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Complexity Analysis of Large-Update Interior-Point Methods for $\mathscr{P}_*(\kappa)$-HLCP Based on a New Parametric Kernel Function

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Abstract This work proposes a primal-dual interior point technique for the $\mathscr{P}_*(\kappa)$-Horizontal Linear Complementarity Problem ($\mathscr{P}_*(\kappa)$-HLCP), based on a novel parameterized kernel function. Our new eligible parametric kernel function's feature produces the following iteration bound $O\left((p+1)n^{\frac{p+2}{2(p+1)}}\log\frac{n}{\varepsilon}\right)$ for the large-update method. Finally, we present numerical results demonstrating the algorithm's practical performance among various parameters.

Keywords $\mathscr{P}_*(\kappa)$-Horizontal Linear Complementarity Problem, interior-point methods, kernel function, large-update method.

AMS 2010 subject classifications 90C05, 90C51

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1. Introduction

Let $\mathscr{M}_n$ be the set of real square matrices of order n. The $\mathscr{P}_*(\kappa)$- horizontal linear complementarity problem involves determining a pair of vectors $(z,w)\in\mathbb{R}^{2n}$ that satisfy the following conditions :

$$\begin{cases} -Mz+Nw=q, \quad (z,w)\geq 0 \\ zw=0 \end{cases} \tag{1.1}$$

where the matrix pair $[M,N]$ belongs to the class of matrices $\mathscr{P}_*$ was originally proposed by Kojima et al. [17], i.e., for some nonnegative κ such that :

$$-Mz+Nw=0\Rightarrow(1+4\kappa)\sum_{i\in I_+(z)}z_iw_i+\sum_{i\in I_-(z)}z_iw_i\geq 0 \tag{1.2}$$

with $I_+(z)=\{i\in I: z_iw_i\geq 0\}$, $I_-(z)=\{i\in I: z_iw_i<0\}$ and $I=1,2,\ldots,n$.

In recent years, researchers have focused on linear complementarity problems (LCPs) and horizontal linear complementarity problems (HLCPs) for two reasons. First, LCPs include primal-dual linear optimization (LO) and convex quadratic optimization (CQO), which can be expanded to semidefinite and $\mathscr{P}_*$cases. Second, LCPs have numerous applications. The $\mathscr{P}_*(\kappa)$-horizontal linear complementarity issue is a specialized mathematical framework with applications in numerous domains. Here are some areas where it is particularly beneficial :

- Theory of Games : This approach is used to address equilibrium issues in non-cooperative games involving several players and sophisticated strategies. These issues are used in optimisation methods and control systems, particularly in settings with limitations and nonlinear dynamics.

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- Problems with Network Flow : $\mathscr{P}_*(\kappa)$-HLCP can be used to solve flow distribution problems in networks, such as traffic or communication systems.
- Engineering Systems : are used to analyse mechanical systems having frictional contact, such as robotics and structural mechanics.
- Economic Modiling : $\mathscr{P}_*(\kappa)$-HLCP is used in market equilibrium analysis to assist estimate pricing and allocations in competitive marketplaces.

This attention has been complemented by significant advancements in the field of LCP solutions. Scientists employed all current methods to solve the LCPs, as well as discovering new iterative methods for the same aim. The interior-point method (IPM) is one of the most efficient numerical approaches for addressing linear complementarity issues, as it has polynomial complexity, can handle very large problems, and is Newton-type. Most researchers use IPMs to solve LCP based on kernel functions developed by Bai et al. [6], such as [14] and [8], which introduced two novel kernel functions for linear complementarity problems, while other functions proposed to solve semidefinite linear complementarity problems (see [1], [2] and [4]). Several interior-point approaches have been employed to solve $\mathscr{P}_*(\kappa)$-HLCP, including those mentioned below.

Wang et al. [22] presented an interior-point algorithm for $\mathscr{P}_*(\kappa)$-HLCP in 2009, and in 2013 presented a feasible inerior-point method for $\mathscr{P}_*(\kappa)$-LCP over symmetric cones ($\mathscr{P}_*(\kappa)$-SCLCP) based on the classical logarithmic barrier function (see [17]). In 2015, Asadi et al. [5] introduced the new kernel function for $\mathscr{P}_*(\kappa)$-HLCP. In 2017, Li et al. [19] investigated IPMs for $\mathscr{P}_*(\kappa)$-LCPs using a novel parametric kernel function with trigonometric barrier term. El Ghami et al. [15] developed a generalized trigonometric barrier function to $\mathscr{P}_*(\kappa)$-LCPs. In 2021, Hazzam et al. [16] expanded and enhanced Bouafia et al. [9]â€™kernel function for LO to $\mathscr{P}_*(\kappa)$-HLCP. In 2023, Bouafia et al. [10] proposed the first efficient multiparameter kernel function with logarithmic barrier term for $\mathscr{P}_*(\kappa)$-HLCP.

This paper examines the difficulty and numerical implementation of a primal-dual IPM for $\mathscr{P}_*(\kappa)$-HLCP, a generic variant of $\mathscr{P}_*(\kappa)$-HLCP employs a novel parametric kernel function $\psi(x)$, which combines the terms of the classical kernel functions. Its growth term $(x-1)^2$ dominates the behaviour of $\psi(x)$ when x approaches infinity, while its barrier term $\psi_b(x)$ dominates its behaviour when x approaches zero, indicating a monotonic decrease in x. We explore our new kernel function and barrier function using basic analytic tools. The advantageous complexity bounds of our small- and large-update primal-dual algorithms are :

$$O\left((p+1)p^{\frac{p+2}{2(p+1)}}\sqrt{n}\log\frac{n}{\varepsilon}\right)$$

and

$$O\left((p+1)n^{\frac{p+2}{2(p+1)}}\log\frac{n}{\varepsilon}\right),$$

respectively, based on the kernel functionâ€™elegant analytic properties.

The paper is organised as follows. In Section 2, we briefly recap the essential principles of IPMs for $\mathscr{P}_*(\kappa)$-HLCP utilised in following parts. Then, a general primal-dual IPM for $\mathscr{P}_*(\kappa)$-HLCP is presented. In Section 3, we introduce the features of $\psi(x)$ and analyse the barrier function $\Psi(v)$. In Section 4, we examine the $\mathscr{P}_*(\kappa)$-HLCP algorithm and determine its complexity bound. Section 5 contains numerical findings. Section 6 has final remarks.

