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Existence of optimal controls for semilinear systems with a nonreflexive control space

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Abstract In this paper, we study the existence of optimal controls that minimize a given functional. We consider a class of infinite-dimensional semilinear systems, and a functional that depends on a control function u and the associated solution of the semilinear equation. The functional is minimized over a set of admissible controls, that is a convex subset of a nonreflexive control space. Under appropriate assumptions, we derive sufficient conditions for the existence of optimal controls, for two classes of semilinear systems. Thereby, we provide two examples of partial differential equations to highlight the obtained results.

Keywords Optimal control, Semilinear systems, Existence theory, Nonreflexive control space.

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1. Introduction and Problem statement

Optimal control theory is a powerful framework to address various problems in engineering, biology, population dynamics, or economics. More importantly, optimal control of infinite dimensional semilinear systems enables the modeling of a wide range of physical problems. Two major questions arise in the study of such problems: Existence of solutions, and optimality conditions. Thereby, a rich literature has been devoted to both questions. Particulary, the question of existence of optimal controls has been investigated for various classes of systems. In [2], Ahmed and Xiang proved the existence of optimal controls for semilinear systems with a linear control term. Meanwhile, in [14], Li and Yong examined the existence of optimal controls for a class of systems with compact semigroups. Bradley and Lenhart proved in [7] the existence of an optimal control for a bilinear Kirchhoff plate, with controls in $L^\infty(Q)$. In [10], the existence of optimal controls is established over a compact set of $L^1(0,T)$.
As for optimality conditions, Bonnans and Casas derived in [6] necessary optimality conditions for elliptic semilinear systems, while in [5], Barbu studied a class of elliptic and parabolic semilinear systems, using the generalized maximum principle. In this respect, Raymond and Zidani also derived in [17] optimality conditions for a class of semilinear parabolic equations. In [14], assumptions on the reachable set were introduced to extend the maximum principle to abstract infinite dimensional semilinear systems.
In [19], optimality conditions have been derived for semilinear systems with real-valued controls. Thereby, the results were extended in [20] to semilinear systems with controls taking values in $L^2(\Omega)$. In addition, Aronna and Tröltzsch derived in [4] first and second order optimality conditions for Fokker-Planck equations.
The present paper focuses on developing sufficient conditions for the existence of optimal controls. We consider

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a new class of abstract semilinear systems, where the control functions belong to a nonreflexive Banach space. Unlike the reflexive case, the existence of a minimizer can be a challenging question, and requires appropriate assumptions on the system, the functional to be minimized, and the set of admissible controls. This is the purpose of the present work, where we will develop sufficient conditions for the existence of (at least) an optimal control.
To be more specific, let us consider the below system:

$$\begin{cases} \dot{z}(t) = Az(t) + f(u(t), z(t)) \\ z(0) = z_0 \end{cases} \tag{1}$$

with the following assumptions:

(A1) $A : \mathcal{D}(A) \to Z$ is the infinitesimal generator of a strongly continuous semigroup $T(t)_{t\geq 0}$, where Z is a reflexive separable Banach space.

(A2) $u \in L^p(0, t_f, U)$ denotes the control, where $1 < p < \infty$, and the control space U is a nonreflexive Banach space. In what follows, $p' \in]1, \infty[$ denotes the conjugate of p, such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

(A3) The mapping $f : U \times Z \to Z$ satisfies:

  i. For every $z \in Z$, the mapping $f(., z) : u \mapsto f(u, z)$ is linear continuous. In what follows, $f^*(., z)$ denotes the adjoint operator of $f(., z)$.

  ii. There exists $\alpha \geq 0$ such that

$$\|f(u, y) - f(u, z)\|_Z \leq \alpha \|u\|_U \|y - z\|_Z, \quad \forall u \in U, \quad \forall y, z \in Z. \tag{2}$$

Denote $\beta = \|f(., 0)\|_{\mathscr{L}(U,Z)}$, then $\|f(u, 0)\|_Z \leq \beta \|u\|_U$, for every $u \in U$. Hence inequality (2) yields

$$\|f(u, z)\|_Z \leq (\alpha \|z\|_Z + \beta) \|u\|_U, \quad \forall u \in U, \quad \forall z \in Z. \tag{3}$$

Due to Proposition 2.5.3 in [14], system (1) has a unique mild solution, written as:

$$z(t) = T(t)z_0 + \int_0^t T(t - s) f(u(s), z(s)) \mathrm{d}s \tag{4}$$

Let U_{ad} be a nonempty and convex set in $L^p(0, t_f; U)$. In what follows, U_{ad} will be referred to as the set of admissible controls. The present optimal control problem consists in finding a control $u^* \in U_{ad}$ that minimizes the following functional

$$J(u) = \mathscr{G}(z(t_f)) + \int_0^{t_f} \mathscr{Q}(z(t)) \mathrm{d}t + \mathscr{R}(u) \tag{5}$$

where z is the solution of (4), associated with u, and $\mathscr{G}, \mathscr{Q} : X \to \mathbb{R}_+$ and $\mathscr{R} : L^p(0, t_f; U) \to \mathbb{R}_+$ are continuous mappings.
Then the present problem is formulated as follows.

$$\begin{cases} \min J(u) \\ u \in U_{ad} \end{cases} \tag{6}$$

The present study focuses solely on the existence of solutions of problem (6). This is the purpose of Section 2, where sufficient conditions for the existence of solutions are developed. Then, Section 3 provides two examples that illustrate the theoretical results.

