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A Novel Hypothesis Testing for EBUC_{mgf} Utilizing Laplace Transform Techniques with Practical Applications

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Abstract In this research, we introduce a novel and comprehensive class of lifetime distributions, termed $\textbf{EBUC}_{\textbf{mgf}}$, specifically designed to effectively model intricate and nuanced aging behaviors observed in reliability systems. Unlike the classic exponential distribution, which assumes a constant failure rate over time and thereby cannot account for more complex failure patterns, the proposed EBUC_{mgf} class offers a flexible framework to address diverse and realistic aging phenomena in practical applications. Our methodological approach is grounded in advanced mathematical principles, particularly leveraging the Laplace transform order, which enables us to rigorously develop a new statistical testing procedure. This innovative test serves to determine whether empirical data align with the theoretical properties of the EBUC_{mgf} distributional class. Moreover, we conduct an in-depth analysis of the statistical test's behavior and efficacy, employing Pitman's concept of asymptotic efficiency. Recognizing the prevalence of incomplete information in real-life reliability studies, we further extend our proposed methodology to accommodate datasets subject to right-censoring. To comprehensively evaluate the reliability and effectiveness of our proposed test, extensive simulation studies are performed under a variety of statistical conditions, capturing a wide range of real-world possibilities. Furthermore, actual case studies involving real datasets are examined, demonstrating the practical utility, versatility, and superiority of our method in contemporary reliability analysis settings. Through these multifaceted investigations, our research offers significant contributions to statistical methodology in reliability engineering and lifetime data analysis.

Keywords Reliability analysis, hypothesis testing, Laplace transform, asymptotic efficiency, statistical simulation.

AMS 2010 subject classifications 60K10, 62E10

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1. Introduction

Reliability theory is a dynamic field that addresses the analysis and prediction of system lifetimes, drawing interest from both engineers and statisticians. Over recent decades, researchers have sought to better understand the diverse patterns in which technical systems and components age, aiming to improve maintenance planning and risk assessment. While the exponential distribution has often served as a standard model for lifetimes due to its mathematical simplicity and memoryless property, it does not always capture the complex aging behaviors observed in practice.

To address this gap, alternative classes of life distributions have been proposed, each attempting to represent different aspects of aging or reliability improvement. These include models accounting for increasing or decreasing failure rates, and notions such as "new better than used" in various statistical senses. The development and study of these classes support more accurate modeling of real-world systems, where performance may gradually deteriorate or, in some cases, improve over time.

From a statistical standpoint, distinguishing whether observed data belong to a specific life distribution class is of

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paramount importance. It enables practitioners to select appropriate reliability strategies and to justify the use of specific probabilistic models. Techniques based on Laplace transform order, for example, provide powerful tools for such discrimination, as they leverage integral properties of lifetime distributions.
This study introduces a newly defined class of life distributions designed to capture nuanced aging properties that the classic exponential model does not reflect. We develop a novel hypothesis testing approach, grounded in the theory of Laplace transforms, to assess exponentiality versus membership in this new class. The test is further evaluated through both theoretical and simulation-based efficiency measures, and its applicability is confirmed using real and right-censored datasets, which are common in applied reliability studies.

Aging is modeled using a non-negative random variable X, which has a cumulative distribution function $F(x)$ and a survival function $\bar{F}(x) = 1 - F(x)$. For practical reasons, X is usually considered to have a continuous probability distribution with an associated density function $f(x)$, though this continuity is not always a necessary assumption.
The exponential better than used (EBU) class was first proposed by Elbatal [12], who explored its fundamental properties such as closure under common reliability operations, associated moment inequalities, and its inheritance under shock models. Since then, this class has attracted extensive attention among statisticians and reliability experts, leading to studies from various analytical perspectives. Several works addressing challenges and characteristics of EBU life distributions can be found in the literature, including those by Hendi et al. [20], Attia et al. [8], AbdulMoniem [1], Gadallah [15], Hendi and Al-Ghufily [21] and Mahmoud and Rady [24].

The Laplace transform is a highly valuable tool in applied mathematics and engineering, with significant applications in probability and statistics. It simplifies mathematical problems by converting differential equations into more manageable algebraic equations. In the field of probability, the Laplace transform is particularly useful for studying random variables, calculating moments, and analyzing stochastic processes such as Markov chains and renewal theory, see Feller [14]. For two non-negative random variables X and Y with distribution functions F and G, (survival functions $\bar{F}$, $\bar{G}$) respectively, the random variable X is smaller than Y in Laplace transform order (denoted by $X \leq_{LT} Y$) if, and only, if

$$\int_0^\infty e^{-sx}\bar{F}(x)dx \leq \int_0^\infty e^{-sy}\bar{G}(y)dy, \quad \text{for all } s \geq 0 \tag{1}$$

In recent years, significant advancements have been made in developing statistical tests for assessing exponentiality against various alternative classes of lifetime distributions, particularly using Laplace transform based techniques. ElArishy et al. [10] proposed a test for exponentiality versus the RNBRUE class, while El-Atfy et al. [11] developed a hypothesis test against the RNBU_{mgf} class. Gadall [16] introduced a testing procedure targeting the EBU_{mgf} class of life distributions, and Gadall et al. [17] extended this work by focusing on the NRBUL class. Al-Gashgari et al. [5] contributed a nonparametric test against the NBUCA class, and Atallah et al. [7] designed a novel method for testing exponentiality versus the NBU_{mgf} class.

Further contributions include Mahmoud et al. [23], who investigated the HNBUE class and the new better than renewal used class within the Laplace transform order framework. Abu-Youssef et al. [3] developed a nonparametric test for the UBAC(2) life distribution class, and Abu-Youssef and El-Toony [4] introduced a new class of life distributions grounded in Laplace transform techniques. More recently, Bakr et al. [9] presented advancements in Laplace transform methods enabling nonparametric hypothesis testing on real-world data through comprehensive statistical analysis. Gadallah et al. [18] also contributed by modeling various survival distributions based on a nonparametric Laplace transform approach with practical applications.
These studies collectively demonstrate the growing versatility and efficacy of Laplace transform-based methodologies in hypothesis testing for lifetime data, providing robust tools to discriminate exponentiality from more complex aging and reliability classes. This extensive body of work establishes a solid foundation and motivates the present study's development of novel testing procedures within this evolving framework.
In this work, we use the class EBUC_{mgf} to present a test statistic based on Laplace transform for testing $H_0 : F$ is

exponential with mean μ against the alternative $H_1 : F$ is EBUC_{mgf} but not exponential.

The remainder of this paper is organized as follows: Section 2 derives and thoroughly discusses the Laplace transform associated with the newly introduced EBUC_{mgf} class of lifetime distributions, highlighting its mathematical properties. In Section 3, we propose a novel statistical testing procedure for distinguishing the exponential distribution from the EBUC_{mgf} class, assuming complete and uncensored data, using the concept of Laplace transform order as a basis for the test. Section 4 is dedicated to evaluating the asymptotic efficiency of the proposed test from Pitman's perspective, comparing performance across several commonly used reliability distributions. Section 5 estimates the critical values of the test statistic through Monte Carlo simulations, analyzes power characteristics, and demonstrates practical application with complete datasets. Section 6 extends the methodology to handle datasets subject to right-censoring, providing corresponding critical value tables and simulated power results to support practitioners. Finally, Section 7 presents a variety of real-world examples to show case the practical effectiveness and applicability of the proposed testing framework in reliability analysis involving both complete and censored lifetime data.

Definition 1: Hendi and AL-Ghufily [21] A life distribution F is said to be **exponential better than used in convex (EBUC)** if

$$\int_{x+t}^{\infty} \overline{F}(u)\, du \leq \mu \overline{F}(t) e^{-x/\mu}, \tag{2}$$

where $\overline{F}(t) = 1 - F(t)$ is the survival function.

Definition 2: Let X and Y be two non-negative random variables with survival functions $\overline{F}$ and $\overline{G}$, respectively. We say that X is less than Y in the **moment-generating function order**, denoted by $X \leq_{\text{mgf}} Y$, if and only if:

$$\int_{0}^{\infty} e^{\lambda x} \overline{F}(x)\, dx \leq \int_{0}^{\infty} e^{\lambda y} \overline{G}(y)\, dy, \quad \forall \lambda > 0. \tag{3}$$

Definition 3: Using Definitions 1 and 2, then X is **exponentially better than used in convex order with respect to the moment-generating function**, denoted by $X \in \text{EBUC}_{\text{mgf}}$, if the following condition is satisfied:

$$\int_{0}^{\infty} \int_{x+t}^{\infty} e^{\lambda x} \overline{F}(u)\, du\, dx \leq \frac{\mu^2}{1 - \mu\lambda} \overline{F}(t). \tag{4}$$

2. Laplace Transform

The upcoming theorems establish the form of the Laplace transform associated with the class of EBUC_{mgf} life distributions.

