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A New Family of Bivariate Alpha Log Power Transformation Model Based on Extreme Shock Models

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Abstract This study presents a new bivariate distribution, referred to as the bivariate alpha log power transformation (BVALPT) model, developed by integrating the alpha log power transformation technique with the Marshall–Olkin extreme shock framework. Closed-form expressions for both the joint probability density function (pdf) and cumulative distribution function (cdf) are derived. The manuscript explores several key statistical properties of the proposed model, including marginal and conditional distributions, as well as survival and hazard rate functions. Parameter estimation is carried out using the maximum likelihood estimation (MLE) method. A notable special case, the bivariate alpha log power transformed exponential (BVALPTE) distribution, is examined in detail. The practical utility of the BVALPT family is demonstrated by fitting the BVALPTE distribution to a real-world dataset. Comparative results reveal that the BVALPTE offers an improved fit and enhanced analytical performance over the benchmark bivariate model considered in the analysis.

Keywords Alpha log power transformation; Survival functions; bivariate distributions; Marginal distributions; Conditional probability.

AMS 2010 subject classifications 62E99, 60E05

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1. Introduction

To better represent complex and varied data structures, there has been an increasing interest in creating new families of univariate probability distributions in recent years. These families of probability distributions are derived using many different techniques such as the exponentiated-G which includes the Lehmann alternative of type 1 and Lehmann alternative of type 2 [1], exponentiated generalised-G [2], the compounding method [3], transmuted-G [4], T-X technique [5], cubic rank transmuted-G [6], Marshall-Olkin-G [7], alpha power transformed-G [8], the generalized flexible-G [9] and the new flexible generalized-G [10]. These techniques have gained interest and have been widely used in literature [11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

Recently, [21] introduced a new technique of creating families of continuous distributions called the alpha log power transformation (ALPT), which generates flexible continuous univariate distributions by modifying the shape of a given baseline distribution without requiring a specific parent form. For any arbitrary baseline cdf distribution $G(t;\eta)$, the cumulative distribution function (cdf) and probability distribution function (pdf) of the ALPT are given by

$$F_{ALPT}(t;\alpha,\eta) = \alpha^{-\log(G(t;\eta))}, \tag{1}$$

and

$$f_{ALPT}(t;\alpha,\eta) = \frac{\log(1/\alpha)g(t;\eta)}{G(t;\eta)\alpha^{\log(G(t;\eta))}}, \tag{2}$$

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respectively, where α is a shape parameter also altering the thickness of the tails of the model, $G(t;\eta)$ is the baseline model, η being a vector of parameters and $t \in \mathbb{R}$. The baseline model $G(x;\gamma)$ is a special case of the ALPT if $\alpha = e^{-1}$. For small values of α (i.e.,$\alpha < e^{-1}$) the resulting distribution has a much heavier tail than $G(t;\eta)$ indicating that extreme values become more probable, for higher values of α (i.e.,$\alpha > e^{-1}$) the tail of the distribution is effectively suppressed, concentrating more mass towards lower values of t.

Extending flexible univariate distributions to the multivariate domain, particularly bivariate cases, remains an essential task in applied probability and statistics. Bivariate distributions are vital in modeling dependencies between random variables in areas such as survival analysis, reliability theory, and risk management. Several techniques have been developed for constructing bivariate distributions, including the use of copula functions [22], conditional specification [23], common latent variables [24], convolution approaches [25], transformation techniques [26] and shock models [27]. Among these, the Marshall–Olkin shock model [27] stands out as a powerful and intuitive method to model dependencies via common or shared shocks affecting multiple components simultaneously. In the literature, applications of the Marshall–Olkin technique [27] are not hard to find. For example the bivariate generalized exponential distribution [28], bivariate exponentiated modified Weibull extension distribution [29], the Marshall-Olkin bivariate exponentiated Lomax [30] and the bivariate generalized geometric [31].

Motivated by these developments, this manuscript introduces a novel bivariate model called the bivariate alpha log power transformation (BVALPT) family. This new family leverages the structure of the ALPT univariate model and incorporates dependency through the Marshall–Olkin framework [27].

The rest of the article is outlined as follows: We introduce the bivariate alpha log power transformation (BVALPT) family in Section 2. Some parameter estimation methods are discussed in Section 3. Section 4 covers the bivariate alpha log power transformed exponential (BVALPTE), a special case of the BVALPT using the exponential as the parent model. In Section 5, we presented some simulations based on the exponential baseline model, therefore, in Section 6, we illustrate the usefulness of the BVALPTE distribution using a real-life data set, followed by concluding remarks in Section 7.

2. Family formulation

Consider three independent sources (source 1, 2, and 3) of extreme shocks affecting a system with two components. Furthermore, assume that the shock from source 1 reaches the system and immediately destroys component 1 only, the shock from source 2 reaches the system and immediately destroys component 2 only, while if the shock from source 3 hits the system it immediately destroys both the components. Let $T_1 \sim ALPT(\alpha_1, \eta)$, $T_2 \sim ALPT(\alpha_2, \eta)$, and $T_3 \sim ALPT(\alpha_3, \eta)$ denote the inter-arrival times between the shocks from sources $1, 2$ and 3 respectively. Suppose X and Y denote the survival times of component 1 and 2, respectively. Following the Marshall–Olkin [27] framework, the joint lifetime of the two components follows the BVALPT with cdf given by

$$F_{BVALPT}(x,y) = \alpha_1^{-\log(G(x;\eta))} \alpha_2^{-\log(G(y;\eta))} \alpha_3^{-\log(G(z;\eta))}, \tag{3}$$

where $\alpha_i \in (0,1)$ for $i = 1,2,3$, η is a parameter vector and $z = min\{X, Y\}$. As a result, the lifetimes of component 1 only, and component 2 only follow the ALPT distribution with cdfs given by

$$F_x(x) = F_{ALPT}(x;\alpha_1\alpha_3,\eta) = (\alpha_1\alpha_3)^{-\log(G(x;\eta))}, \tag{4}$$

and

$$F_y(y) = F_{ALPT}(y;\alpha_2\alpha_3,\eta) = (\alpha_2\alpha_3)^{-\log(G(y;\eta))}, \tag{5}$$

respectively.

As a result, the pdf of the BVALPT is given by

$$f_{BVALPT}(x,y)=\begin{cases} f_{ALPT}(x;\alpha_1\alpha_3,\eta)\times f_{ALPT}(y;\alpha_2,\eta) & x<y\\ f_{ALPT}(y;\alpha_2\alpha_3,\eta)\times f_{ALPT}(x;\alpha_1,\eta) & x>y\\ \frac{\log(\alpha_3)}{\log(\alpha_1\alpha_2\alpha_3)} f_{ALPT}(u;\alpha_1\alpha_2\alpha_3,\eta) & x=y=u,\end{cases} \tag{6}$$

where

$$f_{ALPT}(x;\alpha_1\alpha_3,\eta) = \frac{\log(1/(\alpha_1\alpha_3))g(x;\eta)}{G(x;\eta)(\alpha_1\alpha_3)^{\log(G(x;\eta))}}, \tag{7}$$

$$f_{ALPT}(y;\alpha_2,\eta) = \frac{\log(1/\alpha_2)g(y;\eta)}{G(y;\eta)\alpha_2^{\log(G(y;\eta))}}, \tag{8}$$

$$f_{ALPT}(y;\alpha_2\alpha_3,\eta) = \frac{\log(1/(\alpha_2\alpha_3))g(y;\eta)}{G(y;\eta)(\alpha_2\alpha_3)^{\log(G(y;\eta))}}, \tag{9}$$

$$f_{ALPT}(x;\alpha_1,\eta) = \frac{\log(1/\alpha_1)g(x;\eta)}{G(x;\eta)\alpha_1^{\log(G(x;\eta))}}, \tag{10}$$

and

$$f_{ALPT}(u;\alpha_1\alpha_2\alpha_3,\eta) = \frac{\log(1/(\alpha_1\alpha_2\alpha_3))g(u;\eta)}{G(u;\eta)(\alpha_1\alpha_2\alpha_3)^{\log(G(u;\eta))}}. \tag{11}$$