2. Sufficient conditions of existence

In order to formulate sufficient conditions of existence, we need the following lemmas.

Lemma 2.1
Let $u, v \in U_{ad}$, and denote z_u and z_v the mild solutions of (1) associated with u and v respectively. Then, for every $t \in [0, t_f]$,

$$\|z_u(t) - z_v(t)\| \leq \left\| \int_0^t T(t-s) f(u(s) - v(s), z_v(s)) \mathrm{d}s \right\| \times \exp\left(M e^{|\rho| t_f} \alpha t_f^{\frac{1}{p'}} \|u\|_{L^p(0,t_f,U)} \right) \tag{7}$$

where the constants $M \geq 1$ and $\rho \in \mathbb{R}$ are such that $\|T(t)\| \leq M e^{\rho t}$, for every nonnegative t.

Proof
Let $u, v \in U_{ad}$. Then equation (4) yields

$$\begin{aligned} z_u(t) - z_v(t) &= \int_0^t T(t-s)[f(u(s), z_u(s)) - f(v(s), z_v(s))]\mathrm{d}s \\ &= \int_0^t T(t-s)[f(u(s) - v(s), z_v(s)) \\ &\quad + f(u(s), z_u(s)) - f(u(s), z_v(s))]\mathrm{d}s \end{aligned}$$

By inequality (3), one gets

$$\begin{aligned} \|z_u(t) - z_v(t)\|_Z \leq & \left\| \int_0^t T(t-s)[f(u(s), z_u(s)) - f(v(s), z_v(s))]\mathrm{d}s \right\|_Z \\ & + \int_0^t \|T(t-s)\| (\alpha \|u(s)\|_U \|z_u(s) - z_v(s)\|_Z) \mathrm{d}s \end{aligned}$$

By the Gronwall lemma, one gets

$$\begin{aligned} \|z_u(t) - z_v(t)\|_Z \leq & \left\| \int_0^t T(t-s)[f(u(s), z_u(s)) - f(v(s), z_v(s))]\mathrm{d}s \right\|_Z \\ & \times e^{\int_0^t \|T(t-s)\| \alpha \|u(s)\|_U \mathrm{d}s} \end{aligned}$$

There exist $M \geq 1$ and $\rho \in \mathbb{R}$ are such that $\|T(t)\| \leq M e^{\rho t}$, for every $t \geq 0$. Hence

$$\int_0^t \|T(t-s)\| \alpha \|u(s)\|_U \mathrm{d}s \leq M e^{|\rho| t_f} \alpha t_f^{\frac{1}{p'}} \|u\|_{L^p(0,t_f,U)}$$

which yield (7). □

Lemma 2.2 ([13], p. 250)
Let X be a normed space, Z a separable Banach space, and let $L : X \to Z$ be a continuous linear operator. Denote L^* the adjoint operator of L. The operator L is compact if and only if, for every sequence (y_n) in Z^* such that $y_n \overset{*}{\rightharpoonup} 0$ for $\sigma(Z^*, Z)$, we have $L^* y_n \to 0$ in norm in X^*.

Lemma 2.3 ([14], p. 106)
Let $(T(t))_{t \geq 0}$ be a compact semigroup on a Banach space Z, and let $y_n \in L^2(0, t_f; Z)$ such that $y_n \rightharpoonup y$ weakly in $L^2(0, t_f; Z)$. Then

$$\lim_{n \to \infty} \sup_{0 \leq t \leq t_f} \left\| \int_0^t T(t-s)(y_n(s) - y(s))\mathrm{d}s \right\| = 0 \tag{8}$$

2.1. First class of systems

In addition to assumptions (A1)-(A3), we assume that:

(A1.1) U is the continuous dual space of a separable Banach space U_0.
(A1.2) There exists a Hilbert space H such that the embeddings

$$U \hookrightarrow H \hookrightarrow U_0$$

are dense and continuous, and $\langle v, u\rangle_H = \langle v, u\rangle_{U\times U_0}$, for every $(v, u) \in U \times H$.
(A1.3) For every $z \in Z$, the adjoint operator of $f(., z)$ satisfies

$$f^*(., z)x \in U_0, \quad \forall x \in Z^*$$

(A1.4) The semigroup $(T(t))_{t\geq 0}$ is compact, or the operator $f(., z)$ is compact, for every $z \in Z$.

Then we have the following sufficient conditions.

Proposition 2.4
Let the assumptions (A1.1)-(A1.4) hold. We assume that

- U_{ad} closed in $L^p(0, t_f; H)$ and bounded in $L^p(0, t_f; U)$
- There exist $r \geq 0$ and $s \geq 0$ such that $\mathscr{R}(u) = r\|u\|^s_{L^p(0,t_f;H)}$

Then there exists an optimal control u^*, solution of problem (6).