We can recollect the following well-known notion. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the inner product $\langle .,. \rangle$ and $\|.\|$ as 2-norm. $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ represent the sets of n-dimensional nonnegative and positive vectors, respectively. For $z, w \in \mathbb{R}^n$, z_{min} and zw represent the vectorâ€™smallest component and componentwise product, respectively. $Z = diag(z)$ refers to the $n \times n$ diagonal matrix where the diagonal elements are vector components. Finally, e represents the n-dimensional vector of ones. For $f(z), g(z) : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$, $f(z) = O(g(z))$ if $f(z) \leq C_1 g(z)$ for some positive constant C_1 and $f(z) = \Theta(g(z))$ if $C_2 g(z) \leq f(z) \leq C_3 g(z)$ for some positive constant C_2 and C_3.

2. Preliminaries

2.1. Central path for $\mathscr{P}_*(\kappa)$-HLCP

IPMs replace the second equation in (1.1) with the parameterized nonlinear equation $zw = \mu e$, resulting in the following system :

$$\begin{cases} -Mz+Nw=q, \quad (z,w)\geq 0 \\ zw=\mu e \end{cases} \tag{2.1}$$

where $\mu > 0$, $e = (1,\ldots,1)^T$ and zw in the last equation indicate the componentwise (Hadamard) product of vectors z and w. To solve HLCP, we utilise Newtonâ€™technique to solve the system of equations (2.1). This study focuses on the particular scenario when the HLCP is $\mathscr{P}_*(\kappa)$. In this instance, the matrix pair $[M,N]$ must satisfy both the condition (1.2) and the interior point condition (IPC) of the HLCP. This means that there exists $(z_0,w_0) > 0$ such that :

$$-Mz_0+Nw_0-q=0 \tag{2.2}$$

According to these assumptions, each $\mu > 0$ system (2.1) has a unique solution. $(z(\mu),w(\mu))$ represents the μ-center of $\mathscr{P}_*(\kappa)$-HLCP. The collection of μ-centers determines the central path of $\mathscr{P}_*(\kappa)$-HLCP. Kojima et al. [18] demonstrated that the central path has a limit when μ approaches zero, resulting in the optimum solution (1.1).

2.2. The search directions determined by kernel function

As previously explained, the essential concept of primal-dual IPMs is a straightforward application of Newtonâ€™technique to (6) generates the following system :

$$\begin{cases} -M\Delta z+N\Delta w=0 \\ \Delta z+z\Delta w=\mu e-zw \end{cases} \tag{2.3}$$

To simplify the analysis, we adopt the following notation :

$$v=\sqrt{\frac{zw}{\mu}}, \quad d_z=\frac{v\Delta z}{z} \quad \text{and} \quad d_w=\frac{v\Delta w}{w} \tag{2.4}$$

then the system from (2.3) can be transformed into the following system :

$$\begin{cases} -MZV^(-1)d_z+NWV^(-1)d_w=0 \\ d_z+d_w=v^(-1)-v \end{cases} \tag{2.5}$$

where $Z = diag(z)$, $W = diag(w)$ and $V = diag(v)$. Note that the pair (z,w) coincides with the μ-centre $(z(\mu),w(\mu))$ if and only if $v = e$. The right-hand side of the second equation in (2.5) is equivalent to minus the derivative of the traditional logarithmic barrier function $\Psi_l(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ defined as :

$$\Psi_l(v)=\Psi_l(z,w;\mu)=\sum_{i=1}^{n}\psi_l(v_i)=\sum_{i=1}^{n}\frac{v_i^2-1}{2}-\log v_i \tag{2.6}$$

where we call ψ_l the kernel function of the logarithmic barrier function Ψ_l. Thus, the system (2.5) becomes :

$$\begin{cases} -MZV^(-1)d_z+NWV^(-1)d_w=0 \\ d_z+d_w=-\nabla\Psi_l(v) \end{cases} \tag{2.7}$$

The proximity function $\Psi_l(v)$ measures the distance between the current iteration and the μ-center. We define the norm-based proximity measure $\delta(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ as follows :

$$\delta(v)=\frac{1}{2}\|\nabla\Psi_l(v)\|=\frac{1}{2}\|d_z+d_w\| \tag{2.8}$$

The system (2.7) has a unique solution (d_z,d_w) for any $\mu > 0$ that may be utilised to compute $(\Delta z,\Delta w)$ via (2.4). To create a new pair, we use a step size of $0 < \alpha \leq 1$ along the search directions

$$z_+=z+\alpha\Delta z, w_+=w+\alpha\Delta w \tag{2.9}$$

2.3. *The generic interior-point algorithm for $\mathscr{P}_*(\kappa)$-HLCP*

Input
A proximity function $\Psi_l(v)$; an accuracy parameter $\varepsilon > 0$; a threshold parameter $\tau \geq 1$; a fixed barrier update parameter $0 < \theta < 1$; initial point (z_0, w_0) and the parameter $\mu_0 > 0$.
Iteration
Begin
$z = z_0, w = w_0, \mu = \mu_0, v = \sqrt{\frac{zw}{\mu}}$;
While $(n\mu \geq \varepsilon)$ **do**
Begin
Update of μ and v : $\mu = (1-\theta)\mu$; $v = \frac{v}{\sqrt{1-\theta}}$;
While $(\Psi_l(v) > \tau)$ **do**
Begin
Solve (2.7) and (2.3) to find search directions;
Determine a step size $\alpha > 0$;
$(z, w) = (z, w) + \alpha(\Delta z, \Delta w)$;
$v = \sqrt{\frac{zw}{\mu}}$;
End
End
End

Generic interior-point algorithm for $\mathscr{P}_*(\kappa)$-HLCP.