Theorem 1: Suppose that F is EBUC_{mgf}, and let:

$$\varphi(\lambda) = \mathbb{E}(e^{\lambda X}) = \int_{0}^{\infty} e^{\lambda x}\, dF(x), \quad \varphi(\alpha) = \mathbb{E}(e^{-\alpha X}) = \int_{0}^{\infty} e^{-\alpha x}\, dF(x).$$

Then,

$$\frac{1}{\alpha + \lambda} \left(\frac{1}{\lambda^2} \varphi(\lambda) - \frac{1}{\alpha^2} \varphi(\alpha) - \frac{\alpha + \lambda}{\alpha\lambda} \mu + \frac{\lambda^2 - \alpha^2}{\alpha^2 \lambda^2} \right) \leq \frac{\mu^2}{(1 - \mu\lambda)\alpha} (1 - \varphi(\alpha)). \tag{5}$$

Proof: Let $F(x)$ be EBUC$_{\text{mgf}}$, and:

$$v(x+t)=\int_{x+t}^{\infty}\overline{F}(u)\,du.$$

Then, Eq. (4) becomes:

$$\int_{0}^{\infty} e^{\lambda x}v(x+t)\,dx \leq \frac{\mu^2}{1-\mu\lambda}\overline{F}(t). \tag{6}$$

Making use of Eq. (6), we have:

$$\int_{0}^{\infty}\int_{0}^{\infty} e^{-\alpha t}e^{\lambda x}v(x+t)\,dx\,dt \leq \frac{\mu^2}{1-\mu\lambda}\int_{0}^{\infty} e^{-\alpha t}\overline{F}(t)\,dt. \tag{7}$$

The right-hand side (R.H.S.) of Eq. (7) can be expressed as:

$$\begin{aligned}\frac{\mu^2}{1-\mu\lambda}\int_{0}^{\infty}\overline{F}(t)e^{-\alpha t}\,dt &= \frac{\mu^2}{1-\mu\lambda}\mathbb{E}\left(\int_{0}^{\infty} e^{-\alpha t}I(T>t)\,dt\right)\\ &= \frac{\mu^2}{1-\mu\lambda}\frac{1}{\alpha}\left(1-\mathbb{E}(e^{-\alpha X})\right)\\ &= \frac{\mu^2}{1-\mu\lambda}\frac{1}{\alpha}(1-\varphi(\alpha)).\end{aligned} \tag{8}$$

Similarly, the left-hand side (L.H.S.) of Eq. (7) can be represented as:

$$\begin{aligned}\int_{0}^{\infty}\int_{0}^{\infty} e^{-\alpha t}e^{\lambda x}v(x+t)\,dx\,dt &= \int_{0}^{\infty} e^{-t(\lambda+\alpha)}\int_{t}^{\infty} e^{\lambda y}v(y)\,dy\,dt\\ &= \frac{1}{\alpha+\lambda}\left(\int_{0}^{\infty} e^{\lambda t}v(t)\,dt - \int_{0}^{\infty} e^{-\alpha t}v(t)\,dt\right)\\ &= \frac{1}{\alpha+\lambda}(I_1-I_2),\end{aligned} \tag{9}$$

where:

$$\begin{aligned}I_1 &= \int_{0}^{\infty} e^{\lambda t}v(t)\,dt\\ &= \int_{0}^{\infty} e^{\lambda t}\int_{t}^{\infty}\overline{F}(u)\,du\,dt\\ &= \frac{1}{\lambda^2}\varphi(\lambda)-\frac{1}{\lambda}\mu-\frac{1}{\lambda^2},\end{aligned} \tag{10}$$

$$\begin{aligned}I_2 &= \int_{0}^{\infty} e^{-\alpha t}v(t)\,dt\\ &= \int_{0}^{\infty} e^{-\alpha t}\int_{t}^{\infty}\overline{F}(u)\,du\,dt\\ &= \frac{1}{\alpha^2}\varphi(\alpha)+\frac{1}{\alpha}\mu-\frac{1}{\alpha^2}.\end{aligned} \tag{11}$$

Thus, Eq. (9) can be written as:

$$\int_{0}^{\infty}\int_{0}^{\infty} e^{-\alpha t}e^{\lambda x}v(x+t)\,dx\,dt = \frac{1}{\alpha+\lambda}\left(\frac{1}{\lambda^2}\varphi(\lambda)-\frac{1}{\alpha^2}\varphi(\alpha)-\frac{\alpha+\lambda}{\lambda\alpha}\mu+\frac{\lambda^2-\alpha^2}{\alpha^2\lambda^2}\right). \tag{12}$$

From Eq. (8) and Eq. (12), the theorem is proved.

2.1. Developing a Statistical Test for Exponentiality versus EBUC$_{mgf}$ Class with Complete Data

We consider the problem of testing the null hypothesis:

$$H_0 : F \text{ follows an exponential distribution with mean } \mu,$$

versus the alternative hypothesis:

$$H_1 : F \text{ belongs to EBUC}_{\text{mgf}} \text{ class and is not exponential.}$$

This test is formulated by applying the inequality specified in Eq. (5), utilizing $\delta(\lambda, \alpha)$ as described below:

$$\delta(\lambda,\alpha) = \left(\frac{1}{\alpha^2(\alpha+\lambda)} - \frac{\lambda}{\alpha^2(\alpha+\lambda)}\mu - \frac{1}{\alpha}\mu^2\right)\varphi(\alpha) + \left(\frac{1}{\lambda(\alpha+\lambda)}\mu - \frac{1}{\lambda^2(\alpha+\lambda)}\right)\varphi(\lambda) + \frac{1}{\alpha^2}\mu + \frac{\alpha-\lambda}{\alpha^2\lambda^2}.$$

Under $H_0 : \delta(\lambda, \alpha) = 0$, and it is positive under H_1.

Let $X_1, X_2, \ldots, X_n$ represent a random sample drawn from a population characterized by the distribution function $F(x)$. We denote by $\overline{F}_n(x)$ the empirical survival function estimated from the sample, and by $\delta_n(\lambda, \alpha)$ the empirical estimator of $\delta(\lambda, \alpha)$, where:

$$\overline{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i > x), \quad dF_n(x) = \frac{1}{n},$$

and

$$\delta_n(\lambda,\alpha) = \frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\Bigg[\left(\frac{1}{\alpha^2(\alpha+\lambda)} - \frac{\lambda}{\alpha^2(\alpha+\lambda)}X_i - \frac{1}{\alpha}X_iX_j\right)e^{-\alpha X_k} + \left(\frac{1}{\lambda(\alpha+\lambda)}X_i - \frac{1}{\lambda^2(\alpha+\lambda)}\right)e^{\lambda X_j} + \frac{1}{\alpha^2}X_i + \frac{\alpha-\lambda}{\alpha^2\lambda^2}\Bigg]. \tag{13}$$

To achieve scale invariance for the test, Eq. (13) is expressed in the form:

$$\widehat{\delta}_n(\lambda,\alpha) = \frac{1}{n^3\overline{X}^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\psi(X_i,X_j,X_k), \tag{14}$$

where

$$\psi(X_i,X_j,X_k) = \left(\frac{1}{\alpha^2(\alpha+\lambda)} - \frac{\lambda}{\alpha^2(\alpha+\lambda)}X_i - \frac{1}{\alpha}X_iX_j\right)e^{-\alpha X_k} + \left(\frac{1}{\lambda(\alpha+\lambda)}X_i - \frac{1}{\lambda^2(\alpha+\lambda)}\right)e^{\lambda X_j} + \frac{1}{\alpha^2}X_i + \frac{\alpha-\lambda}{\alpha^2\lambda^2}. \tag{15}$$

The symmetric kernel is given by:

$$\gamma(X_i,X_j,X_k) = \frac{1}{3!}\sum_{\text{R}}\psi(X_i,X_j,X_k),$$

where the summation is over all permutations of X_i, X_j, and X_k. Consequently, the value of $\widehat{\delta}_n(\lambda, \alpha)$ as defined in Eq. (14) can be considered synonymous with the U-statistic given by:

$$U_n = \frac{1}{\binom{n}{3}}\sum\gamma(X_i,X_j,X_k).$$

The next theorem establishes the asymptotic normality of the estimator $\widehat{\delta}_n(\lambda, \alpha)$.

