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A Bivariate Exponential Distribution with q-Exponential Marginals and its Applications

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Abstract Bivariate Gumbel's exponential distribution is one of the most popular continuous bivariate distributions. Comprehensive studies have been done on bivariate Gumbel's exponential model during the past few decades. In this paper, we have derived a generalized version of bivariate Gumbel's exponential model through entropy optimization and we call this model as q-bivariate Gumbel's exponential model. One of the major properties of the q-bivariate Gumbel's exponential model is that its marginal densities are q-exponential distributions. Its survival function, distribution function and density function can be expressed in terms of q-exponential function, which is the q-analogue of exponential function which posses several applications in various fields. Different properties and a characterisation theorem of this distribution have been discussed. For illustrating the use of the proposed model the unknown parameters are estimated using the method of maximum likelihood estimation. A likelihood ratio test is carried out to test the goodness of fit of q-bivariate Gumbel's exponential distribution to verify its compatibility with the existing bivariate Gumbel's exponential model. In order to interpret the practical applicability of q-bivariate Gumbel's exponential model a simulation study and real data application in the field of medicine and finance have been carried out. From this study, we can conclude that q-bivariate Gumbel's exponential model shows a better fit than bivariate Gumbel's exponential model.

Keywords Bivariate distributions, q-exponential function, Entropy, Quantile function.

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1. Introduction

The univariate exponential distribution which is well known in the statistical literature has wide variety of applications in survival analysis, reliability and life testing. In a system consisting of two or more components, occurrence of dependence among the components is natural. In such cases the conventional practise to assume the components to be independent of each other seems to be impractical. As a result several bivariate models have been introduced over the time. A detailed study of bivariate distributions with their applications can be seen in the book [1]. Due to the great importance of bivariate exponential distributions many authors have introduced various generalizations of bivariate exponential models. Some of the popular bivariate exponential distributions are those by [2], [3], [4], [5], [6], [7], [8] and so on.

A family of probability distributions based on non-extensive statistical mechanics, popularly known as q-type distributions, has great importance in several fields of science and engineering. The basic properties of q-exponential, q-Gaussian, and q-Weibull were studied by [9]. In [10] a comparison between q-exponential, q-Weibull, and Weibull distributions to model the frequency distributions of basketball baskets, cyclone victims, brand-name drugs by retail sales, and highway length were carried out. This type of distributions are widely used for modeling complex systems such as cosmic rays ([11]), cyclones ([12]), financial markets ([13]), image

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processing ([14]) etc. The q-type distributions which are obtained by optimizing a particular entropy subject to suitable constraints have the capability to model data sets with extremely large values. These distributions can be considered as the generalization of the existing distributions in literature. The shape parameter q which is also known as the entropic index provides the flexibility to model various data types efficiently.

The concept of information is too broad to be captured completely by a single definition. However, for any probability distribution, we define a quantity called the entropy, which has many properties that agree with the intuitive notion of what a measure of information should be. Entropy then becomes the self-information of a random variable. It is a measure of uncertainty or a measure of information and it was originated in the work of [15]. Since 1948 a number of research papers have been published which simplify and extend Shannon's original work. Entropy is a scientific concept as well as a measurable physical property that is most commonly associated with a state of disorder, randomness, or uncertainty. The term and the concept are used in diverse fields, from classical thermodynamics, where it was first recognized, to the microscopic description of nature in statistical physics, and to the principles of information theory. It has found far-ranging applications in chemistry, physics, biological systems, cosmology, economics, transmission of information in telecommunication, weather science, climate change etc. Entropy optimization methods has been successfully applied to practical problems in many scientific and engineering disciplines. It introduces a unique way of handling information in the form of constraints. It has found applications in Image Processing ([16]), Indirect Imaging ([17]), Calculation of drug absorption rates ([18]), Density Estimation ([19] etc.

Maximum entropy principle which was first introduced by [20], has been applied with varying degree of success in fields such as Statistical Distributions, Statistical Inference, Non Parametric Density Estimation, Speech and Signal Processing, Reliability Theory, Classification and Feature Extraction in Pattern Recognition etc. The principle states that one should look for a distribution, consistent with available information, which maximizes the entropy. The maximum-entropy distribution is the broadest one compatible with the given information. When an inference is made on the basis of incomplete information, it should be drawn from the probability distribution that maximizes the entropy subject to the constraints on the distribution. The resulting maximum entropy probability distribution corresponds to a distribution which is consistent with the given partial information, but has maximum uncertainty or entropy associated with it.

In order to analyze bivariate drought in homogeneous areas of Iran, a maximum entropy copula approach is created in [21]. [22] discussed about a maximum entropy copula-based frequency analysis (MECFA) which was developed through integrating maximum entropy, copulas and frequency analysis for assessing bivariate drought risk. In [23] the parameters of a bivariate Dirichlet distribution are estimated by entropy formalism. Even-though several univariate q-type distributions have been introduced in the literature, much studies have not been done in the case of bivariate q-type distributions. [24] introduced generalized version of bivariate Block and Basu's exponential distribution using entropy optimization and its application to rainfall data, [25] explained the application of q-bivariate Marshal-Olkin exponential distribution in constant stress accelerated life test. [26] discussed the application of bivariate q-Gaussian distribution in modeling trading volume and stock return data. In this paper we construct a generalization of bivariate Gumbel's exponential distribution by using the concept of maximum entropy. To estimate the parameters of the proposed model we used the method of maximum likelihood estimation. A simulation study and and a real data analysis is carried out for illustrating the applications of q-bivariate Gumbel's exponential distribution.

