z}) &= \ln L(p,r,\hat{q};\mathbf{x},\mathbf{z}) \\ &= n\ln\left(\frac{n}{m+n}\right)+m\ln\left(\frac{m}{m+n}\right) \\ &\quad +\left(\sum_{i=1}^{m}x_i\right)\ln p+r(m+n)\ln(1-p)-(m+n)\ln\Gamma(r) \\ &\quad +\sum_{i=1}^{m}\big(\ln\Gamma(x_i+r)-\ln x_i!\big)+\sum_{j=1}^{n}\ln\big(z_j^p-(z_j-1)^p\big)+\sum_{j=1}^{n}\ln\left(\sum_{k=z_j}^{\infty}\frac{\Gamma(k+r)}{k!k^p}p^k\right). \end{aligned} \tag{28}$$

The ML estimate of p is obtained by maximizing (28). To do this, one may use numerical methods; see Section 4.

3.3. *The Poisson distribution*

For the Poisson distribution, the LF is obtained from Equations (16) - (18) as

$$\begin{aligned} L(\lambda,q;\mathbf{x},\mathbf{z}) &= \prod_{i=1}^{m}(1-q)\frac{\lambda^{x_i}e^{-\lambda}}{x_i!}\prod_{j=1}^{n}q\big(z_j^\lambda-(z_j-1)^\lambda\big)\sum_{k=z_j}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!k^\lambda} \\ &= (1-q)^m q^n e^{-(m+n)\lambda}\frac{\lambda^{\sum_{i=1}^{m}x_i}}{\prod_{i=1}^{m}x_i!}\prod_{j=1}^{n}\big(z_j^\lambda-(z_j-1)^\lambda\big)\sum_{k=z_j}^{\infty}\frac{\lambda^k}{k!k^\lambda}. \end{aligned} \tag{29}$$

The LLF is then

$$\begin{aligned} l(\lambda, q; \mathbf{x}, \mathbf{z}) &= \ln L(\lambda, q; \mathbf{x}, \mathbf{z}) \\ &= n \ln q + m \ln(1-q) + (m+n)\lambda + \left(\sum_{i=1}^{m} x_i\right) \ln \lambda - \sum_{i=1}^{m} \ln x_i! \\ &+ \sum_{j=1}^{n} \ln\left(z_j^{\lambda} - (z_j - 1)^{\lambda}\right) + \sum_{j=1}^{n} \ln\left(\sum_{k=z_j}^{\infty} \frac{\lambda^k}{k! k^{\lambda}}\right). \end{aligned} \tag{30}$$

The ML estimate for q is given by Equation (20). So, the logarithm of the profile LF reads

$$\begin{aligned} l(\lambda, \hat{q}; \mathbf{x}, \mathbf{z}) &= \ln L(\lambda, \hat{q}; \mathbf{x}, \mathbf{z}) \\ &= n \ln\left(\frac{n}{m+n}\right) + m \ln\left(\frac{m}{m+n}\right) + (m+n)\lambda + \left(\sum_{i=1}^{m} x_i\right) \ln \lambda - \sum_{i=1}^{m} \ln(x_i!) \\ &+ \sum_{j=1}^{n} \ln\left(z_j^{\lambda} - (z_j - 1)^{\lambda}\right) + \sum_{j=1}^{n} \ln\left(\sum_{k=z_j}^{\infty} \frac{\lambda^k}{k! k^{\lambda}}\right). \end{aligned} \tag{31}$$

The profile ML estimate of the parameter λ is derived numerically by maximizing Equation (31).

3.4. Approximate confidence intervals

Under some regular conditions, the ML estimate of the parameter vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_r)$ has asymptotically the r-variate normal distribution with mean $\boldsymbol{\theta}$ and the covariance matrix $\Sigma = \mathbf{I}^{-1}$, where $\mathbf{I}$ is the Fisher information (FI) matrix (see [8, p.476]). Indeed, $\mathbf{I} = -E_\theta(\mathbf{H})$, where $\mathbf{H} = [[\partial^2/\partial\theta_i\partial\theta_j \log L]]_{1\le i,j\le r}$ is the Hessian matrix of the LLF. Also, one can use the observed FI instead of the FI $\mathbf{I}$. The observed FI is defined by $i(\hat{\theta}) = -\hat{\mathbf{H}}$, where $\hat{\mathbf{H}} = [[\partial^2/\partial\theta_i\partial\theta_j \log L]]|_{\theta=\hat{\theta}}$. Therefore, under the regular conditions, we have

$$\hat{\Sigma}^{-1}(\hat{\theta} - \theta)) \to N_r(\mathbf{0}, \mathbf{J}), \tag{32}$$

where J is the $r \times r$ identity matrix. In the rest, the approximate confidence intervals are derived under a member of the PS family. From Equation (32), the approximate $100(1-\gamma)\%$ equal-tailed confidence intervals for θ_i is obtained as

$$\hat{\theta}_i \pm z_{\gamma/2}\sqrt{\widehat{var}(\hat{\theta}_i)}, \qquad 1 \le i \le r \tag{33}$$

respectively, where z_γ stands for the upper 100γth percentile of the standard normal distribution. The details are illustrated in the next section.

