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Bingham Type Fluids with Tresca Law in 3D: Existence, Asymptotic Analysis, Reynolds Equation

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Abstract In this work, we study a model for incompressible Bingham fluids in a confined three-dimensional domain, Ω^ε, where Tresca boundary conditions are applied on part of the boundary and Dirichlet conditions on another. The domain is perturbed by a small parameter $\varepsilon > 0$. We prove the unique solvability of the problem and carry out an asymptotic analysis as one dimension of the fluid domain diminishes to zero. This approach enables the strong convergence of the velocity field, the derivation of a Reynolds-type limit equation, and the analysis of the asymptotic behavior of the Tresca boundary conditions, while rigorously establishing the uniqueness of the limiting velocity and pressure fields.

Keywords 3D-asymptotic analysis, Variational inequalities, Bingham type fluid, Tresca law, Reynolds equation.

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1. Introduction and motivation

We conduct here a detailed analysis of the Bingham-type non-Newtonian fluid model, described as follows:

$$-\operatorname{Div}\mathbb{S}^\varepsilon + \nabla p^\varepsilon = \boldsymbol{f}^\varepsilon \quad \text{in} \quad \Omega^\varepsilon, \tag{1}$$

$$\begin{cases} \mathbb{S}^\varepsilon = \mu(\|\mathbb{D}\boldsymbol{u}^\varepsilon\|)\mathbb{D}\boldsymbol{u}^\varepsilon + g^\varepsilon \dfrac{\mathbb{D}\boldsymbol{u}^\varepsilon}{\|\mathbb{D}\boldsymbol{u}^\varepsilon\|} & \text{if} \;\; \mathbb{D}\boldsymbol{u}^\varepsilon \neq \mathbf{0}, \\ \|\mathbb{S}^\varepsilon\| \leq g^\varepsilon & \text{if} \;\; \mathbb{D}\boldsymbol{u}^\varepsilon = \mathbf{0}, \end{cases} \tag{2}$$

where f^ε denotes the volume density of applied forces, $u^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^3$ represents the flow velocity, $p^\varepsilon : \Omega^\varepsilon \to \mathbb{R}$ is the pressure, and $\mathbb{S}^\varepsilon : \mathbb{M}^3 \to \mathbb{M}^3$ signifies the extra stress tensor. The plasticity threshold (yield stress) is defined as $g^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^+$. The physical interpretation of these constitutive laws is elaborated in [22, 23, 26].

Recent years have witnessed a growing interest in the mathematical frameworks that elucidate the steady flow of incompressible non-Newtonian fluids of Bingham type within confined domains characterized by complex boundary conditions. This line of inquiry holds considerable practical significance across various technological and industrial sectors, thus attracting notable attention from the scientific community. The mathematical models associated with incompressible Bingham fluids are relatively recent developments and have been the subject of various investigations, as noted in [1, 6, 4].

Asymptotic analysis of these mathematical models is essential for understanding the dynamics of fluids and structures in complex domains. A number of studies have focused on transforming three-dimensional thin domains Ω^ε into two-dimensional representations Ω, independent of the perturbative ε. For example, recent investigations

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have examined the asymptotic properties of Bingham fluids in bounded three-dimensional domains subject to Tresca and Fourier boundary conditions [14], in [14] the Tresca's condition is characterized by zero lower surface velocity. While this paper will analyze a new Bingham model with a Tresca condition, which is characterized by a nonzero lower surface velocity. Additional research has addressed mechanical contact issues, transitioning from three-dimensional configurations to thin domain models in two dimensions [21]. Furthermore, the asymptotic analysis of unilateral contact problems involving Coulomb friction between elastic bodies and thin elastic layers has emerged as a significant area of study [10]. Collectively, these contributions, along with numerous other works in the field [5, 14, 12, 25, 4, 24], underscore the diversity of methodologies that enrich our understanding of complex phenomena related to non-Newtonian fluids and mechanical interactions.

Before detailing the core contributions of this work, we consider the following assumptions:

$(\mathcal{C}_1)$ For any matrices $\mathbf{K}, \mathbf{L} \in \mathbb{M}^{3\times 3}_{\text{sym}}$, we have

$$(\mu(|\mathbf{K}|)\mathbf{K} - \mu(|\mathbf{L}|)\mathbf{L}) : (\mathbf{K} - \mathbf{L}) \geq 0;$$

$(\mathcal{C}_2)$ The function μ is continuous such that

$$0 < \mu_0 < \mu(r) < \mu_1, \quad \forall r \in \mathbb{R}_+;$$

$(\mathcal{C}_3)$ The conditions $g^\varepsilon \in L^2_+(\Omega^\varepsilon)$ and $k^\varepsilon \in L^\infty(\omega)$ hold.

The hypothesis $(\mathcal{C}_1)$-$(\mathcal{C}_2)$ is applicable to conventional models, such as the Carreau-type and power-law models, as evidenced in reference [18]. For instance, the Carreau law is described by

$$\mu(r) = (\mu_0 - \mu_\infty)\left(1 + \alpha r^2\right)^{\frac{t-2}{2}} + \mu_\infty \text{ for all } r \in [0, +\infty)$$

with $\alpha > 0, 1 < t \leq 2$ and $0 < \mu_\infty < \mu_0$. This function satisfies $\mu \in C^1([0, +\infty))$ and

$$\mu_\infty(r - s) \leq \mu(r)r - \mu(s)s \leq \mu_0(r - s) \text{ for all } r \geq s \geq 0. \tag{3}$$

It has been established that if the viscosity μ satisfies condition (3), then the inequalities $(\mathcal{C}_1)$-$(\mathcal{C}_2)$ are valid, with suitable constants $\mu_0, \mu_1 > 0$, as demonstrated in references [7, condition (2.3)] and [8, Lemma 2.1]. Evidently, when the condition $t = 2$ is met, the relationship between $\mu(r)$ and μ_0 is equivalent to the linear Newtonian constitutive relation, as indicated by $\mu(r) = \mu_0$. Moreover, hypothesis $(\mathcal{C}_2)$ is satisfied when μ is a nondecreasing function, for example, $\mu(r) = \sqrt{r} + 1/2$ for $r \in [0, 4]$, and $5/2$ for $r > 4$, or $\mu(r) = (\arctan r)^{1/2} + \mu_0$ for $r \geq 0$, see [9, Remark 3].

The aforementioned problem belongs to a family of problems that have previously been examined in various contexts, particularly in the context of shear flows in narrow films and the theory of lubrication (see [13]). This family of problems includes the Navier-Stokes system, for which $g = 0$.

Continuous experimental studies are underway; however, these studies remain challenging due to the thickness of the gap between the solid surfaces, which can measure as small as 50 nanometers. In such operating conditions, for example a no-slip condition is induced by chemical bonds between the lubricant and the surrounding surfaces. Conversely, tangential stresses are so high that they tend to destroy chemical bonds and induce a slip phenomenon. This phenomenon can be likened to the Tresca free boundary friction model in solid mechanics [15].

Our objective is to examine incompressible Bingham-type models in confined three-dimensional domains, focusing on their reduction to two-dimensional configurations for enhanced understanding and analysis of the underlying physical phenomena. By implementing a small variable transformation, $y = \frac{x_3}{\varepsilon}$, we reformulate the starting problem in the three-dimensional domain Ω^ε into an equivalent problem in a fixed domain Ω, which remains unaffected by the parameter ε. This approach will enable us to establish significant results concerning the strong convergence of velocity, derive a limiting Reynolds-type equation, and characterize the limit of the Tresca free boundary conditions.

The paper is outlined as below: Section 2 introduces the model for an incompressible Bingham-type fluid governed by Tresca's law, deriving its variational formulation and proving its unique solvability. Section 3 provide estimates for the velocity and pressure that are independent of the parameter ε, along with several convergence results. Finally, Section 4 addresses the limit problem, showcasing the uniqueness of the limiting values for both velocity and pressure.

2. Variational Formulation and Unique Solvability

We provide here the fundamental equations of the flow model for a Bingham fluid. Let $\Omega^\varepsilon \subset \mathbb{R}^3$ be a domain characterized by a Lipschitz boundary Γ^ε. We suppose that Γ^ε is partitioned into three distinct parts ω, Γ_1^ε and Γ_L^ε such that $\Gamma^\varepsilon = \overline{\omega} \cup \overline{\Gamma_1^\varepsilon} \cup \overline{\Gamma_L^\varepsilon}$. The area ω signifies a fixed bounded region in the plane, represented by $x = (x_1, x_2) \in \mathbb{R}^2$, serving as the base of the fluid domain. We assume that ω possesses a Lipschitz continuous boundary. Introducing a parameter ε close to zero, we define a positive, smooth, and bounded function $h : \omega \to \mathbb{R}$ that satisfies

$$0 < h_m \leq h(x) \leq h_M, \quad \forall x \in \omega.$$

The upper surface Γ_1^ε is given by the equation $x_3 = \varepsilon h(x)$. The domain Ω^ε can thus be expressed as

$$\Omega^\varepsilon = \left\{(x, x_3) \in \mathbb{R}^3 \,:\, (x, 0) \in \omega,\ 0 < x_3 < \varepsilon h(x)\right\},$$

with its boundary comprising the fixed region ω and the lateral boundary Γ_L^ε. The set Ω^ε is occupied by the incompressible Bingham fluid.