The remaining structure of the paper is given as follows. In Section 2 we describe the construction of q-bivariate Gumbel's exponential model using entropy optimization and some basic properties are discussed. In section 3 we deal with the bivariate quantile function which is helpful in generating random samples from q-bivariate Gumbel's exponential model and we discuss the method of maximum likelihood estimation of the unknown parameters of the model in section 4. A simulation study and real data analysis are presented in Sections 5 and 6 respectively. Finally we conclude the paper in Section 7.

2. Generalized Bivariate Gumbel's Exponential Distribution

In this section we introduce a generalized version of bivariate Gumbel's exponential distribution which we call as q-bivariate Gumbel's exponential distribution (q-BVGED) which attainable through entropy optimization. [15] introduced an uncertainty measure associated with discrete distributions called entropy. The Shannon's entropy of a discrete distribution with cumulative distribution function F(x) and probabilities $p_i's$ obtained from F(x) is defined by,

$$H(F) = -C\sum p_i \ln(p_i) \tag{1}$$

where C is a constant. The continuous analogue of Shannon's entropy for a non negative random variable X with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$ called differential entropy is given by,

$$H(f) = -C\int_0^\infty f(x)\ln(f(x))dx \tag{2}$$

It has found several applications in various fields including Investment Analysis ([27]), Data compression ([28]), Identification of drug targets ([27]), Optimizing data analysis ([29]) etc.
[30] proposed an alternative measure of uncertainty called Cumulative Residual Entropy (CRE) which was obtained by replacing pdf by survival function $\bar{F}(x)$ in (2).

$$\eta(X) = -\int_0^\infty \bar{F}(x)\ln(\bar{F}(x))dx \tag{3}$$

Various properties and applications of CRE have been discussed in [31], [32],[33], [34], [35], [36], [37], [38], and [39].
[40] introduced a generalized version of information measure defined by,

$$M_q(f) = \frac{\int_0^\infty [f(x)]^{2-q}dx - 1}{q-1}; q \neq 1, q < 2. \tag{4}$$

Based on this measure of information we introduce a new measure of uncertainty namely Mathai's Cumulative Residual Entropy given by,

$$M_q(f) = \frac{\int_0^\infty [\bar{F}(X)]^{2-q}dx - 1}{q-1}; q \neq 1, q < 2. \tag{5}$$

when $q \to 1$, equation(5) reduces to equation (3) and satisfies all properties mentioned in [30].
Consider all possible functions $\bar{F}(X)$ such that $\bar{F}(X) \geq 0$ for all X, where X is a vector. We maximize Mathai's Cumulative Residual Entropy (5) with respect to the constraints $\int_0^\infty \bar{F}(X)dX = a_1\ (given)$, $\int_0^\infty g(X)\bar{F}(X)dX = a_2\ (given)$ and $g(X)$ specified. By using method of calculus of variation. We have,

$$U = \bar{F}(X)^{2-q} - \lambda_1\bar{F}(X) + \lambda_2 g(X)\bar{F}(X) \tag{6}$$

Now the Euler equation becomes,

$$\begin{aligned}
\frac{\partial U}{\partial \bar{F}} = 0 &\implies (2-q)(\bar{F}(X))^{1-q} - \lambda_1 + \lambda_2 g(X) = 0 \\
&\implies (\bar{F}(X))^{1-q} = \frac{\lambda_1 - \lambda_2 g(X)}{2-q} \\
&\implies \bar{F}(X) = \left\{\frac{\lambda_1}{2-q}\left[1 - \frac{\lambda_2}{\lambda_1}g(X)\right]\right\}^{\frac{1}{1-q}}
\end{aligned}$$

taking $c_1 = \dfrac{\lambda_1}{2-q}$ and $c_2 = \dfrac{\lambda_2}{\lambda_1}$ we have,

$$\bar{F}(x) = \left\{c_1[1 - c_2 g(X)]\right\}^{\frac{1}{1-q}}, \tag{7}$$

where $c_1 > 0$, $1 - c_2 g(X) > 0$ since $\bar{F}(X) \geq 0$ for all X.
Consider the case when $c_1 = 1$, $c_2 = 1 - q$, with $q > 1$ we get

$$\bar{F}(X) = \left[1 - (1-q)g(X)\right]^{\dfrac{1}{1-q}}, \tag{8}$$

Let $X = (x_1, x_2)$ and $g(x_1, x_2) = \alpha x_1 + \mu x_2 + \theta x_1 x_2$, where α, μ and θ are constants such that $0 \leq x_1, x_2 < \infty$, $\alpha > 0,\ \mu > 0$ and $0 \leq \theta \leq q\alpha\mu$, $1 < q < 2$ equation (8) becomes

$$\bar{F}(x_1, x_2) = \left[1 + (q-1)(\alpha x_1 + \mu x_2 + \theta x_1 x_2)\right]^{\frac{-1}{q-1}}; 0 \leq x_1, x_2 < \infty. \tag{9}$$