Theorem 2: As $n \to \infty$, $\sqrt{n}\left(\widehat{\delta}_n(\lambda, \alpha) - \delta(\lambda, \alpha)\right)$ is asymptotically normal with mean zero and variance σ^2, where:

$$\sigma^2 = \text{Var}\left(\frac{(X-1)(1+\alpha-\lambda)}{\alpha(1+\alpha)(-1+\lambda)\lambda} + \frac{\alpha+\alpha^2+e^{X\lambda}\alpha(1+\alpha)(-1+\lambda)+\alpha\lambda^2-X\alpha\lambda^2+\lambda^3-X\lambda^3}{\alpha(1+\alpha)\lambda^2(\alpha+\lambda)} + \frac{e^{-\alpha X}(-1+e^{\alpha X}-\alpha)(-1+\alpha+\lambda)}{\alpha^2(\alpha+\lambda)}\right). \tag{16}$$

Under H_0, the variance reduces to:

$$\sigma_0^2 = \frac{(1-\lambda)(-10+\lambda(16+\lambda(2\lambda-9)))-2\alpha^2-\alpha(12+\lambda(-22+\lambda(13-2\lambda)))}{(1+\alpha)^2(1+2\alpha)(1+\alpha-\lambda)(\lambda-1)^2(2\lambda-1)}. \tag{17}$$

Proof: Equation (15) can be written as:

$$\psi(X_1, X_2, X_3) = \left(\frac{1}{\alpha^2(\alpha+\lambda)} - \frac{\lambda}{\alpha^2(\alpha+\lambda)}X_1 - \frac{1}{\alpha}X_1X_2\right)e^{-\alpha X_3} + \left(\frac{1}{\lambda(\alpha+\lambda)}X_1 - \frac{1}{\lambda^2(\alpha+\lambda)}\right)e^{\lambda X_2} + \frac{1}{\alpha^2}X_1 + \frac{\alpha-\lambda}{\alpha^2\lambda^2}. \tag{18}$$

Define:

$$\psi(X_1) = \mathbb{E}[\psi(X_1, X_2, X_3) \mid X_1] + \mathbb{E}[\psi(X_2, X_1, X_3) \mid X_1] + \mathbb{E}[\psi(X_3, X_2, X_1) \mid X_1].$$

The three conditional expectations are computed as follows:

$$\mathbb{E}[\psi(X_1, X_2, X_3) \mid X_1] = \frac{(X_1-1)(1+\alpha-\lambda)}{\alpha(1+\alpha)(-1+\lambda)\lambda}, \tag{19}$$

$$\mathbb{E}[\psi(X_2, X_1, X_3) \mid X_1] = \frac{\alpha+\alpha^2+e^{X_1\lambda}\alpha(1+\alpha)(-1+\lambda)+\alpha\lambda^2-X_1\alpha\lambda^2+\lambda^3-X_1\lambda^3}{\alpha(1+\alpha)\lambda^2(\alpha+\lambda)}, \tag{20}$$

$$\mathbb{E}[\psi(X_3, X_2, X_1) \mid X_1] = \frac{e^{-\alpha X_1}(-1+e^{\alpha X_1}-\alpha)(-1+\alpha+\lambda)}{\alpha^2(\alpha+\lambda)}. \tag{21}$$

Thus,

$$\psi(X_1) = \frac{(X_1-1)(1+\alpha-\lambda)}{\alpha(1+\alpha)(-1+\lambda)\lambda} + \frac{\alpha+\alpha^2+e^{X_1\lambda}\alpha(1+\alpha)(-1+\lambda)+\alpha\lambda^2-X_1\alpha\lambda^2+\lambda^3-X_1\lambda^3}{\alpha(1+\alpha)\lambda^2(\alpha+\lambda)} + \frac{e^{-\alpha X_1}(-1+e^{\alpha X_1}-\alpha)(-1+\alpha+\lambda)}{\alpha^2(\alpha+\lambda)}. \tag{22}$$

Hence,

$$\sigma^2 = \text{Var}(\psi(X_1)).$$

Under H_0, the variance reduces to Eq. (17).

2.2. Assessment of Pitman's Asymptotic Efficiency for EBUC$_{mgf}$

Pitman's Asymptotic Relative Efficiency (P.A.E.) is a concept in statistics used to compare the performance of two statistical tests as the sample size approaches infinity. It measures the relative efficiency by examining the ratio of sample sizes required to achieve the same power against local alternatives (i.e., alternatives approaching the null hypothesis).

Key Properties:

- Applies to hypothesis testing problems
- Asymptotic concept (sample size $n \to \infty$)
- Compares tests under local alternatives
- Higher values indicate greater efficiency (fewer observations needed for equivalent power)

This concept is important in choosing statistical methods that remain reliable and powerful as data scales. To assessment of the efficacy of this approach includes the calculation of Pitman asymptotic efficiencies and their comparison with alternative tests for various distributions such as:

1. **Linear Failure Rate (LFR)**: $\overline{F}_1(x) = \exp\left(-x - \frac{\theta x^2}{2}\right), \quad x \geq 0, \theta \geq 0$
2. **Makeham**: $\overline{F}_2(x) = \exp\left(-x - \theta(x + e^{-x} - 1)\right), \quad x \geq 0, \theta \geq 0$
3. **Weibull**: $\overline{F}_3(x) = \exp(-x^\theta), \quad x \geq 0, \theta \geq 1$
4. **Gamma**: $\overline{F}_4(x) = \frac{1}{\Gamma(\theta)} \int_x^\infty e^{-u} u^{\theta-1} du, \quad x \geq 0, \theta \geq 0$

The exponential distribution corresponds to $\theta_0 = 0$ for LFR and Makeham families, and $\theta_0 = 1$ for Weibull and Gamma families.

We utilize the concept of Pitman's asymptotic efficiency (PAE), described as follows, without directly quoting the original text. The Pitman asymptotic efficiency is given by:

$$\text{PAE}(\delta_\theta(\lambda, \alpha)) = \frac{1}{\sigma_0} \left| \frac{\partial}{\partial \theta} \delta_\theta(\lambda, \alpha) \right|_{\theta \to \theta_0} \tag{23}$$

where the derivative term expands to:

$$\begin{aligned}
\frac{\partial}{\partial \theta} \delta_\theta(\lambda, \alpha) &= \left(\frac{1}{\alpha^2(\alpha+\lambda)} - \frac{1}{\alpha}\mu_\theta^2 - \frac{\lambda}{\alpha^2(\alpha+\lambda)}\mu_\theta \right) \int_0^\infty e^{-\alpha x} dF'_\theta(x) \\
&+ \left(-\frac{2}{\alpha}\mu_\theta \mu'_\theta - \frac{\lambda}{\alpha^2(\alpha+\lambda)}\mu'_\theta \right) \int_0^\infty e^{-\alpha x} dF_\theta(x) \\
&+ \left(\frac{1}{\lambda(\alpha+\lambda)}\mu_\theta - \frac{1}{\lambda^2(\alpha+\lambda)} \right) \int_0^\infty e^{\lambda x} dF'_\theta(x) \\
&+ \frac{1}{\lambda(\alpha+\lambda)}\mu'_\theta \int_0^\infty e^{\lambda x} dF_\theta(x) + \frac{1}{\alpha^2}\mu'_\theta
\end{aligned} \tag{24}$$

with $\mu_\theta = \int_0^\infty x dF_\theta(x)$ and $\mu'_\theta = \frac{\partial}{\partial \theta}\mu_\theta$.

To evaluate the performance of the Pitman asymptotic efficiency (PAE) method, we consider several probability models, including the Linear Failure Rate, Makeham, Weibull, and Gamma distributions. The PAE for each distribution family is:

$$\text{PAE}(\delta_\theta(\lambda, \alpha), \overline{F}_1(x)) = \frac{1}{\sigma_0} \left| \frac{3 + \alpha - 3\lambda + \lambda^2}{(1+\alpha)^2(\lambda-1)^2} \right|, \quad \alpha > -1, \lambda < 1 \tag{25}$$

$$\text{PAE}(\delta_\theta(\lambda, \alpha), \overline{F}_2(x)) = \frac{1}{\sigma_0} \left| \frac{5 + \alpha - 4\lambda + \lambda^2}{2(1+\alpha)(2+\alpha)(\lambda-2)(\lambda-1)} \right|, \quad \alpha > -1, \lambda < 1 \tag{26}$$

$$\text{PAE}(\delta_\theta(\lambda, \alpha), \overline{F}_3(x)) = \frac{1}{\sigma_0} \left| \frac{(-1+\alpha+\lambda)\log(1+\alpha)}{\alpha(1+\alpha)(\alpha+\lambda)} + \frac{\log(1-\lambda)}{\lambda(\lambda-1)(\alpha+\lambda)} \right|, \quad \alpha > -1, \lambda < 1 \tag{27}$$

$$\text{PAE}(\delta_\theta(\lambda, \alpha), \overline{F}_4(x)) = \frac{1}{\sigma_0} \left| -\frac{\alpha^2 + (\lambda-1)^2\lambda + \alpha(1-\lambda+\lambda^2)}{\alpha\lambda(\alpha+\lambda)(1+\alpha)(\lambda-1)} + \frac{(\alpha+\lambda-1)\log(1+\alpha)}{\alpha^2(\alpha+\lambda)} + \frac{\log(1-\lambda)}{\lambda^2(\alpha+\lambda)} \right|, \quad \alpha > -1, \lambda < 1 \tag{28}$$

Table 1 presents the efficiency of alternative distributions at various values of the parameters λ and α, compared with the tests conducted by Bakr et al. [9] and El-Sherbin et al. [13]. The values in the table reflect the accuracy and effectiveness of the estimator function $\delta_\theta(\lambda, \alpha)$ in fitting different distributions $\overline{F}_i(x)$ across varying sample sizes. The table also highlights how the performance varies with changes in the regularization parameters and the underlying data.