Hence, the conditional lifetime of component 1, given the lifetime of component 2, is expressed as follows:

$$f(x|y)=\begin{cases} \frac{f_{ALPT}(x;\alpha_1\alpha_3,\eta)\times f_{ALPT}(y;\alpha_2,\eta)}{f_{ALPT}(y;\alpha_2\alpha_3,\eta)} & x<y\\ \frac{f_{ALPT}(y;\alpha_2\alpha_3,\eta)\times f_{ALPT}(x;\alpha_1,\eta)}{f_{ALPT}(y;\alpha_2\alpha_3,\eta)} & x>y\\ \frac{\log(\alpha_3)\times f_{ALPT}(u;\alpha_1\alpha_2\alpha_3,\eta)}{\log(\alpha_1\alpha_2\alpha_3)\times f_{ALPT}(y;\alpha_2\alpha_3,\eta)} & x=y=u,\end{cases} \tag{12}$$

where, $f_{ALPT}(x;\alpha_1\alpha_3,\eta), f_{ALPT}(y;\alpha_2,\eta), f_{ALPT}(y;\alpha_2\alpha_3,\eta), f_{ALPT}(x;\alpha_1,\eta)$, and $f_{ALPT}(u;\alpha_1\alpha_2\alpha_3,\eta)$ are given in equations (7), (8), (9), (10) and (11) respectively. In a similar manner, the conditional lifetime of component 2, given the lifetime of component 1, can be easily derived.

2.1. Survival function and Hazard rate function

The survival function of BVALPT is given by

$$S_{BVALPT}(x,y)=\begin{cases} S_1(x,y) & x<y\\ S_2(x,y) & x>y\\ S_3(u) & x=y=u,\end{cases} \tag{13}$$

where

$$\begin{aligned} S_1(x,y) &= 1-(\alpha_1\alpha_3)^{-\log(G(x;\eta))}-(\alpha_2\alpha_3)^{-\log(G(y;\eta))}\\ &+ (\alpha_1\alpha_3)^{-\log(G(x;\eta))}\alpha_2^{-\log(G(y;\eta))}, \end{aligned} \tag{14}$$

$$\begin{aligned} S_2(x,y) &= 1-(\alpha_1\alpha_3)^{-\log(G(x;\eta))}-(\alpha_2\alpha_3)^{-\log(G(y;\eta))}\\ &+ (\alpha_1)^{-\log(G(x;\eta))}(\alpha_2\alpha_3)^{-\log(G(y;\eta))}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} S_3(u) &= 1-(\alpha_1\alpha_3)^{-\log(G(u;\eta))}-(\alpha_2\alpha_3)^{-\log(G(u;\eta))} \\ &+ (\alpha_1\alpha_2\alpha_3)^{-\log(G(u;\eta))}. \end{aligned} \tag{16}$$

Hence, the BVALPT has its hazard rate function (hrf) as

$$hrf_{BVALPT}(x,y)=\begin{cases} \frac{f_{ALPT}(x;\alpha_1\alpha_3,\eta)\times f_{ALPT}(y;\alpha_2,\eta)}{S_1(x,y)} & x<y \\ \\ \frac{f_{ALPT}(y;\alpha_2\alpha_3,\eta)\times f_{ALPT}(x;\alpha_1,\eta)}{S_2(x,y)} & x>y \\ \\ \frac{\log(\alpha_3)\times f_{ALPT}(u;\alpha_1\alpha_2\alpha_3,\eta)}{\log(\alpha_1\alpha_2\alpha_3)\times S_3(u)} & x=y=u, \end{cases} \tag{17}$$

where, $f_{ALPT}(x;\alpha_1\alpha_3,\eta)$, $f_{ALPT}(y;\alpha_2,\eta)$, $f_{ALPT}(y;\alpha_2\alpha_3,\eta)$, $f_{ALPT}(x;\alpha_1,\eta)$ and $f_{ALPT}(u;\alpha_1\alpha_2\alpha_3,\eta)$ are given in equations (7), (8), (9), (10) and (11) respectively.

3. Estimation Methods

In this section, we present two commonly used methods from the literature for estimating unknown parameters: maximum likelihood estimation (MLE) and Bayesian estimation.

3.1. Maximum likelihood Estimation

In this subsection, we apply the maximum likelihood method to estimate the unknown parameters of the BVALPT family, considering both complete and censored data scenarios.

3.1.1. Complete Data: In this section, we estimate the unknown parameters of the BVALPT using the maximum likelihood estimation (mle) technique. Consider $D=\{(x_1,y_1),(x_2,y_2),...,(x_k,y_k)\}$ be a sample of size k from the BVALPT family. Let $D_1=\{(x_i,y_i);x_i<y_i\}$, $D_2=\{(x_i,y_i);x_i>y_i\}$, $D_3=\{(x_i,y_i);x_i=y_i\}$. Then the sets D_1, D_2, and D_3 are mutually disjoint and partition the set D. Furthermore, suppose $|D_1|=n, |\ D_2|=m$, $|D_3|=p$, and $n+m+p=k$. The likelihood function $L(\Delta)$ of this sample is given by

$$L(\Delta)=\prod_{i=1}^{n} f_1(x,y)\prod_{i=1}^{m} f_2(x,y)\prod_{i=1}^{p} f_3(u). \tag{18}$$

Substituting equation (6) into equation (18) yields

$$\begin{aligned} L(\Delta) &= n\log(\log(1/(\alpha_1\alpha_3)))+\sum_{i=1}^{n}\log(g(x;\eta))-\sum_{i=1}^{n}\log(G(x;\eta))-\log(\alpha_1\alpha_3)\sum_{i=1}^{n}\log(G(x;\eta)) \\ &+ n\log(\log(1/\alpha_2))+\sum_{i=1}^{n}\log(g(y;\eta))-\sum_{i=1}^{n}\log(G(y;\eta))-\log(\alpha_2)\sum_{i=1}^{n}\log(G(y;\eta)) \\ &+ m\log(\log(1/(\alpha_2\alpha_3)))+\sum_{i=1}^{m}\log(g(y;\eta))-\sum_{i=1}^{m}\log(G(y;\eta))-\log(\alpha_2\alpha_3)\sum_{i=1}^{m}\log(G(y;\eta)) \\ &+ m\log(\log(1/(\alpha_1)))+\sum_{i=1}^{m}\log(g(x;\eta))-\sum_{i=1}^{m}\log(G(x;\eta))-\log(\alpha_1)\sum_{i=1}^{m}\log(G(y;\eta)) \\ &+ p\log(\log(1/\alpha_3))+\sum_{i=1}^{p}\log(g(u;\eta))-\sum_{i=1}^{p}\log(G(u;\eta))-\log(\alpha_1\alpha_2\alpha_3)\sum_{i=1}^{p}\log(G(u;\eta)). \end{aligned}$$

To obtain score functions associated with equation (19), we solve the non-linear systems of equations

$$\hat{\boldsymbol{\Delta}} = \left(\frac{\partial \ell(\Delta)}{\partial \alpha_1}, \frac{\partial \ell(\Delta)}{\partial \alpha_2}, \frac{\partial \ell(\Delta)}{\partial \alpha_3}, \frac{\partial \ell(\Delta)}{\partial u}\right)^T = 0 \tag{19}$$

using numerical methods such as the Newton–Raphson technique, bisection method, secant method, fixed point iteration, or false position (regula falsi) technique. The variance-covariance matrix of the maximum likelihood estimators is derived by taking the negative inverse of the matrix of second-order partial derivatives. The standard errors for the parameter estimates are obtained by computing the square roots of the diagonal elements of this matrix. These maximum likelihood estimates, along with their corresponding standard errors, can then be used to calculate asymptotic z-statistics (Wald statistics) or to construct confidence intervals.