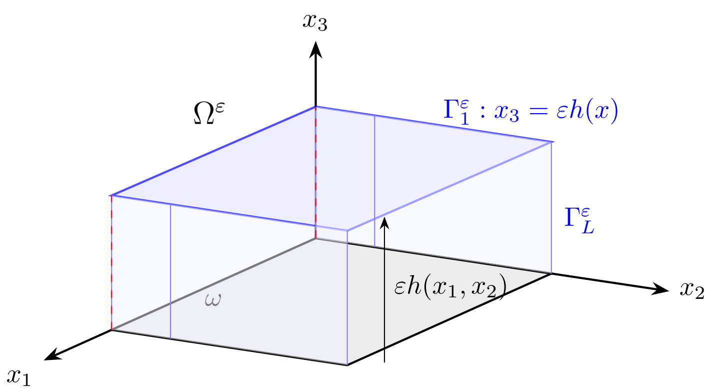

Figure 1.

The Stokes equation embodies the conservation law governing the flow:

$$-\operatorname{Div} \mathbb{S}^\varepsilon + \nabla p^\varepsilon = \boldsymbol{f}^\varepsilon \quad \text{in} \quad \Omega^\varepsilon, \tag{4}$$

where $\mathbb{S}^\varepsilon : \mathbb{M}^3 \to \mathbb{M}^3$ is the extra stress tensor in Ω^ε, defined according to the Bingham constitutive law by

$$\begin{cases} \mathbb{S}^\varepsilon = \mu(\|\mathbb{D}\boldsymbol{u}^\varepsilon\|)\mathbb{D}\boldsymbol{u}^\varepsilon + g^\varepsilon \frac{\mathbb{D}\boldsymbol{u}^\varepsilon}{\|\mathbb{D}\boldsymbol{u}^\varepsilon\|} & \text{if} \quad \mathbb{D}\boldsymbol{u}^\varepsilon \neq \mathbf{0}, \\ \|\mathbb{S}^\varepsilon\| \leq g^\varepsilon & \text{if} \quad \mathbb{D}\boldsymbol{u}^\varepsilon = \mathbf{0}. \end{cases} \tag{5}$$

Equation (5) describes the relationship relating the extra stress tensor $\mathbb{S}^\varepsilon$ to the strain rate tensor $\mathbb{D}\boldsymbol{u}^\varepsilon$, with components defined for $\boldsymbol{u}^\varepsilon = (u_1^\varepsilon, \ldots, u_d^\varepsilon)$, as follows:

$$\mathbb{D}_{ij}(\boldsymbol{u}^\varepsilon) = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right).$$

In this context, μ denotes the viscosity coefficient, and g^ε represents the yield stress of the fluid. The additional stress is limited by a maximum value, denoted as g^ε, known as the yield limit. When the stress is below this threshold, the fluid behaves like a rigid body with no deformations. Conversely, once the stress reaches this limit, the material starts to behave as a fluid. In the case where $g^\varepsilon = 0$ and the viscosity is constant at $\mu(\lambda) = \mu_0$, the constitutive law simplifies to that of a Newtonian fluid within the framework of the Navier–Stokes equations.

The incompressibility of the fluid is conveyed by the solenoidal condition:

$$\operatorname{div} \boldsymbol{u}^\varepsilon = 0 \ \text{ in } \ \Omega^\varepsilon. \tag{6}$$

The homogeneous Dirichlet boundary condition implies that the fluid is in contact with the wall

$$\boldsymbol{u}^\varepsilon = 0 \ \text{ in } \ \Gamma_1^\varepsilon. \tag{7}$$

The velocity on $\Gamma_{\mathrm{L}}^\varepsilon$ is oriented parallel to the ω-plane, indicating that $u^\varepsilon = 0$ on $\Gamma_{\mathrm{L}}^\varepsilon$. On the region ω, there is a no-flux condition, such that

$$u_n^\varepsilon = 0. \tag{8}$$

On the region ω, the tangential velocity adheres to Tresca friction law, where k^ε represents the upper limit for the stress. The law can be expressed as follows:

$$\left.\begin{array}{l} |\sigma_T^\varepsilon| < k^\varepsilon \implies u_T^\varepsilon = s, \\ |\sigma_T^\varepsilon| = k^\varepsilon \implies \exists \lambda \geq s \text{ such that } u_T^\varepsilon = s - \lambda \sigma_T^\varepsilon \end{array}\right\} \quad \text{on} \quad \omega, \tag{9}$$

where $|\cdot|$ represents the Euclidean norm in $\mathbb{R}^2$. Let $n = (n_1, n_2, n_3)$ represent the unit outward normal to Γ^ε. By employing Einstein summation conventions, we obtain:

$$u_n^\varepsilon = u^\varepsilon \cdot n = u_i^\varepsilon n_i, \quad u_{T_i}^\varepsilon = u_i^\varepsilon - u_n^\varepsilon n_i,$$

$$\sigma_n^\varepsilon = (\sigma^\varepsilon \cdot n) \cdot n = \sigma_{ij}^\varepsilon n_i n_j, \quad \sigma_{T_i}^\varepsilon = \sigma_{ij}^\varepsilon n_j - \sigma_n^\varepsilon n_i,$$

where u_n^ε and $u_{T_i}^\varepsilon$ denote the normal and tangential velocities on ω, respectively, while σ_n^ε and $\sigma_{T_i}^\varepsilon$ represent the components of the normal and tangential stress tensors on ω.

In order to obtain the weak formulation of Problem (4)–(9), we introduce some function spaces:

$$K^\varepsilon = \left\{\varphi \in \left(H^1\left(\Omega^\varepsilon\right)\right)^3 : \varphi = 0 \ \text{ on } \ \Gamma_1^\varepsilon \cup \Gamma_{\mathrm{L}}^\varepsilon \ \text{ and } \ \varphi \cdot n = 0 \ \text{ on } \ \omega\right\},$$

$$K_{\mathrm{d}}^\varepsilon = \left\{v \in K^\varepsilon : \operatorname{div}(v) = 0 \ \text{ in } \ \Omega^\varepsilon\right\},$$

and

$$L_0^2(\Omega^\varepsilon) = \left\{q \in L^2\left(\Omega^\varepsilon\right) : \int_{\Omega^\varepsilon} q \, \mathrm{d}x \, \mathrm{d}x_3 = 0 \ \text{ where } \ \mathrm{d}x = \mathrm{d}x_1 \mathrm{d}x_2\right\}.$$

Korn's relation indicates (see [26]), that V equipped with the normal $\|u\|_V = \|\mathbb{D}u\|_{L^2(\Omega)^{d\times d}}$ becomes a separable and reflexive Banach space, and there exists $C_K > 0$ such that

$$C_K \|\phi\|_{L^2(\Omega^\varepsilon)^3} \leq \|\mathbb{D}\phi\|_{L^2(\Omega^\varepsilon)^{3\times 3}}, \quad \forall \, \phi \in K^\varepsilon. \tag{10}$$

To establish the variational formulation, we assume that u, $\mathbb{S}$, and p are sufficiently smooth functions that comply with equations (4) through (9). Consider $\varphi \in K^\varepsilon$ and $u^\varepsilon \in K_d^\varepsilon$. We multiply equation (4) by φ and u^ε, and then integrate over the domain Ω^ε to obtain:

$$\int_{\Omega^\varepsilon} (-\operatorname{Div} \mathbb{S}^\varepsilon) \cdot (\varphi - \boldsymbol{u}^\varepsilon) dx dx_3 + \int_{\Omega^\varepsilon} \nabla p^\varepsilon \cdot (\varphi - \boldsymbol{u}^\varepsilon) dx dx_3 = \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon \cdot (\varphi - \boldsymbol{u}^\varepsilon) dx dx_3. \tag{11}$$

Then, using standard reasoning, the variational formulation of Problem (4)-(9) is given as follows.

Problem (PV.1). Find a velocity $\boldsymbol{u}^{\varepsilon} \in K_d^{\varepsilon}$ and $p^{\varepsilon} \in L_0^2(\Omega^{\varepsilon})$ such that

$$a\left(u^{\varepsilon}, \varphi - u^{\varepsilon}\right) - \left(p^{\varepsilon}, \operatorname{div}\varphi\right) + j^{\varepsilon}(\varphi) - j^{\varepsilon}\left(u^{\varepsilon}\right) \geq \left(f^{\varepsilon}, \varphi - u^{\varepsilon}\right), \quad \forall \varphi \in K^{\varepsilon}, \tag{12}$$

where

$$\begin{aligned} a\left(u^{\varepsilon}, \varphi\right) &= \int_{\Omega^{\varepsilon}} \mu(\|\mathbb{D}\boldsymbol{u}^{\varepsilon}\|)\mathbb{D}\boldsymbol{u}^{\varepsilon} : \mathbb{D}(\varphi) dx dx_3, \\ \left(f^{\varepsilon}, \varphi\right) &= \int_{\Omega^{\varepsilon}} f^{\varepsilon} \cdot \varphi \, \mathrm{d}x dx_3, \\ \left(p^{\varepsilon}, \operatorname{div}\varphi\right) &= \int_{\Omega^{\varepsilon}} p^{\varepsilon} \operatorname{div} \varphi dx dx_3, \\ j^{\varepsilon}(\varphi) &= \int_{\omega} k^{\varepsilon}|\varphi - s| \mathrm{d}x + \int_{\Omega^{\varepsilon}} g^{\varepsilon}\|\mathbb{D}\varphi\| dx dx_3. \end{aligned}$$

If the test function belongs to K_d^{ε}, we obtain the subsequent variational problem.