Table 1. Asymptotic efficiencies of $\delta_\theta(\lambda, \alpha)$ for various combinations of λ and α

Dist.	α	$\delta_\theta(\lambda, \alpha)$				Comparison Tests			
		$\lambda = 0.01$	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.4$	$\delta(s)$			$\delta(s,1)_{\text{EBUCL}}$
						$s = 0.2$	$s = 0.001$	$s = 0.2$	$s = 5$
$\overline{F}_1(x)$	0.2	0.9770	0.9706	0.9600	0.7048	0.9219	0.9818	0.8111	0.8916
	0.5	0.9942	0.9904	0.9831	0.7387				
	2	0.9966	0.9981	0.9972	0.7695				
	4	0.9909	0.9937	0.9942	0.7669				
	5	0.9898	0.9927	0.9932	0.7649				
$\overline{F}_2(x)$	0.2	0.2157	0.2113	0.2049	0.1254	0.1821	0.2181	0.1367	0.1663
	0.5	0.2334	0.2289	0.2223	0.1371				
	2	0.2603	0.2556	0.2486	0.1519				
	4	0.2640	0.2591	0.2518	0.1520				
	5	0.2637	0.2587	0.2513	0.1511				
$\overline{F}_3(x)$	0.2	0.8596	0.8431	0.8194	0.5144	0.7218	0.8550	0.5564	0.6655
	0.5	0.9294	0.9124	0.8878	0.5606				
	2	1.0561	1.0368	1.0084	0.6256				
	4	1.0923	1.0707	1.0393	0.6322				
	5	1.0974	1.0751	1.0426	0.6307				
$\overline{F}_4(x)$	0.2	0.4092	0.3996	0.3862	0.2304	–	–	0.2388	0.2951
	0.5	0.4526	0.4422	0.4276	0.2554				
	2	0.5419	0.5287	0.5099	0.2951				
	4	0.5724	0.5571	0.5356	0.3016				
	5	0.5776	0.5617	0.5393	0.3013				

The table reveals several important findings:

- The first distribution $\overline{F}_1(x)$ achieves the highest efficiency among all alternatives, with values approaching unity (PAE $\approx$ 0.97–0.99). This superior performance consistently outperforms the results from Bakr et al. [9] and El-Sherbin et al. [13], indicating exceptional accuracy.
- The other distributions $\overline{F}_2(x)$, $\overline{F}_3(x)$, and $\overline{F}_4(x)$ demonstrate:
  - Lower overall efficiency scores compared to $\overline{F}_1(x)$
  - Greater variability in performance across parameter values
  - Sensitivity to changes in both λ and α parameters
- Key observations about parameter effects:
  - Efficiency generally decreases as λ increases from 0.01 to 0.4
  - Moderate α values (0.5–2) often yield optimal performance
  - The Gamma distribution ($\overline{F}_4(x)$) shows incomplete comparative data

These results emphasize:

1. The critical importance of distribution selection for achieving high efficiency

2. The need for careful parameter tuning (λ and α balance)
3. The value of comparative benchmarks established in prior studies
4. The exceptional performance of the $\overline{F}_1(x)$ distribution case

3. Monte Carlo Critical Values

Monte Carlo simulation is a computational technique that relies on repeated random sampling to obtain numerical results, often used to approximate the distribution of complex statistics. In this section, the critical values for the null distribution of $\hat{\delta}(\lambda, \alpha)$ are estimated using Monte Carlo simulations. Specifically, 10,000 samples of sizes ranging from 5 to 50 (in increments of 5) were generated from the standard exponential distribution using Mathematica 13. From these simulations, the upper percentile values at confidence levels of 90%, 95%, 98%, and 99% were calculated. These percentile values for the statistic $\hat{\delta}(\lambda, \alpha)$ are displayed in Table 2 and Table 3.

Table 2. Critical Values of the test statistic $\hat{\delta}(\lambda, \alpha)$ for $\lambda = 0.01$ and $\alpha = 2$.

n	90%	95%	98%	99%
5	0.235261	0.261784	0.286409	0.299615
10	0.186350	0.210898	0.233565	0.248351
15	0.158303	0.181641	0.203958	0.217209
20	0.139056	0.162424	0.184875	0.199337
25	0.128272	0.149529	0.172097	0.185261
30	0.118502	0.138517	0.158790	0.171819
35	0.110324	0.127398	0.149318	0.162169
36	0.110291	0.127942	0.146295	0.158447
40	0.104396	0.122901	0.140670	0.150934
45	0.098745	0.116539	0.136559	0.145685
50	0.095549	0.112481	0.129816	0.143214

Table 3. Critical Values of the test statistic $\hat{\delta}(\lambda, \alpha)$ for $\lambda = 0.1$ and $\alpha = 5$.

n	90%	95%	98%	99%
5	0.106268	0.115299	0.123581	0.126413
10	0.082703	0.092668	0.101167	0.107634
15	0.070705	0.080137	0.088809	0.094645
20	0.062318	0.071429	0.080210	0.085588
25	0.057705	0.065958	0.074931	0.081439
30	0.053232	0.061075	0.069988	0.076475
35	0.049273	0.057188	0.066072	0.070915
36	0.049095	0.056984	0.065385	0.070267
40	0.046783	0.054456	0.063164	0.068185
45	0.044293	0.051769	0.059374	0.065292
50	0.042568	0.050044	0.057519	0.061902

Tables 2 and 3, and Figures 1 and 2 shows that the critical values of the test statistic $\hat{\delta}(\lambda, \alpha)$ tend to increase with higher confidence levels, which is expected as wider confidence intervals capture more extreme values in the distribution. The critical values decrease gradually as the sample size n increases, reflecting improved estimation accuracy and reduced random variation with larger samples. This lowers the likelihood of extreme statistic values

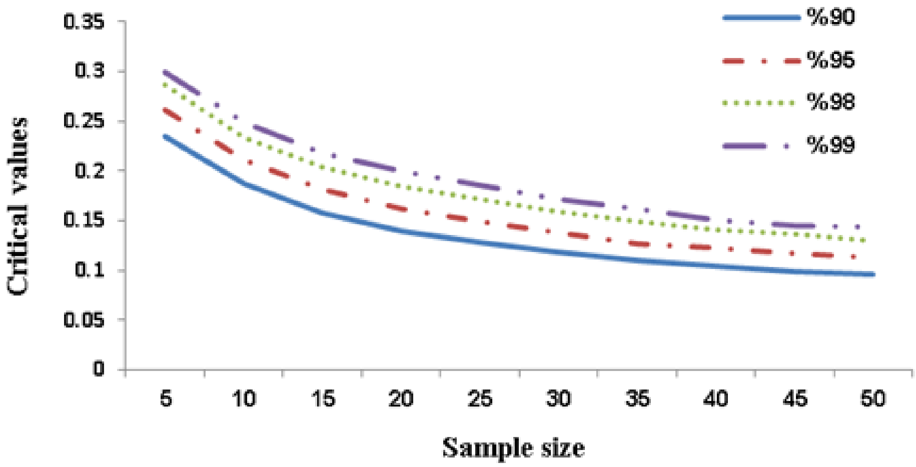

Figure 1. Critical values versus sample size for $\lambda = 0.01$ and $\alpha = 2$.

