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Control Design on a Non-minimum Phase Nonlinear System by Output Redefinition and Particle Swarm Optimization Method

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Abstract In this paper, we study the control design of a non-minimum phase nonlinear system. Here we investigate a nonlinear system in a particular class and use coordinate transformation to determine the normal form of the system. We present some theorems that state a non-minimum phase nonlinear system becomes the minimum phase with a new output. We further use the new output to determine the control variable, and we use particle swarm optimization to determine the desired output of the output that has been selected. From this design, the output of the system can track the desired output of the original system.

Keywords Nonlinear system, non-minimum phase, new output redefinition, technological capability, particle swarm optimization.

AMS 2010 subject classifications 93C10

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1. Introduction

Many technological problems involving control are currently being researched. Zhou et al. have investigated learning control in robot-assisted rehabilitation of motor skills [1]. In [2], Mahmoodabadi and Danesh studied gravitational search algorithm–based fuzzy control for a nonlinear ball and beam system. Next, Zang et al. in [3] have implemented control for the physical security and cybersecurity of multi-agent systems.

Stabilization and output tracking problems recently for non-minimum phase systems have been intensively investigated. In [4], Galeani et al. were able to asymptotically track the output of a nonminimum phase linear system through steady-state compensation. Ahmadin et al. have design control on a nonminimum phase bilinear system by the backstepping method [5]. Khozin et al. have investigated tracking on a bilinear control system with unstable and uncertain internal dynamics using adaptive backstepping [6]. Ho et al. have designed control of non-minimum phase nonlinear systems with input-output linearization on a system with the relative degree 1 [7]. Firman et al. have designed control of a non-minimum phase nonlinear system with a modification of steepest descent control to track desired output. The control design is done on the system with the relative degree $n-1$, where n is a dimension of the system [8]. Furthermore, Naiborhu et al. have discussed output tracking on a non-minimum phase nonlinear system with the relative degree n, where n is a dimension of the system. The input control design is determined through the input-output linearization method [9].

One of the methods to solve the stability problem is by the output redefinition. By redefining the output, it is expected that the system has a minimum phase or can be linearized exactly. Some researchers in designing controls begin by redefining the output, as can be seen in [7, 8, 9, 10]. Next, Yang et al. in [11] have investigated tip-trajectory

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tracking control of single-link flexible robots via output redefinition. In [12], Moallem et al. have designed vibration control of flexible link manipulators using the output definition method. Riccardo Marino and Patrizio Tomei in [13] have discussed the stabilization of a non-minimum phase non-linear system. The control design begins with redefining the output so that the system is a minimum phase with respect to the new output. Furthermore, in [14], Zili Li et al. have designed a control to stabilize a non-minimum phase non-linear system with uncertain parameters by using the output redefinition. The particle swarm optimization (PSO) method has many uses in the control area. Heo et al. in [15] used PSO on multi-objective control of power plants. Optimization of a fuzzy logic control-based maximum power point tracking algorithm using the particle swarm optimization is investigated by Cheng et al. in [16]. Chabra et al. in [17] have designed a precise tracking controller with minimum control effort for a robotic manipulator with the use of PSO. Next, Liren Zou has designed a reactive power optimization control for an electromechanical system based on fuzzy with a used particle swarm optimization algorithm [18].

In this paper, we design control for a specific class of nonlinear systems are development from [10, 19] by choose a new output. There are some theorems that represent the nonlinear system has a minimum phase with the new output. Furthermore, we find the desired new output based on the desired output of the original system by PSO. The good result of this approach is demonstrated by two examples: the nonlinear system with the relative degrees 2 and 3.

2. Definition and Problem Statements

Considering the nonlinear control system as single input-single output, we obtain:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \tag{1}$$

$$y = h(\mathbf{x}). \tag{2}$$

where $\mathbf{x} \in R^n$ denotes the state vector, $u \in R$ denotes the control input, and $y \in R$ denotes the measured output. $\mathbf{f} : R^n \rightarrow R^n$, $\mathbf{g} : R^n \rightarrow R^n$, and $h : R^n \rightarrow R$ denote smooth functions.
Hereafter, we define the relative degree of the nonlinear system (1)-(2) [20].

Definition 1. Nonlinear system (1)-(2) is said to have relative degree r at $\mathbf{x}_0$, if

(i) $L_g L_f^k h(\mathbf{x}) = 0$ for all $\mathbf{x}$ in a neighborhood of $\mathbf{x}_0$ and $k = 0, 1, 2, ..., r-2$, and
(ii) $L_g L_f^{r-1} h(\mathbf{x}_0) \neq 0$,

where
$L_g L_f^k h(\mathbf{x}) = \dfrac{\partial y^{(k)}}{\partial \mathbf{x}} g(\mathbf{x})$ and $L_g L_f^{r-1} h(\mathbf{x}_0) = \dfrac{\partial y^{(r-1)}}{\partial \mathbf{x}} g(\mathbf{x}_0)$. Definition 1 indicates that we need to differentiate the output y of system (1)-(2) with respect to t r times to obtain an explicit relationship between the output y and input u.
Suppose that the relative degree of the system is r, where $r < n$, then we have

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(\mathbf{x}).$$

The following lemma is referred from [20]:

Lemma 2. Suppose the non linear system (1)-(2) have relative degree r at $\mathbf{x}_0, r \leq n$.

$$\begin{array}{rcl} \phi_1(\mathbf{x}) & = & h(\mathbf{x}) \\ \phi_2(\mathbf{x}) & = & L_f h(\mathbf{x}) \\ & \vdots & \\ \phi_r(\mathbf{x}) & = & L_f^{r-1} h(\mathbf{x}). \end{array}$$

If r is strictly less than n , it is possible to determine $n - r$ more functions $\phi_{r+1}(\mathbf{x}), \phi_{r+2}(\mathbf{x}), ..., \phi_n(\mathbf{x})$ such that the mapping

$$\phi(\mathbf{x}) = \begin{pmatrix} \phi_1(\mathbf{x}) \\ \phi_2(\mathbf{x}) \\ \vdots \\ \phi_n(\mathbf{x}) \end{pmatrix}$$

has a jacobian matrix, which is nonsingular at $\mathbf{x}_0$ and therefore qualifies as local coordinates transformation in the neighborhood of $\mathbf{x}_0$. The value at $\mathbf{x}_0$ of these additional functions can be fixed arbitrarily. Moreover, it becomes possible to choose $\phi_{r+1}(\mathbf{x}), \phi_{r+2}(\mathbf{x}), ..., \phi_n(\mathbf{x})$ such that $L_g\phi_i(\mathbf{x}) = 0\ \forall r+1 \leq i \leq n$ and $\forall\mathbf{x}$ around $\mathbf{x}_0$.