The foregoing explanation clearly indicates that the method described below acts as follows: It starts with the parameters $\varepsilon > 0$, $\tau \geq 1$, $0 < \theta < 1$, the parameter $\mu > 0$, a beginning point (z_0, w_0), and the scaled vector v (should be set in such a way that the algorithmâ€™s necessary number of iterations is as low as possible). To determine the search directions $(\Delta z, \Delta w)$ using the two systems (2.3) and (2.7), compute the step size α and then use (2.9) to produce new iterations. Therefore, in each outer iteration, we update the values of μ and v by the fraction $1-\theta$. If the condition $\Psi_l(v) > \tau$ is met, we enter the inner iteration. We proceed in this fashion until μ is small enough and v agree on $\Psi_l(v) \leq \tau$, at which point we claim to have uncovered the optimum solution to the $\mathscr{P}_*(\kappa)$-HLCP.

3. Properties of the new parametric kernel function

Our goal in this study is to offer a primal-dual interior-point method for a $\mathscr{P}_*(\kappa)$-HLCP based on a novel parameterized kernel function with a logarithmic barrier term of the form :

$$\psi_L(x) = (x-1)^2 + \psi_b(x) = (x-1)^2 + x\log x + \frac{x^{-p}-1}{p}, \quad p > 0 \tag{3.1}$$

which we use in our algorithm's complexity analysis, thus we are fully aware that a properly parameterized kernel function may enhance the numerical characteristics of the matrices and systems solved during iterations, hence stabilizing the interior-point approach. In this part, we also examine some basic aspects of the new kernel function. We must first review the definition that follows.

Definition 3.1. $\psi_L(x) : \mathbb{R}_{++} \to \mathbb{R}_+$ is a kernel function if ψ_L is twice differentiable and satisfies the following conditions :

- $\psi_L(1) = \psi_L'(1) = 0$.
- $\psi_L''(x) > 0, \quad \forall x > 0$.
- $\lim_{x\to+\infty} \psi_L(x) = \lim_{x\to 0^+} \psi_L(x) = +\infty$.

The first three derivatives of (3.1) with regard to x are as follows :

$$\psi_L^{'}(x) = 2(x-1) + \log(x) + 1 - x^{-p-1} \tag{3.2}$$

$$\psi_L^{''}(x) = 2 + \frac{1}{x} + (p+1)x^{-p-2} \tag{3.3}$$

$$\psi_L^{'''}(x) = \frac{-1}{x^2} - (p+1)(p+2)x^{-p-3} \tag{3.4}$$

Lemma 3.1
Let $\psi_L(x)$ be as defined in (3.1). Then :

$$x\psi_L^{''}(x) - \psi_L^{'}(x) > 0, \quad \forall x > 0 \tag{3.5}$$

$$x\psi_L^{''}(x) + \psi_L^{'}(x) > 0, \quad \forall x > 0 \tag{3.6}$$

$$2(\psi_L^{''})^2(x) - \psi_L^{'}(x)\psi_L^{'''}(x) > 0, \quad \forall x > 0 \tag{3.7}$$

$$\psi_L^{'''}(x) < 0, \quad \forall x > 0 \tag{3.8}$$

$$\psi_L^{''} > 2 > 0, \quad \forall x > 0 \tag{3.9}$$

Proof
The three first inequalities holds due to (3.2),(3.3) and (3.4) :

$x\psi_L^{''}(x) - \psi_L^{'}(x) = 2 - \log(x) + (p+1)x^{-p-1} > 0,$

$x\psi_L^{''}(x) + \psi_L^{'}(x) = 4x + \log(x) + px^{-p-1} > 0,$

$$\begin{aligned} 2(\psi_L^{''})^2(x) - \psi_L^{'}(x)\psi_L^{'''}(x) &= 8 + \frac{10}{x} + \frac{\log(x)}{x^2} + 4(p+1)x^{-p-1} + 2(p+1)(p+6)x^{-p-2} \\ &\quad + ((p+1)(p+2)(\log(x)-1) - 1)x^{-p-3} + p(p+1)x^{2p-4} > 0. \end{aligned}$$

From (3.4), the positivity of x and p, we obtain $\psi_L^{'''}(x)$. Thus, the desired result (3.8). Additionally, (3.3) indicate that we have :

$\psi_L^{''}(x) = 2 + \frac{1}{x} + (p+1)x^{-p-2} \geq 2 > 0$

which completes the proof. □

The four first inequalities leads us to the conclusion that our kernel function, $\psi_L(x)$ is an eligible kernel function based on the aforementioned lemma.

Lemma 3.2
For all $p > 0$, the kernel function $\psi_L(x)$ hold the following result :

$$(x-1)^2 \leq \psi_L(x) \leq \frac{1}{4}(\psi_L^{'}(x))^2, \quad \forall x > 0 \tag{3.10}$$

$$\psi_L(x) \leq \frac{p+4}{2}(x-1)^2, \quad \forall x \geq 1 \tag{3.11}$$

Proof
For (3.10). From the first condition in Definition 3.1. and (3.9), we get :

$$\psi_L(x) = \int_1^x \int_1^y \psi_L^{''}(\zeta)d\zeta dy, \quad \forall x > 0 \tag{3.12}$$

Using (3.12), we obtain :

$\psi_L(x) = \int_1^x \int_1^y \psi_L^{''}(\zeta)d\zeta dy \geq \int_1^x \int_1^y 2(\zeta)d\zeta dy = (x-1)^2$

and

$$\psi_L(x) = \int_1^x \int_1^y \psi_L''(\zeta)d\zeta dy \leq \int_1^x \int_1^y \frac{1}{2}\psi_L''(y)\psi_L''(\zeta)d\zeta dy = \frac{1}{2}\int_1^x \psi_L''(y)\psi_L'(y)dy = \frac{1}{4}(\psi_L'(x))^2$$

For (3.11), using Taylorâ€™theorem, the first condition of definition 3.1, (3.3) and (3.8) for some ζ, $1 \leq \zeta \leq x$ we get :

$$\begin{aligned}\psi_L(x) &= \psi_L(1) + \psi_L'(1)(x-1) + \frac{1}{2}\psi_L''(1)(x-1)^2 + \frac{1}{6}\psi_L'''(\zeta)(\zeta-1)^3 \\ &\leq \frac{1}{2}\psi_L''(1)(x-1)^2 \\ &= \frac{p+4}{2}(x-1)^2\end{aligned}$$

□

following lemma establishes a relationship between $\delta(v)$ and $\Psi_L(v)$.