Proof
Denote $J^* = \inf\limits_{u\in U_{ad}} J(u)$, and let $(u_n)_n$ be a sequence in U_{ad} such that $J(u_n) \to J^*$. The set U_{ad} is bounded in $L^p(0, t_f; U)$, then, by the Alaoglu-Bourbaki theorem, there exists a subsequence, still denoted $(u_n)_n$, that is convergent for the weak $*$ topology of $L^p(0, t_f; U)$. Denote u^* its limit, then, for every $v \in L^{p'}(0, t_f; U_0)$, we have

$$\langle u_n, v\rangle_{L^p(0,t_f;U)\times L^{p'}(0,t_f;U_0)} \to \langle u^*, v\rangle_{L^p(0,t_f;U)\times L^{p'}(0,t_f;U_0)}$$

By assumption (A1.2), the embedding $L^{p'}(0, t_f; H) \hookrightarrow L^{p'}(0, t_f; U_0)$ is continuous. Then, for every $v \in L^{p'}(0, t_f; H)$, we get

$$\langle u_n, v\rangle_{L^p(0,t_f;H)\times L^{p'}(0,t_f;H)} \to \langle u^*, v\rangle_{L^p(0,t_f;H)\times L^{p'}(0,t_f;H)}$$

It follows that $u_n \rightharpoonup u^*$ weakly in $L^p(0, t_f; H)$. Moreover, U_{ad} is convex and strongly closed in $L^p(0, t_f; H)$. Thus U_{ad} is weakly closed in $L^p(0, t_f; H)$. Therefore, $u^* \in U_{ad}$.
Let z_n and z^* be the mild solutions of (1), associated with u_n and u^* respectively. Denote $\mu = \sup \|u_n\|_{L^p(0,t_f;U)}$. Then inequality (7) yields

$$\|z_n(t) - z^*(t)\|_Z \leq \|L_t(u_n - u^*)\|_Z \exp\left(Me^{|\rho|t_f}\alpha t_f^{\frac{1}{p'}}\mu\right) \tag{9}$$

where the linear operator $L_t : L^p(0, T; U) \to Z$ is defined as:

$$L_t u = \int_0^t T(t - s)f(u(s), z^*(s))\mathrm{d}s. \tag{10}$$

Assumption (A1.4) leads to two cases, which are discussed below.
Case 1: We assume that the operator $f(., z)$ is compact, for every $z \in Z$.
Let us prove that L_t is compact. Given that Z is reflexive, Z can be identified with its bidual Z^{**}. It follows that L_t is the adjoint operator of $L_0 : Z^* \to L^{p'}(0, T; U_0)$, given by:

$$L_0 y = \begin{cases} f_0(., z^*(s))T^*(t - s)y & \text{if } 0 \leq s < t, \\ 0 & \text{otherwise,} \end{cases}$$

for every $y \in Z^*$, where operator $f_0(., z^*(s)) : Z^* \to U_0$ is written as:

$$f_0(., z^*(s)) : y \mapsto f^*(., z^*(s))y$$

By assumption (A1.3), operator $f_0(., z^*(s))$ is well defined. Additionally, the adjoint operator of $f_0(., z^*(s))$ is $f(., z^*(s))$. By the compactness of $f(., z^*(s))$, it follows that $f_0(., z^*(s))$ is compact too. Let $(y_m)_m$ be a sequence in Z^* such that $y_m \rightharpoonup 0$ weakly in Z^*. To prove the compactness of L_0, it suffices to show that $L_0 y_m \to 0$ strongly in $L^{p'}(0, T; U_0)$, which is separable.
Let $s \in [0, t]$. By the compactness of $f_0(., z^*(s))$, operator $f_0(., z^*(s))T^*(t-s)$ is also compact. Hence

$$\lim_{m\to\infty} \|f_0(., z^*(s))T^*(t-s)y_m\|_{U_0} = 0, \qquad \forall s \in [0, t].$$

It follows that $\lim\limits_{m\to\infty} \|(L_0 y_m)(s)\|_{U_0} = 0$, for almost every $s \in [0, t_f]$. Using appropriate bounds, the dominated convergence theorem yields

$$\lim_{m\to\infty} \|(L_0 y_m)\|_{L^{p'}(0,T;U_0)} = 0.$$

Hence L_0 is compact, which proves the compactness of $L_0^* = L_t$. Now, by applying Lemma 2.2 to L_0 and L_t, the weak $*$ convergence $(u_n - u^*) \overset{*}{\rightharpoonup} 0$ leads to

$$L_t(u_n - u^*) \to 0 \ \text{ strongly in } Z.$$

Then inequality (9) yields, for every $t \in [0, t_f]$,

$$\lim_{n\to\infty} \|z_n(t) - z^*(t)\|_Z = 0.$$

By the continuity of $\mathscr{G}$ and $\mathscr{Q}$, one gets

$$\mathscr{G}(z^*(t_f)) = \lim_{n\to\infty} \mathscr{G}(z_n(t_f)), \quad \mathscr{Q}(z^*(t)) = \lim_{n\to\infty} \mathscr{Q}(z_n(t)).$$

Hence, the dominated convergence theorem yields

$$\int_0^{t_f} \mathscr{Q}(z^*(t))\mathrm{d}t = \lim_{n\to\infty} \int_0^{t_f} \mathscr{Q}(z_n(t))\mathrm{d}t.$$

Finally, $\mathscr{R}(u_n) = r\|u_n\|^s_{L^p(0,t_f;H)}$. By the lower semicontinuity of norms, the weak convergence $u_n \rightharpoonup u^*$ in $L^p(0, t_f; H)$ yields

$$\mathscr{R}(u^*) \leq \liminf_{n\to\infty} \mathscr{R}(u_n).$$

Therefore $J(u^*) \leq \liminf\limits_{n\to\infty} J(u_n)$. Hence u^* is a minimizer of J over U_{ad}.
Case 2: We assume that the semigroup $(T(t))_{t\geq 0}$ is compact.
Inequality (9) can be written as:

$$\|z_n(t) - z^*(t)\|_Z \leq \left\| \int_0^t T(t-s) f(u_n(s) - u^*(s), z^*(s)) \right\|_Z \times \exp\left(M e^{|\rho| t_f} \alpha t_f^{\frac{1}{p'}} \mu \right) \tag{11}$$