3.1.2. Censored Data: We now consider that the data is subject to right censoring. Specifically, let $(x_1, y_1), (x_2, y_2), ..., (x_k, y_k)$ represent a random sample of size k drawn from a bivariate lifetime distribution. To account for the censoring, we introduce the following indicator variables.

$$\begin{cases} \gamma_{1i} = 1 & \text{if} \quad X_i < c_{1i} \quad \text{and} \quad 0 \quad \text{otherwise} \\ \gamma_{2i} = 1 & \text{if} \quad Y_i < c_{2i} \quad \text{and} \quad 0 \quad \text{otherwise,} \end{cases} \tag{20}$$

where (c_{1i}, c_{2i}) are the right censoring times $(i = 1, 2, ..., k)$. As such, we have four possible situations, as described in the following

- c_1: Both X_i and Y_i are complete observations $(\gamma_{1i} = 1, \gamma_{2i} = 1)$;
- c_2: X_i is complete and Y_i is censored $(\gamma_{1i} = 1, \gamma_{2i} = 0)$;
- c_3: X_i is censored and Y_i is complete $(\gamma_{1i} = 0, \gamma_{2i} = 1)$;
- c_1: Both X_i and Y_i are censored observations $(\gamma_{1i} = 0, \gamma_{2i} = 0)$.

In every case, the observed data are defined as $t_{1i} = min(X_i, c_{1i})$ and $t_{2i} = min(Y_i, c_{2i})$. Therefore, the likelihood contribution from the i^{th} observation is expressed as:

$$L(\Delta) = \prod_{i \in c_1} f(t_{1i}, t_{2i}) \prod_{i \in c_2} \left(-\frac{\partial S(t_{1i}, t_{2i})}{\partial t_{1i}}\right) \prod_{i \in c_3} \left(-\frac{\partial S(t_{1i}, t_{2i})}{\partial t_{2i}}\right) \prod_{i \in c_4} S(t_{1i}, t_{2i}), \tag{21}$$

where $\partial S(t_{1i}, t_{2i})/\partial t_{1i}$ and $\partial S(t_{1i}, t_{2i})/\partial t_{2i}$ are the derivatives of the joint survival functions of T_1 and T_2 respectively.

3.1.3. Bayesian approach Suppose an independent Uniform or Gamma prior distributions for the model parameters and suppose that $\beta_j \sim \Gamma(a_j, b_j)(j = 1, 2, 3)$. Hence,

$$\pi_j(\beta_j) = \frac{b_j^{a_j}}{\Gamma(a_j)} \beta_j^{a_j - 1} e^{-b_j \beta_j}, \beta_j > 0, a_j > 0, b_j > 0.$$

As such , the joint posterior density of β can be expressed as

$$\pi_j(\beta | t) = \frac{L(\Delta) \prod_{j=1}^{3} \pi_j(\beta_j)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\Delta) \prod_{j=1}^{3} \pi_j(\beta_j) d\beta_j}. \tag{22}$$

Using equation (22), a Bayes estimator of any function of β, say $\rho(\beta)$, assuming the squared error loss function, is given by

$$\hat{\rho}(B) = \frac{\int_0^\infty \int_0^\infty \int_0^\infty \rho(\beta) L(\Delta) \prod_{j=1}^{3} \pi_j(\beta_j) d\beta_j}{\int_0^\infty \int_0^\infty \int_0^\infty L(\Delta) \prod_{j=1}^{3} \pi_j(\beta_j) d\beta_j}. \tag{23}$$

Since posterior equation (22) cannot be computed analytically, MCMC methods should be considered [32, 33] to obtain posterior summaries from equation (23).

4. Bivariate alpha log-power transformation exponential

The formulation of the bivariate alpha log-power transformed exponential (BVALPTE) is obtained by taking the exponential distribution with scale parameter "a" as a baseline model for Equation (3). Thus the BVALPTE has its cdf given by

$$F_{BVALPTE}(x,y) = \alpha_1^{-\log\left(1-e^{-ax}\right)}\alpha_2^{-\log\left(1-e^{-ay}\right)}\alpha_3^{-\log\left(1-e^{-az}\right)}, \tag{24}$$

where $z = min\{X, Y\}$. Figure 1 shows the cdf plots of the BVALPTE for different parameter values.

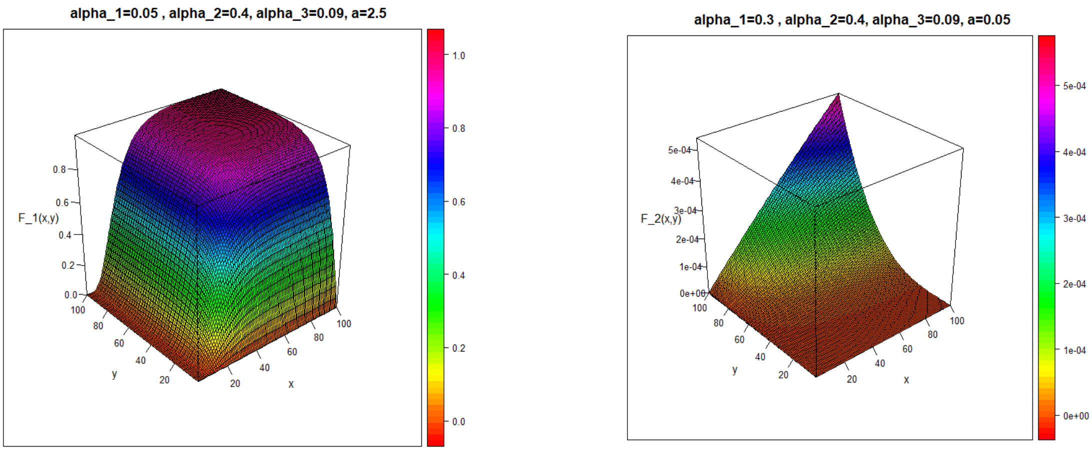

Figure 1. The cdf plots of the BVALPTE for different parameter values

4.1. *Joint probability distribution function*

The pdf of the BVALPTE is given by

$$f_{BVALPTE}(x,y) = \begin{cases} f_1(x,y) & x < y \\ f_2(x,y) & x > y \\ f_3(u) & x = y = u, \end{cases} \tag{25}$$

where

$$\begin{aligned} f_1(x,y) &= f_{ALPTE}(x;\alpha_1\alpha_3,a) \times f_{ALPTE}(y;\alpha_2,a) \\ &= \log(1/(\alpha_1\alpha_3))a^2e^{-ax}\left(1-e^{-ax}\right)^{-1}(\alpha_1\alpha_3)^{-\log\left(1-e^{-ax}\right)} \\ &\times \log(1/\alpha_2)e^{-ay}\left(1-e^{-ay}\right)^{-1}\alpha_2^{-\log\left(1-e^{-ay}\right)}, \end{aligned} \tag{26}$$

$$\begin{aligned} f_2(x,y) &= f_{ALPTE}(y;\alpha_2\alpha_3,a) \times f_{ALPTE}(x;\alpha_1,a) \\ &= \log(1/(\alpha_2\alpha_3))a^2e^{-ay}\left(1-e^{-ay}\right)^{-1}(\alpha_2\alpha_3)^{-\log\left(1-e^{-ay}\right)} \\ &\times \log(1/\alpha_1)e^{-ax}\left(1-e^{-ax}\right)^{-1}\alpha_1^{-\log\left(1-e^{-ax}\right)} \end{aligned} \tag{27}$$

and

$$\begin{aligned} f_3(u) &= \frac{\log(\alpha_3)}{\log(\alpha_1\alpha_2\alpha_3)} f_{ALPTE}(y;\alpha_1\alpha_2\alpha_3,a) \\ &= \log(1/\alpha_3)ae^{-au}\left(1-e^{-au}\right)^{-1}(\alpha_1\alpha_2\alpha_3)^{-\log\left(1-e^{-au}\right)}. \end{aligned} \tag{28}$$

For various parameter values, the pdf plots of the BVALPTE are given in Figure 2.