Problem (PV.2). Find $u^{\varepsilon} \in K_d^{\varepsilon}$ such that

$$a\left(u^{\varepsilon}, \varphi - u^{\varepsilon}\right) + j^{\varepsilon}(\varphi) - j^{\varepsilon}\left(u^{\varepsilon}\right) \geq \left(f^{\varepsilon}, \varphi - u^{\varepsilon}\right), \quad \forall \varphi \in K_d^{\varepsilon}\left(\Omega^{\varepsilon}\right). \tag{13}$$

The subsequent theorems provide a proof of unique solvability for both Problems (**PV.2**) and (**PV.1**).

Theorem 2.1
Suppose that $(\mathcal{C}_1)$-$(\mathcal{C}_3)$ and $f^{\varepsilon} \in \left(L^2\left(\Omega^{\varepsilon}\right)\right)^3)$ hold, Thus, Problem (**PV.2**) possesses a unique solution. In addition, when $s = 0$ the weak solution u^{ε} satisfies the energy equality

$$\int_{\Omega^{\varepsilon}} \mu\left(\|\mathbb{D}\left(u^{\varepsilon}\right)\|\right)\|\mathbb{D}\left(u^{\varepsilon}\right)\|^2 dx dx_3 + \int_{\Omega^{\varepsilon}} g^{\varepsilon}\|\mathbb{D}\left(u^{\varepsilon}\right)\| dx dx_3 + \int_{\omega} k^{\varepsilon}\left|u^{\varepsilon}\right| dx = \int_{\Omega^{\varepsilon}} f^{\varepsilon} \cdot u^{\varepsilon} dx dx_3.$$

Proof
According to [11], it is sufficient to verify that the bilinear form a is continuous and coercive on $K_d^{\varepsilon} \times K_d^{\varepsilon}$. We recall that the functional j^{ε} is convex and continuous on K_d^{ε}. The bilinear form a is continuous and coercive. In fact, from condition $(\mathcal{C}_2)$, we have

$$\begin{aligned} |a(u, v)| &= \left|\int_{\Omega^{\varepsilon}} \mu(\|\mathbb{D}u\|)\mathbb{D}(u) : \mathbb{D}(v) dx dx_3\right| \\ &\leq \mu_1 ||\mathbb{D}(u)||_{L^2(\Omega^{\varepsilon}, \mathbb{M}^3)} ||\mathbb{D}(v)||_{L^2(\Omega^{\varepsilon}, \mathbb{M}^3)} = \mu_1 ||u||_{k_d^{\varepsilon}} ||v||_{k_d^{\varepsilon}}, \end{aligned}$$

and

$$a(u, u) = \int_{\Omega\varepsilon} \mu\|\mathbb{D}(u)\|^2 dx dx_3 \geq \mu_0\|u\|_{K_d^{\varepsilon}}^2, \quad \text{for all } u \in K_d^{\varepsilon}.$$

The convexity of j^{ε} is a direct consequence of the convexity of $\varphi \mapsto g^{\varepsilon}\|\mathbb{D}(\varphi)\|$. moreover, j^{ε} is continuous. In fact, from hypothesis $(\mathcal{C}_3)$ and the continuity of the trace operator, we have :

$$\begin{aligned} \left|j^{\varepsilon}(u) - j^{\varepsilon}(v)\right| &\leq \left\|k^{\varepsilon}\right\|_{\infty,\omega}|\omega|^{\frac{1}{2}}\|u - v\|_{L^2(\omega)} + \|g^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})}\|\mathbb{D}(u - v)\|_{L^2(\Omega^{\varepsilon}, \mathbb{M}^3)} \\ &\leq \left\|k^{\varepsilon}\right\|_{\infty,\omega}|\omega|^{\frac{1}{2}} C_0\|u - v\|_{H^1(\Omega^{\varepsilon})} + \|g^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})}\|u - v\|_{K_d^{\varepsilon}} \end{aligned}$$

By applying Korn's inequality (10), we can write:

$$\left|j^{\varepsilon}(u) - j^{\varepsilon}(v)\right| \leq \left(\left\|k^{\varepsilon}\right\|_{\infty,\omega}|\omega|^{\frac{1}{2}}\frac{C_0}{C_K} + \|g^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})}\right)\|u - v\|_{K_d^{\varepsilon}}.$$

Next, we demonstrate that the energy equality holds for any solution u^ε of Problem (**PV.2**). Specifically, by substituting $\varphi = 2u^\varepsilon$ into Problem (**PV.2**), we obtain:

$$\int_{\Omega^\varepsilon} \mu\left(\|\mathbb{D}\left(u^\varepsilon\right)\|\right) \|\mathbb{D}\left(u^\varepsilon\right)\|^2 \, dx dx_3 + \int_{\Omega^\varepsilon} g^\varepsilon \, \|\mathbb{D}\left(u^\varepsilon\right)\| \, dx dx_3 + \int_\omega k^\varepsilon \, |u^\varepsilon| \, dx \geq \int_{\Omega^\varepsilon} f^\varepsilon \cdot u^\varepsilon \, dx dx_3.$$

On the other hand, selecting $\varphi = \mathbf{0}$ in Problem (**PV.2**) yields:

$$-\int_{\Omega^\varepsilon} \mu\left(\|\mathbb{D}\left(u^\varepsilon\right)\|\right) \|\mathbb{D}\left(u^\varepsilon\right)\|^2 \, dx dx_3 - \int_{\Omega^\varepsilon} g^\varepsilon \, \|\mathbb{D}\left(\mathbf{u}\right)\| \, dx dx_3 - \int_\omega k^\varepsilon \, |u^\varepsilon| \, dx \geq -\int_{\Omega^\varepsilon} f^\varepsilon \cdot u^\varepsilon \, dx dx_3.$$

Clearly, by combining the last two inequalities, we obtain the energy equation. □

Theorem 2.2
Under the conditions of Theorem 2.1, the problem (**PV.1**) admits a unique solution $(u^\varepsilon, p^\varepsilon)$ in $K^\varepsilon \times L_0^2(\Omega^\varepsilon)$.

Proof
Given that the test function is part of K_d^ε, Theorem 2.1 guarantees the unique solvability $u^\varepsilon \in K_d^\varepsilon$ for the variational Problem (**PV.1**). To obtain p^ε, we will utilize the duality results from convex optimization [16]. First, note that we can rewrite Problem (**PV.1**) to ensure it is defined over K^ε. To do this, we introduce the indicator functions:

$$\phi_{K^\varepsilon} : \left(L^2\left(\Omega^\varepsilon\right)\right)^3 \to \overline{\mathbb{R}} \quad \text{with} \quad u \mapsto \phi_{K^\varepsilon}(u) = \begin{cases} 0 & \text{if } u \in K^\varepsilon, \\ +\infty & \text{if } u \notin K^\varepsilon, \end{cases}$$

and

$$\mathcal{R} : L^2\left(\Omega^\varepsilon\right) \to \overline{\mathbb{R}} \quad \text{with} \quad g \mapsto \mathcal{R}(g) = \begin{cases} 0 & \text{if } g = 0, \\ +\infty & \text{if } g \neq 0, \end{cases}$$

Then, we can therefore express (13) as follows:

$$a(u^\varepsilon, \varphi - u^\varepsilon) + j^\varepsilon(\varphi) - j^\varepsilon(u^\varepsilon) + \phi_{K^\varepsilon}(\varphi) - \phi_{K^\varepsilon}(u^\varepsilon) \geq \left(f^\varepsilon, \varphi - u^\varepsilon\right), \quad \forall\, \varphi \in K^\varepsilon \text{ with } \operatorname{div}(\varphi) = 0,$$

and the specific solution identified in Theorem 2.1 minimizes the functional

$$\inf_{\varphi \in K^\varepsilon} \left\{ (1/2)\, a(\varphi, \varphi) - (f^\varepsilon, \varphi) + j^\varepsilon(\varphi) + \mathcal{R}(\operatorname{div}(\varphi)) + \phi_{K^\varepsilon}(\varphi) \right\}. \tag{14}$$

This can be represented as below:

$$\inf_{\varphi \in K^\varepsilon} F(\varphi) + G(A(\varphi)),$$

where

$$F : K^\varepsilon \to \mathbb{R}, \quad \psi \mapsto F(\psi) = \frac{1}{2} a(\psi, \psi) - (f^\varepsilon, \psi),$$

$$A : K^\varepsilon \to X = L^2(\omega) \times L^2\left(\Omega^\varepsilon\right) \times K^\varepsilon, \quad \psi \mapsto A(\psi) = (A_1\psi, A_2\psi, \psi) = \left(\psi|_\omega, \operatorname{div}(\psi), \psi\right),$$

and

$$G : X \to \overline{\mathbb{R}}, \quad \psi \mapsto G(\psi) = j\left(\psi_1\right) + \mathcal{R}\left(\psi_2\right) + \phi_{K^\varepsilon}\left(\psi_3\right).$$