If system (1)-(2) have a relative degree r at $\mathbf{x}_0$, where $r < n$, then system (1)-(2) can be transformed into

$$\dot{z}_k = z_{k+1}, \quad k = 1, 2, ..., r-1, \tag{3}$$

$$\dot{z}_r = a(z, \eta) + b(z, \eta)u, \tag{4}$$

$$\dot{\eta} = q(z, \eta), \tag{5}$$

where, $y = z_1, a(z, \eta) = L_f^r h(\mathbf{x})$ and $b(z, \eta) = L_g L_f^{r-1} h(\mathbf{x})$. The system (3)-(4) is called external dynamics. Meanwhile, the internal dynamics is $\dot{\eta} = q(z, \eta)$ and the zero dynamics is $\dot{\eta} = q(0, \eta)$.

Next, we refer to [19]. Here in, we examine a special form of the system (1)-(2), by replacing $f(\mathbf{x}) = A\mathbf{x}$, $g(\mathbf{x}) = B(\mathbf{x})$, and $h(\mathbf{x}) = C\mathbf{x}$, where $A \in R^{n\times n}$, $B \in R^{n\times n}$, and $C \in R^{1\times n}$.

Consider the following single input–single output nonminimum phase nonlinear control system:

$$\dot{\mathbf{x}} = A\mathbf{x} + B(\mathbf{x})u, \tag{6}$$
$$y = C\mathbf{x}. \tag{7}$$

where, $\mathbf{x} \in R^n$; $n \geq 4$; $u, y \in R$;

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & \cdots & a_{2(n-1)} & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & \cdots & a_{3(n-1)} & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & 0 & \cdots & a_{4(n-1)} & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(n-3)1} & a_{(n-3)2} & \cdots & & & & & a_{(n-3)(n-1)} & a_{(n-3)n} \\ a_{(n-2)1} & a_{(n-2)2} & \cdots & & & & & a_{(n-2)(n-1)} & a_{(n-2)n} \\ a_{(n-1)1} & 0 & \cdots & & & & & a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{n1} & 0 & \cdots & & & & & a_{n(n-1)} & a_{nn} \end{pmatrix};$$

$B(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 & \cdots & g_{n-2}(\mathbf{x}) & 0 & 0 \end{pmatrix}^T$; $\quad C = \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \end{pmatrix}$; $\quad a_{ij} \in R$; $\quad i, j = 1, 2, \cdots, n$; $a_{(n-1)(n-1)}, a_{nn} > 0$; $c_1 \in R, c_1 \neq 0$.

We use a coordinate transformation of the system (6)-(7), so that the zero dynamics $\dot{\eta} = q(0, \eta)$ is unstable; therefore, the system (6)-(7) is a nonminimum phase. Next, we perform a redefinition of the output to ensure that the system's minimum phase and design control align with the desired output.

3. Results and Discussion

In the following section, we constructed some theorems that show that the system has a non-minimum phase, and the system's non-minimum phase becomes a minimum phase with a new output. Next, we identify the desired new output using PSO, allowing us to track the original desired outcome.

3.1. The Control Design on Nonlinear System with Redefinition Output

The following theorem states that a non-minimum phase nonlinear system becomes a minimum phase with a new output.The following lemma and theorem relate to system (6)-(7):

Lemma 3. System (6)-(7) has a relative degree of $n-2$ at $\mathbf{x}_0$.
Proof.
The output of system (6)-(7) is $y = c_1 x_1$. As the elements of $B(x)$ in the $1, 2, \cdots, n-3$ row are zero, then by Definition 1, $L_{\mathbf{g}} h(\mathbf{x}) = L_{\mathbf{g}} L_{\mathbf{f}} h(\mathbf{x}) = L_{\mathbf{g}} L_{\mathbf{f}}^2 h(\mathbf{x}) = \cdots = L_{\mathbf{g}} L_{\mathbf{f}}^{n-4} h(\mathbf{x}) = 0$, for each $\mathbf{x}$ in neighborhood of $\mathbf{x}_0$ and $L_{\mathbf{g}} L_{\mathbf{f}}^{n-3} h(\mathbf{x}_0) = c_1 a_{12} a_{23} a_{34} \cdots a_{(n-3)(n-2)} g_{n-2}(\mathbf{x})$. Therefore, system (6)-(7) has relative degree of $n-2$. Qed

The following theorem indicates that system (6)-(7) belongs a non-minimum phase.
Theorem 4. System (6)-(7) with $a_{(n-1)(n-1)}, a_{nn} > 0$ possess a non-minimum phase.
Proof.
As the relative degree of the system (6)-(7) is $n-2$, based on Lemma 2, we can execute coordinate transformation, so that (6)-(7) can be expressed as

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ &\vdots \\ \dot{z}_{n-3} &= z_{n-2}, \\ \dot{z}_{n-2} &= a(z,\eta) + b(z,\eta)u, \end{aligned}$$

where,

$$\begin{aligned} z_1 &= c_1 x_1, \\ a(z,\eta) &= c_1\big(a_{11}{x_1}^{(n-3)} + a_{1(n-1)}{x_{n-1}}^{(n-3)} + a_{1n}{x_n}^{(n-3)}\big) \\ &\quad + c_1 a_{12}\big(a_{21}{x_1}^{(n-4)} + a_{22}{x_2}^{(n-4)} + a_{2(n-1)}{x_{n-1}}^{(n-4)} + a_{2n}{x_n}^{(n-4)}\big) \\ &\quad + c_1 a_{12} a_{23}\big(a_{31}{x_1}^{(n-5)} + a_{32}{x_2}^{(n-5)} + a_{33}{x_3}^{(n-5)} + a_{3(n-1)}{x_{n-1}}^{(n-5)} \\ &\quad + a_{3n}{x_n}^{(n-5)}\big) + c_1 a_{12} a_{23} a_{34}\big(a_{41}{x_1}^{(n-6)} + a_{42}{x_2}^{(n-6)} + a_{43}{x_3}^{(n-6)} \\ &\quad + a_{44}{x_4}^{(n-6)} + a_{4(n-1)}{x_{n-1}}^{(n-6)} + a_{4n}{x_n}^{(n-6)}\big) + \cdots \\ &\quad + c_1 a_{12} a_{23} a_{34} \cdots a_{(n-3)(n-2)}\big(a_{(n-2)1}x_1 + a_{(n-2)2}x_2 + \cdots + a_{(n-2)n}x_n\big), \end{aligned}$$