Let $f(u_n - u^*, z^*)$ denote the mapping

$$s \mapsto f(u_n(s) - u^*(s), z^*(s)).$$

Then $f(u_n - u^*, z^*) \in L^p(0, t_f; Z)$. Let us prove that $f(u_n - u^*, z^*) \rightharpoonup 0$ weakly in $L^p(0, t_f; Z)$. To this end, let $y \in L^{p'}(0, t_f; Z^*)$. By assumption (A1.3), $f^*(., z^*(s))y(s) \in U_0$, a.e. on $[0, t_f]$. Then

$$\langle f(u_n(s) - u^*(s), z^*(s)), y(s) \rangle_{Z\times Z^*} = \langle u_n(s) - u^*(s), f^*(., z^*(s))y(s) \rangle_{U\times U_0}$$

In addition, by inequality (3), we have

$$\begin{aligned}\|f(.,z^*(s))\|_{\mathscr{L}(Z,U)} &= \|f^*(.,z^*(s))\|_{\mathscr{L}(U^*,Z^*)}\\ &\leq \alpha\|z^*(s)\|+\beta.\end{aligned}$$

Hence

$$\|f^*(.,z^*(s))y(s)\|_{U_0} \leq (\alpha\|z^*(s)\|_Z+\beta)\,\|y(s)\|_{Z^*}$$

which implies $f^*(.,z^*)y \in L^{p'}(0,t_f;U_0)$. Thereby

$$\langle f(u_n-u^*,z^*),y\rangle_{L^p(0,t_f;Z)\times L^{p'}(0,t_f;Z^*)} = \langle u_n-u^*, f^*(.,z^*)y\rangle_{L^p(0,t_f;U)\times L^{p'}(0,t_f;U_0)}.$$

The weak $*$ convergence $u_n \overset{*}{\rightharpoonup} u^*$ yields

$$\lim_{n\to\infty}\langle u_n-u^*, f^*(.,z^*)y\rangle_{L^p(0,t_f;U)\times L^{p'}(0,t_f;U_0)} = 0.$$

Consequently

$$\lim_{n\to\infty} \quad \langle f(u_n-u^*,z^*),y\rangle_{L^p(0,t_f;Z)\times L^{p'}(0,t_f;Z^*)} = 0, \quad \forall y\in L^{p'}(0,t_f;Z^*),$$

which means that $f(u_n-u^*,z^*)\rightharpoonup 0$ weakly in $L^p(0,t_f;Z)$. Now, the semigroup $(T(t))_{t\geq 0}$ is compact, then Lemma 2.3 yields

$$\lim_{n\to\infty}\sup_{0\leq t\leq t_f}\left\|\int_0^t T(t-s)f(u_n(s)-u^*(s),z^*(s))\mathrm{d}s\right\| = 0.$$

By inequality 11, it follows that (z_n) converges uniformly to z^* in $C([0,t_f];Z)$. Finally, using similar arguments to Case 1, we obtain $J(u^*)\leq \liminf_{n\to\infty} J(u_n)$. Therefore, u^* minimizes J over U_{ad}. □

Proposition 2.5
Let assumptions (A1.1)-(A1.4) hold. We assume that

- U_{ad} is closed in $L^p(0,t_f;H)$.
- There exist $r>0$ and $s>0$ such that $\mathscr{R}(u) = r\|u\|^s_{L^p(0,t_f;H)}$.

Then the optimal control problem (6) has at least a solution.

Proof
Let $(u_n)_n$ be a sequence in U_{ad} such that $J(u_n)\to J^* = \inf_{u\in U_{ad}} J(u)$. Since $r,s>0$, and

$$\mathscr{R}(u_n) = r\|u_n\|^s_{L^p(0,t_f,H)} \leq J(u_n),$$

then sequence $(u_n)_n$ is bounded in $L^p(0,t_f,H)$. Thereby, there exists a subsequence still denoted $(u_n)_n$, such that $u_n\rightharpoonup u^*$ weakly in $L^p(0,t_f,H)$. In addition, U_{ad} is convex and closed in $L^p(0,t_f,H)$, hence weakly closed in $L^p(0,t_f,H)$, which yields $u^*\in U_{ad}$. Thereby, for every $v\in L^{p'}(0,t_f,H)$, we have

$$\begin{aligned}\langle u_n,v\rangle_{L^p(0,t_f,U)\times L^{p'}(0,t_f,U_0)} &= \langle u_n,v\rangle_{L^p(0,t_f,H)\times L^{p'}(0,t_f,H)}\\ &\to \langle u^*,v\rangle_{L^p(0,t_f,H)\times L^{p'}(0,t_f,H)}\end{aligned}$$