Next, the following represents the dual problem to (14), i.e.,

Find $p^\star$ in $X^\star = L^2(\omega) \times L^2\left(\Omega^\varepsilon\right) \times K^{\star,\varepsilon}$ such that:

$$\sup_{q^\star \in Y^\star} \left\{ -F^\star\left(A^\star q^\star\right) - G^\star\left(-q^\star\right) \right\},$$

where

$$F^\star(A^\star q^\star) = \sup_{\varphi\in K^\varepsilon} \big\{ \langle A_1^\star q_1^\star, \varphi\rangle + \langle A_2^\star q_2^\star, \varphi\rangle + \langle A_3^\star q_3^\star, \varphi\rangle - F(\varphi)\big\},$$

$$\begin{aligned} G^\star(-q^\star) &= \sup_{q\in X} \big\{ \langle -q^\star, q\rangle - G(q)\big\} \\ &= \sup_{q_1\in L^2(\omega)} \big\{ \langle -q_1^\star, q_1\rangle - j(q_1) \big\} + \sup_{q_2\in L^2(\Omega^\varepsilon)} \big\{ \langle -q_2^\star, q_2\rangle - \mathcal{R}(q_2) \big\} \\ &\quad + \sup_{q_3\in K^\varepsilon} \big\{ \langle -q_3^\star, q_3\rangle - \phi_{K^\varepsilon}(q_3) \big\}. \end{aligned}$$

Since the function $G : X \to \overline{\mathbb{R}}$ is continuous, there is $p^\star \in X^\star$ that satisfies the following relation, as stated in [21]:

$$\{F(u^\varepsilon) + G(A(u^\varepsilon))\} + \{F^\star(A^\star p^\star) + G^\star(-p^\star)\} = 0,$$

this can be formulated as

$$\{F(u^\varepsilon) + j(A_1 u^\varepsilon) + \mathcal{R}(A_2 u^\varepsilon) + \phi_{K^\varepsilon}(A_3 u^\varepsilon)\} + \big\{F^\star(A^\star p^\star) + j^\star(-p_1^\star) + (\phi_{K^\varepsilon})^\star(-p_3^\star)\big\} = 0.$$

By subtracting $\langle p_2^\star, A_2 u^\varepsilon\rangle$ from both sides, we obtain

$$\begin{aligned} &F(u^\varepsilon) - F(\varphi) + j(A_1 u^\varepsilon) - j(q_1) + \phi_{K^\varepsilon}(A_3 u^\varepsilon) - \phi_{K^\varepsilon}(q_3) + \langle A_1^\star p_1^\star, \varphi\rangle \\ &+ \langle A_2^\star p_2^\star, \varphi\rangle + \langle A_3^\star p_3^\star, \varphi\rangle + \langle -q_1^\star, q_1\rangle + \langle -q_3^\star, q_3\rangle - \langle p_2^\star, A_2 u^\varepsilon\rangle + \mathcal{R}(A_2 u^\varepsilon) = -\big\langle p_2^\star, A_2 u^\varepsilon\big\rangle. \end{aligned} \tag{15}$$

Based on the definition of $\mathcal{R}$, for any $q = (q_1, q_2, q_3)$ in $X = L^2(\omega) \times L^2(\Omega^\varepsilon) \times K^\varepsilon$, we have

$$G^\star(-q^\star) \geq \{\langle -q_1^\star, q_1\rangle - j(q_1)\} + \{\langle -q_3^\star, q_3\rangle - \phi_{K^\varepsilon}(q_3)\}. \tag{16}$$

By combining (15) and (16), utilizing the definition of $\mathcal{R}$ and tacking $q = A\varphi$ for φ in K^ε, we obtain

$$\begin{aligned} &F(u^\varepsilon) - F(\varphi) + j(A_1 u^\varepsilon) - j(A_1\varphi) + \phi_{K^\varepsilon}(A_3 u^\varepsilon) \\ &- \phi_{K^\varepsilon}(A_3\varphi) + \langle p_2^\star, A_2\varphi\rangle - \langle p_2^\star, A_2 u^\varepsilon\rangle \leq \big\{ -\mathcal{H}(A_2 u^\varepsilon) - \langle p_2^\star, \operatorname{div}(u^\varepsilon)\rangle \big\} \leq 0, \end{aligned}$$

which corresponds that for all $\varphi \in K^\varepsilon$, we have

$$a(u^\varepsilon, \varphi - u^\varepsilon) + j(\varphi) - j(u^\varepsilon) + \phi_{K^\varepsilon}(A_3\varphi) - \phi_{K^\varepsilon}(A_3 u^\varepsilon) - \langle p_2^\star, \operatorname{div}(\varphi - u^\varepsilon)\rangle \geq (f^\varepsilon, \varphi - u^\varepsilon).$$

Since u^ε is unique in K^ε, it follows that $p_2^\star$ is also unique in $L^2(\Omega^\varepsilon)$. Thus, Theorem 2.2 has been proven. □

Lemma 2.3
Let u^ε be a solution of problem (12), then

$$a(u^\varepsilon, u^\varepsilon) + g^\varepsilon \int_{\Omega^\varepsilon} |D(u^\varepsilon)|\, \mathrm{d}x\, \mathrm{d}x_3 + \int_\omega k^\varepsilon |u^\varepsilon - s|\, \mathrm{d}x \leq \frac{1}{2}\mu C_k \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{(\varepsilon h_M)^2}{2\mu C_k} \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}. \tag{17}$$

Proof
By selecting $\varphi = 0$ as the test function in inequality (13), we obtain

$$a(u^\varepsilon, u^\varepsilon) + g^\varepsilon \int_{\Omega^\varepsilon} |D(u^\varepsilon)|\, \mathrm{d}x\, \mathrm{d}x_3 + \int_\omega k^\varepsilon |u^\varepsilon - s|\, \mathrm{d}x \leq (f^\varepsilon, u^\varepsilon). \tag{18}$$

Applying the Poincaré inequality [10], we find

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq \varepsilon h_M \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}.$$

Using the Young inequality, we get

$$\begin{aligned}(f^\varepsilon, u^\varepsilon) &\leq \varepsilon h_M \left\|\nabla u^\varepsilon\right\|_{L^2(\Omega^\varepsilon)} \left\|f^\varepsilon\right\|_{L^2(\Omega^\varepsilon)} \\ &\leq (\mu C_k)^{\frac{1}{2}} \left\|\nabla u^\varepsilon\right\|_{L^2(\Omega^\varepsilon)} \frac{\varepsilon h_M}{(\mu C_k)^{\frac{1}{2}}} \left\|f^\varepsilon\right\|_{L^2(\Omega^\varepsilon)} \\ &\leq \frac{1}{2}\mu C_K \left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + \frac{(\varepsilon h_M)^2}{2\mu C_k} \left\|f^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)}.\end{aligned} \tag{19}$$

Thus, from (18) and (19), we deduce

$$\begin{aligned}&a\left(u^\varepsilon, u^\varepsilon\right) + g^\varepsilon \int_{\Omega^\varepsilon} \left|D\left(u^\varepsilon\right)\right| \mathrm{d}x\, \mathrm{d}x_3 + \int_\omega k^\varepsilon \left|u^\varepsilon - s\right| \mathrm{d}x \\ &\qquad \leq \frac{1}{2}\mu C_K \left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + \frac{(\varepsilon h_M)^2}{2\mu C_k} \left\|f^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)}.\end{aligned} \tag{20}$$

□

3. Boundedness and weak convergences

For the asymptotic analysis of Problem (**PV.1**), we transform the problem from the domain Ω^ε, which relies on a small parameter ε, to an equivalent problem in the fixed domain Ω that is independent of ε. This is done by applying a scaling technique on the x_3 coordinate, introducing the variable change $y = \frac{x_3}{\varepsilon}$. Hence, we specify the domain as

$$\Omega = \left\{(x, y) \in \mathbb{R}^3 : (x, 0) \in \omega, \quad 0 < y < h(x)\right\}.$$

We represent its boundary by $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_L \cup \bar{\omega}$ and proceed to define the following functions in Ω:

$$\begin{cases} \hat{u}_i^\varepsilon(x, y) = u_i^\varepsilon\left(x, x_3\right) \quad (i = 1, 2), \\ \hat{u}_3^\varepsilon(x, y) = \dfrac{1}{\varepsilon} u_3^\varepsilon\left(x, x_3\right), \\ \hat{p}^\varepsilon(x, y) = \varepsilon^2 p^\varepsilon\left(x, x_3\right). \end{cases}$$

The vector independent of ε must first be defined:

$$\hat{f}(x, y) = \left(\hat{f}_1(x, y), \hat{f}_2(x, y), \hat{f}_3(x, y)\right).$$

Next, we make the following assumption regarding the dependence of the data on ε:

$$\hat{f}(x, y) = \varepsilon^2 f^\varepsilon\left(x, x_3\right), \quad \hat{g} = \varepsilon g^\varepsilon \quad \text{and} \quad \hat{k} = \varepsilon k^\varepsilon. \tag{21}$$

We then introduce the following useful sets and spaces:

$$K(\Omega) = \left\{\hat{\varphi} \in \left(H^1(\Omega)\right)^3 : \hat{\varphi} = 0 \text{ on } \Gamma_1 \cup \Gamma_L,\ \hat{\varphi} \cdot n = 0 \text{ on } \omega\right\},$$

$$K_{\mathrm{d}}(\Omega) = \left\{\hat{\varphi} \in K(\Omega) : \operatorname{div} \hat{\varphi} = 0 \text{ in } \Omega\right\},$$

and

$$V_y = \left\{v = (v_1, v_2) \in \left(L^2(\Omega)\right)^2 : \frac{\partial v_i}{\partial y} \in L^2(\Omega)\, (i = 1, 2),\ v = 0 \text{ on } \Gamma_1\right\}.$$

The space V_y, equipped with the following norm, is a Banach space.