where $\alpha, \mu > 0, 0 \leq \theta \leq q\alpha\mu, 1 < q < 2$, as $q \to 1$ (9) becomes,

$$\bar{F}(x_1, x_2) = exp\{-\alpha x_1 - \mu x_2 - \theta x_1 x_2\}; 0 \leq x_1, x_2 < \infty, \alpha, \mu > 0, 0 \leq \theta \leq \alpha\mu. \tag{10}$$

which is the survival function of the well known bivariate exponential distribution proposed by, [2]. The obtained model (9) can be considered as a generalized version of the model (10)
The cumulative distribution function (CDF) of (10) is given by,

$$F(x_1, x_2) = 1 - exp\{-\alpha x_1\} - exp\{-\mu x_2\} + exp\{-\alpha x_1 - \mu x_2 - \theta x_1 x_2\}; 0 \leq x_1, x_2 < \infty \tag{11}$$

where $\alpha,\ \mu > 0$ and $0 \leq \theta \leq \alpha\mu$.
The corresponding joint probability density function is given by,

$$f(x_1, x_2) = \begin{cases} [(\mu + \theta x_1)(\alpha + \theta x_2) - \theta] exp\{-\alpha x_1 - \mu x_2 - \theta x_1 x_2\}; 0 \leq x_1, x_2 < \infty \\ 0 \qquad ; otherwise. \end{cases} \tag{12}$$

where $\alpha,\ \mu > 0$ and $0 \leq \theta \leq \alpha\mu$, which is the association parameter.
Further in this section we define the q-bivariate Gumbel's exponential distribution (q-BVGED) and discuss some of its important properties.

Definition 2.1. *Let X=(X_1, X_2) be a vector random variable following q-BVGED with parameters $q, \alpha,\ \mu$ and θ then the survival function of X is given by,*

$$\bar{F}(x_1, x_2) = \left[1 + (q-1)(\alpha x_1 + \mu x_2 + \theta x_1 x_2)\right]^{\frac{-1}{q-1}}; 0 \leq x_1, x_2 < \infty. \tag{13}$$

where $\alpha, \mu > 0, 0 \leq \theta \leq q\alpha\mu, 1 < q < 2$.

The plots of the survival function for the q-BVGED for various values of the parameters are depicted in Figure 2.1.

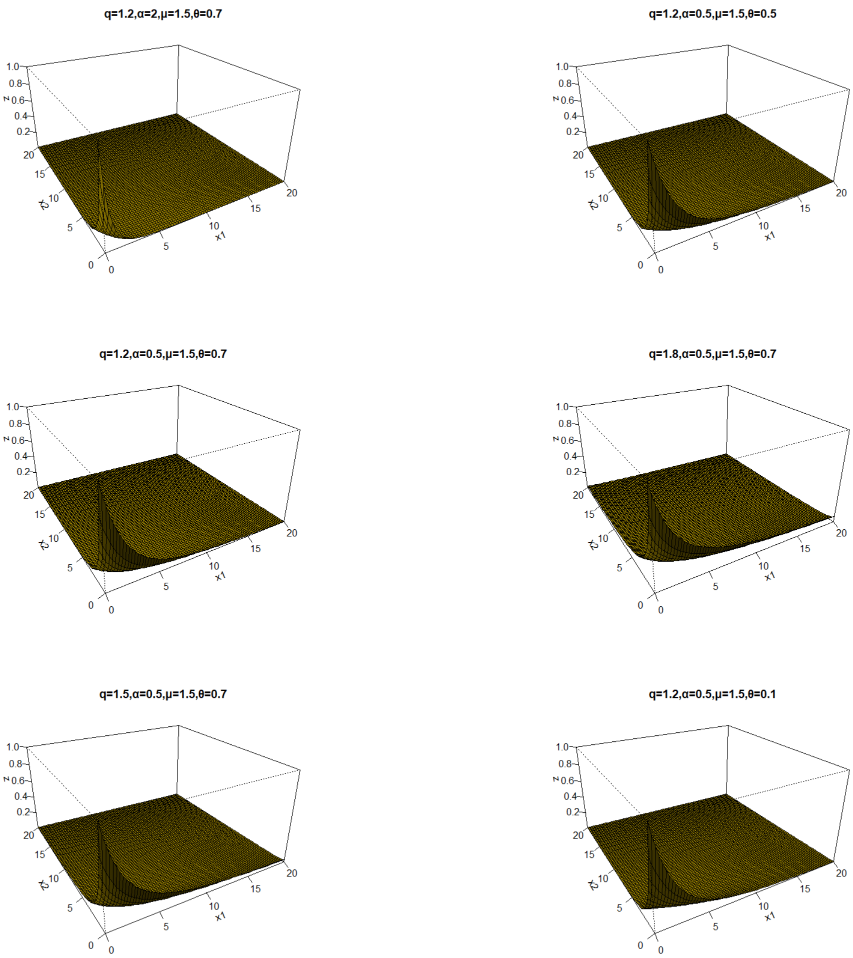

Figure 2.1 Survival plots for q-BVGED

Definition 2.2. *Let X=(X_1, X_2) be a vector random variable following q-BVGED with parameters q, α, μ and θ then the cumulative distribution function (CDF) of X is given by,*

$$F(x_1, x_2) = 1 - \left[1 + (q-1)\alpha x_1\right]^{\frac{-1}{q-1}} - \left[1 + (q-1)\mu x_2\right]^{\frac{-1}{q-1}} + \left[1 + (q-1)(\alpha x_1 + \mu x_2 + \theta x_1 x_2)\right]^{\frac{-1}{q-1}}, 0 \leq x_1, x_2 < \infty \quad (14)$$

where $\alpha, \mu > 0,\ 0 \leq \theta \leq q\alpha\mu,\ 1 < q < 2$.