Using the density of the embedding

$$L^{p'}(0,t_f,H)\hookrightarrow L^{p'}(0,t_f,U_0),$$

and by the Moore-Osgood theorem for interchanging limits, one gets

$$\langle u_n,v\rangle_{L^p(0,t_f,U)\times L^{p'}(0,t_f,U_0)} \to \langle u^*,v\rangle_{L^p(0,t_f,U)\times L^{p'}(0,t_f,U_0)},$$

for every $v \in L^{p'}(0, t_f, U_0)$, which yields $u_n \overset{*}{\rightharpoonup} u^*$ weakly $*$ in $L^p(0, t_f, U_0)$.
Let z_n and z^* be the mild solutions of (1), associated with u_n and u^* respectively. By similar arguments to the proof of Proposition 2.4, we prove that, for every $t \in [0, t_f]$,

$$\lim_{n\to\infty} \|z_n(t) - z^*(t)\|_Z = 0.$$

Consequently

$$\mathscr{G}(z^*(t_f)) = \lim_{n\to\infty} \mathscr{G}(z_n(t_f)),$$

$$\int_0^{t_f} \mathscr{Q}(z^*(t))\mathrm{d}t = \lim_{n\to\infty} \int_0^{t_f} \mathscr{Q}(z_n(t))\mathrm{d}t,$$

$$\mathscr{R}(u^*) \leq \liminf_{n\to\infty} \mathscr{R}(u_n),$$

which yields $J(u^*) \leq \liminf_{n\to\infty} J(u_n)$. □

2.2. *Second class of systems*

Now, in addition to (A1)-(A3), we assume that:

(A2.1) U is the continuous dual space of a Banach space U_0, and the embedding $U_0 \hookrightarrow U$ is continuous.
(A2.2) The semigroup $(T(t))_{t\geq 0}$ is compact, or the operator $f(., z)$ is compact, for every $z \in Z$.

Then we have the following result.

Proposition 2.6
We assume that

- U_{ad} is a closed and bounded subset of $L^p(0, t_f; U_0)$.
- The mapping $u \mapsto \mathscr{R}(u)$ is convex and continuous on $L^p(0, t_f; U_0)$.

Then there exists an optimal control u^*, solution of (6).

Proof
Let $(u_n)_n$ be a a sequence in U_{ad} such that $J(u_n) \to J^* = \inf_{u\in U_{ad}} J(u)$. Considering the canonical embedding $U_0 \hookrightarrow U_0^{**} = U^*$, U_{ad} is a bounded subset of $L^p(0, t_f, U^*)$. It follows that $(u_n)_n$ is bounded in $L^p(0, t_f, U^*)$. Thereby, there exists a subsequence, denoted (u_n) as well, such that $u_n \overset{*}{\rightharpoonup} u^*$ weakly $*$ in $L^p(0, t_f; U^*)$. Namely

$$\langle u_n, v\rangle_{L^p(0,t_f;U^*)\times L^{p'}(0,t_f;U)} \to \langle u_n, v\rangle_{L^p(0,t_f;U^*)\times L^{p'}(0,t_f;U)},$$

for every $v \in L^{p'}(0, t_f; U)$. It follows that

$$\langle u_n, v\rangle_{L^p(0,t_f;U_0)\times L^{p'}(0,t_f;U)} \to \langle u_n, v\rangle_{L^p(0,t_f;U_0)\times L^{p'}(0,t_f;U)}, \quad \forall v \in L^{p'}(0, t_f; U),$$

which means that $u_n \rightharpoonup u^*$ weakly in $L^p(0, t_f; U_0)$. Now, U_{ad} is convex and closed in $L^p(0, t_f; U_0)$, then U_{ad} is weakly closed in $L^p(0, t_f; U_0)$. Therefore, $u^* \in U_{ad}$.
Let z_n and z^* be the mild solutions of (1), associated with u_n and u^* respectively, and denote $\mu = \sup \|u_n\|_{L^p(0,t_f;U)}$. Then

$$\|z_n(t) - z^*(t)\|_Z \leq \left\| \int_0^t T(t-s) f(u_n(s) - u^*(s), z^*(s)) \right\|_Z \times \exp\left(M e^{|\rho| t_f} \alpha t_f^{\frac{1}{p'}} \mu \right). \tag{11}$$

Two cases arise from assumption (A2.2), and are discussed hereafter.
Case 1: We assume that $f(., z)$ is compact.

We consider the following linear operator

$$\begin{array}{rcl} \Lambda_t : L^p(0,t_f;U_0) & \to & Z \\ u & \mapsto & \displaystyle\int_0^t T(t-s)f(u(s),z^*(s)) \end{array} \qquad (12)$$

Λ_t is well defined, since the embedding $U_0 \hookrightarrow U$ is continuous, and its adjoint $\Lambda_t^* : Z^* \to L^{p'}(0,t_f;U)$ is given by:

$$(\Lambda_t^* y)(s) = \begin{cases} \varphi_s^* T^*(t-s)y & \text{if } 0 \le s \le t, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varphi_s^* : Z^* \to U$ denotes the adjoint operator of

$$\begin{array}{rcl} \varphi_s : U_0 & \to & Z \\ v & \mapsto & f(v, z^*(s)) \end{array}$$

It is clear that φ_s is the restriction of $f(., z^*(s))$ on U_0. By the continuity of the embedding $U_0 \hookrightarrow U$, and the compactness of $f(., z^*(s))$, it follows that φ_s is compact. Subsequently, operator $\varphi_s^* T^*(t-s)$ is compact, for every $s \in [0,t]$.

Now, let $(y_m)_m$ be a sequence in Z^* such that $y_m \rightharpoonup 0$ weakly. The compactness of $\varphi_s^* T^*(t-s)$ yields

$$\lim_{m\to\infty} \|\varphi_s^* T^*(t-s) y_m\|_U = 0, \qquad \forall s \in [0,t].$$

Then, $\lim\limits_{m\to\infty} \|(\Lambda_t^* y_m)(s)\|_U = 0$ a.e. on $[0,t_f]$.