Lemma 3.3
Let $\delta(v)$ be defined as in (2.8). Then we have :

$$\delta(v) \geq \sqrt{\Psi_L(v)} \tag{3.13}$$

Proof
(2.6),(2.8) and (3.10) are caused by :

$$\Psi_L(v) = \sum_{i=1}^n \psi_L(v_i) \leq \sum_{i=1}^n \frac{1}{4}\psi_L'(v_i)^2 = \frac{1}{4}\|\nabla\psi_L(v)\|^2 = \delta^2(v).$$

this implies the desired result. □

Let $\gamma(y) : [0,+\infty] \to [1,+\infty]$ be the inverse function of $\psi_L(x)$ for all $x \geq 1$ and $\rho(y) : [0,+\infty] \to [0,1]$ be the inverse function of $\frac{-1}{2}\psi_L'(x)$ for all $x \in [0,1]$. Then, we have the following lemma.

Lemma 3.4
For $p > 0$, we have :

$$1 + \sqrt{\frac{2y}{p+4}} \leq \gamma(y) \leq 1 + \sqrt{y} \tag{3.14}$$

$$\rho(y) \geq \left(\frac{1}{2y+1}\right)^{\frac{1}{p+1}} \tag{3.15}$$

Proof
For (3.14), let $\psi_L(x) = y$, $x \geq 1$ i.e., $x = \gamma(y)$, $y \geq 0$. By using the left part of the inequality (3.10), we have :

$$y = \psi_L(x) \geq (x-1)^2 \quad \Leftrightarrow \quad x = \gamma(y) \leq 1 + \sqrt{y}$$

and from (3.11), we get :

$$y = \psi_L(x) \leq \frac{p+4}{2}(x-1)^2 \quad \Leftrightarrow \quad x = \gamma(y) \geq 1 + \sqrt{\frac{2y}{p+4}}.$$

For (3.15), let $y = \frac{-1}{2}\psi_L'(x)$ for $0 < x \leq 1$. Due to the definition of ρ and (3.2), we get :

$$\begin{aligned}y = \frac{-1}{2}\psi_L'(x) &\Leftrightarrow y = \frac{-1}{2}\left(2(x-1) + log(x) + 1 - x^{-p-1}\right) \\ &\Leftrightarrow 2y = -2(x-1) - log(x) - 1 + x^{-p-1} \\ &\Leftrightarrow x^{-p-1} = 2y + 2x - 1 + log(x) \\ &\Leftrightarrow x^{-p-1} \leq 2y + 1 \\ &\Leftrightarrow x \leq \left(\frac{1}{2y+1}\right)^{\frac{1}{p+1}}\end{aligned}$$

this completes the proof. □

The following theorem gives an estimate of the effect of a μ-update on the value of $\Psi(v)$.

Theorem 3.1
(Theorem 3.2, [6]) Let $\gamma(y) : [0,+\infty[\to [1,+\infty[$ be the inverse function of $\psi(x)$ for all $x \geq 1$. Then we have for any positive vector v and any $\beta \geq 1$ that

$$\Psi(\beta v) \leq n\psi\left(\beta\gamma\left(\frac{\Psi(v)}{n}\right)\right). \tag{3.16}$$

We now compute an upper bound for the effect of a μ-update on the value of $\Psi_L(v)$ in the following lemma.

Lemma 3.5
Let $0 < \theta < 1$ and $v_+ = \dfrac{v}{\sqrt{1-\theta}}$. If $\Psi_L(v) \leq \tau$, then :

$$\Psi_L(v_+) \leq \frac{p+4}{2(1-\theta)}(\theta\sqrt{n}+\sqrt{\tau})^2 \tag{3.17}$$

Proof
Since $\dfrac{1}{\sqrt{1-\theta}} \geq 1$ and $\gamma\left(\dfrac{\Psi_L(v)}{n}\right) \geq 1$, we have $\dfrac{1}{\sqrt{1-\theta}}\gamma\left(\dfrac{\Psi_L(v)}{n}\right) \geq 1$. Using Theorem 3.16 with $\beta = \dfrac{1}{\sqrt{1-\theta}}$, $\Psi_L(v) \leq \tau$, (3.11) we obtain :

$$\begin{aligned}\Psi_L(v_+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\gamma\left(\frac{\Psi_L(v)}{n}\right)\right)\\ &\leq n\frac{p+4}{2}\left(\frac{1}{\sqrt{1-\theta}}\gamma\left(\frac{\Psi_L(v)}{n}\right)-1\right)\end{aligned}$$

and due to the second inequality in (3.14), we have :

$$\begin{aligned}n\frac{p+4}{2}\left(\frac{1}{\sqrt{1-\theta}}\gamma\left(\frac{\Psi_L(v)}{n}\right)-1\right) &\leq n\frac{p+4}{2}\left(\frac{1+\frac{\Psi_L(v)}{n}}{\sqrt{1-\theta}}-1\right)^2\\ &= n\frac{p+4}{2}\left(\frac{1+\frac{\Psi_L(v)}{n}-\sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2\end{aligned}$$

because $1\hat{a}e\text{“}\sqrt{1-\theta} = \dfrac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, we have :

$$\begin{aligned}n\frac{p+4}{2}\left(\frac{1+\frac{\Psi_L(v)}{n}-\sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 &\leq n\frac{p+4}{2}\left(\frac{\theta+\frac{\tau}{n}}{\sqrt{1-\theta}}\right)^2\\ &= \frac{p+4}{2(1-\theta)}(\theta\sqrt{n}+\sqrt{\tau})^2\end{aligned}$$

this completes the proof. □

For all $p > 0$, we use $\tilde{\Psi}_{L0}$ to denote an upper bound of $\Psi_L(v)$:

$$\tilde{\Psi}_{L0} = \frac{p+4}{2(1-\theta)}(\theta\sqrt{n}+\sqrt{\tau})^2. \tag{3.18}$$

This section provides a thorough examination of our kernel function by demonstrating a few fundamental characteristics that are crucial to the complexity analysis and by calculating the upper bound for the impact of a μ-update on the value of $\Psi_L(v)$.