Definition 2.3. *Let X=(X_1, X_2) be a vector random variable following q-BVGED with parameters q, α, μ, θ then the joint probability density function (PDF) of X is given by,*

$$f(x_1, x_2) = \begin{cases} \Big[q(\alpha+\theta x_2)(\mu+\theta x_1) - \theta(1+(q-1)(\alpha x_1+\mu x_2+\theta x_1 x_2))\Big] & \\ \Big[1+(q-1)(\alpha x_1+\mu x_2+\theta x_1 x_2)\Big]^{\frac{-1}{q-1}-2} & ; 0 \le x_1, x_2 < \infty \\ 0 & ; otherwise \end{cases} \tag{15}$$

where $\alpha, \mu > 0,\ 0 \le \theta \le q\alpha\mu,\ 1 < q < 2.$
The plots of the probability density function of q-BVGED for various values of the parameters are depicted in Figure 2.2.

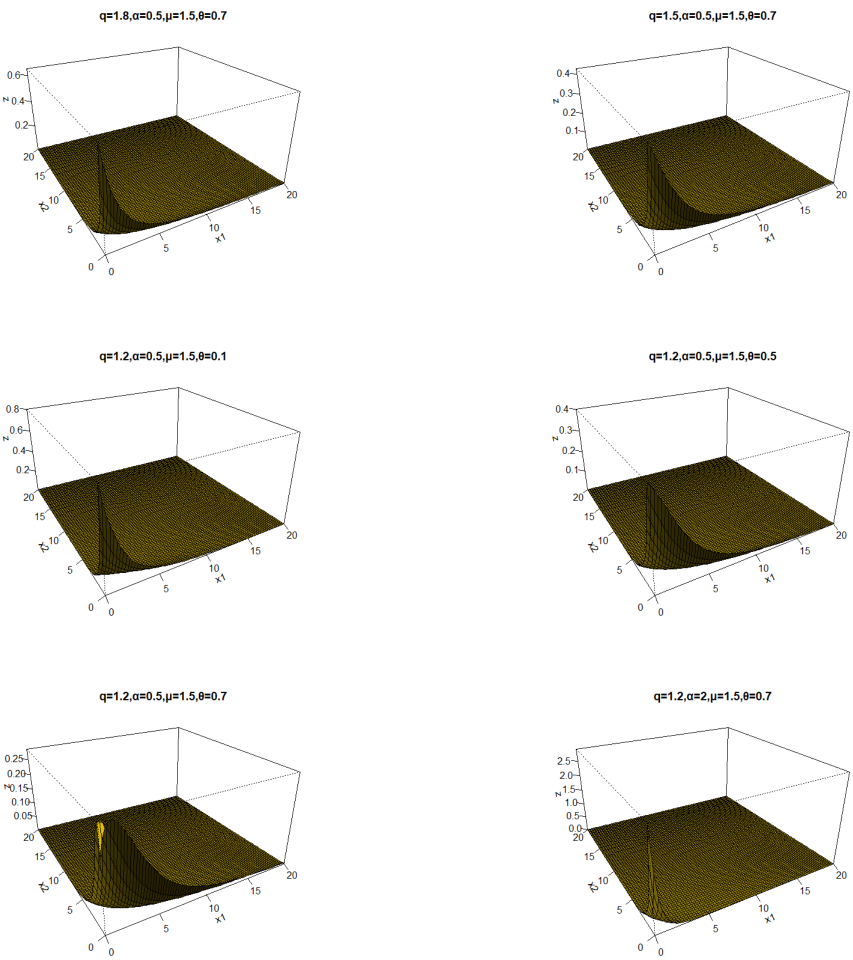

Figure 2.2 Probability density plots for q-BVGED

The following theorom deals with the marginal distributions of a q-BVGED.

Theorem 2.1
Let $X = (X_1, X_2)$ be a bivariate random variable following a q-BVGED with parameter $q, \alpha, \mu,$ and θ then the marginals distributions of X_1 and X_2 follows q-exponential distributions with parameter (q,α) and (q,μ) respectively.

proof: The marginal distribution of X_1 is given by,

$$f(x_1) = \int_0^\infty f(x_1, x_2)\, dx_2 \tag{16}$$

substituting equation (15) in (16) and solving we get,

$$f(x_1) = \begin{cases} \alpha\Big[1 + (q-1)\alpha x_1\Big]^{\frac{-1}{q-1}-1} & ; 0 \leq x_1 < \infty\ , \alpha > 0\ , 1 < q < 2. \\ 0 & ; otherwise. \end{cases} \tag{17}$$

Similarly we get the marginal distribution of X_2 and is given by

$$f(x_2) = \begin{cases} \mu\Big[1 + (q-1)\mu x_2\Big]^{\frac{-1}{q-1}-1} & ; 0 \leq x_2 < \infty,\ \mu > 0,\ 1 < q < 2. \\ 0 & ; otherwise. \end{cases} \tag{18}$$