Using appropriate bounds, and applying the dominated convergence theorem, one gets

$$\lim_{m\to\infty} \|(\Lambda_t^* y_m)\|_{L^{p'}(0,t_f;U)} = 0.$$

Applying Lemma 2.2 to operator Λ_t, it follows that Λ_t is compact.

Thereby, the weak convergence $u_n \rightharpoonup u^*$ in $L^p(0,t_f;U_0)$ leads to $\lim\limits_{n\to\infty} \|\Lambda_t(u_n - u^*)\|_Z = 0$. Then, from inequality (11), it results that:

$$\lim_{n\to\infty} \|z_n(t) - z^*(t)\|_Z = 0, \qquad \forall t \in [0,t_f].$$

Therefore

$$\mathscr{G}(z^*(t_f)) = \lim_{n\to\infty} \mathscr{G}(z_n(t_f)).$$

$$\mathscr{Q}(z^*(t)) = \lim_{n\to\infty} \mathscr{Q}(z_n(t)), \qquad \text{a.e. on } [0,t_f].$$

By the dominated convergence theorem, we get

$$\int_0^{t_f} \mathscr{Q}(z^*(t))\mathrm{d}t = \lim_{n\to\infty} \int_0^{t_f} \mathscr{Q}(z_n(t))\mathrm{d}t.$$

Finally, $u \mapsto \mathscr{R}(u)$ is convex and continuous on $L^p(0,t_f;U_0)$. Then $\mathscr{R}$ is lower semicontinuous for the weak topology of $L^p(0,t_f;U_0)$. Then the weak convergence $u_n \rightharpoonup u^*$ in $L^p(0,t_f;U_0)$ leads to $\mathscr{R}(u^*) \le \liminf\limits_{n\to\infty} \mathscr{R}(u_n)$. Consequently, $J(u^*) \le \liminf\limits_{n\to\infty} J(u_n)$, hence u^* is an optimal control.

Case 2: We assume that $(T(t))_{t\ge 0}$ is compact.

Let us define the following linear operator

$$\begin{array}{rcl} f_1(., z^*) : L^p(0,t_f;U_0) & \to & L^p(0,t_f;Z) \\ u & \mapsto & f(u(.), z^*(.)) \end{array} \qquad (13)$$

Since the embedding $U_0 \hookrightarrow U$ is continuous, then there exists a constant $\gamma \geq 0$ such that $\|v\|_U \leq \gamma \|v\|_{U_0}$, for every $v \in U_0$. If $u(s) \in U_0$, then inequality (3) implies

$$\|f(u(s), z^*(s))\|_Z \leq \gamma(\alpha \|z^*(s)\|_Z + \beta)\|u(s)\|_{U_0}$$

which yields

$$\|f(u(.), z^*(.))\|_{L^p(0,t_f;Z)} \leq \gamma(\alpha \|z^*\|_{C([0,t_f];Z)} + \beta)\|u\|_{L^p(0,t_f;U_0)},$$

for every $u \in L^p(0, t_f; U_0)$. It follows that operator $f_1(., z^*)$, given by (13), is continuous. Hence $f_1(., z^*)$ is continuous for the weak topology. Thereby, the weak convergence $u_n - u^* \rightharpoonup 0$ in $L^p(0, t_f; U_0)$ yields

$$f_1(., z^*)(u_n - u^*) = f(u_n - u^*, z^*) \rightharpoonup 0$$

weakly in $L^p(0, t_f; Z)$. By Lemma 2.3 and the compacity of $(T(t))_{t\geq 0}$, we obtain

$$\lim_{n\to\infty} \sup_{0\leq t\leq t_f} \left\| \int_0^t T(t-s) f(u_n(s) - u^*(s), z^*(s)) \mathrm{d}s \right\| = 0.$$

Thereby, by inequality (11), $(z_n)_n$ converges uniformly to z^* in $C([0, t_f]; Z)$.
By similar arguments to Case 1, one gets $J(u^*) \leq \liminf_{n\to\infty} J(u_n)$, which proves that u^* is a minimizer of functional J on U_{ad}. □

3. Examples

In this section, the previous results will be applied to two examples of semilinear partial differential equations, a heat equation, and a wave equation.

Example 1

On $I = [0, 1]$, we consider the below heat equation, with Neumann boundary conditions.

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \Delta z(x,t) + u(x,t)B(z)(x,t), & \text{in } I \times [0, t_f] \\ \dfrac{\partial z}{\partial x}(0,t) = \dfrac{\partial z}{\partial x}(1,t) = 0, & \text{on } [0, t_f] \\ z(x,0) = z_0(x), & \text{on } I \end{cases} \tag{14}$$

where $z_0 \in L^2(I)$ and $B : L^2(I) \to L^2(I)$ is a Lipschitz operator. The control function u is such that $u(., t) \in L^\infty(I)$.
We set $Z = L^2(I)$, $z(t) = z(., t)$, $u(t) = u(., t)$, and

$$A = \Delta \text{ with } \mathcal{D}(A) = \left\{ z \in H^2(I) : \frac{\partial z}{\partial x}(0) = \frac{\partial z}{\partial x}(1) = 0 \right\}.$$