$$b(z,\eta) = c_1 a_{12} a_{23} a_{34} \cdots a_{(n-3)(n-2)} g_{n-2}(\mathbf{x})$$
$$(z,\eta) = (z_1, z_2, \cdots, z_{n-2}, \eta_1, \eta_2).$$

Next, we choose $\eta_1(\mathbf{x}) = x_{n-1}, \eta_2(\mathbf{x}) = x_n$, so that $\dfrac{\partial \eta_i(\mathbf{x})}{\partial \mathbf{x}} B(\mathbf{x}) = 0; i = 1, 2$. We obtain the internal dynamics of (6)-(7), which can be expressed as

$$\dot{\eta} = \begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} \dfrac{a_{(n-1)1}}{c_1} z_1 + a_{(n-1)(n-1)}\eta_1 + a_{(n-1)n}\eta_2 \\ \dfrac{a_{n1}}{c_1} z_1 + a_{n(n-1)}\eta_1 + a_{nn}\eta_2 \end{pmatrix}.$$

By substituting the value of $z = 0$ into the internal dynamics, we obtain zero dynamics as

$$\dot{\eta} = \begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{n(n-1)} & a_{nn} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

Furthermore, the eigenvalue is obtained as

$$\begin{vmatrix} \lambda - a_{(n-1)(n-1)} & -a_{(n-1)n)} \\ -a_{n(n-1)} & \lambda - a_{nn} \end{vmatrix} = 0,$$

thus, we obtained

$$\lambda_{1,2} = \frac{(a_{(n-1)(n-1)} + a_{nn})}{2} \pm \frac{\sqrt{(a_{(n-1)(n-1)} + a_{nn})^2 - 4(a_{(n-1)(n-1)}a_{nn} - a_{(n-1)n}a_{n(n-1)})}}{2}.$$

. Therefore $a_{(n-1)(n-1)}, a_{nn} > 0$, then λ_1 or λ_2 is positive or λ_1 and λ_2 possess a positive real part. Consequently the zero dynamics of the system (6)-(7) become unstable, so that system (6)-(7) possess a non-minimum phase. Qed

Lemma 5. System (6) with a new output, i.e.,$y_b = c_{n-1}x_{n-1}, c_{n-1} \in R, c_{n-1} \neq 0$, has a relative degree of $n-1$ at $\mathbf{x}_0$.

Proof.

The new output of system (6) is $y_b = c_{n-1}x_{n-1}$. As $\dot{x}_1$ appears on $\ddot{y}$ and the elements of $B(x)$ in the $1, 2, \cdots, n-3$ row and $n-1$ row are zero, then by Definition 1,$L_{\mathbf{g}}h(\mathbf{x}) = L_{\mathbf{g}}L_{\mathbf{f}}h(\mathbf{x}) = L_{\mathbf{g}}L_{\mathbf{f}}^2h(\mathbf{x}) = \cdots = L_{\mathbf{g}}L_{\mathbf{f}}^{n-3}h(\mathbf{x}) = 0$, for each $\mathbf{x}$ in neighborhood of $\mathbf{x}_0$ and $L_gL_f^{n-2}h(\mathbf{x}_0) = c_{n-1}a_{(n-1)1}a_{12}a_{23}a_{34}\cdots a_{(n-3)(n-2)}g_{n-2}(\mathbf{x})$. Therefore, system (6) with a new output $y_b = c_{n-1}x_{n-1}$ has a relative degree of $n-1$. Qed

Theorem 6. Suppose that system (6)-(7) with $a_{(n-1)(n-1)}, a_{nn} > 0$. If system (6) is assigned a new output $y_b = c_{n-1}x_{n-1}, c_{n-1} \in R, c_{n-1} \neq 0, a_{(n-1)1} > 0$ and $a_{(n-1)1}a_{nn} < a_{n1}a_{(n-1)n}$, system (6) with the new output $y_b = c_{n-1}x_{n-1}$ possesses minimum phase phase.

Proof.

As the relative degree of system (6) with $\mathrm{y}_b = c_{n-1}x_{n-1}$ is $n-1$, then based on Lemma 2, coordinate transformation can be performed, so that system (6) with $\mathrm{y}_b = c_{n-1}x_{n-1}$ can be expressed in the form

$$\begin{aligned}
\dot{z}_1 &= z_2,\\
\dot{z}_2 &= z_3,\\
&\vdots\\
\dot{z}_{n-2} &= z_{n-1},\\
\dot{z}_{n-1} &= a(z,\eta) + b(z,\eta)u,
\end{aligned}$$

where

$$\begin{aligned}
z_1 &= c_{n-1}x_{n-1},\\
a(z,\eta) &= c_{n-1}(a_{(n-1)(n-1)}x_{n-1}{}^{(n-2)} + a_{(n-1)n}x_n{}^{(n-2)})\\
&\quad + c_{n-1}a_{(n-1)1}(a_{11}x_1{}^{(n-3)} + a_{1(n-1)}x_{n-1}{}^{(n-3)} + a_{1n}x_n{}^{(n-3)})\\
&\quad + c_{n-1}a_{(n-1)1}a_{12}(a_{21}x_1{}^{(n-4)} + a_{22}x_2{}^{(n-4)} + a_{2(n-1)}x_{n-1}{}^{(n-4)}\\
&\quad + a_{2n}x_n{}^{(n-4)}) + c_{n-1}a_{(n-1)1}a_{12}a_{23}(a_{31}x_1{}^{(n-5)} + a_{32}x_2{}^{(n-5)}\\
&\quad + a_{33}x_3{}^{(n-5)} + a_{3(n-1)}x_{n-1}{}^{(n-5)} + a_{3n}x_n{}^{(n-5)}) + \cdots\\
&\quad + c_{n-1}a_{(n-1)1}a_{12}a_{23}a_{34}\cdots a_{(n-3)(n-2)}(a_{(n-2)1}x_1 + a_{(n-2)2}x_2\\
&\quad + \cdots + a_{(n-2)n}x_n),\\
b(z,\eta) &= c_{n-1}a_{(n-1)1}a_{12}a_{23}a_{34}\cdots a_{(n-3)(n-2)}g_{n-2}(\mathbf{x}).
\end{aligned}$$