Remark 2.1. The mean value and variance of X_1 are given by,

$$E(X_1) = \frac{1}{\alpha(2-q)}; q < 2. \tag{19}$$

$$\mathrm{Var}(X_1) = \frac{1}{\alpha^2(2-q)^2(3-2q)}; q < 3/2. \tag{20}$$

Remark 2.2. The mean value and variance of X_2 are given by,

$$E(X_2) = \frac{1}{\mu(2-q)}; q < 2. \tag{21}$$

$$\mathrm{Var}(X_2) = \frac{1}{\mu^2(2-q)^2(3-2q)}; q < 3/2. \tag{22}$$

Among the probability models used in reliability, the exponential and Weibull distributions are the most used ones. Recently, the q-exponential distribution, proposed by [41], has emerged as an alternative. However, this probability model presents some features that should be investigated so as to enable its wide use in the reliability context.
According to [42], the q-exponential distribution is obtained by maximizing the Tsallis entropy subject to the constraint that the first moment is fixed (for more details about these constraints see [42], p.89). As other q-distributions, it has been applied to a variety of problems in many research areas including the field of complex systems. [10] bring in their work a summary of its basic properties, like the success of q-distributions in describing some systems to be in part due to its ability of exhibit heavy-tails and model power law behavior. For instance, [43] showed that the population of a country is well described by a q-exponential distribution with Probability Density Function (PDF) presenting a power law behavior [10]. q is the shape parameter in q-exponential distribution. As compared to the exponential distribution that has just one parameter, the q-exponential distribution has more flexibility regarding the decay of the PDF. Indeed, the exponential probability distribution is a special case of the q-exponential when q $\rightarrow$ 1. Indeed, $1 < q < 2$ characterizes a power law behavior for the q-exponential PDF, whereas a shape parameter between 0 and 1 indicates a stretched exponential behavior for the Weibull PDF([10],[44] As pointed out by [44] a stretched exponential PDF has a tail that is heavier than the exponential PDF but lighter than a pure power law PDF. The stretched exponential provides a compromise between exponential and power law behaviors. Thus, we expect a superior performance of q-exponential over Weibull distribution in the characterization of data sets with extremely large values. The q-BVGED having q-exponential marginals exhibits heavy tail behaviours as compared to the BVGED which improves its applicability in different fields.

Result 2.1. *Let $X = (X_1, X_2)$ be a bivariate random variable following a q-BVGED with parameters q, α, μ,and θ then $E(X_1, X_2)$ is given by,*

$$E(X_1X_2) = \frac{\alpha}{\mu}\left[\frac{1}{(q-1)\alpha}\right]^2 B\left(2, \frac{2-q}{q-1}\right) {}_2F_1\left(1, 2; \frac{q}{q-1}; 1 - \frac{\theta}{(q-1)\alpha\mu}\right)$$
$$+ \frac{\theta}{\mu^2(2-q)}\left[\frac{1}{(q-1)\alpha}\right]^2 B\left(2, \frac{2-q}{q-1}\right) {}_2F_1\left(2, 2; \frac{q}{q-1}; 1 - \frac{\theta}{(q-1)\alpha\mu}\right) \quad (23)$$

where $|1 - \frac{\theta}{(q-1)\alpha\mu}| < 1,\ 1 < q < 2,\ \alpha,\ \mu > 0,\ 0 \le \theta \le q\alpha\mu$.

The correlation coefficient can be calculated by using, (19), (20), (21), (22) and (23).
In the following theorems we define the conditional properties of a q-BVGED with parameters $q,\ \alpha,\ \mu$, and θ.

Theorem 2.2
Let $X = (X_1, X_2)$ be a bivariate random variable following a q-BVGED with parameters $q,\ \alpha,\ \mu$, and θ then the conditional PDF of X_2 given X_1 is given by,

$$f(x_2|x_1) = [q(\alpha + \theta x_2)(\mu + \theta x_1) - \theta[1 + (q-1)(\alpha x_1 + \mu x_2 + \theta x_1 x_2)]] \times \frac{[1 + (q-1)(\alpha x_1 + \mu x_2 + \theta x_1 x_2)]^{\frac{-1}{q-1} - 2}}{\alpha[1 + (q-1)\alpha x_1]^{\frac{-1}{q-1} - 1}} \quad (24)$$

and 0 otherwise. Where $1 < q < 2,\ \alpha,\ \mu > 0,\ 0 \le \theta \le q\alpha\mu$.
proof: We have,

$$f(x_2|x_1) = \frac{f(x_1, x_2)}{f(x_1)} \quad (25)$$

substituting equation (15) and (17) in (25) we get (24).

Remark 2.3. The conditional mean value and variance of X_2 given X_1 are given by,

$$E(X_2|X_1) = \frac{1 + (q-1)\alpha x_1}{\mu + \theta x_1}\left[1 + \frac{\theta(1 + (q-1)\alpha x_1)}{\alpha(2-q)(\mu + \theta x_1)}\right]; q < 2$$
$$Var(X_2|X_1) = \frac{(1 + (q-1)\alpha x_1)^2}{(2-q)(\mu + \theta x_1)^2}\left[q + \frac{2\theta(5-2q)(1 + (q-1)\alpha x_1)}{\mu(3-2q)(\mu + \theta x_1)} - \frac{\theta^2(1 + (q-1)\alpha x_1)^2}{\mu^2(\mu + \theta x_1)^2}\right]; q < 3/2$$

Similarly we can find the distribution of X_1 given X_2 also.