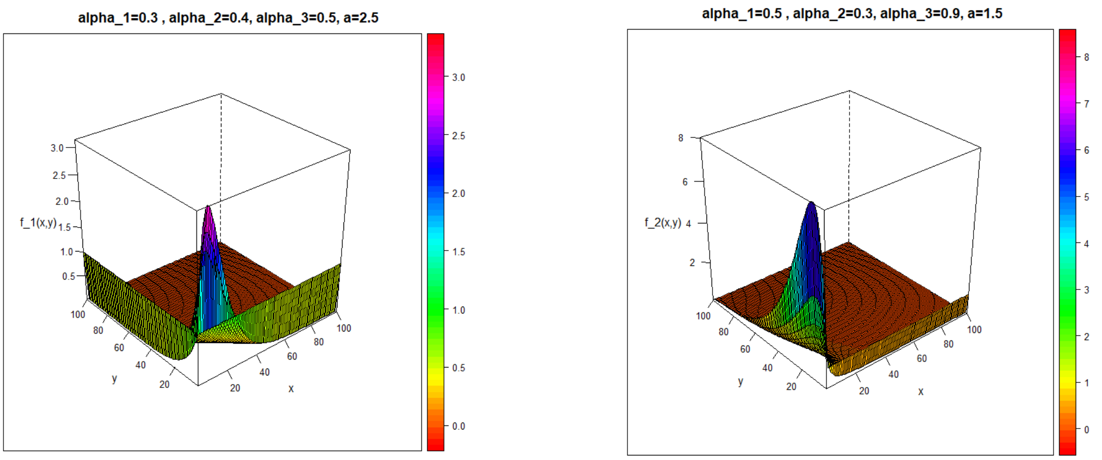

Figure 2. The pdf plots of the BVALPTE for different parameter values

4.2. Marginal distributions

The marginal cdfs of X and Y are respectively given by:

$$F_x(x) = F_{ALPTE}(x; \alpha_1\alpha_3, a) = (\alpha_1\alpha_3)^{-\log\left(1-e^{-ax}\right)} \tag{29}$$

and

$$F_y(y) = F_{ALPTE}(y; \alpha_2\alpha_3, a) = (\alpha_2\alpha_3)^{-\log\left(1-e^{-ay}\right)}. \tag{30}$$

Correspondingly, the marginal pdfs of X and Y are respectively expressed as:

$$f_x(x) = f_{ALPTE}(x; \alpha_1\alpha_3, a) = \log(1/(\alpha_1\alpha_3))ae^{-ax}\left(1-e^{-ax}\right)^{-1}(\alpha_1\alpha_3)^{-\log\left(1-e^{-ax}\right)}, \tag{31}$$

and

$$f_y(y) = f_{ALPTE}(y; \alpha_2\alpha_3, a) = \log(1/(\alpha_2\alpha_3))ae^{-ay}\left(1-e^{-ay}\right)^{-1}(\alpha_2\alpha_3)^{-\log\left(1-e^{-ay}\right)}. \tag{32}$$

4.3. Survival functions

The survival function of the BVALPE is expressed as:

$$S_{BVALPTE}(x, y) = \begin{cases} S_1(x, y) & x < y \\ S_2(x, y) & x > y \\ S_3(u) & x = y = u, \end{cases} \tag{33}$$

where

$$\begin{aligned} S_1(x, y) &= 1 - (\alpha_1\alpha_3)^{-\log\left(1-e^{-ax}\right)} - (\alpha_2\alpha_3)^{-\log\left(1-e^{-ay}\right)} \\ &+ (\alpha_1\alpha_3)^{-\log\left(1-e^{-ax}\right)}\alpha_2^{-\log\left(1-e^{-ay}\right)}, \end{aligned} \tag{34}$$

$$\begin{aligned} S_2(x, y) &= 1 - (\alpha_1\alpha_3)^{-\log\left(1-e^{-ax}\right)} - (\alpha_2\alpha_3)^{-\log\left(1-e^{-ay}\right)} \\ &+ (\alpha_1)^{-\log\left(1-e^{-ax}\right)}(\alpha_2\alpha_3)^{-\log\left(1-e^{-ay}\right)}, \end{aligned} \tag{35}$$

and

$$S_3(u) \;=\; 1-(\alpha_1\alpha_3)^{-\log\left(1-e^{-au}\right)}-(\alpha_2\alpha_3)^{-\log\left(1-e^{-au}\right)} + (\alpha_1\alpha_2\alpha_3)^{-\log\left(1-e^{-au}\right)}. \tag{36}$$

The plots the survival function of the BVALPTE for different parameter values are given in Figure 3.

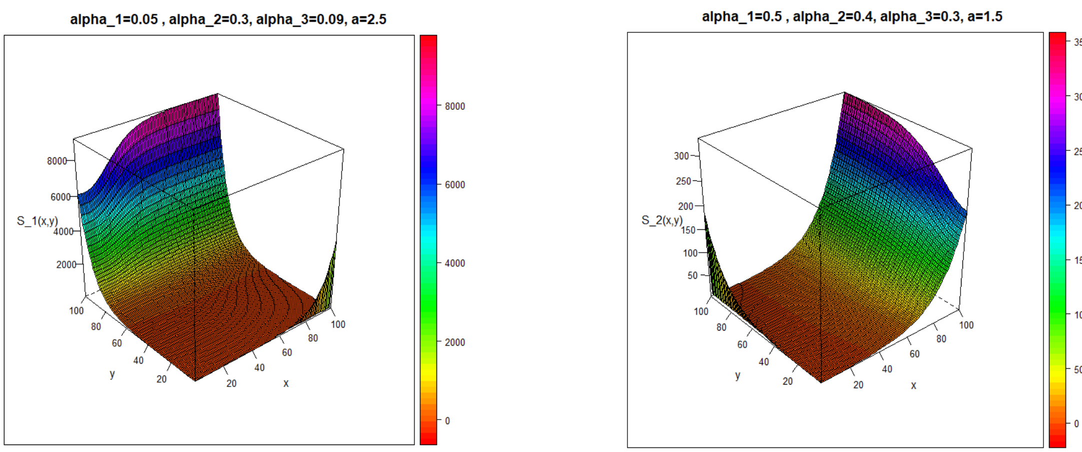

Figure 3. The plots of the survival function of the BVALPTE for different parameter values

4.4. Hazard rate functions

The BVALPTE has its hrf as

$$hrf_{BVALPTE}(x,y)=\begin{cases}\frac{f_1(x,y)}{S_1(x,y)} & x<y\\[2ex] \frac{f_2(x,y)}{S_2(x,y)} & x>y\\[2ex] \frac{f_3(u)}{S_3(u)} & x=y=u,\end{cases} \tag{37}$$

where, $f_1(x,y)$, $f_2(x,y)$, $f_3(u)$, $S_1(x,y)$, $S_2(x,y)$, and $S_3(u)$, are as given in Equations (26), (27), (28), (34), (35), and (36), respectively. The hrf plots of the BVALPTE for different parameter values are given in Figure 4.