Next, we choose $\eta(\mathbf{x}) = x_n$, so that $\dfrac{\partial\eta(\mathbf{x})}{\partial\mathbf{x}}B(\mathbf{x}) = 0$. We obtain the internal dynamics of system (6) with $y_b = c_{n-1}x_{n-1}$, which can be expressed as

$$\dot{\eta} = \dot{x}_n = \frac{a_{n1}}{c_{n-1}a_{(n-1)1}}(z_2 - c_{n-1}a_{(n-1)(n-1)}z_1 - c_{n-1}a_{(n-1)n}\eta) + a_{n(n-1)}z_1 + a_{nn}\eta.$$

By substituting $z = 0$ into the internal dynamics, we obtain zero dynamics, which can be expressed as

$$\dot{\eta} = \frac{a_{(n-1)1}a_{nn} - a_{n1}a_{(n-1)n}}{a_{(n-1)1}}\eta.$$

So that, we get $\eta = ke^{\frac{a_{(n-1)1}a_{nn} - a_{n1}a_{(n-1)n}}{a_{(n-1)1}}t}, k \in R$ is an arbitrary constant.

Therefore $a_{(n-1)1} > 0$ and $a_{(n-1)1}a_{nn} < a_{n1}a_{(n-1)n}$, then $a_{(n-1)1}a_{nn} - a_{n1}a_{(n-1)n} < 0$. This means,

$\dfrac{a_{(n-1)1}a_{nn} - a_{n1}a_{(n-1)n}}{a_{(n-1)1}} < 0$, consequently the zero dinamcs the system (6) with a new output $y_b = c_{n-1}x_{n-1}$ is asymptotically stable, so that the system (6) with a new output $y_b = c_{n-1}x_{n-1}$ has a minimum phase. Qed

Lemma 7. System (6) with a new output, i.e.,$y_b = c_n x_n, c_n \in R, c_n \neq 0$, has a relative degree of $n-1$ at $\mathbf{x}_0$.
Proof.
The new output of system (6) is $y_b = c_n x_n$. As $\dot{x}_1$ appears on $\ddot{y}$ and the elements of $B(x)$ in the $1, 2, \cdots, n-3$ row and n row are zero, then by Definition 1,$L_{\mathbf{g}}h(\mathbf{x}) = L_{\mathbf{g}}L_{\mathbf{f}}h(\mathbf{x}) = L_{\mathbf{g}}L_{\mathbf{f}}^2 h(\mathbf{x}) = \cdots = L_{\mathbf{g}}L_{\mathbf{f}}^{n-3}h(\mathbf{x}) = 0$, for each $\mathbf{x}$ in neighborhood of $\mathbf{x}_0$ and $L_g L_f^{n-2}h(\mathbf{x}_0) = c_{n-1}a_{(n-1)1}a_{12}a_{23}a_{34}\cdots a_{(n-3)(n-2)}g_{n-2}(\mathbf{x})$. Therefore, system (6) with a new output $y_b = c_n x_n$ has a relative degree of $n-1$. Qed

Theorem 8. Suppose that system (6)-(7) with $a_{(n-1)(n-1)}, a_{nn} > 0$. If system (6) is assigned a new output $y_b = c_n x_n, c_n \in R, c_n \neq 0, a_{n1} > 0$ and $a_{n1}a_{(n-1)(n-1} < a_{(n-1)1}a_{n(n-1)}$, system (6) with the new output $y_b = c_n x_n$ possesses minimum phase phase.
Proof.
As the relative degree of system (6) with $\mathrm{y}_b = c_n x_n$ is $n-1$, then based on Lemma 2, coordinate transformation can be performed, so that system (6) with $\mathrm{y}_b = c_n x_n$ can be expressed in the form

$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= z_3, \\
&\vdots \\
\dot{z}_{n-2} &= z_{n-1}, \\
\dot{z}_{n-1} &= a(z,\eta) + b(z,\eta)u,
\end{aligned}$$

where

$$\begin{aligned}
z_1 &= c_n x_n, \\
a(z,\eta) &= c_n(a_{n(n-1)}x_{n-1}{}^{(n-2)} + a_{nn}x_n{}^{(n-2)}) \\
&\quad + c_n a_{n1}(a_{11}x_1{}^{(n-3)} + a_{1(n-1)}x_{n-1}{}^{(n-3)} + a_{1n}x_n{}^{(n-3)}) \\
&\quad + c_n a_{n1}a_{12}(a_{21}x_1{}^{(n-4)} + a_{22}x_2{}^{(n-4)} + a_{2(n-1)}x_{n-1}{}^{(n-4)} + a_{2n}x_n{}^{(n-4)}) \\
&\quad + c_n a_{n1}a_{12}a_{23}(a_{31}x_1{}^{(n-5)} + a_{32}x_2{}^{(n-5)} + a_{33}x_3{}^{(n-5)} \\
&\quad + a_{3(n-1)}x_{n-1}{}^{(n-5)} + a_{3n}x_n{}^{(n-5)} + \cdots \\
&\quad + c_n a_{n1}a_{12}a_{23}a_{34}\cdots a_{(n-3)(n-2)}(a_{(n-2)1}x_1 + a_{(n-2)2}x_2 + \cdots + a_{(n-2)n}x_n), \\
b(z,\eta) &= c_n a_{n1}a_{12}a_{23}a_{34}\cdots a_{(n-3)(n-2)}g_{n-2}(\mathbf{x}).
\end{aligned}$$