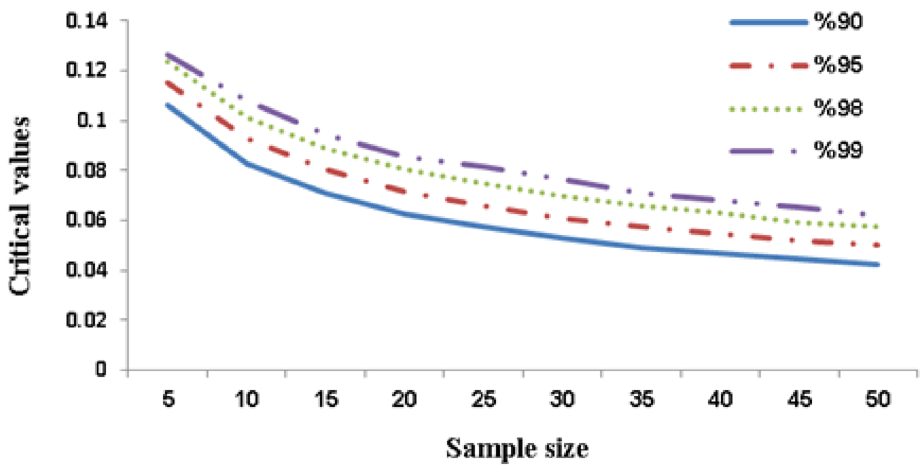

Figure 2. Critical values versus sample for $\lambda = 0.1$ and $\alpha = 5$.

under the null hypothesis. Table 2 show critical values for a small regularization parameter $\lambda = 0.01$ and $\alpha = 2$, where critical values are generally higher compared to those in Table 3. Table 3 presents critical values for $\lambda = 0.1$ and $\alpha = 5$, showing generally smaller critical values, indicating that stronger regularization leads to reduced critical thresholds and can affect test sensitivity and discriminative power. In summary, the Monte Carlo simulation results confirm the inverse relationship between sample size and critical value magnitude, as well as the influence of regularization parameters on the distribution of the test statistic. These insights are instrumental in designing tests with optimized accuracy and reliability based on the desired level of confidence.

3.1. Power Estimates of the Test $\hat{\delta}(\lambda, \alpha)$

Power in statistical testing refers to the probability that a test correctly rejects a false null hypothesis. In other words, it measures the test's ability to detect a true effect or difference when it actually exists. A higher power (closer to 1) means the test is more sensitive and less likely to commit a Type II error (failing to reject a false null hypothesis).Several factors influence the power of a test, including sample size, the magnitude of the effect, significance level (confidence level), and the underlying distribution of the data. Power is often estimated through simulation methods such as Monte Carlo experiments or analytically in simpler cases. Assessing power is essential when designing experiments and choosing appropriate tests to ensure sufficient sensitivity to detect meaningful effects. Power estimation in statistical testing refers to the procedure used to determine how likely a test is to correctly reject a false null hypothesis, thus gauging its sensitivity to detect actual effects. In this section, we assess the power of the test statistic $\hat{\delta}(\lambda, \alpha)$ at a $(1-\alpha)\%$ confidence level, using selected values of θ, for sample sizes of $n = 10, 20, 30$. The power analysis is performed relative to three alternative distributions: the Linear Failure Rate (LFR), Weibull, and Gamma distributions. This evaluation is carried out by running 10,000 simulated replicates for each scenario.

Table 4. Power Estimates at $\lambda = 0.01$, $\alpha = 2$

Distribution	n	$\theta = 2$	$\theta = 3$	$\theta = 4$
LFR	10	0.0873	0.0681	0.0501
	20	0.2397	0.2263	0.1993
	30	0.3998	0.4262	0.4220
Gamma	10	0.4330	0.7581	0.8935
	20	0.6786	0.9572	0.9951
	30	0.8088	0.9923	0.9997
Weibull	10	0.7091	0.9851	0.9998
	20	0.9708	1.0000	1.0000
	30	0.9978	1.0000	1.0000

Table 5. Power Estimates at $\lambda = 0.1$, $\alpha = 5$

Distribution	n	$\theta = 2$	$\theta = 3$	$\theta = 4$
LFR	10	0.2028	0.2300	0.2406
	20	0.3802	0.4560	0.4944
	30	0.5652	0.6547	0.7118
Gamma	10	0.4188	0.8074	0.9618
	20	0.6774	0.9677	0.9974
	30	0.8363	0.9950	0.9998
Weibull	10	0.7653	0.9935	0.9999
	20	0.9835	1.0000	1.0000
	30	0.9993	1.0000	1.0000

The tables demonstrate that the test statistic $\hat{\delta}(\lambda, \alpha)$ becomes increasingly effective and efficient as the sample size grows, with a marked improvement in the ability to detect true differences when moving from smaller samples ($n = 10$) to larger ones ($n = 30$) across all studied distributions. The test shows particularly high sensitivity to the Gamma and Weibull distributions, where power values rise significantly with increasing θ and approach near-perfect performance even at moderate sample sizes. In contrast, the Linear Failure Rate (LFR) distribution exhibits comparatively lower power levels, with a more gradual increase across θ values and sample sizes. Moreover, increasing the regularization parameters λ and α notably enhances power, especially for the LFR distribution, suggesting that tuning these parameters can improve the test's sensitivity in more challenging detection scenarios. Among the distributions, the Weibull alternatives consistently yield the best power and stability in estimates, confirming the superior ability of the test to detect alternatives following this distribution. Overall, these results highlight the importance of selecting an appropriate sample size, carefully tuning regularization parameters, and understanding the nature of alternative distributions to optimize the accuracy and effectiveness of the $\hat{\delta}(\lambda, \alpha)$ test in practical applications.

4. Applications Using Complete (Uncensored) Data

Here, we provide several significant real-world examples to demonstrate how our test statistic $\hat{\delta}_n(\lambda, \alpha)$ can be applied to non-censored data at the 95% confidence level.

4.1. Example 1: Leukemia Patient Lifespans

A data set comprises the recorded lifespans (in years) of 40 patients diagnosed with leukemia, sourced from a Ministry of Health hospital in Saudi Arabia as referenced in Abouammoh [2]. The ordered lifetimes (in years) are:

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162
2.211	2.370	2.532	2.693	2.805	2.910	2.912	3.192	3.263	3.348
3.348	3.427	3.499	3.534	3.767	3.751	3.858	3.986	4.049	4.244
4.323	4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074	4.381

Based on the analysis, the computed values of the test statistic are:

- $\hat{\delta}_n(\lambda, \alpha) = 0.257821$ when $\lambda = 0.01$ and $\alpha = 2$
- $\hat{\delta}_n(\lambda, \alpha) = 0.115343$ when $\lambda = 0.1$ and $\alpha = 5$

Both values exceed their respective critical values (Tables 2 and 3), providing sufficient evidence to reject the null hypothesis H_0 at the 0.05 significance level. The data do not follow an exponential distribution, but rather belong to the EBUC_{mgf} class.

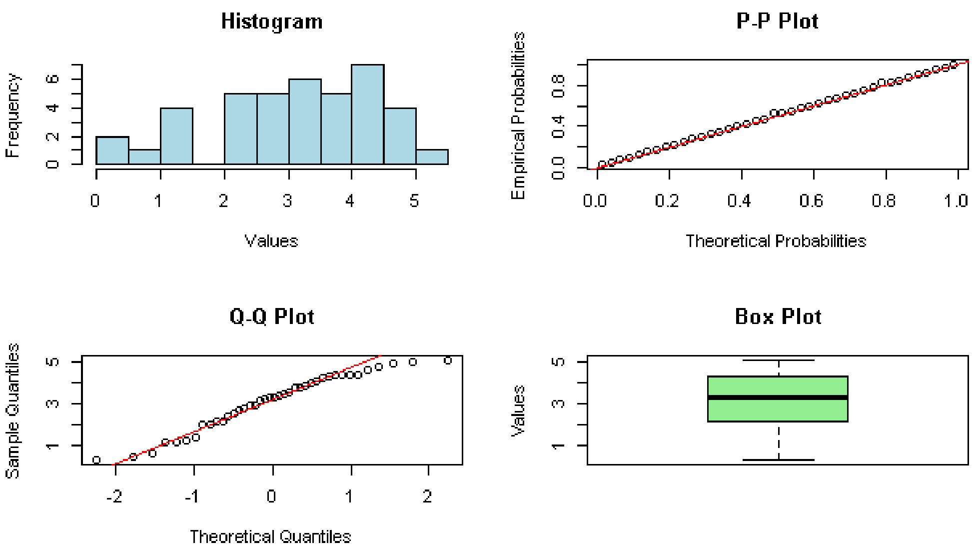

Figure 3. Diagnostic plots for Example 1: (a) Histogram, (b) P-P plot, (c) Q-Q plot, (d) Box plot

To assess the performance of the test asset of nonparametric plots are shown as: The histogram shows the distribution of the data, which appears moderately spread and does not closely resemble the classic shape of the exponential distribution. The P-P plot and Q-Q plot indicate deviations from the straight line, suggesting that the empirical data do not perfectly align with the theoretical exponential distribution. The box plot reveals the spread and central tendency of the values, with no strong skewness or presence of extreme outliers. The middle 50% of the data is concentrated in the interquartile range. Together, these graphical summaries visually reinforce the statistical finding that the data deviate from exponential behavior, supporting the conclusion to reject H_0 in favor of the alternative distribution class.