4. Case study

In this section a real data set is analysed to demonstrate applications of the findings of this paper.

4.1. The dataset

Table 3 presents a real-world dataset pertaining to ARC-1 VHF communication transmitter-receivers used by a commercial airline. The dataset captures the number of operational hours until failure (see [10]). When failures occurred, the affected units were removed from the aircraft for maintenance. However, in some instances, the reported failures were unconfirmed, as the units demonstrated satisfactory functionality upon inspection at the maintenance center. Consequently, the failure times can be categorized into two distinct groups: **confirmed failures** ($\mathbf{x}$) and **unconfirmed failures** ($\mathbf{z}$). The dataset comprises $m = 218$ observations for confirmed failures (X) and $n = 107$ observations for unconfirmed failures (Z).

Table 3. ARC-1 VHF communication transmitter-receivers of a single commercial number of hours to failure

x	16, 224, 16, 80, 128, 168, 144, 176, 176, 568, 392, 576, 128, 56, 112, 160, 384, 600, 40, 416,408, 384, 256, 246, 184, 440, 64, 104, 168, 408, 304, 16, 72, 8, 88, 160, 48, 168, 80, 512,208, 194, 136, 224, 32, 504, 40, 120, 320, 48, 256, 216, 168, 184, 144, 224, 488, 304, 40, 160, 488, 120, 208, 32, 112, 288, 336, 256, 40, 296, 60, 208, 440, 104, 528, 384, 264, 360, 80, 96, 360, 232, 40, 112, 120, 32, 56, 280, 104, 168, 56, 72, 64, 40, 480, 152, 48, 56, 328, 192, 168, 168, 114, 280, 128, 416, 392, 160, 144, 208, 96, 536, 400, 80, 40, 112, 160, 104, 224, 336, 616, 224, 40, 32, 192, 126, 392, 288, 248, 120, 328, 464, 448, 616, 168, 112, 448, 296, 328, 56, 80, 72, 56, 608, 144, 408, 16, 560, 144, 612, 80, 16, 424, 264, 256, 528, 56, 256, 112, 544, 552, 72, 184, 240, 128, 40, 600, 96, 24, 184, 272, 152, 328, 480, 96, 296, 592, 400, 8, 280, 72, 168, 40, 152, 488, 480, 40, 576, 392, 552, 112, 288, 168, 352, 160, 272, 320, 80, 296, 248, 184, 264, 96, 224, 592, 176, 256, 344, 360, 184, 152, 208, 160, 176, 72, 584, 144, 176.
z	368, 136, 512, 136, 472, 96, 144, 112, 104, 104, 344, 246, 72, 80, 312, 24, 128, 304, 16, 320, 560, 168, 120, 616, 24, 176, 16, 24, 32, 232, 32, 112, 56, 184, 40, 256, 160, 456, 48, 24, 200, 72, 168, 288, 112, 80, 584, 368, 272, 208, 144, 208, 114, 480, 114, 392, 120, 48, 104, 272, 64, 112, 96, 64, 360, 136, 168, 176, 256, 112, 104, 272, 320, 8, 440, 224, 280, 8, 56, 216, 120, 256, 104, 104, 8, 304, 240, 88, 248, 472, 304, 88, 200, 392, 168, 72, 40, 88, 176, 216, 152, 184, 400, 424, 88, 152, 184.