Result 2.2. *Let $X = (X_1, X_2)$ be a bivariate random variable following a q-BVGED with parameters $q,\ \alpha,\ \mu$ and θ then the conditional survival function is given by,*

$$P(X_2 > x_2|X_1 > x_1) = \left[1 + \frac{(q-1)x_2(\mu + \theta x_1)}{1 + (q-1)\alpha x_1}\right]^{\frac{-1}{q-1}}; 0 \le x_1, x_2 < \infty \quad (26)$$

where $1 < q < 2,\ \alpha,\ \mu > 0,\ 0 \le \theta \le q\alpha\mu$.

proof: We have,

$$P(X_2 > x_2|X_1 > x_1) = \frac{P(X_1 > x_1, X_2 > x_2)}{P(X_1 > x_1)} \quad (27)$$
$$= \frac{\left[1 + (q-1)(\alpha x_1 + \mu x_2 + \theta x_1 x_2)\right]^{\frac{-1}{q-1}}}{[1 + (q-1)(\alpha x_1)]^{\frac{-1}{q-1}}} \quad (28)$$
$$= \left[1 + \frac{(q-1)x_2(\mu + \theta x_1)}{1 + (q-1)\alpha x_1}\right]^{\frac{-1}{q-1}}; 0 \le x_1, x_2 < \infty \quad (29)$$

where $1 < q < 2,\ \alpha,\ \mu > 0,\ 0 \le \theta \le q\alpha\mu$.

Theorem 2.3
If $X = (X_1, X_2)$ follows a q-bivariate Gumbel's exponential distribution with parameters $q,\ \alpha,\ \mu\ and\ \theta$ then the conditional distribution of X_2 given $X_1 > x_1$ and X_1 given $X_2 > x_2$ follows a q-exponential distribution.

proof: We have,

$$\begin{aligned} P(X_2|X_1 > x_1) &= \frac{P(X_1 > x_1, X_2)}{P(X_1 > x_1)} \\ &= \frac{\int_{x_1}^{\infty} f(x_1, x_2)\, dx_1}{\int_{x_1}^{\infty} f(x_1)\, dx_1}. \end{aligned} \tag{30}$$

substituting (15) and (17) in (30) and simplifying we get,

$$P(X_2|X_1 > x_1) = \frac{(\mu + \theta x_1)[1 + (q-1)(\alpha x_1 + \mu x_2 + \theta x_1 x_2]^{\frac{-1}{q-1}-1}}{[1 + (q-1)\alpha x_1]^{\frac{-1}{q-1}}}; 0 \leq x_1, x_2 < \infty. \tag{31}$$

where $1 < q < 2$, $\alpha,\ \mu > 0$, $0 \leq \theta \leq q\alpha\mu$. equation (31) can be written as

$$P(X_2|X_1 > x_1) = \frac{(\mu + \theta x_1)}{1 + (q-1)\alpha x_1}\left[1 + \frac{(q-1)x_2(\mu + \theta x_1)}{1 + (q-1)\alpha x_1}\right]^{\frac{-1}{q-1}-1}; 0 \leq x_1, x_2 < \infty. \tag{32}$$

where $1 < q < 2$, $\alpha,\ \mu > 0$, $0 \leq \theta \leq q\alpha\mu$.
From equation (32) we can see that the conditional distribution of X_2 given $X_1 > x_1$ follows a q-exponential distribution. Similarly we can show that X_1 given $X_2 > x_2$ also follows a q-exponential distribution.
In the following theorem we discuss a characterization theorom for q-BVGED.

Theorem 2.4
Let $X = (X_1, X_2)$ be a vector random variable which admits probability density function with respect to Lebesgue measure in R_2^+ then X has a q-BVGED with parameters $q,\ \lambda_1,\ \lambda_2$ and θ with probability density function as given in (15) if and only if for i=1,2, the conditional densities of X_i given $X_{3-i} > t_{3-i}$ follows q-exponential distribution with parameters (q, $\xi_i(t_{3-i})$) where $\xi_i' s$ are non decreasing functions for all $t_{3-i} \geq 0$ satisfying $\xi_i(0) = \lambda_i$.

proof: Assume that X follows a q-BVGED with parameters $q,\ \lambda_1,\ \lambda_2$ and θ with joint density function given in (15). From (32) we have the conditional distribution of $X_i|X_{3-i} > t_{3-i}$ given by,

$$f(x_i|X_{3-i} > t_{3-i}) = \frac{(\lambda_i + \theta t_{3-i})}{1 + (q-1)\lambda_{3-i}t_{3-i}}\left[1 + \frac{(q-1)x_i(\lambda_i + \theta t_{3-i})}{1 + (q-1)\lambda_{3-i}t_{3-i}}\right]^{\frac{-1}{q-1}-1}. \tag{33}$$

where $\xi_i(t_{3-i}) = \frac{(\lambda_i + \theta t_{3-i})}{1+(q-1)\lambda_{3-i}t_{3-i}}$ is a non decreasing function for all $t_{3-i} \geq 0$ and satisfies $\xi_i(0) = \lambda_i$, $1 < q < 2$, $\lambda_1,\ \lambda_2 > 0$, $0 \leq \theta \leq q\lambda_1\lambda_2$. Thus the condition of the theorem is satisfied.
In order to prove the converse part we assume that,

$$f(x_i|X_{3-i} > t_{3-i}) = \xi_i(t_{3-i})\left[1 + (q-1)\xi_i(t_{3-i})x_i\right]^{\frac{-1}{q-1}-1}. \tag{34}$$