4.2. *Example 2: COVID-19 Mortality Rates*

A data set presented in Almetwally [6], covers a 36-day period from April 10 to May 15, 2020, and presents COVID-19 statistics recorded in Canada. Specifically, the data represent the drought mortality rate observed during this time frame, as illustrated in Figure 4. The mortality rates are as follows:

3.1091	3.3825	3.1444	3.2135	2.4946	3.5146
4.9274	3.3769	6.8686	3.0914	4.9378	3.1091
3.2823	3.8594	4.0480	4.1685	3.6426	3.2110
2.8636	3.2218	2.9078	3.6346	2.7957	4.2781
4.2202	1.5157	2.6029	3.3592	2.8349	3.1348
2.5261	1.5806	2.7704	2.1901	2.4141	1.9048

Test statistic values:

- $\hat{\delta}_n(\lambda, \alpha) = 0.293812$ ($\lambda = 0.01$, $\alpha = 2$)
- $\hat{\delta}_n(\lambda, \alpha) = 0.125227$ ($\lambda = 0.1$, $\alpha = 5$)

both values exceed the corresponding critical values found in Tables 2 and 3, there is sufficient evidence to reject the null hypothesis. This result leads us to accept the alternative hypothesis H_1, indicating that the data set possesses the EBUC_{mgf} property and does not follow the exponential distribution.

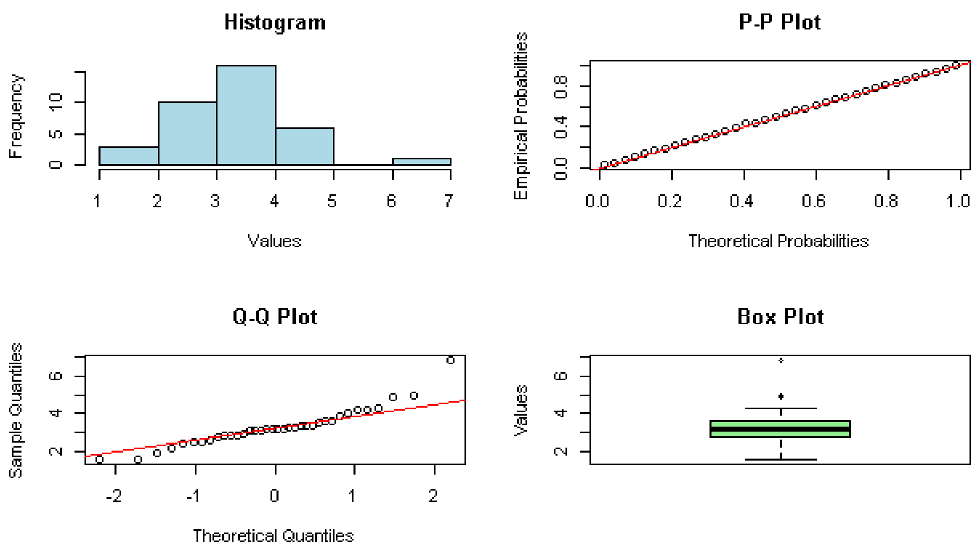

Figure 4. Diagnostic plots for Example 2: (a) Histogram, (b) P-P plot, (c) Q-Q plot, (d) Box plot

It is easily to show that from figure 4: The histogram reveals that the data distribution does not strictly follow the typical exponential shape, displaying a moderate spread among values. The P-P plot and Q-Q plot both show some deviation from the straight reference lines, illustrating differences between the empirical data and the theoretical exponential distribution. These patterns further support the rejection of exponentiality. The box plot demonstrates the data's central tendency and variability. While there are a few mild outliers, most values remain within a relatively consistent range, reinforcing the findings from the other plots. These graphical representations visually strengthen the statistical conclusion that the data set does not have an exponential distribution but instead exhibits the characteristics described by the EBUC$_{\text{mgf}}$ class.

4.3. Example 3: Customer Arrival Intervals

A data set sourced from Grubbs [19] and records the time intervals between the arrivals of 25 customers at a specific facility. The data represent the durations (in time units) between consecutive customer arrivals, as illustrated in Figure 5. The intervals are as follows:

1.80	2.89	2.93	3.03	3.15
3.43	3.48	3.57	3.85	3.92
3.98	4.06	4.11	4.13	4.16
4.23	4.34	4.53	4.62	4.65
4.73	4.84	4.91	4.99	5.17

Test statistic values:

- $\hat{\delta}_n(\lambda, \alpha) = 0.299909$ ($\lambda = 0.01$, $\alpha = 2$)
- $\hat{\delta}_n(\lambda, \alpha) = 0.134581$ ($\lambda = 0.1$, $\alpha = 5$)

It is found that these values exceed the critical threshold mentioned in Tables 2 and 3, supporting the acceptance of the alternative hypothesis H_1. This indicates that the dataset exhibits the EBUC$_{\text{mgf}}$ property and does not follow an exponential distribution.

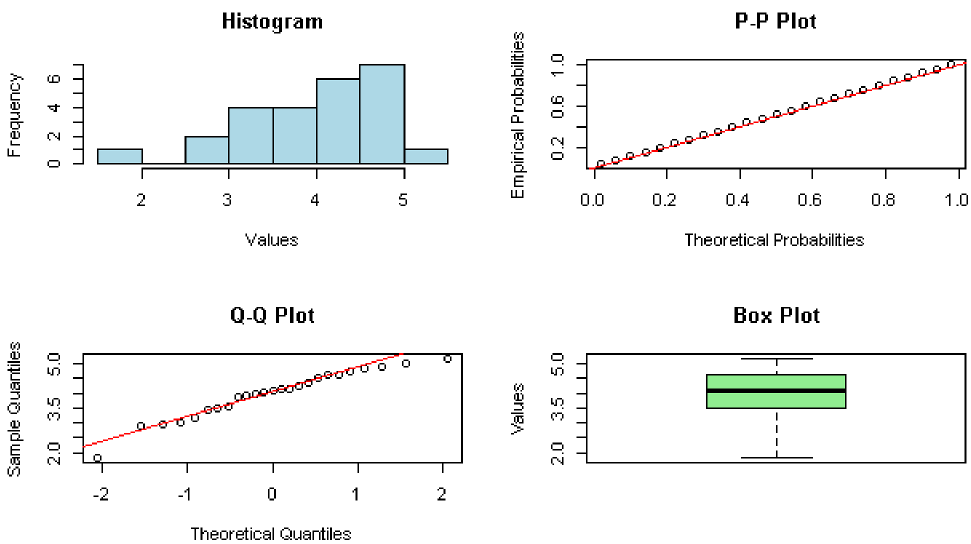

Figure 5. Diagnostic plots for Example 3: (a) Histogram, (b) P-P plot, (c) Q-Q plot, (d) Box plot

To evaluate the test Figure 5 indicate that the graph displays the data distribution through multiple elements, including a histogram showing frequency values, a P-P plot comparing theoretical probabilities with observed distributions, and a box plot summarizing key statistical measures. Together, these visualizations aid in analyzing the dataset's characteristics and reinforce the statistical conclusions drawn in the study. The results align with the numerical findings, confirming the non-exponential nature of the data.

5. Testing Against EBUC$_{\text{mgf}}$ Class for Censored Data

In many life-testing models and clinical studies, it is common to encounter censored data, where full information about the time to event (such as device failure or patient death) is not available for every unit in the study. One of the most frequent scenarios is random right-censoring.

Statistical Model with Random Right-Censoring Suppose n units are placed under observation, with their actual failure times denoted as $X_1, X_2, \ldots, X_n$ which are assumed to be independent and identically distributed (i.i.d.) according to a continuous lifetime distribution F. The censoring times $Y_1, Y_2, \ldots, Y_n$ are also i.i.d. from another continuous distribution G, and are independent of the X's.

We observe the pairs (Z_j, δ_j) for $j = 1, \ldots, n$ where:

- $Z_j = \min(X_j, Y_j)$: the observed time (event time or censoring time)
- δ_j: indicator variable ($\delta_j = 1$ if event observed, $\delta_j = 0$ if censored)

Significance of Handling Censored Data Specialized statistical methods such as survival analysis, Kaplan-Meier estimation, and the log-rank test are necessary for proper analysis of censored data. Traditional methods that ignore censoring often result in biased conclusions.