Next, we choose $\eta(\mathbf{x}) = x_{n-1}$, so that $\dfrac{\partial\eta(\mathbf{x})}{\partial\mathbf{x}}B(\mathbf{x}) = 0$. We obtain the internal dynamics of system (6) with $y_b = c_n x_n$, which can be expressed as

$$\dot{\eta} = \dot{x}_{n-1} = \frac{a_{(n-1)1}}{c_n a_{n1}}(z_2 - c_n a_{nn}z_1 - c_n a_{n(n-1)}\eta) + a_{(n-1)n}z_1 + a_{(n-1)(n-1)}\eta.$$

By substituting $z = 0$ into the internal dynamics, we obtain zero dynamics, which can be expressed as

$$\dot{\eta} = \dot{x}_{n-1} = \frac{a_{n1}a_{(n-1)(n-1)} - a_{(n-1)1}a_{n(n-1)}}{a_{n1}}\eta.$$

So that, we get $\eta = ke^{\frac{a_{n1}a_{(n-1)(n-1)} - a_{(n-1)1}a_{n(n-1)}}{a_{n1}}t}, k \in R$ is an arbitrary constant.
Therefore $a_{n1} > 0$ dan $a_{n1}a_{(n-1)(n-1)} < a_{(n-1)1}a_{n(n-1)}$, then $a_{n1}a_{(n-1)(n-1)} - a_{(n-1)1}a_{n(n-1)} < 0$. This means, $\dfrac{a_{n1}a_{(n-1)(n-1)} - a_{(n-1)1}a_{n(n-1)}}{a_{n1}} < 0$, consequently the zero dinamcs the system (6) with a new output $y_b = c_n x_n$ is asymptotically stable, so that the system (6) with a new output $y_b = c_n x_n$ has a minimum phase. Qed

Next, the input control can be written as a static control law

$$u = \frac{1}{b(z,\eta)}(-a(z,\eta) + v), \tag{8}$$

where
$b(z,\eta) \neq 0, \forall t$ and $v = -k_0 z_1 - k_1 z_2 - \cdots - k_{n-2} z_{n-1}$. Thus, the external dynamics of the system (6) with the new output can be written as

$$\dot{z} = A_1 z, \tag{9}$$

where $A_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -k_0 & -k_1 & -k_2 & \cdots & -k_{n-2} \end{pmatrix}$.

The characteristic polynomial of the matrix A_1 is $p(\lambda) = k_0 + k_1\lambda + \cdots + k_{n-2}\lambda^{n-2} + \lambda^{n-1}$, where λ is the characteristic value of the matrix A_1. By using the Hurwith criterion,then can be chosen the value of $k_0, k_1, \cdots, k_{n-2}$ so that the characteristic value of the matrix A_1 has a negative real part, so that the system (9) becomes asymptotically stable. Next, the controller value u that causes the system (9) to become asymptotically stable is substituted into the original system, so that the original system with the new output becomes asymptotically stable.

3.2. Particle Swarm Optimization

In this section we will approximate the $x_{(n-1)d}(t)$ and $x_{nd}(t)$ using PSO. Let $x_{(n-1)d}(t)$ and $x_{nd}(t)$ in the form of Fourier series:

$$\begin{array}{lcl} x_{(n-1)d}(t) & = & \alpha + \sum_{i=1}^{(n-1)} \beta_i sin(t) + \gamma_i cos(t), \\ x_{nd}(t) & = & \alpha + \sum_{i=1}^{(n-1)} \beta_i sin(t) + \gamma_i cos(t). \end{array} \tag{10}$$

Furthemore, we use PSO to determine the value of parameters $\alpha, \beta_i, \gamma_i; i = 1, 2, \cdots, n-1$ such that $\int_0^T (y(t) - y_d(t))dt$ tend to zero.

Let NM is the number of parameter which is determined by PSO. NS is the number of swarm which is used. NIT is the number of iteration maximum which admissible. Next, X is array of swarm position that it become candidate solution of parameter which be find, then X can be written in the form

$$X = Array(NM, NS, NIT).$$

V is array velocity swarm wherev can be written in the form

$$V = Array(NM, NS, NIT).$$

Pb and Gb are personal best array and global best array can be state as

$$Pb = Array(NM, NS, NIT),$$

$$Gb = Array(NM, 1, NIT).$$

F is array ofthe fitness value of swarm. The fitness value is integral error of $e(t) = y(t) - y_d(t)$. Next, F can be written as

$$F = Array(1, NS, NIT).$$

The fitness value is defined as

$$IAE = \int_{t_0}^{t_f} |(y(t) - y_d(t))|dt$$

PSO Procedure

1. Initialization.
   generate the initial position and velocity of swarm randomly.
$$X_1^i = X_{min} + rand(0,1) \cdot (X_{max} - X_{min})$$
$$V_1^i = V_{min} + rand(0,1) \cdot (V_{max} - V_{min})$$
   Generate the value of inertia weight
$$W = W_{max} - \frac{W_{max} - W_{min}}{k_{max}} \cdot k$$
2. Calculate the fitness value of X
   (i) Solve the external dynamics of the system (6) with the new output with control law (8).
   (ii) Calculate the fitness value:
$$IAE = \int_{t_0}^{t_f} |(y(t) - y_d(t))|dt.$$
3. Determine the personal and global best.
4. Update the swarm position
$$X_{k+1}^i = X_k^i + V_{k+1}^i,$$
   where
$$V_{k+1}^i = V_{k+1}^i + \psi \cdot c_1(Pb_k^i - X_k^i) + \psi \cdot c_2(Gb_k^i - X_k^i).$$
5. Repeat the second step until stopping creteria is satisfied.

3.3. Examples

In this section, we give two examples of design control on nonlinear systems with relative degrees of two and three for tracking the desired output. We design control use by redefining the new output and Particle Swarm Optimization method. We design the system for tracking the desired output, $y_d(t) = sin(t)$.