4.5. Conditional probability density functions

The conditional probability of X, given Y is provided by

$$f(x|y)=\begin{cases}\frac{f_1(x,y)}{f_{ALPTE}(y;\alpha_2\alpha_3,a)} & x<y\\[2ex] \frac{f_2(x,y)}{f_{ALPTE}(y;\alpha_2\alpha_3,a)} & x>y\\[2ex] \frac{f_3(u)}{f_{ALPTE}(y;\alpha_2\alpha_3,a)} & x=y=u,\end{cases} \tag{38}$$

where, $f_1(x,y)$, $f_2(x,y)$, $f_3(u)$, and $f_{ALPTE}(y;\alpha_2\alpha_3,a)$ are as given in Equations (26), (27), (28), and (32), respectively. In a similar manner, the conditional probability of Y, given X, can be easily derived.

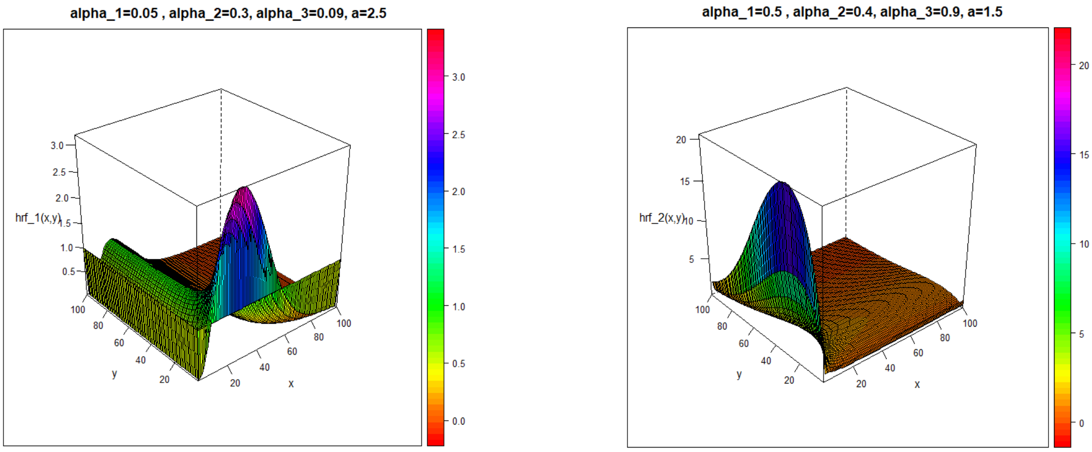

Figure 4. The hrf plots of the BVALPTE for different parameter values

5. Simulation Study

A BVALPTE Monte Carlo simulation is carried out in this section using **R** software. Generating random numbers from a joint distribution was discussed by [22]. We can generate a bivariate sample by using the conditional approach using the following steps [34];

- U and V are generated independently from uniform $(0, 1)$ distribution.
- Set $x = Q_E(u) = -a^{-1} \log(1 - U)$.
- Set $F_{(y|x)} = V$, to obtain y by numerical analysis.
- Repeat above steps (n) items to get (x_i, y_i), $i = 1, ..., n$.

A simulation experiment was carried out based on the data generated from the BVALPTE distribution for different sample size $(n = 25, 100, 200, 400, 500, 800, 1000)$ with 2000 replications. The simulation methods are evaluated based on their effectiveness in parameter estimation. This comparison is carried out by computing and recording the estimate, the root mean squared error (RMSE) and the bias for all sample sizes. Tables 1 and 2 shows that as the sample size increases, the estimates approaches the initial parameter values and both the RMSE and Bias decreases.

6. Application

In this section, we analyse the computer systems data [35] and diabetic nephropathy [36] to show the practicality of the BVALPTE model.

6.1. Computer System Data

The data was extracted from [35]. The dataset consists of a simulated rudimentary computer system comprising a processor and memory across $n = 50$ units. The system functions only if both components operate properly. Over time, the system undergoes a latent deterioration process, with degradation accelerating rapidly within a short period (measured in hours). As the system weakens, it becomes increasingly susceptible to shocks, any of which can randomly cause the failure of one or both components. A critical aspect of this process is that a catastrophic shock can simultaneously disable both components, challenging the assumption of component independence. To address this, we applied the BVALPTE extreme shock model to analyze system failure dynamics. The data set is given as follows:

Table 1. **BVALPTE Model's Simulations Results 1**

		$(0.5, 0.5, 0.5, 1.0)$			$(0.2, 0.5, 0.5, 1.0)$		
Parameter	Sample Size	MLE	RMSE	Bias	MLE	RMSE	Bias
α_1	75	0.9690	562.9358	48.469	0.8027	769.8674	55.6027
	100	0.7603	534.7081	27.2603	0.8030	112.7451	11.8304
	200	0.6098	23.0799	1.5980	0.9511	4.9372	0.7511
	400	0.5954	1.7639	0.0954	0.2777	0.9097	0.0777
	500	0.5037	0.4574	0.0037	0.2146	0.1772	0.0146
	800	0.4166	0.1954	-0.0834	0.1823	0.0853	-0.0177
	1000	0.4063	0.2099	-0.0937	0.1706	0.0941	-0.0294
α_2	75	0.6626	32.4791	5.1626	0.8238	39.4054	5.7384
	100	0.7353	16.8174	2.2353	0.8749	14.6501	2.2491
	200	0.8334	2.7695	0.8336	0.8206	2.1583	0.7057
	400	0.8035	0.6371	0.3035	0.7881	0.5712	0.2881
	500	0.7645	0.5510	0.2645	0.7609	0.4448	0.2609
	800	0.7086	0.3562	0.2086	0.7068	0.3011	0.2068
	1000	0.6616	0.2633	0.1616	0.6585	0.2343	0.1585
α_3	75	0.7059	12.7124	4.2059	0.9053	13.1934	4.5533
	100	0.9155	9.0755	2.4155	0.9016	8.0206	2.4016
	200	0.8292	2.9656	0.7916	0.8215	2.3587	0.7150
	400	0.7822	1.0041	0.2822	0.7571	0.8135	0.2571
	500	0.7153	0.5548	0.2153	0.7054	0.4103	0.2054
	800	0.6554	0.2788	0.1554	0.6497	0.2402	0.1497
	1000	0.6140	0.2142	0.1140	0.6088	0.1929	0.1088
a	75	0.8715	0.6329	-0.1285	0.8355	0.6286	-0.1645
	100	0.8837	0.5647	-0.1163	0.8576	0.5702	-0.1424
	200	0.8597	0.4318	-0.1403	0.8490	0.4249	-0.1510
	400	0.8602	0.3068	-0.1398	0.8535	0.2881	-0.1465
	500	0.8688	0.2876	-0.1312	0.8494	0.2753	-0.1506
	800	0.8702	0.2363	-0.1298	0.8571	0.2325	-0.1429
	1000	0.8940	0.2010	-0.1060	0.8885	0.1875	-0.1115

Processor lifetime (X): 1.9292 3.6621 3.6621 3.6621 1.0833 1.0833 0.3309 0.3309 0.5784 0.5520 1.9386 2.1000 0.9867 0.9867 1.3989 2.3757 3.5202 2.3364 0.8584 4.3435 1.1739 1.3482 3.0935 2.1396 1.3288 0.1115 0.8503 0.1955 0.4614 3.3887 0.1181 5.0533 1.6465 0.9096 1.7494 0.1058 0.1058 0.9938 5.7561 5.7561 0.6270 0.7947 0.5079 2.5913 2.5372 1.1917 1.5254 1.0986 1.0051 1.3640.
Memory lifetime (Y): 3.9291 0.0026 0.0026 0.0026 3.3059 3.3059 0.3309 0.3309 1.8795 0.5520 4.0043 2.0513 0.9867 0.9867 4.1268 2.7953 1.4095 0.1624 1.9556 1.0001 3.3857 1.9705 3.0935 2.1548 0.9689 0.1115 2.8578 0.1955 0.8584 1.9796 0.0884 2.3238 2.0197 0.6214 2.3643 0.1058 0.1058 1.7689 0.3212 0.3212 1.7289 0.7947 5.3535 2.5913 2.4923 0.0801 4.4088 1.0986 1.0051 1.3640.