We have,

$$P(X_i > t_i|X_{3-i} > t_{3-i}) = \left[1 + (q-1)\xi_i(t_{3-i})t_i\right]^{\frac{-1}{q-1}}, \tag{35}$$

from (35) as t_{3-i} tends to zero,

$$P(X_i > t_i) = [1 + (q-1)\lambda_i t_i]^{\frac{-1}{q-1}} \tag{36}$$

by the conditions of the theorem $\lim_{t_{3-i}\to 0} \xi_i(t_{3-i})$ exists and is equal to λ_i.
From (35) and (36) we get two equations for the expression $P(X_i > t_i, X_{3-i} > t_{3-i})$ given by,

$$P(X_1 > t_1, X_2 > t_2) = \left[1 + (q-1)\xi_1(t_2)t_1\right]^{\frac{-1}{q-1}}\left[1 + (q-1)\lambda_2 t_2\right]^{\frac{-1}{q-1}}, \tag{37}$$

and,

$$P(X_1 > t_1, X_2 > t_2) = [1 + (q-1)\xi_2(t_1)t_2]^{\frac{-1}{q-1}} [1 + (q-1)\lambda_1 t_1]^{\frac{-1}{q-1}}. \tag{38}$$

equating the equations (37) and (38) yields,

$$\begin{aligned} [1 + (q-1)\xi_1(t_2)t_1] [1 + (q-1)\lambda_2 t_2] &= [1 + (q-1)\xi_2(t_1)t_2] [1 + (q-1)\lambda_1 t_1] \\ \lambda_2 t_2 + [1 + (q-1)\lambda_2 t_2] \xi_1(t_2)t_1 &= \lambda_1 t_1 + [1 + (q-1)\lambda_1 t_1] \xi_2(t_1)t_2 \\ ([1 + (q-1)\lambda_2 t_2] \xi_1(t_2) - \lambda_1) t_1 &= ([1 + (q-1)\lambda_1 t_1] \xi_2(t_1) - \lambda_2) t_2 \end{aligned} \tag{39}$$

to solve (39), we write it in the form,

$$([1 + (q-1)\lambda_2 t_2] \xi_1(t_2) - \lambda_1) t_2^{-1} = ([1 + (q-1)\lambda_1 t_1] \xi_2(t_1) - \lambda_2) t_1^{-1}. \tag{40}$$

The equation (40) holds for all $t_1, t_2 \geq 0$ implies,

$$([1 + (q-1)\lambda_{3-i} t_{3-i}] \xi_i(t_{3-i}) - \lambda_i) t_{3-i}^{-1} = \theta; i = 1, 2. \tag{41}$$

where θ is a constant. Thus we get,

$$\xi_i(t_{3-i}) = \frac{\lambda_i + \theta t_{3-i}}{[1 + (q-1)\lambda_{3-i} t_{3-i}]}. \tag{42}$$

we have $\xi_i(t_{3-i})$ is non decreasing and $\xi_i(0) = \lambda_i > 0$ hence $\theta \geq 0$. Using (42) in (37) we get the required form of q-BVGED.

The quantile function corresponding to a particular distribution efficiently describes the statistical properties of a random variable. They can be considered as an alternative to distribution function and can be applied in different forms of statistical analysis. One may refer to [45], [46] for details regarding the advantages, flexibility to modelling and generating methods of quantile functions. Works have been done in literature which extends the concept of quantile function to higher dimensions such as [47], [48], [49], [50] etc. In the following section we define the bivariate quantile function of a q-BVGED.

3. Bivariate Quantile Function of q-BVGED

Let $X = (X_1, X_2)$ be non negative vector random variable having absolutely continuous probability density function, distribution function and survival function. [50] provided the definition of bivariate quantile function in terms of quantile functions of $P(X_1 > x_1)$ and $P(X_2 > x_2 | X_1 > x_1)$ such that the joint survival function is given by,

$$\bar{F}(x_1, x_2) = P(X_1 > x_1) P(X_2 > x_2 | X_1 > x_1). \tag{43}$$

Definition 3.1. *The bivariate quantile function of (X_1, X_2) is given by the pair,$(Q_1(u_1), Q_{21}(u_2|u_1))$, where Q_1 is given by,*

$$Q_1(u_1) = \inf\{x_1 | F_1(x_1) \geq u_1\}, 0 \leq u_1 \leq 1, \tag{44}$$

and

$$Q_{21}(u_2|u_1)) = \inf\{x_2 | P(X_2 \leq x_2 | X_1 > Q_1(u_1) > u_2\}, 0 \leq u_2 \leq 1. \tag{45}$$