4. Analysis of the interior-point algorithm for $\mathscr{P}_*(\kappa)$-HLCP

Based on our kernel function (3.1), we calculate the iteration boundaries for large-update method in this section. Prior to expressing the reduction of the proximity function during an inner iteration, it must first compute a default step size α. Every step of our algorithm results in a new iteration (z_+, w_+), and it is widely known that the value of μ is fixed throughout an inner iteration. Using (2.4) and (2.9), we therefore obtain :

$$z_+ = z + \alpha\Delta z = z\left(e + \alpha\frac{\Delta z}{z}\right) = z\left(e + \alpha\frac{d_z}{v}\right) = \frac{z}{v}(v + \alpha d_z),$$
$$w_+ = w + \alpha\Delta w = w\left(e + \alpha\frac{\Delta w}{w}\right) = w\left(e + \alpha\frac{d_w}{v}\right) = \frac{w}{v}(v + \alpha d_w),$$
$$v_+ = \sqrt{\frac{z_+ w_+}{\mu}} = \sqrt{(v + \alpha d_z)(v + \alpha d_w)}.$$

from (3.6), it is clear that :

$$\Psi_L(v_+) = \Psi_L\left(\sqrt{(v + \alpha d_z)(v + \alpha d_w)}\right) \leq \frac{1}{2}\left(\Psi_L(v + \alpha d_z) + \Psi_L(v + \alpha d_w)\right)$$

For $\alpha > 0$, we define :

$$f(\alpha) = \Psi_L(v_+) - \Psi_L(v) \tag{4.1}$$
$$f_1(\alpha) = \frac{1}{2}\left(\Psi_L(v + \alpha d_z) + \Psi_L(v + \alpha d_w)\right) - \Psi_L(v) \tag{4.2}$$

we have by taking the two successive derivatives of $f_1(\alpha)$ with respect to α :

$$f_1^{'}(\alpha) = \sum_{i=1}^{n}\left(\psi_L^{'}(v_i + \alpha d_{zi})d_{zi} + \psi_L^{'}(v_i + \alpha d_{wi})d_{wi}\right) \tag{4.3}$$
$$f_1^{''}(\alpha) = \sum_{i=1}^{n}\left(\psi_L^{''}(v_i + \alpha d_{zi})d_{zi}^2 + \psi_L^{''}(v_i + \alpha d_{wi})d_{wi}^2\right) \tag{4.4}$$

Therefore, by using the second equation in (2.7) and (2.8), we have :

$$f(\alpha) \leq f_1(\alpha), \quad f(0) = f_1(0) = 0$$

and

$$f_1^{'}(0) = \frac{1}{2}\langle\nabla\Psi_L(v), (d_z + d_w)\rangle = \frac{-1}{2}\|\nabla\Psi_L(v)\|^2 = -2\delta^2(v) \tag{4.5}$$

We denote $v_{min} = \min_{i\in\{1,\ldots,n\}} v_i$, $\delta = \delta(v)$ and $\Psi_L = \Psi_L(v)$. We assume this study. Using Lemma 3.3, we obtain $\delta \geq \sqrt{\tau} \geq 1$. We shall reiterate the following axioms

Lemma 4.1
(Lemma 4.2, [6]) $f_1^{'}(\alpha) \leq 0$ certainly holds if α satisfies the inequality

$$-\psi^{'}(v_{min} - 2\alpha\delta) + \psi^{'}(v_{min}) \leq 2\delta. \tag{4.6}$$

Lemma 4.2
(Lemma 4.3, [6]) Let ρ be as defined in Lemma 3.4. Then, the largest step size α of the worst case is given by $\overline{\alpha} := \dfrac{\rho(\delta) - \rho(2\delta)}{2\delta}$.

Lemma 4.3
(Lemma 4.4, [6]) Let ρ and $\overline{\alpha}$ be as defined in Lemma 4.2. Then

$$\alpha^{\star} = \frac{1}{\psi^{''}(\rho(2\delta))} \leq \overline{\alpha}. \tag{4.7}$$

Lemma 4.4
(Lemma 4.5, [6]) If the step size α is such that $\alpha \leq \overline{\alpha}$, then $f(\alpha) \leq -\alpha\delta^2$.

Theorem 4.1
(Theorem 4.6, [6]) With $\alpha^\star$ being the default step size as given by (4.7), one has

$$f(\alpha^\star) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))}.$$

We will now use the aforementioned axioms to describe our findings.

Lemma 4.5
Let $\rho(y) : [0,+\infty[\rightarrow]0,1]$ denote the inverse function of $\frac{-1}{2}\psi_L'(x)$ and the largest step size $\overline{\alpha}$ of the worst case associated with $\psi_L(x)$ be as defined in Lemma 4.2. We obtain :

$$\overline{\alpha} \geq \frac{1}{2+(p+2)(4\delta)^{\frac{p+2}{p+1}}}, \quad p>0 \tag{4.8}$$

Proof
According to (3.3) with $x=\rho(2\delta)\in]0,1]$, we have :

$$\begin{aligned}\psi_L''(\rho(2\delta)) &= 2+\frac{1}{\rho(2\delta)}+(p+1)\rho(2\delta)^{(}-p-2)\\ &\leq 2+(4\delta+1)^{\frac{1}{p+1}}+(p+1)(4\delta+1)^{\frac{p+2}{p+1}}\end{aligned}$$

Since $(4\delta+1)^{\frac{1}{p+1}} \leq (4\delta+1)^{\frac{p+2}{p+1}}$, we get :
$\psi_L''(\rho(2\delta)) \leq 2+(p+2)(4\delta+1)^{\frac{p+2}{p+1}}$
hence :

$$\overline{\alpha} \geq \frac{1}{\psi_L''(\rho(2\delta))} \geq \frac{1}{2+(p+2)(4\delta+1)^{\frac{p+2}{p+1}}}.$$

□

The default step size α^* ($\alpha^* \leq \overline{\alpha}$) for our algorithm is defined as follows :

$$\alpha^* = \frac{1}{2+(p+2)(4\delta+1)^{\frac{p+2}{p+1}}} \tag{4.9}$$

Now, we use the default step size α^* to indicate the reduction in the proximity function Ψ_L during an inner iteration . We arrived at the results shown below to do this.