$$\|v\|_{V_y} = \left(\sum_{i=1}^{2} \|v_i\|^2_{L^2(\Omega)} + \left\| \frac{\partial v_i}{\partial y} \right\|^2_{L^2(\Omega)} \right)^{1/2}$$

and define its linear subspace, which is equipped with the same topology

$$\tilde{V}_y = \left\{ v \in V_y : v \text{ satisfies condition } (D') \right\},$$

where the condition (D′) is given as follows:

$$\int_\Omega (\hat{\phi}_1 \frac{\partial \theta}{\partial x_1} + \hat{\phi}_2 \frac{\partial \theta}{\partial x_2}) \,\mathrm{d}x \,\mathrm{d}y = 0 \text{ for all } (\hat{\phi}, \theta) \in \left(L^2(\Omega)\right)^2 \times C_0^\infty(\Omega).$$

By incorporating new data and unknowns into Problem (**PV.1**) and multiplying by ε, we obtain:

$$\hat{a}\left(\hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon\right) - \left(\hat{p}^\varepsilon, \operatorname{div}(\hat{\varphi})\right) + \hat{j}(\hat{\varphi}) - \hat{j}\left(\hat{u}^\varepsilon\right) \geq \left(\hat{f}, \hat{\varphi} - \hat{u}^\varepsilon\right), \quad \forall \hat{\varphi} \in K, \tag{22}$$

Where

$$\begin{aligned}
\hat{a}\left(\hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon\right) &= \frac{1}{2}\mu\varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right) \frac{\partial}{\partial x_j} \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon\right) dxdy \\
&+ \frac{1}{2}\mu \sum_{i=1}^{2} \int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial y} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right) \frac{\partial}{\partial y} \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon\right) dxdy \\
&+ \frac{1}{2}\mu\varepsilon^2 \sum_{j=1}^{2} \int_\Omega \left(\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial y} \right) \frac{\partial}{\partial x_j} \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon\right) dxdy \\
&+ \mu\varepsilon^2 \int_\Omega \frac{\partial \hat{u}_3^\varepsilon}{\partial y} \cdot \frac{\partial}{\partial y} \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon\right) dxdy.
\end{aligned}$$

$$\left(\hat{p}^\varepsilon, \operatorname{div}(\hat{\varphi})\right) = \int_\Omega \hat{p}^\varepsilon \left(\frac{\partial \hat{\varphi}_1}{\partial x_1} + \frac{\partial \hat{\varphi}_2}{\partial x_2} + \frac{\partial \hat{\varphi}_3}{\partial y} \right) dxdy,$$

$$\hat{j}(\hat{\varphi}) = \int_\omega \hat{k}|\hat{\varphi} - s| dx + \hat{g} \int_\Omega \left| \tilde{\mathbb{D}}(\hat{\varphi}) \right| \mathrm{d}x \,\mathrm{d}y,$$

$$\left(\hat{f}, \hat{\varphi} - \hat{u}^\varepsilon\right) = \sum_{i=1}^{2} \int_\Omega \hat{f}_i \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon\right) \mathrm{d}x \,\mathrm{d}y + \int_\Omega \varepsilon \hat{f}_3 \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon\right) \mathrm{d}x \,\mathrm{d}y,$$

and

$$\left| \tilde{\mathbb{D}}(\hat{u}^\varepsilon) \right| = \left(\frac{1}{4}\varepsilon^2 \sum_{i,j=1}^{2} \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right)^2 + \frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial y} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right)^2 + \varepsilon^2 \left(\frac{\partial \hat{u}_3^\varepsilon}{\partial y} \right)^2 \right)^{\frac{1}{2}}.$$

In the following part of this section, we will establish the estimates and convergence results for the velocity field $\hat{u}^\varepsilon$ and the pressure $\hat{p}^\varepsilon$ within the domain Ω.

Theorem 3.1
Under the conditions of Theorem 2.1, if $(\hat{u}^\varepsilon, \hat{p}^\varepsilon) \in K_{\mathrm{d}}(\Omega) \times L_0^2(\Omega)$ is the solution to problem (22), then there exists a constant $C > 0$, independent of ε, for wich we have

$$\sum_{i,j=1}^{2} \left\| \varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \right\|^2_{L^2(\Omega)} + \left\| \varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial y} \right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \left(\left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial y} \right\|^2_{L^2(\Omega)} + \left\| \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right\|^2_{L^2(\Omega)} \right) \leq C \tag{23}$$

Proof
After multiplying (17) by ε and utilizing the relation

$$\left\|\hat{f}\right\|^2_{L^2(\Omega)} = \varepsilon^3 \left\|f^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)},$$

we obtain:

$$\varepsilon a\left(u^\varepsilon, u^\varepsilon\right) + \hat{g}\int_\Omega \left|\tilde{D}\left(\hat{u}^\varepsilon\right)\right| \mathrm{d}x\, \mathrm{d}y + \int_\omega \hat{k}\left|\hat{u}^\varepsilon - s\right| \mathrm{d}x \leq \frac{1}{2}\mu C_k \varepsilon \left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + \frac{h_M^2}{2\mu C_k}\left\|\hat{f}\right\|^2_{L^2(\Omega)}. \tag{24}$$

According to Korn's inequality, there is $C_K > 0$ that is independent of ε for wich we have

$$a(u^\varepsilon, u^\varepsilon) \geq \mu C_k \left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)}. \tag{25}$$

By combining (24) and (25), we obtain

$$\varepsilon\frac{1}{2}\mu C_k \varepsilon \left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + \hat{g}\int_\Omega \left|\tilde{D}\left(\hat{u}^\varepsilon\right)\right| \mathrm{d}x\, \mathrm{d}y \quad + \int_\omega \hat{k}\left|\hat{u}^\varepsilon - s\right| \mathrm{d}x \leq \frac{h_M^2}{2\mu C_k}\left\|\hat{f}\right\|^2_{L^2(\Omega)}, \tag{26}$$

and

$$\varepsilon\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} = \sum_{i,j=1}^{2}\left\|\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}\right\|^2_{L^2(\Omega)} + \left\|\varepsilon\frac{\partial \hat{u}_3^\varepsilon}{\partial y}\right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2}\left(\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial y}\right\|^2_{L^2(\Omega)} + \left\|\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right\|^2_{L^2(\Omega)}\right).$$

Then, we find

$$\sum_{i,j=1}^{2}\left\|\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}\right\|^2_{L^2(\Omega)} + \left\|\varepsilon\frac{\partial \hat{u}_3^\varepsilon}{\partial y}\right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2}\left(\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial y}\right\|^2_{L^2(\Omega)} + \left\|\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right\|^2_{L^2(\Omega)}\right) \leq C,$$

where

$$C := \left(\frac{h_M}{\mu C_k}\right)^2 \left\|\hat{f}\right\|^2_{L^2(\Omega)}.$$

□

Theorem 3.2
Suppose the conditions of Theorem 2.1 hold, if $(\hat{u}^\varepsilon, \hat{p}^\varepsilon) \in K_{\mathrm{d}}(\Omega) \times L_0^2(\Omega)$ represents the solution to problem (22), then there is $C' > 0$, which does not depend on ε, for wich we have

$$\left\|\frac{\partial \hat{p}^\varepsilon}{\partial x_i}\right\|_{H^{-1}(\Omega)} \leq C' \; (i = 1, 2) \quad \text{and} \quad \left\|\frac{\partial \hat{p}^\varepsilon}{\partial y}\right\|_{H^{-1}(\Omega)} \leq \varepsilon C'.$$

Proof
By selecting an arbitrary $\psi \in (H_0^1(\Omega))^3$ and substituting $\varphi = u^\varepsilon + \psi$ into (12), we obtain:

$$(p^\varepsilon, div\psi) \leq a(u^\varepsilon, \psi) + \int_{\Omega^\varepsilon} g\left|D(\psi)\right| dx dx_3 + (f^\varepsilon, -\psi). \tag{27}$$

According to [10] we find

$$a(u^\varepsilon, \psi) \leq \mu_1 \left\|\nabla u^\varepsilon\right\|_{L^2(\Omega^\varepsilon)} \left\|\nabla\psi\right\|_{L^2(\Omega^\varepsilon)}.$$

Then, after multiplying (27) by ε and applying Hölder's inequality, we obtain:

$$(\hat{p}^\varepsilon, div\psi) \leq \mu_1 \varepsilon \left\|\nabla u^\varepsilon\right\|_{L^2(\Omega^\varepsilon)} \left\|\nabla\psi\right\|_{L^2(\Omega^\varepsilon)} + \hat{g}\left|\Omega\right|^{\frac{1}{2}} \left\|D\psi\right\|_{L^2(\Omega,\mathbb{M}^3)} + \left\|\hat{f}\right\|_{L^2(\Omega)} \left\|\psi\right\|_{L^2(\Omega)}.$$

Thus,

$$\begin{aligned}(\hat{p}^{\varepsilon}, div\psi) &\leq \mu_1 \varepsilon \left\| \nabla u^{\varepsilon} \right\|_{L^2(\Omega^{\varepsilon})} \left\| \psi \right\|_{H^1(\Omega^{\varepsilon})} + \hat{g} \left| \Omega \right|^{\frac{1}{2}} \alpha \left\| \psi \right\|_{H^1(\Omega)} + \left\| \hat{f} \right\|_{L^2(\Omega)} \left\| \psi \right\|_{H^1(\Omega)} \\ &\leq \left(\mu_1 C + \hat{g} \left| \Omega \right|^{\frac{1}{2}} \alpha + \left\| \hat{f} \right\|_{L^2(\Omega)} \right) \left\| \psi \right\|_{H^1(\Omega)} .\end{aligned} \tag{28}$$