Before analysing the data using the bivariate alpha log power transformed exponential model, we first fit the marginals X, Y and $min(X, Y)$ separately on the same data. Figures 5 to 9 show the fitted probability density

Table 2. **BVALPTE Model's Simulations Results 2**

		$(0.2, 0.5, 0.2, 1.0)$			$(0.5, 0.5, 0.3, 1.0)$		
Parameter	Sample Size	MLE	RMSE	Bias	MLE	RMSE	Bias
α_1	75	0.7274	4.7582	1.0742	0.7335	50.8870	4.8352
	100	0.7620	3.3107	0.5620	0.7057	10.2519	1.5565
	200	0.2414	0.3949	0.0414	0.7145	1.3175	0.2145
	400	0.1594	0.1791	-0.0406	0.4584	0.4405	-0.0416
	500	0.1385	0.1504	-0.0615	0.4383	0.3716	-0.0617
	800	0.1380	0.1422	-0.0620	0.3856	0.2816	-0.1144
	1000	0.1132	0.1261	-0.0868	0.3747	0.2428	-0.1253
α_2	75	0.9369	48.4408	9.4369	0.9098	36.1855	8.5980
	100	0.8485	27.9628	4.9854	0.7503	21.3124	4.5325
	200	0.8450	8.1284	0.9496	0.7508	7.5746	1.0081
	400	0.8029	0.6859	0.3029	0.7584	0.7801	0.2584
	500	0.8045	0.4781	0.3045	0.7241	0.7318	0.2241
	800	0.7308	0.3554	0.2308	0.6706	0.3372	0.1706
	1000	0.8453	0.4711	0.3453	0.6581	0.2770	0.1581
α_3	75	0.7039	3.5245	0.8395	0.9577	5.4632	1.6577
	100	0.5930	1.5780	0.3930	0.6134	2.7890	0.8337
	200	0.3341	0.2922	0.1341	0.5295	0.7476	0.2295
	400	0.2888	0.1423	0.0888	0.4053	0.2394	0.1053
	500	0.2931	0.1353	0.0931	0.3893	0.1953	0.0893
	800	0.2742	0.1165	0.0742	0.3739	0.1442	0.0739
	1000	0.3050	0.1354	0.1050	0.3696	0.1261	0.0696
a	75	0.8104	0.4665	-0.1896	0.8413	0.5303	-0.1587
	100	0.8006	0.4227	-0.1994	0.8457	0.4775	-0.1543
	200	0.8192	0.3384	-0.1808	0.8631	0.3651	-0.1369
	400	0.8290	0.2765	-0.1710	0.8866	0.2750	-0.1134
	500	0.8065	0.2773	-0.1935	0.8900	0.2498	-0.1100
	800	0.8379	0.2461	-0.1621	0.8926	0.2122	-0.1074
	1000	0.7673	0.2978	-0.2327	0.8924	0.1999	-0.1076

function (pdf) plots, fitted emperical cumulative distribution function (ECDF), Kaplain Meirer (KM) and the probability plots (ppp). We can observe that the marginals fit the dataset well, therefore the bivariate alpha log-power transformed exponential model can be used to analyse this dataset.

Table 3 discuss MLE estimator of marginal parameters with standard errors in parenthesis, also different measures of goodness of fit (GoF) statistics such as the -2 log-likelihood (-2 log L) Akaike information criterion (AIC), the corrected AIC (CAIC) and the Bayesian information criterion (BIC) for models fitted the processor and memory lifetime data. Models fitted are the novel bivariate alpha log power transformation exponential (BVALPTE) model, bivariate exponential modified Weibull (BVEMW) model [37] and the bivariate exponentiated modified Weibull extension (BVEMWE) model [29]. The novel BVALPTE outperforms other comparative bivariate models discussed in this paper as it has the lowest G0F statistics values.

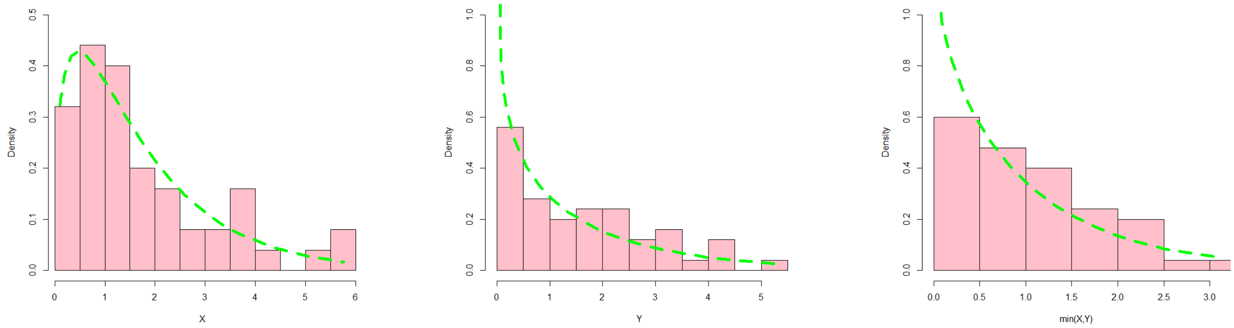

Figure 5. Fitted pdf plots for X, Y and $min(X, Y)$

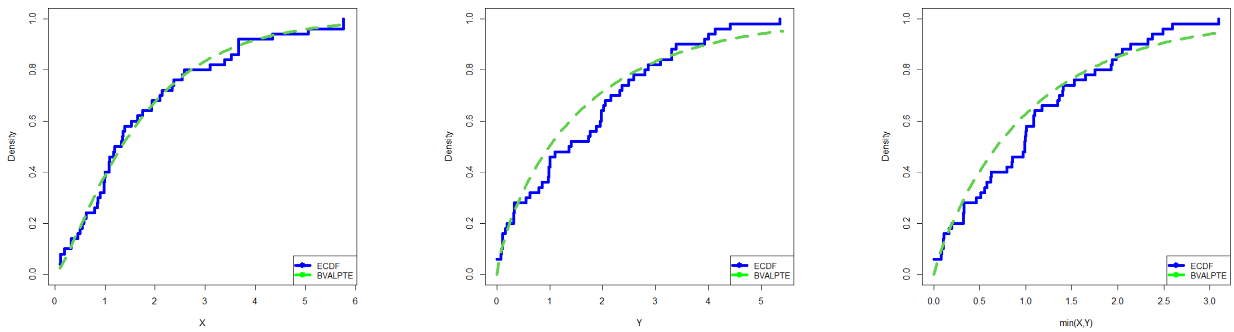

Figure 6. Fitted ECDF plots for X, Y and $min(X, Y)$

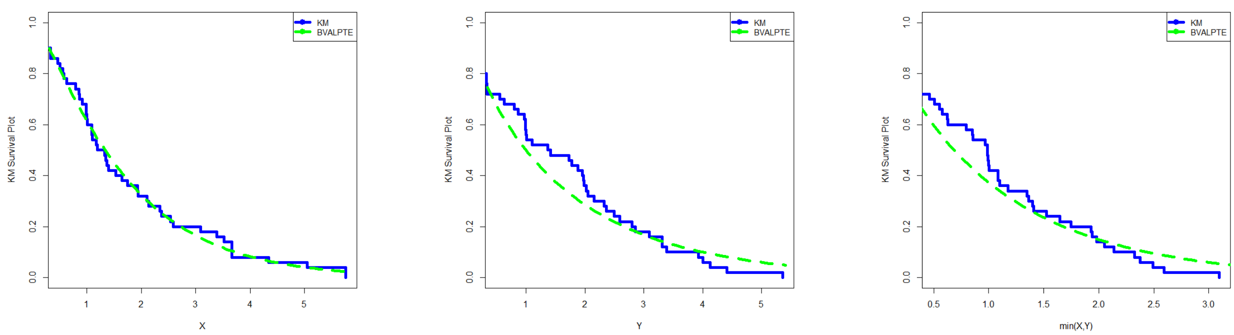

Figure 7. Fitted KM plots for X, Y and $min(X, Y)$

6.2. Diabetic Nephropathy

In this subsection, we consider both serum creatinine $(SrCr)$ levels and the duration of diabetes. Since all patients were already diagnosed with diabetes, our focus is on assessing potential complications arising from it. Based on $SrCr$ levels, patients were categorized into two groups: those with diabetic nephropathy (DN), defined as $SrCr \geq 1.4mg/dl$, and those without DN, with $SrCr < 1.4mg/dl$. $SrCr$ data were available for 200 patients, with reports collected a pathology lab between January 2012 and August 2013. The dataset includes the