1. Consider the following single input-single output nonlinear control system:

$$\begin{array}{rcl} \dot{x}_1 &=& x_1 + x_2 + x_3 - 5x_4, \\ \dot{x}_2 &=& x_1 + x_2 - 2x_3 + x_4 + (x_1 + x_2 - x_3^2 + x_4)u, \\ \dot{x}_3 &=& x_1 + 3x_3 + 4x_4, \\ \dot{x}_4 &=& x_1 + 7x_3 + 4x_4, \end{array} \tag{11}$$

$$y = x_1. \tag{12}$$

Using coordinate transformation, system (11)-(12) become

$$\begin{array}{rcl} \dot{z}_1 &=& z_2, \\ \dot{z}_2 &=& a(z,\eta) + b(z,\eta)u, \\ \dot{\eta}_1 &=& z_1 + 3\eta_1 + 4\eta_2, \\ \dot{\eta}_2 &=& z_1 + \eta_1 + 2\eta_2, \end{array} \tag{13}$$

where

$$\begin{array}{rcl} z_1 &=& x_1, \\ a(z,\eta) &=& -2x_1 + 2x_2 - 33x_3 - 10x_4, \\ b(z,\eta) &=& x_1 + x_2 - x_3^2 + x_4, \\ \eta_1 &=& x_3, \\ \eta_2 &=& x_4. \end{array}$$

We can express zero dynamics as

$$\dot{\eta} = \begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 7 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

so that the following eigen values can be obtained: $\lambda_1 = \frac{5}{2} + \frac{\sqrt{113}}{2} > 0$ and $\lambda_2 = \frac{5}{2} - \frac{\sqrt{113}}{2} < 0$.
As the eigen value is positive, the zero dynamics become unstable; thus, system (11)-(12) non-minimum phase.

Next, we define a new output

$$y_b = x_3. \tag{14}$$

Using coordinate transformation, system (11)-(14) can be expressed as

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ \dot{z}_3 &= a(z,\eta) + b(z,\eta)u, \\ \dot{\eta} &= 4z_1 + z_2 - 2\eta. \end{aligned} \tag{15}$$

where

$$\begin{aligned} z_1 &= x_3, \\ a(z,\eta) &= 62x_1 + 9x_2 + 225x_3 + 143x_4, \\ b(z,\eta) &= x_1 + x_2 - x_3^2 + x_4, \\ \eta &= x_4 \end{aligned}$$

We can express the zero dynamics as $\dot{\eta} = -2\eta$; thus, we obtain $\eta = ke^{-2t}$, where $k \in R$ denotes an arbitrary constant.

Therefore, the zero dynamics of system (11)-(14) become asymptotically stable, so that system (11) with $y_b = x_3$ exhibits a minimum phase.
Hence, The control

$$u = \frac{1}{b(z,\eta)}(-a(z,\eta) + v), \tag{16}$$

where
$v = \dot{z}_{3d} + \frac{1}{k_3}(-k_0(z_1 - z_{1d}) - k_1(z_2 - z_{2d}) - k_2(z_3 - z_{3d}))$ so that can track the new output desire y_b $(= x_{3d}(t))$.

The simulation is done as follows.
First, We approximate the $x_{3d}(t)$ as a fourier series.
We define

$$x_{3d}(t) = \alpha + \sum_{i=1}^{3} \beta_i sin(t) + \gamma_i cos(t).$$

By PSO (Initial value of the system is $[0,0,0,0]$;$t = [0,100]$; $NS = 40$; $NM = 7$; $c_1 = 1.5$ and $c_2 = 1.5$; $NIT = 100$), we obtained the value of parameters $\alpha, \beta_i, \gamma_i; i = 1,2,3$, see Table 1.
Second, We calculate IAE of y and y_d. The results is shown in Figure 1.

Figure 1 shows the change of IAE with the number of iterations. The IAE value converges to a relatively small value. A small IAE value indicates that the control system can track the desired trajectory very well. In the figure 2, we obtained the output x_3 can track x_{3d} for the system relative degree 2. Then, in the figure 3, we obtained the output y can track y_d for the system relative degree 2.

2. Consider the following single input-single output nonlinear control system:

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 + x_4 + 5x_5, \\ \dot{x}_2 &= x_1 + x_2 - 2x_3 + x_4 + x_5, \\ \dot{x}_3 &= x_1 - x_2 + x_3 + 3x_4 + x_5 + (2x_1 + x_2^2 + x_3x_4 + x_4 + x_5)u, \\ \dot{x}_4 &= x_1 + 4x_4 - x_5, \\ \dot{x}_5 &= x_1 + 7x_4 + 3x_5, \end{aligned} \tag{17}$$

Table 1. The table value of parameters

No	Parameter	Value
1	α	5.4293e-07
2	β_1	-0.1000
3	β_2	0.0996
4	β_3	-0.0916
5	γ_1	-0.0234
6	γ_2	0.1000
7	γ_3	-0.1000

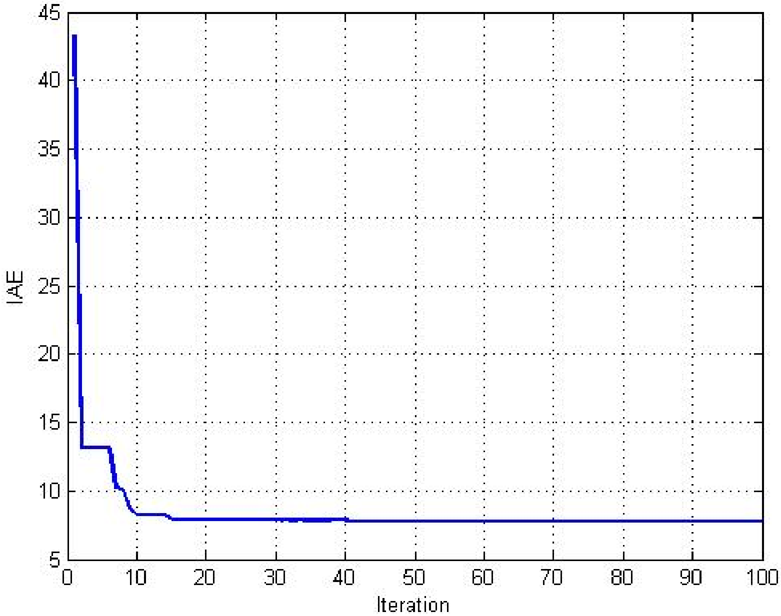

Figure 1. IAE evolution of y and y_d for the system relative degree 2

$$y = x_1. \tag{18}$$

Using coordinate transformation, system (17)-(18) become

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ \dot{z}_3 &= a(z,\eta) + b(z,\eta)u, \\ \dot{\eta}_1 &= z_1 + 4\eta_1 - \eta_2, \\ \dot{\eta}_2 &= z_1 + 7\eta_1 + 3\eta_2, \end{aligned} \tag{19}$$

where

$$\begin{aligned} z_1 &= x_1, \\ a(z,\eta) &= x_1 - x_2 + x_3 + 3x_4 + x_5, \\ b(z,\eta) &= 2x_1 + x_2^2 + x_3x_4 + x_4 + x_5, \\ \eta_1 &= x_4, \\ \eta_2 &= x_5. \end{aligned}$$