Similarly, by choosing $\varphi = u^{\varepsilon} - \psi$ in (12), we find

$$-(\hat{p}^{\varepsilon}, div\psi) \leq \left(\mu_1 C + \hat{g} \left| \Omega \right|^{\frac{1}{2}} \alpha + \left\| \hat{f} \right\|_{L^2(\Omega)} \right) \left\| \psi \right\|_{H^1(\Omega)} . \tag{29}$$

Then, by utilizing the inequalities (28) and (29), we obtain

$$\left| (\hat{p}^{\varepsilon}, div\psi) \right| \leq \left(\mu_1 C + \hat{g} \left| \Omega \right|^{\frac{1}{2}} \alpha + \left\| \hat{f} \right\|_{L^2(\Omega)} \right) \left\| \psi \right\|_{H^1(\Omega)} , \quad \forall \psi \in H_0^1(\Omega). \tag{30}$$

Substituting $\psi = (\phi, 0, 0)$ and $\psi = (0, \phi, 0)$ into (30), and applying Green's formula, we deduce:

$$\left| \int_{\Omega} \frac{\partial \hat{p}^{\varepsilon}}{\partial x_i} \phi dx dy \right| \leq \left(\mu_1 C + \hat{g} \left| \Omega \right|^{\frac{1}{2}} \alpha + \left\| \hat{f} \right\|_{L^2(\Omega)} \right) \left\| \phi \right\|_{H^1(\Omega)} \quad \text{for } i = 1, 2.$$

On the other hand, by substituting $\psi = (0, 0, \varepsilon \phi)$ into (30), we obtain

$$\left| \int_{\Omega} \frac{\partial \hat{p}^{\varepsilon}}{\partial y} \phi dx dy \right| \leq \varepsilon \left(\mu_1 C + \hat{g} \left| \Omega \right|^{\frac{1}{2}} \alpha + \left\| \hat{f} \right\|_{L^2(\Omega)} \right) \left\| \phi \right\|_{H^1(\Omega)}$$

□

Corollary 3.3
If assumptions of Theorem 2.1 hold, there exist $u_i^{\star} \in V_y$ for $i = 1, 2$ and $p^{\star} \in L_0^2(\Omega)$ for wich we have

$$\hat{u}_i^{\varepsilon} \to u_i^{\star} \quad \text{weakly in } V_y \quad (i = 1, 2), \tag{31}$$

$$\varepsilon \frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j} \to 0 \quad \text{weakly in } L^2(\Omega) \quad (i, j = 1, 2), \tag{32}$$

$$\varepsilon \frac{\partial \hat{u}_3^{\varepsilon}}{\partial y} \to 0 \quad \text{weakly in } L^2(\Omega), \tag{33}$$

$$\varepsilon^2 \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i} \to 0 \quad \text{weakly in } L^2(\Omega) \quad (i = 1, 2), \tag{34}$$

$$\varepsilon \hat{u}_3^{\varepsilon} \to 0 \quad \text{weakly in } L^2(\Omega), \tag{35}$$

$$\hat{p}^{\varepsilon} \to p^{\star} \quad \text{weakly in } L_0^2(\Omega). \tag{36}$$

Proof
To begin, from equation (23), we get a constant C that is independent of ε satisfying:

$$\left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial y} \right\|_{L^2(\Omega)}^2 \leq C \quad \text{for } i = 1, 2. \tag{37}$$

Applying the Poincaré inequality [4] in conjunction with condition (7), we derive:

$$\left\| \hat{u}_i^{\varepsilon} \right\|_{L^2(\Omega)}^2 \leq 2 h_M^2 \left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial y} \right\|_{L^2(\Omega)}^2 \quad \text{for } i = 1, 2. \tag{38}$$

From inequalities (37) and (38), we can conclude the result presented in (31). To demonstrate the convergence in (32), we utilize both the inequality in (23) and the convergence established in (31). Furthermore, we rely on the previously obtained results along with the condition $\operatorname{div}(\hat{u}_i^\varepsilon) = 0$, which yields:

$$\sum_{i=1}^{2} \frac{\partial \hat{u}_i^\varepsilon}{\partial x_i} = -\frac{\partial \hat{u}_3^\varepsilon}{\partial y}.$$

Thus, from (32), the convergence (33) holds, and from (23), there exists a constant $C > 0$ for which

$$\left\| \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 \leq C \quad \text{for } i = 1, 2. \tag{39}$$

Utilizing (38), we derive:

$$\|\varepsilon \hat{u}_3^\varepsilon\|_{L^2(\Omega)}^2 \leq 2h_M^2 \left\| \varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial y} \right\|_{L^2(\Omega)}^2. \tag{40}$$

From (23), we also have:

$$\left\| \varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial y} \right\|_{L^2(\Omega)}^2 \leq C \quad \text{for } i = 1, 2. \tag{41}$$

Combining results from (39), (40), and (41), we conclude that:

$$\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \to 0 \quad \text{weakly in } L^2(\Omega, \quad (i = 1, 2).$$

From (40) and (41), we can assert that there is $C > 0$ such that:

$$\|\varepsilon \hat{u}_3^\varepsilon\|_{L^2(\Omega)}^2 \leq C. \tag{42}$$

Consequently, there exists $u_3^\star \in L^2(\Omega)$ such that:

$$\varepsilon \hat{u}_3^\varepsilon \to u_3^\star \quad \text{weakly in } L^2(\Omega). \tag{43}$$

This implies:

$$\varepsilon \hat{u}_3^\varepsilon \to u_3^\star \quad \text{in } D'(\Omega). \tag{44}$$

Given that $\operatorname{div}(\hat{u}^\varepsilon) = 0$ in Ω, for any $\Phi \in L_0^2(\Omega)$, we have:

$$\int_\Omega \Phi \operatorname{div}(\hat{u}^\varepsilon)\, dx\, dy = 0. \tag{45}$$

We select Φ such that $\Phi(x, y) = y\varphi(x) - \beta$, where $\varphi \in \mathcal{C}_0^\infty(\omega)$ and:

$$\beta = \frac{\int_\Omega y\varphi\, dx\, dy}{\int_\Omega dx\, dy}.$$

Using (45), the Green formula, and the boundary conditions on Γ, we obtain:

$$-\sum_{i=1}^{2} \int_\Omega y\varepsilon \hat{u}_i^\varepsilon \frac{\partial \varphi}{\partial x_i}\, dx\, dy - \int_\Omega \varphi \varepsilon \hat{u}_3^\varepsilon\, dx\, dy = 0.$$

As $\hat{u}_i^\varepsilon \rightharpoonup u^\star$ in V_y for $i = 1, 2$, then as ε tends to zero, (35) holds. Finally, we have (see [17]):

$$\|\hat{p}^\varepsilon\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla \hat{p}^\varepsilon\|_{H^{-1}(\Omega)}.$$

Since $L_0^2(\Omega)$ is weakly closed in $L^2(\Omega)$, from Theorem 4.2, we conclude (see (36)):

$$\hat{p}^\varepsilon \to p^\star \quad \text{weakly in } L_0^2(\Omega).$$

This completes the proof. □

4. Study of Limiting Problems

We analyze the limit behavior of Problem $(\mathbf{PV.1})$ as ε approaches zero. We will prove the theorem below, establishing the equations that the limits $p^\star$ and $u^\star$ of $\hat{p}^\varepsilon$ and $\hat{u}^\varepsilon$ satisfy in Ω, along with the inequalities for the trace of the velocity $u^\star(x,0)$.

Theorem 4.1
Assuming the conditions of Theorem 3.1 are satisfied, the limit functions $(u^\star, p^\star)$ fulfill the following conditions:

$$p^\star \in H^1(\omega), \tag{46}$$

$$-\frac{1}{2}\mu\frac{\partial^2 u_i^\star}{\partial y^2} + \frac{\partial p^\star}{\partial x_i} = \hat{f}_i \quad (\text{ for } i=1,2) \text{ in } L^2(\Omega). \tag{47}$$

Proof
To begin the proof, we choose $\varphi_3 = \hat{u}_3^\varepsilon \pm \psi$ and $\varphi_i = \hat{u}_i^\varepsilon$ for $i=1,2$, where ψ belongs to $H_0^1(\Omega)$. This selection in equation (22) yields the following result:

$$\frac{1}{2}\mu\varepsilon^2\sum_{j=1}^{2}\int_\Omega\left(\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial\hat{u}_j^\varepsilon}{\partial y}\right)\frac{\partial\psi}{\partial x_j}dxdy + \int_\Omega \mu\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial y}\frac{\partial\psi}{\partial y}dxdy - \int_\Omega \hat{p}^\varepsilon\frac{\partial\psi}{\partial y}dxdy = \int_\Omega \varepsilon f_3\psi dxdy. \tag{48}$$

By applying equations (31), (33), (34), and (36), we obtain for $\varepsilon\to 0$, the following results :

$$\int_\Omega p^\star\frac{\partial\psi}{\partial y}dxdy = 0, \quad \forall\psi\in H_0^1(\Omega).$$

Therefore, by using green's formula, we find

$$\frac{\partial p^\star}{\partial y} = 0 \quad \text{ in } H^{-1}(\Omega). \tag{49}$$