The control space is $U = L^\infty(I)$, and $u \in L^2(0, t_f; L^\infty(I))$. We define the mapping $f : U \times Z \to Z$ as:

$$f(u, z)(x) = u(x)(B(z))(x), \quad \text{a.e. on } I.$$

Then assumptions (A1)-(A3) are satisfied. In addition, A is the generator of the below compact semigroup

$$T(t)(z) = \sum_{n=0}^{\infty} e^{-\pi^2 n^2} \langle z, e_n \rangle_{L^2(I)} e_n$$

where $e_0(x) = 1$, $e_n(x) = \sqrt{2}\cos(n\pi x)$, if $n \geq 1$ (See Chapter VI, Example 8.9 in [12] for further details).
We consider the below functional:

$$J(u) = a_0 \int_I z(x, t_f)^2 \mathrm{d}x + a_1 \int_0^{t_f} \int_I z(x,t)^2 \mathrm{d}x\mathrm{d}t \tag{15}$$

where $a_0, a_1 > 0$. Let the set of admissible controls be

$$U_{ad} = \{u \in L^2(0, t_f; L^\infty(I)) \ : \ m \leq u(x,t) \leq M \text{ a.e. on } I \times [0, t_f]\} \tag{16}$$

such that $m < M$. By the boundedness of I, the embeddings

$$L^\infty(I) \hookrightarrow L^2(I) \hookrightarrow L^1(I)$$

are continuous. The density of the above embeddings follows from the density of $C_c(I)$ in $L^2(I)$ and $L^1(I)$. Then assumptions (A1.1) and (A1.2) hold, for $U = L^\infty(I)$, $H = L^2(I)$, and $U_0 = L^1(I)$. In addition, the adjoint of $f(.,z)$ is given by:

$$(f^*(.,z)y)(x) = y(x)(B(z))(x), \quad \forall y \in L^2(I).$$

Then $f^*(.,z)y \in L^1(I) = U_0$, for every $z \in L^2(I)$. Hence assumption (A1.3) holds. Finally, (A1.4) is satisfied since the semigroup $(T(t))_{t\geq 0}$ is compact.
Now, The set U_{ad}, given by (16), is closed in $L^2(0, t_f; L^2(I))$ and bounded in $L^2(0, t_f; L^\infty(I))$. Considering functional (15), we have $\mathscr{R}(u) = 0$. Therefore, by virtue of Proposition 2.4, there exists an optimal control u^* that minimizes (15) over U_{ad}.

Example 2

On $I = [0, 1]$, we consider the following wave equation.

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) = \Delta y(x,t) + \displaystyle\int_I y(x,t)\mathrm{d}\mu(t), & \text{in } I \times [0, t_f] \\ y(0,t) = y(1,t) = 0, & \text{on } [0, t_f] \\ y(x,0) = y_0(x), \ \dfrac{\partial y}{\partial t}(x,0) = y_1(x) & \text{on } I \end{cases} \tag{17}$$

where $y_0 \in H_0^1(I)$, $y_1 \in L^2(I)$, and $\mu(t) \in \mathcal{M}(I)$, where $\mathcal{M}(I)$ is the space of real-valued Radon measures over I. Hence the control space is set to be $U = \mathcal{M}(I)$. It follows that U is the dual space of $U_0 = C(I)$, the space of continuous functions on I.
We set $Z = H_0^1(I) \times L^2(I)$, and

$$z(t) = \begin{pmatrix} y(.,t) \\ \dfrac{\partial y}{\partial t}(.,t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta & 0 \end{pmatrix},$$

with $\mathcal{D}(A) = [H^2(I) \cap H_0^1(I)] \times H_0^1(I)$. Besides, we define the mapping $f : \mathcal{M}(I) \times Z \to Z$ as:

$$f(\mu, z) = \begin{pmatrix} 0 \\ \displaystyle\int_I z_1(x)\mathrm{d}\mu \end{pmatrix}.$$

Then equation (17) has the form of system (1). Z is endowed with the norm $\|z\|_Z^2 = \|z_1\|_{H_0^1(I)}^2 + \|z_2\|_{L^2(I)}^2$, for every $z = (z_1, z_2) \in Z$. In addition, we have

$$\|f(\mu,z)\|_Z \leq \left| \int_I z_1(x)\mathrm{d}\mu \right| \leq \|\mu\|_{\mathcal{M}(I)} \|z_1\|_{C(I)}$$

Since $z_1(0) = 0$ then, applying the Cauchy-Schwarz inequality yields, for every $x \in I$,

$$|z_1(x)| = \left| \int_0^x z_1'(\xi) \mathrm{d}\xi \right| \leq \left(\int_0^x z_1'(\xi)^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \leq \|z_1\|_{H_0^1(I)}.$$

Then

$$\|z_1\|_{C(I)} = \sup_{x \in I} |z_1(x)| \leq \|z_1\|_{H_0^1(I)}.$$

It follows that

$$\begin{aligned} \|f(\mu, z)\|_Z &\leq \|\mu\|_{\mathcal{M}(I)} \|z_1\|_{H_0^1(I)} \\ &\leq \|\mu\|_{\mathcal{M}(I)} \|z\|_Z. \end{aligned}$$

Therefore, the mapping f satisfies assumption (A3).
Operator A is the infinitesimal generator of a continuous semigroup $(T(t))_{t \geq 0}$ (For further details, see Example 2.41 in [9]).
We consider the following functional:

$$J(u) = a_0 \int_0^{t_f} \int_I y(x,t)^2 \mathrm{d}x\mathrm{d}t + a_1 \int_0^{t_f} \int_I \frac{\partial y}{\partial t}(x,t)^2 \mathrm{d}x\mathrm{d}t + a_2 \|u\|_{\mathcal{M}(I)} \tag{18}$$

where $a_0, a_1 > 0$ and $a_2 \geq 0$. The set of admissible controls is defined as follows.