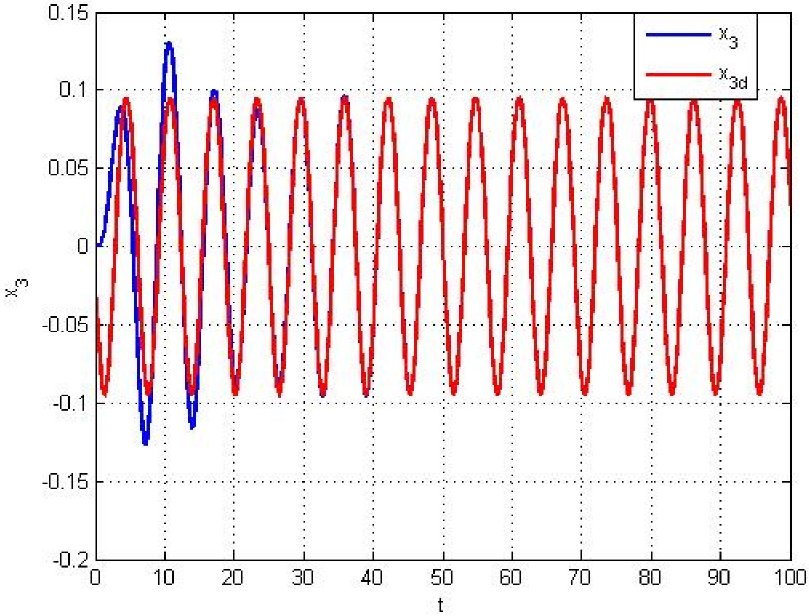

Figure 2. Output tracking x_3 to x_{3d} for the system relative degree 2

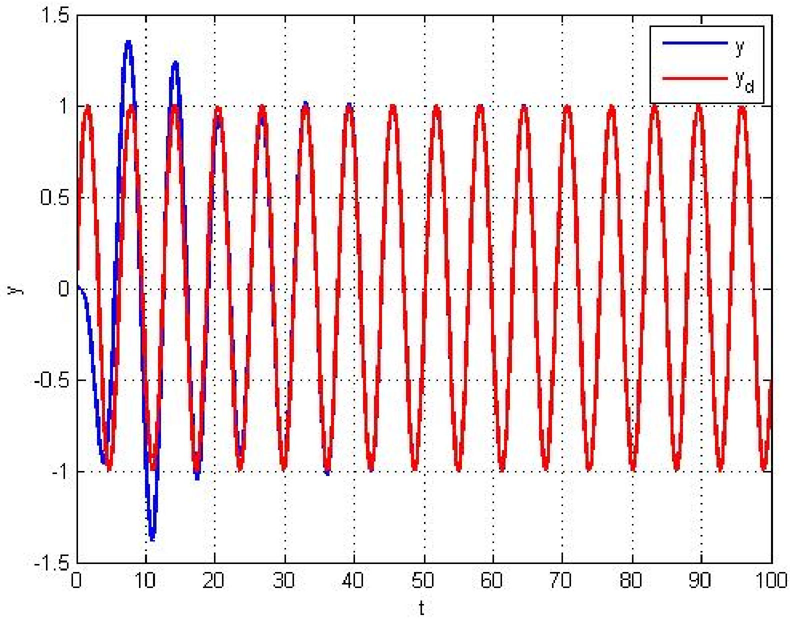

Figure 3. Output tracking y to y_d for the system relative degree 2

We can express zero dynamics as

$$\dot{\eta} = \begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

so that the following eigen values can be obtained: $\lambda_1 = \frac{7}{2} + \frac{3\sqrt{3}}{2} > 0$ and $\lambda_2 = \frac{7}{2} - \frac{3\sqrt{3}}{2} < 0$.

As the eigen value is positive, the zero dynamics become unstable; thus, system (17)-(18) non-minimum phase. Next, we define a new output

$$y_b = x_5. \tag{20}$$

Using coordinate transformation, system (17)-(20) can be expressed as

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ \dot{z}_3 &= z_4, \\ \dot{z}_4 &= a(z,\eta) + b(z,\eta)u, \\ \dot{\eta} &= -z_1 + z_2 - 3\eta. \end{aligned} \tag{21}$$

where

$$\begin{aligned} z_1 &= x_5, \\ a(z,\eta) &= 341x_1 - 81x_2 + 26x_3 + 1272x_4 + 141x_5, \\ b(z,\eta) &= 4x_1 + 2x_2^2 + 2x_3x_4 + 2x_4 + 2x_5, \\ \eta &= x_4 \end{aligned}$$

We can express the zero dynamics as $\dot{\eta} = -3\eta$; thus, we obtain $\eta = ke^{-3t}$, where $k \in R$ denotes an arbitrary constant.

Therefore, the zero dynamics of system (17)-(20) become asymptotically stable, so that system (17) with $y_b = x_5$ exhibits a minimum phase.

Hence, The control

$$u = \frac{1}{b(z,\eta)}(-a(z,\eta) + v), \tag{22}$$

where

$v = \dot{z}_{4d} + \frac{1}{k_4}(-k_0(z_1 - z_{1d}) - k_1(z_2 - z_{2d}) - k_2(z_3 - z_{3d}) - k_3(z_4 - z_{4d}))$ so that can track the new output desire y_b $(= x_{5d}(t))$.

The simulation is done as follows.

First, We approximate the $x_{5d}(t)$ as a fourier series.

We define

$$x_{5d}(t) = \alpha + \sum_{i=1}^{4} \beta_i sin(t) + \gamma_i cos(t).$$

By PSO (Initial value of the system is$[0,0,0,0,0]$;$t = [0,100]$; $NS = 40$; $NM = 9$; $c_1 = 1.5$ and $c_2 = 1.5$; $NIT = 100$), we obtained the value of parameters α, β_i and $\gamma_i, i = 1,2,3,4$, see table 2.