Let $Z_{(0)} = 0 < Z_{(1)} < Z_{(2)} < \cdots < Z_{(n)}$ denote the ordered Z's, and $\delta_{(j)}$ is the δ_j corresponding to $Z_{(j)}$. The Kaplan-Meier [22] proposed the product limit estimator as:

$$\bar{F}_n(x) = 1 - F_n(x) = \prod_{j:Z_{(j)} \le x} \left(\frac{n-j}{n-j+1} \right)^{\delta_{(j)}}, \quad x \in [0, Z_j]$$

For testing $H_0 : \Delta_{\text{EBUC}^c_{\text{mgf}}} = 0$ versus $H_1 : \Delta_{\text{EBUC}^c_{\text{mgf}}} > 0$, we propose:

$$\Delta^c_{\text{EBUC}_{\text{mgf}}} = \frac{1}{\mu^2} \left\{ \left(\frac{1}{\alpha^2(\alpha+\lambda)} - \frac{\lambda}{\alpha^2(\alpha+\lambda)}\mu - \frac{1}{\alpha}\mu^2 \right) \phi(\alpha) + \left(\frac{1}{\lambda(\alpha+\lambda)}\mu - \frac{1}{\lambda^2(\alpha+\lambda)} \right) \phi(\lambda) + \frac{1}{\alpha^2}\mu + \frac{\alpha-\lambda}{\alpha^2\lambda^2} \right\}$$

The computable form is:

$$\hat{\Delta}^c_{\text{EBUC}_{\text{mgf}}} = \frac{1}{\Omega^2} \left\{ \left(\frac{1}{\alpha^2(\alpha+\lambda)} - \frac{\lambda}{\alpha^2(\alpha+\lambda)}\Omega - \frac{1}{\alpha}\Omega^2 \right) \Phi + \left(\frac{1}{\lambda(\alpha+\lambda)}\Omega - \frac{1}{\lambda^2(\alpha+\lambda)} \right) \psi + \frac{1}{\alpha^2}\Omega + \frac{\alpha-\lambda}{\alpha^2\lambda^2} \right\} \tag{29}$$

Where:

$$\Omega = \sum_{i=1}^{n} \prod_{m=1}^{i-1} C_m^{\delta(m)} (Z_{(i)} - Z_{(i-1)})$$

$$\Phi = \sum_{j=1}^{n} e^{-\alpha Z_{(j)}} \left(\prod_{v=1}^{j-2} C_v^{\delta(v)} - \prod_{v=1}^{j-1} C_v^{\delta(v)} \right)$$

$$\psi = \sum_{k=1}^{n} e^{\lambda Z_{(j)}} \left(\prod_{r=1}^{k-2} C_r^{\delta(r)} - \prod_{r=1}^{k-1} C_r^{\delta(r)} \right)$$

$$dF_n(Z_j) = \bar{F}_n(Z_{j-1}) - \bar{F}_n(Z_j), C_k = \frac{n-k}{n-k+1}$$

Table 6 gives the critical values percentiles of $\hat{\Delta}^c_{\text{EBUC}_{\text{mgf}}}$ test for sample sizes $n = 5(5)30(10)70, 81, 86$ based on 10000 replications by using Mathematica v. 13. The empirical results can be sketched in Figure 6.

Table 6 and Figure 6 illustrates how the upper percentile values of the test statistic $\hat{\Delta}^c_{\text{EBUC}_{\text{mgf}}}$ vary with different sample sizes n and confidence levels (90%, 95%, 98%, and 99%) at the parameters $\lambda = 0.1$ and $\alpha = 5$. It is evident

Table 6. Upper percentiles of $\hat{\Delta}^{c}_{\text{EBUC}_{\text{mgf}}}$ at $\lambda = 0.1$, $\alpha = 5$

n	90%	95%	98%	99%
5	2.84261	3.26867	3.73314	4.03664
10	2.53984	2.92774	3.38003	3.63530
15	2.21713	2.55282	2.94718	3.21128
20	2.01794	2.32631	2.68769	2.91219
25	1.84536	2.14377	2.47771	2.72944
30	1.72029	1.98339	2.30469	2.51428
40	1.53165	1.76083	2.01903	2.19377
50	1.40474	1.61752	1.85308	2.02714
60	1.27128	1.47154	1.69589	1.86008
70	1.19781	1.38917	1.62414	1.75106
81	1.13492	1.31466	1.49748	1.64965
86	1.10650	1.28521	1.46918	1.58630

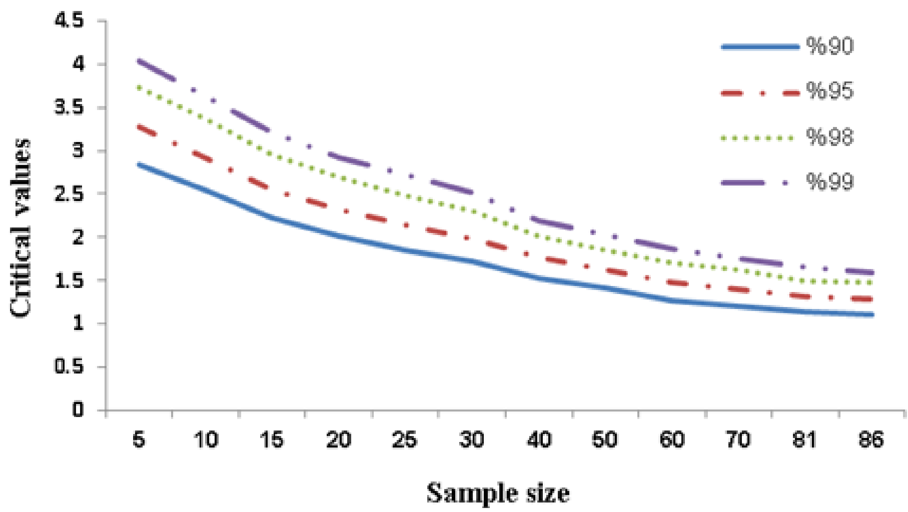

Figure 6. Variation of critical values with sample size and confidence level

that the critical values tend to decrease gradually as the sample size increases. This behavior is consistent with improved estimation accuracy and reduced random fluctuations associated with larger sample sizes. Conversely, the critical values increase as the confidence level rises, reflecting the higher thresholds required to reject the null hypothesis with greater certainty. This direct relationship means that to achieve higher confidence, the test statistic must exceed a larger critical value. Furthermore, the rate of decrease in critical values diminishes for larger samples beyond size 50, suggesting relative stability and consistency of the test statistic distribution at larger sample sizes.

5.1. Power Estimates for $\hat{\Delta}^{c}_{EBUC_{mgf}}$

We evaluate the statistical power for testing exponentiality against the EBUC_{mgf} alternative. The assessment is conducted at a given significance level, using parameter values $\theta = 2, 3, 4$ and sample sizes $n = 10, 20, 30$. The analysis considers three alternative distributions: Linear Failure Rate (LFR), Gamma, and Weibull families.

The power estimates in Table 7 indicate an exceptionally high sensitivity of the test $\hat{\Delta}^{c}_{\text{EBUC}_{\text{mgf}}}$ across different sample sizes and alternative distributions. For the Linear Failure Rate (LFR) and Weibull distributions, the power

Table 7. Power estimates for $\hat{\Delta}^{c}_{\mathrm{EBUC_{mgf}}}$ at $\lambda = 0.1$, $\alpha = 5$

Distribution	n	$\theta = 2$	$\theta = 3$	$\theta = 4$
LFR	10	1.0000	1.0000	1.0000
	20	1.0000	1.0000	1.0000
	30	0.9999	1.0000	1.0000
Gamma	10	0.8886	0.8259	0.7310
	20	0.9989	0.9412	0.8278
	30	0.9909	0.8512	0.7035
Weibull	10	1.0000	1.0000	1.0000
	20	1.0000	1.0000	1.0000
	30	1.0000	1.0000	1.0000

consistently reaches or is extremely close to 1.0000 at all tested sample sizes ($n = 10, 20, 30$) and values of θ (2, 3, 4), demonstrating that the test is almost perfectly capable of detecting deviations from the null hypothesis in these cases.In contrast, the power for the Gamma distribution, while still relatively high, shows somewhat lower values that tend to decrease with increasing θ, especially noticeable at larger θ values and smaller sample sizes. For instance, power values range roughly from about 0.73 to 0.999, indicating good but less robust sensitivity compared to the LFR and Weibull cases. Overall, these results confirm that the test $\hat{\Delta}^{c}_{\mathrm{EBUC_{mgf}}}$, under the specified regularization parameters, is highly effective for detecting alternatives corresponding to LFR and Weibull distributions. This suggests strong applicability of the test in practical situations involving lifetime or reliability data.