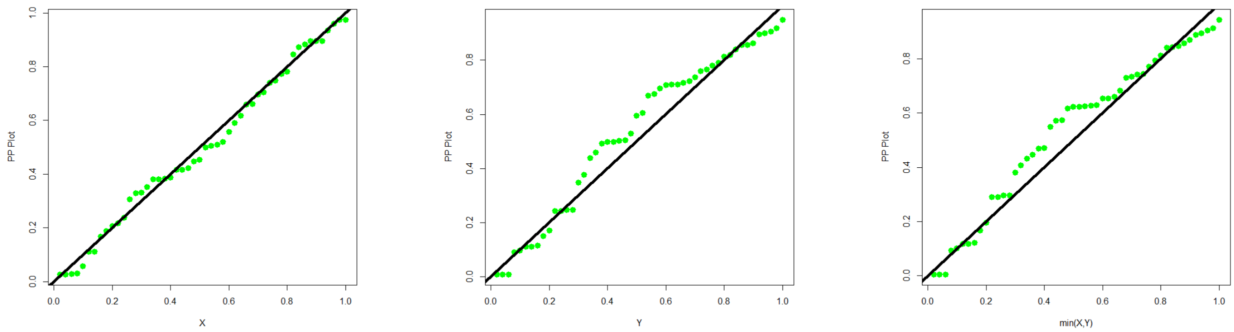

Figure 8. Fitted PPP plots for X, Y and $min(X, Y)$

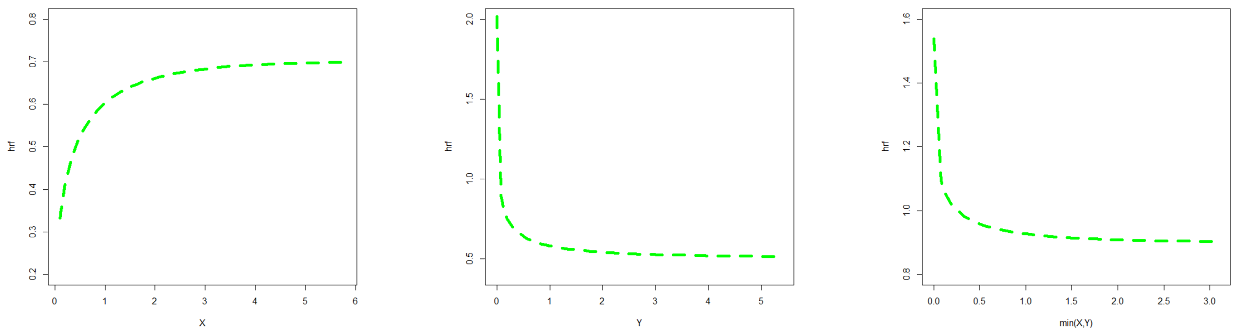

Figure 9. Fitted hrf plots for X, Y and $min(X, Y)$

Table 3. **Estimates, Standard errors and GoF statistics values**

Model	Parameter Estimates						$-2\ \log\ L$	AIC	CAIC	BIC
	α_1	α_2	α_3	a	-	-				
BVALPTE	0.6127	0.6391	0.4635	0.5900	-	-	14376.8300	14384.8300	14385.3633	14383.6258
	(0.0106)	(0.0094)	(0.0108)	(0.0116)	-	-				
	α_1	α_2	α_3	θ	β	γ				
BVEMW	2.4584	2.0544	3.3373	0.6714	0.1706	1.0373	14890.6500	14896.6500	14898.6000	14900.8438
	(0.5379)	(0.4526)	(0.7488)	(0.0173)	(0.0219)	(0.1922)				
	γ_1	γ_2	γ_3	α	β	λ				
BVEMWE	0.3089	0.2497	0.4017	0.1372	1.4777	0.5384	14787.4100	14793.4100	14795.3634	14797.6038
	(0.0546)	(0.0432)	(0.0699)	(0.0324)	(0.2336)	(0.1508)				

average duration of diabetes among 132 individuals diagnosed with type 2 diabetic nephropathy over varying time intervals, as reported [36, 38]. The data is as follows:

Duration of diabetes (X): 7.4, 9, 10, 11, 12, 13, 13.75, 14.92, 15.8286, 16.9333, 18, 19, 20, 21, 22, 23, 24, 26, 26.6.
Serum Creatinine (Y): 1.925, 1.5, 2, 1.6, 1.7, 1.7533, 1.54, 1.694, 1.8843, 1.8433, 1.832, 1.59, 1.7833, 1.2, 1.792, 1.5, 1.5033, 2, 2.14.

Before applying the bivariate alpha log power transformed exponential model to the diabetic nephropathy data, we begin by fitting the marginal distributions X, Y and $\min(X, Y)$ separately to the same dataset. Likewise, Figures 10 to 14 display the fitted pdfs, ECDFs, KM curves, and PPP plots. The results indicate that the marginal distributions provide a good fit to the data, supporting the use of the bivariate alpha log-power transformed exponential model for further analysis.