Second, We calculate IAE of y and y_d. The results is shown in Figure 4

Figure 1 shows the change of IAE with the number of iterations. The IAE value converges to a relatively small value. A small IAE value indicates that the control system can track the desired trajectory very well. Figure 5, we obtained the output x_5 can track x_{5d} for the system relative degree 3. Then, in the figure 6, we obtained the output y can track y_d for the system relative degree 3.

Table 2. The table value of parameters

No	Parameter	Value
1	α	2.8099e-06
2	β_1	-0.20000
3	β_2	-0.20000
4	β_3	-0.0759
5	β_4	-0.04764
6	γ_1	0.2000
7	γ_2	-0.2000
8	γ_3	0.11035
9	γ_4	0.2000

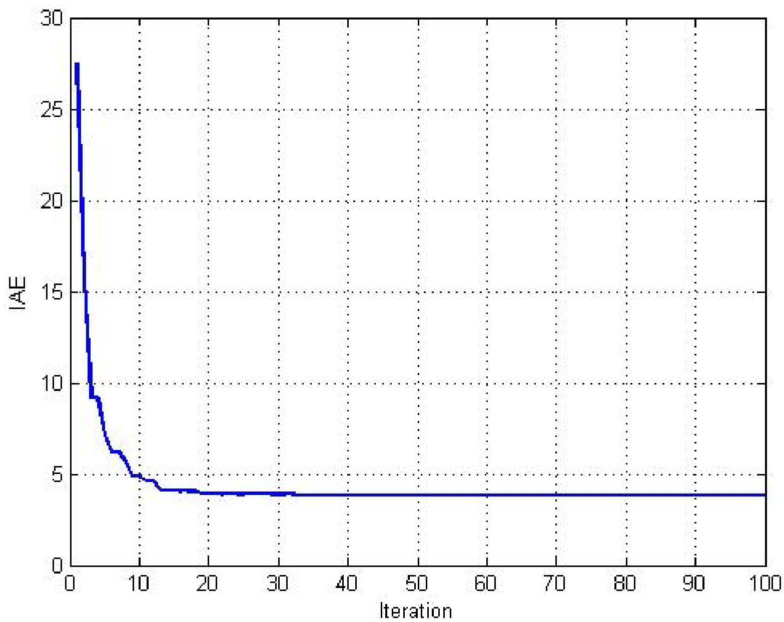

Figure 4. IAE evolution of y and y_d for the system relative degree 3

4. Conclusion

In this paper, we have investigated a control design on a certain class of nonlinear systems by redefining new output. We have constructed some theorems that state the non-minimum phase nonlinear system; the non-minimum phase nonlinear system becomes the minimum phase with the new output. From the results of the discussion, the desired output of the output that has been selected can be determined by PSO. Besides that, the output of the system can track the desired output.

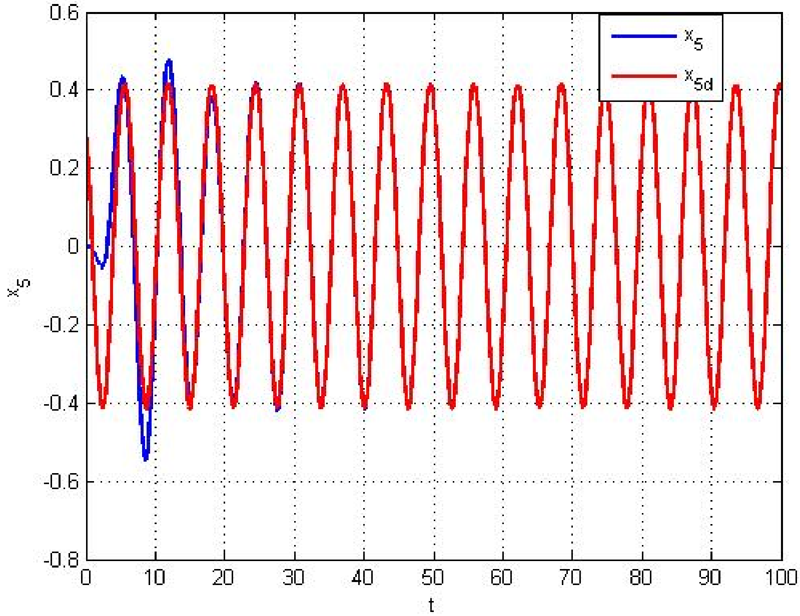

Figure 5. Output tracking x_5 to x_{5d} for the system relative degree 3

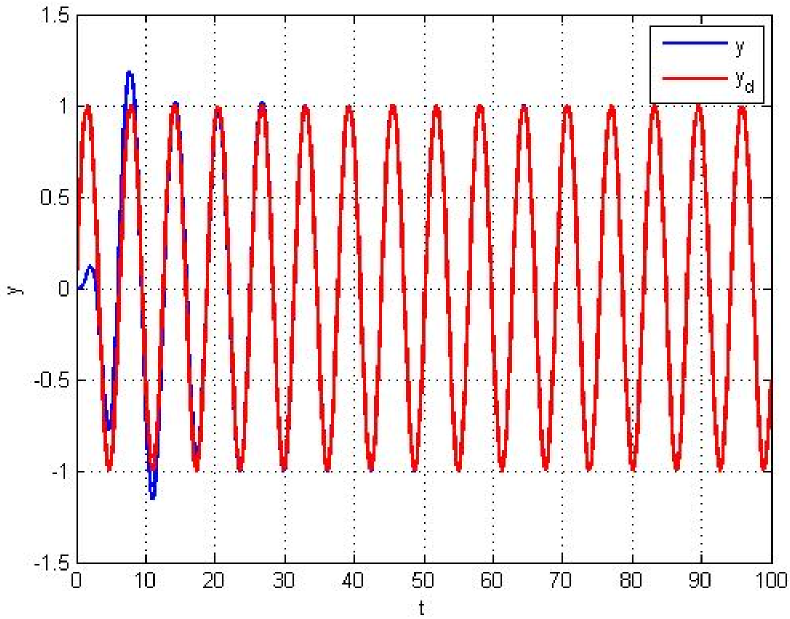

Figure 6. Output tracking y to y_d for the system relative degree 3

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