Lemma 4.6
Let α^* defined as in (4.9). Then we have :

$$f(\alpha^*) \leq -\frac{\Psi_L^{\frac{p}{2(p+1)}}}{75(p+2)} \tag{4.10}$$

Proof
By combining (4.9) and $\delta \geq \sqrt{\Psi_L}$ with Theorem 4.1, we may demonstrate this lemma. We discover :

$$\begin{aligned}f(\alpha^*) \leq \frac{-\delta^2}{\Psi_L''(\rho(2\delta))} &\leq \frac{-\delta^2}{2+(p+2)(4\delta+1)^{\frac{p+2}{p+1}}}\\ &\leq \frac{-\delta^2}{2\delta^{\frac{p+2}{p+1}}+(p+2)(4\delta+\delta)^{\frac{p+2}{p+1}}}\\ &\leq \frac{-\delta^{2-\frac{p+2}{p+1}}}{2+(p+2)5^{\frac{p+2}{p+1}}}\\ &\leq \frac{-\delta^{\frac{p}{p+1}}}{75(p+2)} \leq \frac{-\Psi_L^{\frac{p}{2(p+1)}}}{75(p+2)}\end{aligned}$$

□

At this stage, we count how many inner iterations our algorithm requires to achieve the situation $\Psi_L \leq \tau$. Let Ψ_{L0} denote the value of Ψ_L after μ-update and subsequent values in the same outer iteration are denoted as Ψ_{Lk} $k = 1, \ldots, K$, where K denotes the total number of inner iterations in the outer iteration and we use the following lemma to do so.

Lemma 4.7
(Proposition 1.3.2, [20]) Let a sequence $x_k > 0$, $k = 0, \ldots, K$ that verifies $x_{k+1} \leq x_k - \kappa x_k^{1-\nu}$ with $\kappa > 0$, $0 < \kappa \leq 1$ and $k = 0, \ldots, K$, then :

$$K \leq \frac{x_0^{\nu}}{\kappa\nu} \tag{4.11}$$

In the following theorem, we calculate the total number of iterations that our algorithm has undergone.

Theorem 4.2
The total number of iterations required to obtain the optimal solution for $\mathscr{P}_*(\kappa)$-HLCP is bounded by :

$$150(p+1)\Psi_{L0}^{\frac{p+2}{2(p+1)}} \frac{\log \frac{n}{\varepsilon}}{\theta} \tag{4.12}$$

Proof
Using Lemma 4.7 for $x_k = \Psi_{Lk}$, Lemma 4.4 with the definition of $f(\alpha)$, we obtain:
$f(\kappa) = \Psi_{Lk+1} - \Psi_{Lk} \leq -\kappa\delta^2$
then we suppose that they exist $\kappa > 0$ and $0 < \nu \leq 1$, such that :

$$\Psi_{Lk+1} - \Psi_{Lk} \leq -\kappa\Psi_{Lk}^{1-\nu} \tag{4.13}$$

Lemma 4.7 states that we may get $K \leq \frac{\Psi_{L0}^{\nu}}{\kappa\nu}$ by setting $x_k = \Psi_{Lk}$. From (4.13) and (4.10), we conclude that $\nu = \frac{p+2}{2(p+1)}$ and $\kappa = \frac{1}{75(p+2)}$, then one can easily deduce the total number of inner iterations in the outer iteration is given by :
$K \leq 150(p+1)\Psi_{L0}^{\frac{p+2}{2(p+1)}}$
The above limit on the number of outer iterations is $\frac{1}{\theta}\log\frac{n}{\varepsilon}$ (see [18]). We obtain by multiplying the quantity of inner iterations by the quantity of outer iterations
$K\frac{\log\frac{n}{\varepsilon}}{\theta} \leq 150(p+1)\Psi_{L0}^{\frac{p+2}{2(p+1)}}\frac{\log\frac{n}{\varepsilon}}{\theta}$,
which completes the proof. □

It is well known that for small-update methods, we use $\tau = O(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{\theta}}\right)$ on (3.18), we obtain $\tilde{\Psi}_{L0} = O(p)$, and that for large-update methods, we use $\tau = O(n)$ and $\theta = \Theta(1)$ on (3.18), which yields $\widehat{\Psi}_{L0} = O(n)$.
The following table summarises the complexity result of small- and large-update techniques by substituting the aforementioned notations $\tilde{\Psi}_{L0}$ and $\widehat{\Psi}_{L0}$ in (4.12).

Methods	Complexity results
Small-update	$O\left((p+1)p^{\frac{p+2}{2(p+1)}}\sqrt{n}\log\frac{n}{\varepsilon}\right)$ with $p > 0$.
Large-update	$O\left((p+1)n^{\frac{p+2}{2(p+1)}}\log\frac{n}{\varepsilon}\right)$ with $p > 0$.

Table 1. Complexity results for small- and large-update methods.

The iteration complexity result for the small-update approach is $O(\sqrt{n}\log\frac{n}{\varepsilon})$ if we choose any constant value for p. The iteration complexity for the large-update approach is $O\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$ if $p = \frac{\log n}{2} - 1$. The most well-known complexity results are these.

5. Numerical results

We provide some numerical results on a few $\mathscr{P}_*(\kappa)$- horizontal linear complementarity problems in this section to illustrate the performance of our proposed function (3.1) when the experiments were modified in *Dev-Cpp 5.11 TDM-GCC 4.9.2* Setup and run on a PC. We assume that the accuracy parameter is 10^{-6} throughout the process because the parameters τ, θ, and α should be selected so that the algorithm is "optimized" in the sense that it requires the fewest number of iterations possible. In order to do this, we choose these criteria in a thorough, distinct manner that is suitable for the scope of any issue we are researching in the manner described below : $\tau = \sqrt{n}$, $\theta = \{0.1, 0.2, 0.3, \ldots, 0.9\}$, and $\alpha = \rho \min(\alpha_z, \alpha_w)$ where

$$\alpha_z = \min_{i=1,\ldots,n} \begin{cases} -\dfrac{z_i}{\Delta z_i}, & \text{if } \Delta z_i < 0 \\ 1 & else \end{cases}, \quad \alpha_w = \min_{i=1,\ldots,n} \begin{cases} -\dfrac{w_i}{\Delta w_i}, & \text{if } \Delta w_i < 0 \\ 1 & else \end{cases}$$

with $\rho \in (0,1)$. Iter and CPU stand for the quantity of iterations and the duration generated by our approach, respectively. Our primary goal is to compare the computation time and iteration counts of the approach for the kernel functions given below with our new function, in which we select the barrier parameter p for each parametrized function that provides the optimal complexity for large-updates.