Table 4. ML estimates of parameters based on ARC-1 VHF data set

Distribution	$G(t)$	$MLestimate$	$\widehat{LLF}$	AIC
logarithmic	t^p	$\hat{p} = 0.9996$	-2558.88	5119.76
Poisson	t^λ	$\hat{\lambda} = 3.535867$	-29783.56	59565.12
negative binomial	t^p	$r = 1$, $\hat{p} = 0.9963$	-2300.538	4603.076
		$r = 2$, $\hat{p} = 0.9923$	-2276.055	4554.11
		$r = 3$, $\hat{p} = 0.9884$	-2297.733	4597.466
		$r = 4$, $\hat{p} = 0.9845$	-2335.408	4672.816
		$r = 5$, $\hat{p} = 0.9806$	-2380.789	4763.578
		$r = 6$, $\hat{p} = 0.9767$	-2430.464	4862.928
		$r = 10$, $\hat{p} = 0.9616$	-2646.737	5295.474
		$r = 20$, $\hat{p} = 0.9257$	-3197.791	6397.582

4.2. *The repair alert model selection*

For the analysis of this dataset the logarithmic, negative binomial, and Poisson distributions were considered for modeling the lifetime X. Numerical computations were performed using R (version 4.4.0) and Mathematica (version 9) software programs .The source code is publicly available at: https://fumdrive.um.ac.ir/index.php/s/S8pBKfQFWYJKrmJ. The empirical cumulative distribution function (ECDF) and the fitted parametric distribution functions as well the histogram of the data set and the corresponding fitted parametric PMFs are shown in Figures 2 and 1, respectively. These figures motivates that the negative binomial distribution with parameters $r = 2$ and $\hat{p} = 0.9923$ is superior to the logarithmic and Poisson distributions for this data set; For more information about ECDF based on RSCD, see [9]. The Akaike Information Criterion

(AIC) was employed to compare the fitted models. The AIC is defined as: AIC $= -2\widehat{LLF} + 2M$, where $\widehat{LLF}$ represents the maximized log-likelihood function (LLF) evaluated at the maximum likelihood (ML) estimates, and M denotes the number of estimated parameters. The results are summarized in Table 4. As illustrated in Table 4, the negative binomial distribution with parameters $r = 2$ and $\hat{p} = 0.9923$ exhibits a lower Akaike Information Criterion (AIC) value, indicating its superiority as the better-fitting model. An approximate 95 % confidence interval for p is $p \in (0.9917, 0.9929)$.

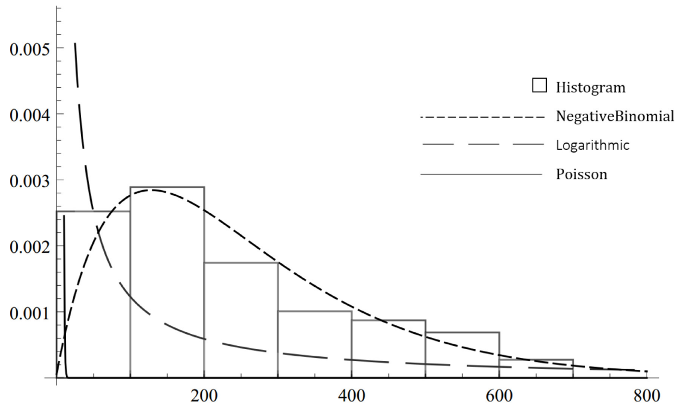

Figure 1

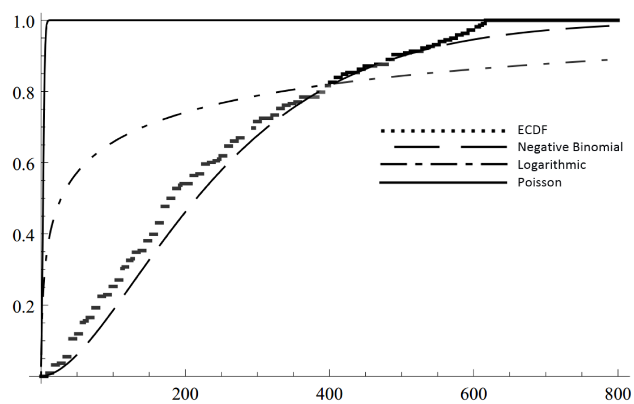

Figure 2

4.3. An application in system availability evaluation

To apply these findings, suppose we have a three-component system as depicted in Figure 3. The component lifetimes are identically distributed with the common negative binomial distribution with parameters $r = 2$ and $\hat{p} = 0.9923$. So, the mean time to failure (MTTF) is MTTF $= \frac{r\hat{p}}{1-\hat{p}} = \frac{2\times 0.9923}{0.0076687} = 258.8$. As in [4, page 193], it is also assumed that the components operate independently of one another. Indeed, while the replacement of a failed component is occurring in one position, the components in the remaining positions continue to operate. Moreover, we consider the mean time to repair (replace) (MTTR) equal to 24 time units. Therefore, the availability of Component i, for $i = 1, 2, 3$, denoted by A_i is

$$A_i = \frac{MTTF}{MTTF + MTTR} = \frac{258.8}{258.8 + 24} = 0.9151. \tag{34}$$

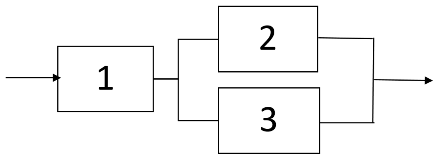

Figure 3. Three-component series-parallel system

Following [4, page 193], the system's availability is given by

$$A_{sys} = A_1\big(1-(1-A_2)(1-A_3)\big) \tag{35}$$
$$= 0.9151\big(1-(1-0.9151)(1-0.9151)\big)$$
$$= 0.9085$$