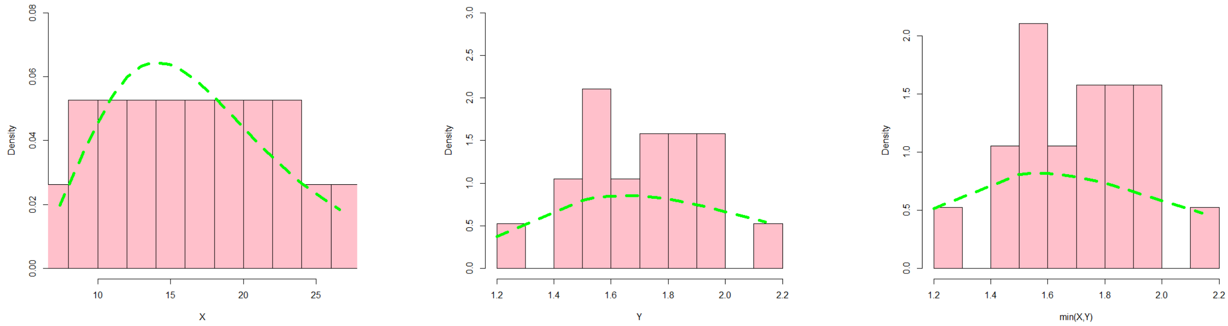

Figure 10. Fitted pdf plots for X, Y and $min(X, Y)$

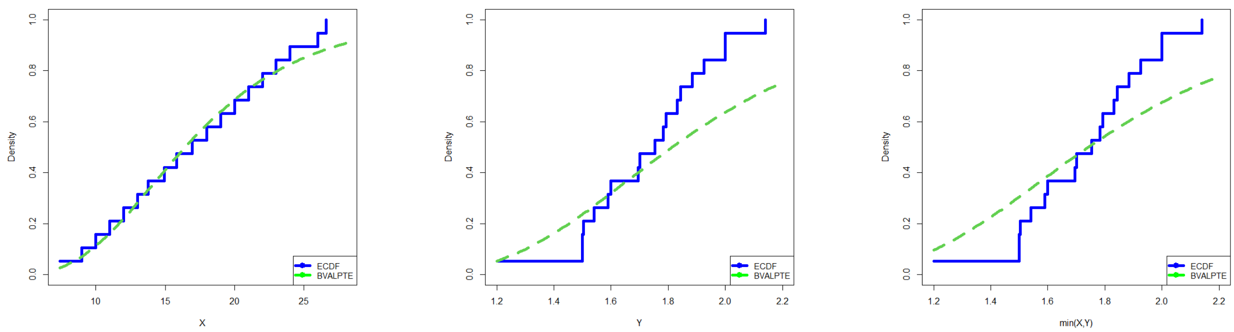

Figure 11. Fitted ECDF plots for X, Y and $min(X, Y)$

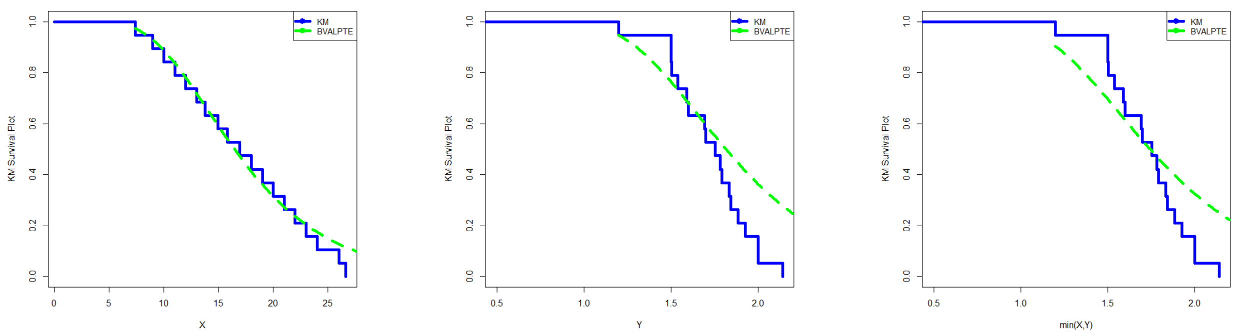

Figure 12. Fitted KM plots for X, Y and $min(X, Y)$

Table 4 presents the MLE estimates of the marginal parameters, with standard errors shown in parentheses, along with various GoF statistics for models applied to the processor and memory lifetime data. The BVALPTE model

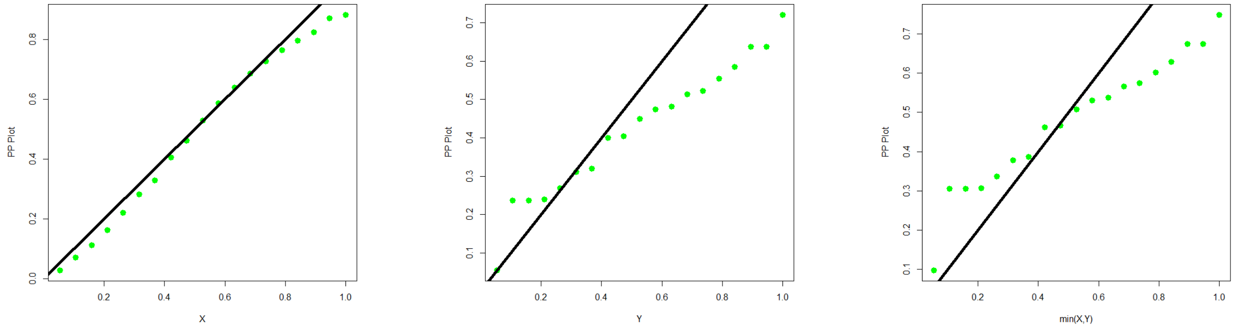

Figure 13. Fitted PPP plots for X, Y and $min(X, Y)$

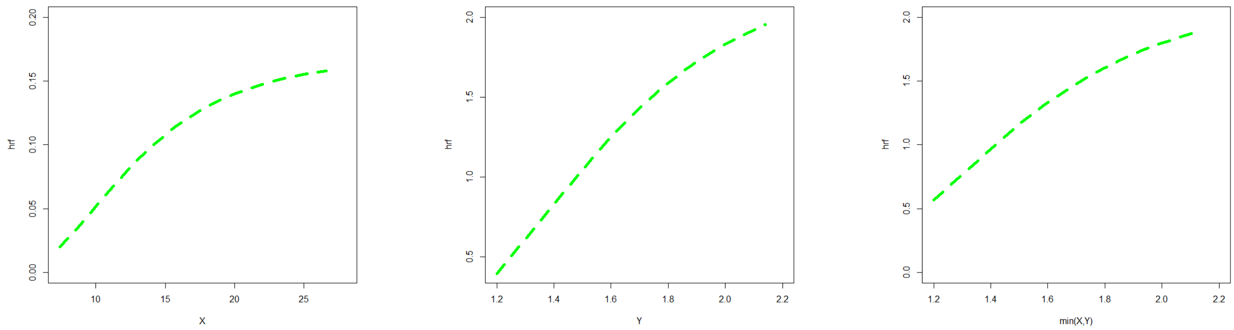

Figure 14. Fitted hrf plots for X, Y and $min(X, Y)$

demonstrates superior performance, as indicated by its consistently lower GoF statistic values compared to the other models.

Table 4. **Estimates, Standard errors and GoF statistics values**

Model	Parameter Estimates						$-2\ \log\ L$	AIC	CAIC	BIC
	α_1	α_2	α_3	a	-	-				
BVALPTE	8.8624×10^{-5}	0.5515	0.3836	0.1657	-	-	3774.8500	3782.8500	3785.7071	3779.9650
	(9.2243×10^{-5})	(0.0197)	(4.0416×10^{-5})	(0.0067)	-	-				
	α_1	α_2	α_3	θ	β	γ				
BVEMW	2.1372	0.0120	1.0120	0.0878	0.5446	0.1013	4341.5620	4353.5620	4359.5620	4349.2345
	(0.1318)	(0.0460)	(0.0460)	(0.0030)	(0.0643)	(0.0130)				
	γ_1	γ_2	γ_3	α	β	λ				
BVEMWE	1.6377	1.0470	2.0470	0.0365	0.1044	3.0079	7635.7930	7647.7930	7653.7930	7643.4655
	(0.0863)	(0.0877)	(0.0877)	(0.0045)	(0.0090)	(0.1675)				

7. Conclusions

In this research, we introduced the BVALPT distribution, developed by incorporating the ALPT into the Marshall–Olkin extreme shock model. This new family offers enhanced flexibility in modeling dependent lifetime data, especially for systems that are susceptible to several shock sources.

For the survival functions, hazard rate functions, conditional densities, and joint and marginal distributions, we provided closed-form expressions. The BVALPTE, a special case of this model was explored in detail. We produced parameter estimates using maximum likelihood estimation and showed that the BVALPTE is applicable to real-world data.

A thorough goodness-of-fit analysis was performed to evaluate the model's performance in comparison to other well-known bivariate models. The findings demonstrate that the BVALPTE model fits the data better and more accurately depicts the underlying dependency structure.

For practitioners in risk assessment, reliability engineering, and survival analysis, the suggested BVALPT family may be a useful substitute. Future research could look into using the model in larger empirical contexts with censored or truncated data, investigating Bayesian estimation techniques, or expanding the model to higher dimensions.

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