i	The kernel function $\psi_i(x)$	Ref
cl	$\dfrac{x^2-1}{2} - \log(x)$	[6]
1	$(p+1)x^2 - \dfrac{1}{x^p} - (p+2)x, \quad p > 4$	[14]
2	$x^2 - 1 - \log(x) + \dfrac{x^{-p}-1}{p}, \quad p \geq 1$	[11]
3	$x^2 - 1 - \dfrac{x^{-2p+1}-1}{-2p+1} - \dfrac{x^{-p+1}-1}{-p+1}, \quad p > 1$	[7]
4	$2x^2 - 2 - 2\log(x) + \dfrac{\sum_{i=1}^m x^{-p_i}}{a} + \dfrac{x^{-a}}{a} - \dfrac{m+1}{a}, \quad p_i > 0, \quad a = \sum_{i=1}^m p_i$	[6]

Table 2. SOME KERNEL FUNCTIONS.

Since the majority of researchers in this field approve of these challenges, we now assume three distinct problems, which were selected at random.

Example 5.1. (Problem 1, [3]) Consider the HLCP where M, N and q are given by :

$$M = \begin{pmatrix} 9.5 & 0 & -2.5 & 0.5 & -20.1 \\ 1 & 0.75 & 0 & 4 & 24.5 \\ 0 & -5.6 & -1 & 7.6 & -10.5 \\ 0 & 7.7 & 0 & -7 & 1 \\ 2 & 5 & -4 & 0 & 10 \end{pmatrix}, N = \begin{pmatrix} 6.5 & 1 & 0.5 & 1.5 & -19.9 \\ 1 & 1.25 & 2 & 4 & 25.5 \\ 2 & -4.4 & 1 & 8.4 & -9.5 \\ 0 & 8.3 & 2 & -5 & 1 \\ 4 & 5 & -2 & 0 & 10 \end{pmatrix}, q = e \qquad \text{(P1)}$$

The strictly feasible starting point is :
$z_0 = \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{pmatrix}^T$
and
$w_0 = \begin{pmatrix} 0.192520 & 0.171470 & 0.185830 & 0.145980 & 0.104440 \end{pmatrix}^T$
The numerical results obtained by our algorithm are shown in the below Table.
The unique solution of the HLCP is given by :
$z_* = \begin{pmatrix} 0.000021 & 0.000002 & 0.000007 & 0.000002 & 0.012232 \end{pmatrix}^T$ and
$w_* = \begin{pmatrix} 0.026540 & 0.231548 & 0.070883 & 0.210378 & 0.000031 \end{pmatrix}^T$

	0.1		0.2		0.3		0.4		0.5		0.6		0.7		0.8		0.9	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_L(x)$	**125**	**1.17**	**61**	**1.05**	49	0.98	42	**0.55**	37	**0.50**	30	0.53	**21**	0.41	21	**0.37**	21	**0.33**
$\psi_{cl}(x)$	145	1.71	79	1.32	56	1.04	45	0.94	40	0.67	34	0.81	30	0.76	29	0.67	26	0.62
$\psi_1(x)$	127	1.26	73	1.12	54	0.99	43	0.90	41	0.72	31	0.67	25	0.61	23	0.61	22	0.58
$\psi_2(x)$	133	1.32	67	0.93	54	0.98	43	0.64	38	0.62	33	0.60	28	0.56	27	0.53	24	0.42
$\psi_3(x)$	135	1.33	69	1.16	56	1.05	44	0.94	39	0.87	34	0.62	30	0.56	28	0.52	25	0.40
$\psi_4(x)$	127	1.24	**61**	1.07	**48**	**0.70**	**37**	0.57	**31**	0.51	**26**	**0.42**	22	**0.39**	**20**	0.39	**20**	0.35

Table 3. Numerical experiments of (P1).

Example 5.2. (Problem 4, [3]) Let us consider the following HLCP with the following data :

$$M = \begin{pmatrix} 4 & -1 & 0 & \dots & 0 \\ -1 & 4 & -1 & \dots & 0 \\ 0 & -1 & 4 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 4 \end{pmatrix}, \quad N = I, \quad q = -e \tag{P2}$$

The starting point for our algorithm is :
$z_0 = e$ and $w_0 = \begin{pmatrix} 2 & 1 & \dots & 1 & 2 \end{pmatrix}^T$.
The unique solution of the HLCP is given by :
$z_* = \begin{pmatrix} 0.365385 & 0.461539 & 0.840769 & \dots & 0.840769 & 0.461539 & 0.365385 \end{pmatrix}^T$
and $w_* = \begin{pmatrix} 0.000001 & 0 & \dots & 0 & 0.000001 \end{pmatrix}^T$
The numerical results obtained by our algorithm are shown in the below Table.

	0.1		0.2		0.3		0.4		0.5		0.6		0.7		0.8		0.9	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_L(x)$	**168**	**1.05**	**81**	**0.99**	**51**	**0.86**	**35**	**0.86**	**26**	**0.73**	**20**	**0.68**	**16**	**0.66**	**12**	**0.66**	**9**	**0.62**
$\psi_{cl}(x)$	205	2.68	116	0.68	86	2.58	71	2.49	63	2.45	57	2.35	52	2.32	51	2.2.31	45	2.30
$\psi_1(x)$	193	2.26	104	2.08	74	2.06	59	2.03	50	1.98	44	1.91	39	1.91	36	1.82	30	1.74
$\psi_2(x)$	180	1.51	91	1.44	61	1.35	46	1.35	37	1.33	31	1.32	26	1.30	23	1.14	22	1.07
$\psi_3(x)$	182	1.46	93	1.45	63	1.47	48	1.43	39	1.43	33	1.36	28	1.34	26	1.29	21	1.23
$\psi_4(x)$	170	1.13	**81**	1.11	**51**	1.10	36	1.04	27	0.94	21	0.82	**16**	0.73	13	0.69	**9**	0.65

Table 4. Numerical experiments of (P2) with $n = 50$.