In Subsection 4.2, an approximate 95% confidence interval for p was obtained as $(0.9917, 0.9929)$. So, an approximate 95% confidence interval for the system availability can be derived by transforming the confidence bounds for p. Using the lower bound and the upper bound, the corresponding component availabilities are $A_L = 0.9087$ and $A_U = 0.9209$, respectively. Substituting these values into equation 35. Hence, an approximate 95% confidence interval for the system availability is derived as (0.9011,0.9150). This means that the system is available at least approximately 90.12%.

4.4. Extension to Bridge System Configuration

To address more complex system architectures, we consider a five-component bridge system as depicted in Figure 4. Following the same assumptions as in the previous example, each component has an identical availability of $A_i = 0.9151$.

The steady-state availability of the bridge system, using minimal path sets, is given by:

$$\begin{aligned} A_{\text{bridge}} =& A_1A_4 + A_2A_5 + A_1A_3A_5 + A_2A_3A_4 \\ &- A_1A_2A_4A_5 - A_1A_3A_4A_5 - A_1A_2A_3A_5 \\ &- A_1A_2A_3A_4 - A_2A_3A_4A_5 + 2A_1A_2A_3A_4A_5. \end{aligned}$$

With identical components, this simplifies to:

$$A_{\text{bridge}} = 2A^2 + 2A^3 - 5A^4 + 2A^5. \tag{36}$$

Substituting $A = 0.9151$ yields:

$$\begin{aligned} A_{\text{bridge}} &= 2(0.9151)^2 + 2(0.9151)^3 - 5(0.9151)^4 + 2(0.9151)^5 \\ &= 1.6748 + 1.5326 - 3.5060 + 1.2834 \\ &= 0.9848. \end{aligned}$$

This high availability (0.9848) indicates that the bridge structure with these highly reliable components results in a nearly perfect system availability, which is higher than the three-component series system due to the built-in redundancy. This example demonstrates how the proposed approach can be applied to non-trivial configurations, potentially informing maintenance scheduling by identifying critical components (e.g., the bridge component) that have a greater impact on overall system availability.

Following the same approach as in Section 4.3, an approximate 95% confidence interval for the bridge system availability can be derived by transforming the confidence bounds for p. Using the lower bound $\hat{p}_L = 0.9917$ and the upper bound $\hat{p}_U = 0.9929$, the corresponding component availabilities are $A_L = 0.9087$ and $A_U = 0.9209$, respectively. Substituting these values into equation 36. An approximate 95% confidence interval for the bridge system availability is $(0.9822, 0.9867)$. This narrow interval indicates that despite the increased system complexity, the high reliability of individual components results in a stable and precisely estimated system availability.

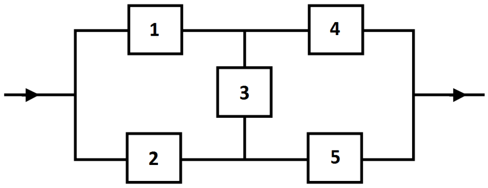

Figure 4. Five-component bridge system configuration

5. Conclusion

In this paper, we examined repair alert models for analyzing discrete lifetimes derived from maintenance programs. We focused on developing these models under the assumption that the lifetimes of the devices are discrete. We considered a broad class of discrete lifetime distributions known as the power series family. Furthermore, we addressed the challenges associated with parameter estimation for the logarithmic and negative binomial distributions within this family. In analyzing a real dataset, RAM selection and its application in system availability evaluation were demonstrated. This paper may be extended in various directions. For example, RSCD may be analyzed by the Bayesian approach under symmetric and asymmetric loss functions such as square error and linear-exponential ones. Prediction of the component lifetime X based on the observed RSCD is also worth consideration.

Declarations

Conflict of Interest The authors declare that they have no conflict of interest. All authors read and approved the manuscript.

Ethical Conduct This article does not contain any studies with human participants performed by any of the authors.

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Data Availability Statements All data supporting the findings of this study are available within this paper.

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