Alternatively, selecting $\varphi_i = \hat{u}_i^\varepsilon \pm \psi_i$ where $\psi_i\in H_0^1(\Omega)$ $(i=1,2)$, and setting $\varphi_3 = \hat{u}_3^\varepsilon$ in (22) to obtain:

$$\begin{aligned} &\frac{1}{2}\mu\varepsilon\sum_{i,j=1}^{2}\int_\Omega\left(\varepsilon\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j} + \varepsilon\frac{\partial\hat{u}_j^\varepsilon}{\partial x_i}\right)\frac{\partial\psi_i}{\partial x_j}dxdy \\ &+ \sum_{i=1}^{2}\int_\Omega\frac{1}{2}\mu\left(\frac{\partial\hat{u}_i^\varepsilon}{\partial y} + \varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\right)\frac{\partial\psi_i}{\partial y}dxdy \\ &- \int_\Omega \hat{p}^\varepsilon\frac{\partial\psi_i}{\partial x_j}dxdy = \sum_{i=1}^{2}\int_\Omega \hat{f}_i\psi_i dxdy. \end{aligned} \tag{50}$$

Employing equations (31), (32), (34), and (36), we deduce that as ε approaches zero, first with $\psi_1 = 0$ and $\psi_2\in H_0^1(\Omega)$, and subsequently with $\psi_2 = 0$ and $\psi_1\in H_0^1(\Omega)$, the following equality holds:

$$\sum_{i=1}^{2}\int_\Omega\frac{1}{2}\mu\frac{\partial u_i^\star}{\partial y}\frac{\partial\psi_i}{\partial y}dxdy - \sum_{i=1}^{2}\int_\Omega p^\star\frac{\partial\psi_i}{\partial x_i}dxdy = \sum_{i=1}^{2}\int_\Omega \hat{f}_i\psi_i\, dxdy, \tag{51}$$

then, by using Green's formula, we get

$$-\frac{1}{2}\mu\frac{\partial^2 u_i^\star}{\partial y^2} + \frac{\partial p^\star}{\partial x_i} = \hat{f}_i \quad (\text{for } i=1,2) \text{ in } H^{-1}(\Omega). \tag{52}$$

Let's recall from (49) that $p^\star$ is a function that depends solely on $x \in \omega$. By substituting ψ_i into (51), where $\psi_i(x,y) = y(y-h(x))\varphi(x)$ with $\varphi \in H_0^1(\omega)$, and applying Green's formula, we obtain:

$$\frac{1}{6}\int_\omega p^\star \frac{\partial\left(h^3\varphi\right)}{\partial x_i} dx - \mu \int_\omega h\tilde{u}_i^\star \varphi dx = \int_\omega \tilde{f}_i \varphi dx$$

where

$$\tilde{u}_i^\star(x) = \frac{1}{h(x)}\int_0^{h(x)} u_i^\star(x,y)\, dy$$

and

$$\tilde{f}_i(x) = \int_0^{h(x)} y(y-h(x))\hat{f}_i(x,y)\, dy,$$

which, upon applying Green's formula, yields

$$-\frac{1}{6}h^3\frac{\partial p^\star}{\partial x_i} - \mu h \tilde{u}_i^\star = \tilde{f}_i \quad (\text{ for } i=1,2) \text{ in } H^{-1}(\Omega). \tag{53}$$

Since $f_i \in L^2(\Omega)$, it follows that $\tilde{f}_i \in L^2(\omega)$. Similarly, because $u_i^\star \in V_y$, we also have $\tilde{u}_i^\star \in L^2(\omega)$. From (53), we then obtain $p^\star \in H^1(\omega)$. Furthermore, since $f_i \in L^2(\Omega)$, it follows from (52) that $\frac{\partial^2 u_i^\star}{\partial y^2} \in L^2(\Omega)$. Hence, (47) holds. We also deduce that $\frac{\partial u_i^\star}{\partial y} \in V_y$. Thus, the proof is complete. □

We now introduce the limiting form of the Tresca boundary conditions. The following notations will be used:

$$s^\star(x) = u^\star(x,0) \quad \text{and} \quad \tau^\star(x) = \left(\frac{\partial u^\star}{\partial y}\right)(x,0).$$

Since $\frac{\partial u^\star}{\partial y}$ belongs to V_y, it follows that $\tau^\star \in L^2(\omega)$.

Theorem 4.2

Under the same hypotheses as Theorem 4.1, the pair $(s^\star, \tau^\star)$ satisfies the following inequalities:

$$\int_\omega \hat{k}\left(|\psi + s^\star - s| - |s^\star - s|\right) dx - \int_\omega \frac{1}{2}\mu\tau^\star\psi\, dx \geq 0, \quad \forall \psi \in \left(L^2(\omega)\right)^2,$$

and

$$\begin{cases} \dfrac{1}{2}\mu\,|\tau^\star| = \hat{k} \implies \exists \lambda \geq 0 \text{ such that } s^\star = s + \lambda\tau^\star, \\ \dfrac{1}{2}\mu\,|\tau^\star| < \hat{k} \implies s^\star = s \text{ a.e. in } \omega. \end{cases}$$

Proof

By choosing $\hat{\varphi} = \left(\hat{u}_1^\varepsilon + \psi_1, \hat{u}_2^\varepsilon + \psi_2, \varepsilon\hat{u}_3^\varepsilon\right)$, where $\psi_i \in H^1_{\Gamma_1\cup\Gamma_L}(\omega)$ for $i=1,2$, and

$$H^1_{\Gamma_1\cup\Gamma_L}(\omega) = \left\{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1 \cup \Gamma_L\right\}.$$

Substituting this into (22) leads to

$$\frac{1}{2}\mu\varepsilon^2 \sum_{i,j=1}^{2}\int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}\right)\frac{\partial\psi_i}{\partial x_j}\, dx\, dy + \frac{1}{2}\mu\sum_{i=1}^{2}\int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial y} + \varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right)\frac{\partial \psi_i}{\partial y} dxdy$$

$$-\sum_{i=1}^{2}\int_\Omega \hat{p}^\varepsilon \frac{\partial\psi_i}{\partial x_i} dxdy + \int_\omega \hat{k}\left(|\psi + \hat{u}^\varepsilon - s| - |\hat{u}^\varepsilon - s|\right) dx + \hat{g}\int_\Omega \left(\left|\tilde{\mathbb{D}}(\hat{\varphi})\right| - \left|\tilde{\mathbb{D}}(\hat{u}^\varepsilon)\right|\right) dxdy$$

$$\geq \sum_{i=1}^{2}\int_\Omega \hat{f}_i\psi_i\, dx\, dy.$$

By applying Corollary 3.3, we conclude that as ε tends to zero, the following holds

$$\begin{aligned}&\frac{1}{2}\mu\sum_{i=1}^{2}\int_{\Omega}\frac{\partial u_i^\star}{\partial y}\frac{\partial \psi_i}{\partial y}\,dx\,dy-\sum_{i=1}^{2}\int_{\Omega}p^\star\frac{\partial \psi_i}{\partial x_i}\,dx\,dy+\int_{\omega}\hat{k}\left(|\psi+s^\star-s|-|s^\star-s|\right)\,dx\\&+\hat{g}\int_{\Omega}\left(\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\psi_i-u_i^\star}{\partial y}\right)^2\right)^{\frac{1}{2}}-\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^\star}{\partial y}\right)^2\right)^{\frac{1}{2}}\right)\,dx\,dy\\&\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\psi_i\,dx\,dy.\end{aligned}\tag{54}$$

By applying Green's formula along with equation (47) and the condition that $\psi_i=0$ on $\Gamma_1\cap\Gamma_L$, we obtain:

$$\begin{aligned}&\int_{\omega}\hat{k}\left(|\psi+s^\star-s|-|s^\star-s|\right)\,dx-\int_{\omega}\frac{1}{2}\mu\tau^\star\psi\,dx\\&\quad+\hat{g}\int_{\Omega}\left(\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\psi_i+u_i^\star}{\partial y}\right)^2\right)^{\frac{1}{2}}-\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^\star}{\partial y}\right)^2\right)^{\frac{1}{2}}\right)\,dy\,dx\\&\geq 0,\quad\forall\psi\in\left(H^1_{\Gamma_1\cup\Gamma_L}(\omega)\right)^2.\end{aligned}\tag{55}$$

Since (55) holds for all ψ in $\mathcal{D}(\omega)^2$, extended also to $\left(L^2(\omega)\right)^2$ due to the density of $\mathcal{D}(\omega)$ in $L^2(\omega)$. Thus, we infer

$$\int_{\omega}\hat{k}\left(|\psi+s^\star-s|-|s^\star-s|\right)dx-\int_{\omega}\frac{1}{2}\mu\tau^\star\psi dx\geq 0,\quad\forall\psi\in\left(L^2(\omega)\right)^2.\tag{56}$$

By substituting $\psi=\pm(s^\star-s)$ into equation (56), we obtain

$$\int_{\omega}\left(\hat{k}\,|s^\star-s|-\frac{1}{2}\mu\tau^\star\left(s^\star-s\right)\right)dx=0.\tag{57}$$

Let $\psi=\varphi-(s^\star-s)$ with $\varphi\in\left(L^2(\omega)\right)^2$. By inserting this expression into equation (56), we obtain

$$\int_{\omega}\left(\hat{k}|\varphi|-\frac{1}{2}\mu\tau^\star\varphi\right)dx\geq\int_{\omega}\left(\hat{k}\,|s^\star-s|-\frac{1}{2}\mu\tau^\star\left(s^\star-s\right)\right)dx.$$