	0.1		0.2		0.3		0.4		0.5		0.6		0.7		0.8		0.9	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_L(x)$	**171**	**1.12**	**82**	**1.09**	**52**	**1.08**	**37**	**0.92**	**28**	**0.82**	**22**	**0.77**	**17**	**0.72**	**14**	**0.72**	**11**	**0.72**
$\psi_{cl}(x)$	227	3.85	138	3.62	108	3.58	93	3.56	85	3.56	80	3.50	74	3.41	75	3.40	68	3.33
$\psi_1(x)$	206	2.68	117	2.66	87	2.51	72	2.47	63	2.43	57	2.43	52	2.41	49	2.39	41	2.35
$\psi_2(x)$	181	1.56	94	1.48	64	1.45	48	1.44	41	1.42	34	1.41	30	1.36	26	1.18	23	1.06
$\psi_3(x)$	178	2.26	85	2.24	54	2.22	39	2.21	29	2.21	23	2.21	18	2.16	**14**	2.16	**11**	1.96
$\psi_4(x)$	177	1.70	84	1.63	53	1.62	38	1.61	**28**	1.61	23	1.56	18	1.50	**14**	1.35	**11**	1.26

Table 5. Numerical experiments of (P2) with $n = 100$.

Example 5.3. ([10]) We consider the following problem :

$$M(i,j) = \begin{cases} -1 & \text{if } i = j \\ -2 & \text{if } i < j \end{cases}, \quad N(i,j) = -I_n(i,j), \quad q(i) = 1, \quad 1 \le i, j \le n. \tag{P3}$$

The starting point for our algorithm is :

$z_0 = \begin{pmatrix} 2 & 2 & \dots & 2 & 2 \end{pmatrix}^T$ and
$w_0 = \begin{pmatrix} 2n-3 & 2n-7 & 2n-11 & \dots & 2n-(2n-5) & 2n-(2n-1) \end{pmatrix}^T$
The numerical results obtained by our algorithm are shown in the below Table.

Table 6. Numerical experiments of (P3) with $n = 50$.

	0.1		0.2		0.3		0.4		0.5		0.6		0.7		0.8		0.9	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_L(x)$	**220**	**0.99**	**105**	**0.93**	**66**	**0.92**	**47**	**0.91**	**35**	**0.91**	**27**	**0.87**	**21**	0.86	**16**	**0.73**	**11**	0.71
$\psi_{cl}(x)$	296	4.64	182	4.57	143	4.44	125	4.36	113	4.28	106	4.26	100	4.20	96	4.11	87	4.07
$\psi_1(x)$	241	2.04	126	1.97	87	1.94	68	1.89	56	1.86	48	1.84	42	1.84	37	1.80	32	1.80
$\psi_2(x)$	236	2.14	122	1.78	83	1.71	64	1.68	52	1.67	44	1.64	38	1.60	33	1.55	27	1.49
$\psi_3(x)$	238	1.99	123	1.83	84	1.76	65	1.75	52	1.71	45	1.70	40	1.66	34	1.63	28	1.59
$\psi_4(x)$	**220**	1.94	**105**	1.65	**66**	1.64	**47**	1.13	**35**	0.93	**27**	0.88	**21**	**0.80**	**16**	0.79	**11**	**0.67**
	0.1		0.2		0.3		0.4		0.5		0.6		0.7		0.8		0.9	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_L(x)$	**221**	**1.38**	**106**	**1.30**	**67**	**1.02**	48	**0.97**	**36**	**0.87**	**28**	**0.80**	**22**	**0.68**	17	**0.61**	**12**	**0.60**
$\psi_{cl}(x)$	307	5.09	192	4.99	153	4.97	136	4.93	124	4.95	117	4.86	112	4.84	107	4.83	98	4.83
$\psi_1(x)$	253	2.46	138	2.46	99	2.45	80	2.42	68	2.41	60	2.40	54	2.36	49	2.35	45	2.25
$\psi_2(x)$	239	1.83	124	1.80	85	1.77	66	1.73	54	1.72	46	1.70	41	1.71	36	1.69	30	1.60
$\psi_3(x)$	239	2.80	124	2.29	85	1.94	66	1.90	54	1.83	46	1.79	41	1.78	36	1.74	30	1.73
$\psi_4(x)$	222	1.76	**106**	1.35	**67**	1.12	**47**	0.99	**36**	0.89	**28**	0.82	**22**	0.70	**16**	0.66	**12**	0.65

Table 7. Numerical experiments of (P3) with $n = 100$.

The unique solution of the HLCP is given by :
$z_* = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}^T$ and $w_* = \begin{pmatrix} 0.999999 & 0.999999 & \dots & 0.999999 & 0 \end{pmatrix}^T$

Comments. Remember that the numerical results were obtained by applying the kernel functions listed in Table 1 to seven test problems using Algorithm 1. The shortest iteration number, or the best one, was displayed for each case using strong type. The following observations may be made based on the tables above :

- Values of barrier degrees p, step size α, and parameters θ define the number of iterations in the method.
- We accomplish the fewest number of iterations in the least amount of time for each θ to be around 1.
- In any situation, independent of θ and α, we obtain the most number of iterations for our kernel function $\psi_L(x)$ with the barrier parameter p that provides the best complexity for large-updates.
- For $i = cl, 1, \dots, 3$, our new kernel function ψ_L outperforms the functions ψ_i in terms of the number of iterations "Iter" and the time spent "CPU."
- It is challenging to compare functions ψ_L and ψ_4 since, as can be observed, the number of iterations is typically the same, particularly in Problem 5.3. Generally speaking, however, function ψ_L takes less time to obtain the solution than function ψ_4.
- For an algorithm, "Iter" and "CPU" are both crucial performance indicators. However, because time is directly tied to real-world restrictions, it frequently takes primacy in practical applications.

6. Conclusions and various proposals for further research

In this article, we provide a primal-dual interior-point method for the $\mathscr{P}_*(\kappa)$- horizontal linear complementarity problem, according to a new parametric kernel function with a logarithmic barrier term. As a result, we examined our algorithm using the new kernel function (3.1) that were presented, and we came up with the following complexity bound for large-update methods $O\left((p+1)n^{\frac{p+2}{2(p+1)}}\log\frac{n}{\varepsilon}\right)$. Lastly, we show some numerical experiments on the some problems to demonstrate the effectiveness of our function. The generalization of the complexity bound based on a new kernel function with a new barrier term for primal-dual interior-point methods in semidefinite optimization problems is one of the most significant areas of study that catches our attention.

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