$$U_{ad} = \{u \in L^2(0, t_f; C(I)) \ : \ |u(x,t)| \leq M \text{ on } I \times [0, t_f]\} \tag{19}$$

Every $u \in U_0 = C(I)$ can be identified with a measure, still denoted u, in $U = \mathcal{M}(I)$. Moreover, we have

$$\|u\|_{\mathcal{M}(I)} = \int_I |u(x)| \mathrm{d}x \leq \sup_{x \in I} |u(x)| = \|u\|_{C(I)}$$

It follows that the embedding $U_0 \hookrightarrow U$ is continuous. Hence assumption (A2.1) holds. Additionally, the operator

$$f(., z) : \mu \mapsto \begin{pmatrix} 0 \\ \displaystyle\int_I z_1(x) \mathrm{d}\mu \end{pmatrix}$$

is compact, since $\mu \mapsto \displaystyle\int_I z_1(x) \mathrm{d}\mu$ is a continuous linear form. It follows that assumption (A2.2) holds too.
Finally, U_{ad} is a closed and bounded subset of $L^2(0, t_f; U_0)$, and $\mathscr{R} : u \mapsto a_2 \|u\|_{\mathcal{M}(I)}$ is convex and continuous on $L^2(0, t_f; U_0)$. Therefore, by Proposition 2.6, the optimal control problem (6) has a solution u^*, that minimizes functional (18) over the set (19).

4. Conclusion

In this paper, we have investigated an optimal control problem, governed by an important class of semilinear systems. The problem consists in minimizing an abstract functional, over a given set of admissible controls in a nonreflexive Banach space. By introducing specific assumptions, and discussing two classes of systems, the question of existence of optimal controls has been addressed. Then two examples of partial differential equations have been provided to demonstrate the applicability of the obtained results. This work may be extended to larger classes of nonlinear systems. This is under consideration for future research papers.

REFERENCES

1. A. Addou, and A. Benbrik, *Existence and uniqueness of optimal control for a distributed paramete bilinear systems*, J. Dyn. Control Syst., vol. 8, no. 2, pp. 141–152, 2002.
2. N. U. Ahmed, and X. Xiang, *Optimal control of infinite-dimensional uncertain systems*, J. Optimiz. Theory Appli., vol. 80, no. 2, pp. 261–272, 1994.
3. N. U. Ahmed, and X. Xiang, *Necessary conditions of optimality for infinite dimensional uncertain systems*, Math. Probl. Eng., vol. 1, no. 3, pp. 179–191, 1995.
4. M.S. Aronna, and F. Tröltzsch, *First and second order optimality conditions for the control of Fokker-Planck equations*, ESAIM: COCV, vol. 27, 2021.
5. V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems*, Mathematics in Science and Engineering, Vol. 190, Academic Press, 1993.
6. J. F. Bonnans, and E. Casas, *Un principe de Pontryagine pour le contrôle des systèmes semilinéaires elliptiques*, J. Differ. Equat., vol. 90, pp. 288–303, 1991.
7. M. E. Bradley, and S. Lenhart, *Bilinear optimal control of a Kirchhoff plate*, Syst. Control Lett., vol. 22, pp. 27–38, 1994.
8. P. Cannarsa, and H. Frankowska, *Value function and optimality condition for semilinear control problems. II: Parabolic case*, Appl. Math. Optim., vol. 26, pp. 139–169, 1992.
9. R.F. Curtain, and A.J. Pritchard, *Infinite dimensional linear systems theory*, Lect. Notes Control Inf. Sci., Springer, Berlin Heidelberg New York, 1978.
10. D. Deng, and W. Wei, *Existence and stability analysis for nonlinear optimal control problems with 1-mean equicontinuous controls*, J. Ind. Manam. Optim., vol. 11, no. 4, pp. 1409–1422, 2015.
11. N. El Boukhari, and E.H. Zerrik, *Optimal control of distributed semilinear systems with essentially bounded measurable control functions*, Int. J. Dynam. Control, vol. 11, pp. 2809–2819, 2023.
12. J. C. Engel, and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, Springer, New York, 2000.
13. L. V. Kantorovich, and G. P. Akilov, *Functional Analysis*, 2nd edition, Pergamon Press: Oxford, 1982. Translated by H. L. Silcock.
14. X. Li, and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems, Systems & Control : Foundations & Applications*, Birkhäuser Basel, 1995.
15. L. Liang, *Bilinear optimal control for a wave equation*, Math. Models Methods Appl. Sci., vol. 9, no. 1, pp. 45–68, 1999.
16. M. Ouzahra, *Finite-time control for the bilinear heat equation*, Eur. J. Control, vol. 57, pp. 284–293, 2021.
17. J. P. Raymond, and H. Zidani, *Hamiltonian Pontryagin's Principles for Control Problems Governed by Semilinear Parabolic Equations*, Appl. Math. Optim., vol. 39, pp. 143–177, 1999.
18. F. Tröltzsch, *Optimal Control of Partial Differential Equations*, Graduate Studies in Mathematics 112, AMS, 2010.
19. E. Zerrik, and N. El Boukhari, *Constrained optimal control for a class of semilinear infinite dimensional systems*, J. Dyn. Control Syst., vol. 24, no. 1, pp. 65–81, 2018.
20. E. Zerrik, and N. El Boukhari, *Regional optimal control for a class of semilinear systems with distributed controls*, Int. J. Control, vol. 92, no. 4, pp. 896–907, 2019.