Then, by using (57), we deduce

$$\int_{\omega}\left(\hat{k}|\varphi|-\frac{1}{2}\mu\tau^\star\varphi\right)dx\geq 0,\quad\forall\varphi\in\left(L^2(\omega)\right)^2.\tag{58}$$

By taking $\varphi=(\varphi_1,\varphi_2)$ with $\varphi_i\geq 0$ for $i=1,2$, we substitute into equation (58) to obtain

$$\int_{\omega}\left(\hat{k}|\varphi|-\frac{1}{2}\mu\,|\tau^\star|\cdot|\varphi|\cos\left(\tau^\star,\varphi\right)\right)dx=\int_{\omega}\left(\hat{k}-\frac{1}{2}\mu\,|\tau^\star|\cos\left(\tau^\star,\varphi\right)\right)|\varphi|dx\geq 0.$$

Thus,

$$\frac{1}{2}\,\mu\,|\tau^\star|\cos\left(\tau^\star,\varphi\right)\leq\hat{k}\qquad\text{a.e. on}\quad\omega.\tag{59}$$

Now, by considering $-\varphi$, where $\varphi=(\varphi_1,\varphi_2)$ and $\varphi_i\geq 0$ for $i=1,2$, in equation (4.13), we find

$$\int_{\omega}\left(\hat{k}|\varphi|+\frac{1}{2}\mu\,|\tau^\star|\cdot|\varphi|\cos\left(\tau^\star,\varphi\right)\right)dx=\int_{\omega}\left(\hat{k}+\frac{1}{2}\mu\,|\tau^\star|\cos\left(\tau^\star,\varphi\right)\right)|\varphi|dx\geq 0.$$

Then,

$$-\frac{1}{2}\mu\,|\tau^\star|\cos\left(\tau^\star,\phi\right)\leq\hat{k}\quad\text{ a.e. on }\ \omega. \tag{60}$$

Using (59) and (60) we get

$$\frac{1}{2}\mu\,|\tau^\star|\leq\hat{k}\quad\text{ a.e. on }\ \omega. \tag{61}$$

Hence,

$$\hat{k}\,|s^\star-s|\geq\frac{1}{2}\mu\,|\tau^\star|\cdot|s^\star-s|\geq\frac{1}{2}\mu\tau^\star\cdot(s^\star-s)\quad\text{ a.e. on }\ \omega,$$

and

$$\hat{k}\,|s^\star-s|-\frac{1}{2}\mu\tau^\star\cdot(s^\star-s)\geq 0\quad\text{ a.e. on }\ \omega,$$

Then, it follows from (57) that a.e. on ω, we have

$$\hat{k}\,|s^\star-s|-\frac{1}{2}\mu\tau^\star\cdot(s^\star-s)=0. \tag{62}$$

If $\frac{1}{2}\mu\,|\tau^\star|=\hat{k}$, then from equation (62), we have

$$\mu\,|\tau^\star|\cdot|s^\star-s|=\mu\tau^\star\cdot(s^\star-s)\quad\text{ a.e. on }\ \omega,$$

which implies $\cos\left(s^\star-s,\mu\tau^\star\right)=1$ and leads to $s^\star=s+\lambda\mu\tau^\star$ for some $\lambda\geq 0$. Conversely, if $\frac{1}{2}\mu\,|\tau^\star|<\hat{k}$, then we derive from (62) that a.e. on ω, we have

$$\hat{k}\,|s^\star-s|-\frac{1}{2}\mu\tau^\star\cdot(s^\star-s)=0\geq\left(\hat{k}-\frac{1}{2}\mu\,|\tau^\star|\right)|s^\star-s|\,.$$

Consequently, we have $s^\star=s$ almost everywhere on ω. □

Theorem 4.3
Let us consider the same hypotheses as in Theorem 4.1, and assume that $\hat{f}$ is a function of x only. Then, we have

$$\frac{h^2}{2}\nabla p^\star+\frac{1}{2}\mu s^\star+\frac{h}{2}\mu\tau^\star-\frac{h^2}{2}\hat{f}=0\quad\text{ a.e. on }\ \omega, \tag{63}$$

$$\int_\omega\left(\frac{h}{2}s^\star-\frac{h^3}{6\mu}\nabla p^\star+\frac{h^3}{6\mu}\hat{f}\right)\nabla\varphi\,dx=\int_{\partial\omega}\varphi\ell\cdot n\ \text{ for all }\ \varphi\in H^1(\omega), \tag{64}$$

$$\int_\omega\left(4hs^\star(x)+h^2\tau^\star\right)\nabla\varphi\,dx=6\int_{\partial\omega}\varphi\ell\cdot n\ \text{ for all }\ \varphi\in H^1(\omega). \tag{65}$$

Proof
By Theorem 5.1, we have the following relationship:

$$-\frac{1}{2}\mu\frac{\partial^2u_i^\star}{\partial y^2}+\frac{\partial p^\star}{\partial x_i}=\hat{f}_i\quad\text{ for } i=1,2.$$

Integrating this equation twice from 0 to y, we obtain:

$$-\frac{1}{2}\mu u_i^\star(x,y)+\frac{1}{2}\mu u_i^\star(x,0)+\frac{y^2}{2}\frac{\partial p^\star(x)}{\partial x_i}+\frac{1}{2}\mu y\frac{\partial u_i^\star(x,0)}{\partial y}=\frac{y^2}{2}\hat{f}_i(x)\quad\text{ for } i=1,2. \tag{66}$$

Setting $y=h$, we find that (63) holds, since $u_i^\star(x,h)=0$. Next, integrating (66) from 0 to h, we obtain:

$$h\tilde{u}^\star(x)=hs^\star(x)+\frac{h^3}{3\mu}\nabla p^\star(x)+\frac{h^2}{2}\tau^\star-\frac{h^3}{3\mu}\hat{f}(x), \tag{67}$$

where

$$\tilde{u}^\star(x) = \frac{1}{h(x)} \int_0^{h(x)} u^\star(x,y)\,dy, \quad \forall x \in \omega.$$

On the other hand, for every $\varphi \in H^1(\omega)$, we have:

$$\int_\Omega \varphi \,\mathrm{div}(\hat{u}^\varepsilon)\,dx\,dy = 0.$$

Thus, it follows that:

$$\int_\omega \varphi(x) \sum_{i=1}^{2} \left(\frac{\partial \left(h\tilde{\hat{u}}_i^\varepsilon\right)}{\partial x_i} + \hat{u}_3^\varepsilon(x,h) - \hat{u}_3^\varepsilon(x,0) \right) dx = 0.$$

Since $\hat{u}_3^\varepsilon = 0$ on $\partial\Omega = \bar{\omega} \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_L$, we then have:

$$\int_\omega \varphi(x) \sum_{i=1}^{2} \frac{\partial \left(h\tilde{\hat{u}}_i^\varepsilon\right)}{\partial x_i}\,dx = 0.$$

Applying Green's formula, we obtain:

$$-\sum_{i=1}^{2} \int_\omega h\tilde{\hat{u}}_i^\varepsilon \frac{\partial \varphi}{\partial x_i}\,dx + \sum_{i=1}^{2} \int_{\partial\omega} h\tilde{\hat{u}}_i^\varepsilon \varphi n_i\,d\Gamma = 0.$$

As $\hat{u}_i^\varepsilon \rightharpoonup u_i^\star$ in V_y, we find that $\tilde{\hat{u}}_i^\varepsilon \rightharpoonup \tilde{u}_i^\star$ in $L^2(\omega)$. Thus, we have:

$$\sum_{i=1}^{2} \int_\omega h\tilde{u}_i^\star \frac{\partial \varphi}{\partial x_i}\,dx = \sum_{i=1}^{2} \int_{\partial\omega} \varphi(x)\ell_i(x) n_i\,d\Gamma, \quad \forall \varphi \in H^1(\omega),$$

where

$$\ell_i = h\tilde{\hat{u}}_i^\varepsilon \quad \text{on} \quad \partial\omega.$$

From (67), we derive:

$$\int_\omega \left(hs^\star + \frac{h^3}{3\mu}\nabla p^\star + \frac{h^2}{2}\tau^\star - \frac{h^3}{3\mu}\hat{f} \right) \nabla\varphi\,dx = \int_{\partial\omega} \varphi \ell \cdot n\,d\Gamma, \quad \forall \varphi \in H^1(\omega). \tag{68}$$

The weak formulation of Reynolds equation (64) follows from (63) and (68). So, to get (65), we use (63)-(64). □

Remark 4.4. The uniqueness of $(u^\star, p^\star)$ follows from (64)-(54), using the same arguments as in [11, Theorem 5.3].

Conclusion

In this work, we studied an incompressible Bingham fluid model in a perturbed three-dimensional domain with Tresca and Dirichlet boundary conditions. We proved the unique solvability of the problem and conducted an asymptotic analysis as one dimension of the domain tends to zero. Our approach established the strong convergence of the velocity field, derived a Reynolds-type limit equation, and analyzed the asymptotic behavior of the Tresca boundary conditions, rigorously proving the uniqueness of the limiting velocity and pressure fields. These results not only provide a deeper understanding of the fluid's behavior in confined geometries but also open avenues for exploring more complex non-Newtonian fluid models and boundary conditions. Future research could extend this framework to account for additional physical effects, such as temperature dependence or more intricate rheological properties, as well as investigate the applicability of the derived limit equations in real-world engineering scenarios, such as lubrication or flow through porous media.

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