6. Applications Using Incomplete (Censored) Data

6.1. Example 1: Melanoma Survival Times

Consider the dataset from Susarla and Vanryzin [28], which includes 81 survival times (in weeks) of melanoma patients with 46 uncensored observations:

13	14	19	19	20	21	23	23	25	26
26	27	27	31	32	34	34	37	38	38
40	46	50	53	54	57	58	59	60	65
65	66	70	85	90	98	102	103	110	118
124	130	136	138	141	234				

The ordered censored observations:

16	21	44	50	55	67	73	76
80	81	86	93	100	108	114	120
124	125	129	130	132	134	140	147
138	151	152	152	158	181	190	193
194	213	215					

The test statistic computed via formula in Eq. (29) yielded:

$$\hat{\Delta}^{c}_{\mathrm{EBUC_{mgf}}} = -2.45409 \times 10^{-34}$$

This value is significantly below the critical threshold Table 6, leading to rejection of the $\mathrm{EBUC_{mgf}}$ property.

To assess the behavior of the data, a variety of nonparametric plots were utilized, as shown in Figure 7 These include histogram, P-P plot, Q-Q plot, and box plot. The histogram shows irregular frequency distribution, while the

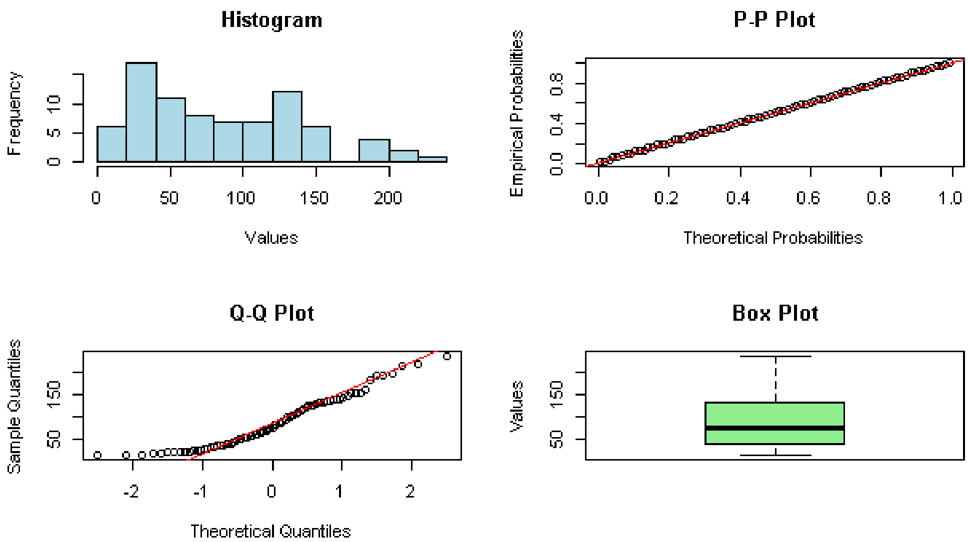

Figure 7. Diagnostic plots for melanoma survival data: (a) Histogram, (b) P-P plot, (c) Q-Q plot, (d) Box plot

P-P plot indicates deviations from theoretical probabilities, suggesting non-exponential behavior. The symmetric box plot further supports this conclusion. Together, these visual elements align with the statistical rejection of the EBUC_{mgf} property, confirming the dataset does not follow the expected distribution.

6.2. Example 2: Lung Cancer Survival Data

This study examines survival patterns using clinical data from Pena's [27] research on lung cancer patients. The dataset contains:

- 86 total observations (months)
- 22 right-censored cases (25.6%)
- 64 complete observations

Uncensored data:

0.99	1.28	1.77	1.97	2.17	2.63	2.66	2.76
2.79	2.86	2.99	3.06	3.15	3.45	3.71	3.75
3.81	4.11	4.27	4.34	4.40	4.63	4.73	4.93
4.93	5.03	5.16	5.17	5.49	5.68	5.72	5.85
5.98	8.15	8.62	8.48	8.61	9.46	9.53	10.05
10.15	10.94	10.94	11.24	11.63	12.26	12.65	12.78
13.18	13.47	13.96	14.88	15.05	15.31	16.13	16.46
17.45	17.61	18.20	18.37	19.06	20.70	22.54	23.36

Censored observations:

11.04	13.53	14.23	14.65	14.91	15.47
16.49	17.05	17.28	17.88	17.97	18.83
19.55	19.58	19.75	19.78	19.95	20.04
20.24	20.73	21.55	21.98		

The test statistic value:

$$\hat{\Delta}^{c}_{\text{EBUC}_{\text{mgf}}} = -1.09531 \times 10^{-6}$$

is significantly lower than the critical value from Table 6 at ($\alpha = 0.05$). This result provides sufficient evidence to reject the alternative hypothesis H_1, indicating that the dataset does not possess the EBUC_{mgf} property.

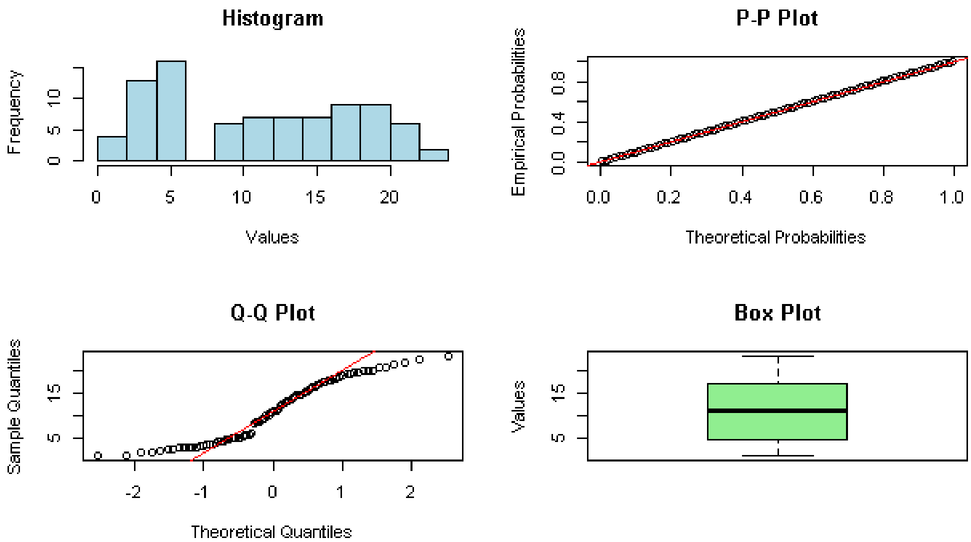

Figure 8. Diagnostic plots for lung cancer data: (a) Histogram, (b) P-P plot, (c) Q-Q plot, (d) Box plot

To examine the characteristics of the data, several nonparametric plots such as box plots, histograms, Q-Q plots, and P-P plots have been constructed, as illustrated in Figure 8. The histogram bimodal peaks and the Q-Q, P-P plots' significant deviations from reference lines indicate non-exponential distribution. Combined with the box plot's right-skewed outliers, these visual patterns strongly support rejecting the EBUC_{mgf} property at $\alpha = 0.05$. The graphical evidence aligns perfectly with the statistical test results.

7. Conclusions

In this study, we introduced a novel statistical testing procedure for evaluating the exponentiality of lifetime data against a new class of life distributions termed the "Exponential Better than Used in Convex order based on the Moment Generating Function" (EBUC_{mgf}). The proposed test exploits properties of the Laplace transform to develop a robust, mathematically grounded method for distinguishing the exponential distribution from this broader, more flexible family of lifetime models.

Our work encompassed theoretical derivations establishing the asymptotic properties of the test statistic, including its asymptotic normality and Pitman asymptotic efficiency across various alternative distributions common in reliability and survival analysis. The critical values of the test statistic were successfully estimated through extensive Monte Carlo simulations, accounting for both complete and right-censored data scenarios, thus extending the practical applicability of the method to real-world datasets where censoring is common.

Simulation results demonstrated that the proposed test possesses high statistical power in detecting departures from exponentiality towards relevant alternative distributions such as Linear Failure Rate, Weibull, and Gamma families. Moreover, the analysis of empirical case studies from medical survival data and reliability engineering datasets validated the effectiveness and versatility of our method in practical applications.

The following key contributions summarize the impact of this research:

- Development of a novel, nonparametric hypothesis test based on Laplace transform order for discriminating exponential from a complex class of lifetime distributions

- Comprehensive theoretical examination including asymptotic behavior and efficiency comparisons with existing tests
- Extension of the testing procedure to right-censored data, addressing a critical challenge in lifetime and survival analysis
- Confirmation of the test's practical utility through illustrative applications on real, uncensored and censored datasets

Future research directions include:

- Exploring extensions to other forms of censoring (e.g., left or interval censoring)
- Enhancing computational efficiency for large-scale data
- Broadening empirical validation across diverse real-life reliability and survival contexts
- Examining robustness under model misspecification
- Developing software implementations for wider adoption

Overall, this work provides a significant advancement in the toolkit for reliability theory and lifetime data analysis, offering researchers a powerful, theoretically sound, and practically relevant method to assess deviation from exponentiality in complex systems.

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