The bivariate quantile function of a vector random variable $X = (X_1, X_2)$ following a q-BVGED with parameters $q,\ \alpha,\ \mu$ and θ is given by the pair $(Q_1(u_1), Q_{21}(u_2|u_1))$ where,

$$Q_1(u_1) = \frac{(1-u_1)^{-q+1} - 1}{\alpha(q-1)}, 1 < q < 2, 0 \leq u_1 \leq 1, \alpha > 0, \tag{46}$$

and

$$Q_{21}(u_2|u_1)) = \frac{\alpha[(1-u_2)^{-q+1} - 1](1-u_1)^{-q+1}}{\alpha(q-1)\mu + \theta[(1-u_1)^{-q+1} - 1]}, \tag{47}$$

where $1 < q < 2,\ 0 \leq u_i \leq 1,\ i = 1, 2,\ \alpha > 0,\ \mu > 0,\ 0 \leq \theta \leq q\alpha\mu$.
An important advantage of quantile functions is that they can be used to generate random variables from a particular distribution. One may refer to [51], [52], [53] etc. for further details. Extending the applicability of quantile functions to higher dimensions enables the simulation of multivariate random variables much more easier. In our present study we have used bivariate quantile function for generating random samples from q-BVGED.
The following section explains the method used for estimating the parameters of a q-BVGED.We have employed the method of maximum likelihood estimation for estimating the parameters of a q-BVGED.

4. Maximum Likelihood Estimation

Let $(x_{1i}, x_{2i}),\ i = 1, 2, ...n$ denote a random sample of size n taken from the q-BVGED with parameters $q,\ \alpha,\ \mu$ and θ. Then the likelihood function is given by,

$$L(X_1, X_2|q, \alpha, \mu, \theta) = \prod_{i=1}^{n} \Bigg\{ [1 + (q-1)(\alpha x_{1i} + \mu x_{2i} + \theta x_{1i}x_{2i})]^{\frac{-1}{q-1}-2}\, q(\alpha + \theta x_{2i})(\mu + \theta x_{1i}) - \theta[1 + (q-1)(\alpha x_{1i} + \mu x_{2i} + \theta x_{1i}x_{2i})]^{\frac{-1}{q-1}-1} \Bigg\} \quad (48)$$

The log likelihood function is given by,

$$\ln L = \sum_{i=1}^{n} \ln \Bigg\{ [1 + (q-1)(\alpha x_{1i} + \mu x_{2i} + \theta x_{1i}x_{2i})]^{\frac{-1}{q-1}-2}\, q(\alpha + \theta x_{2i})(\mu + \theta x_{1i}) -\theta[1 + (q-1)(\alpha x_{1i} + \mu x_{2i} + \theta x_{1i}x_{2i})]^{\frac{-1}{q-1}-1} \Bigg\} \quad (49)$$

The maximum likelihood estimates are obtained by differentiating (49) with respect to each of the parameters and solving the resulting system of equations. Thus the MLE's $(\hat{q}, \hat{\alpha}, \hat{\mu}, \hat{\theta})$ of the parameters are obtained by solving, $\frac{\partial \ln L}{\partial q} = 0,\ \frac{\partial \ln L}{\partial \alpha} = 0,\ \frac{\partial \ln L}{\partial \mu} = 0,\ \frac{\partial \ln L}{\partial \theta} = 0$ respectively. The first derivatives of the log-likelihood function w.r.t. parameters are nonlinear, and analytical solutions are very difficult to be obtained. A constrained optimization method can be applied to tackle this problem. This optimization problem can be carried out using constrOptim() or optim() function in R software.

5. Simulation Study

Random samples were generated from q-bivariate Gumbel's exponential distribution using bivariate quantile functions. We considered random samples of sizes n=200, 400, 800 and the procedure was replicated 1000 times. Maximum likelihood estimates were computed using the optim() function in R. The results are given in table 1 and 2. We can see that the bias and MSE decreases as the sample size increases.

6. Real Data Analysis

6.1. Medical Data

The data set that we consider addresses the recurrence time of infections of 38 kidney patients using a portable dialysis machine which was taken from [54]. For each patient, two times to recurrence of an infection at the site of insertion of the catheter placements (in days), (T_1, T_2) are recorded. Our objective here is to check whether the proposed bivariate exponential model is appropriate for the data. The data is given in table 3.

Table 1. ML estimates, bias and MSE's for the parameter values (q,α, μ, θ)=(1.5,1.3,1.5,0.9)

n	$\hat{q}$	Abs. Bias	MSE	$\hat{\alpha}$	Abs. Bias	MSE
200	1.48623	0.01377	0.00019	1.29310	0.00690	0.00005
400	1.49031	0.00969	0.00009	1.29356	0.00644	0.00004
800	1.49313	0.00687	0.00005	1.29632	0.00368	0.00001
n	$\hat{\mu}$	Abs. Bias	MSE	$\hat{\theta}$	Abs. Bias	MSE
200	1.46808	0.03192	0.00102	0.96533	0.06533	0.00427
400	1.4733	0.02669	0.00071	0.95141	0.051411	0.00264
800	1.47685	0.02315	0.00054	0.94901	0.04901	0.00240

Table 2. ML estimates, bias and MSE's for the parameter values (q,α, μ, θ)=(1.3,1.5,1.4,0.6)

n	$\hat{q}$	Abs. Bias	MSE	$\hat{\alpha}$	Abs. Bias	MSE
200	1.28801	0.01199	0.00014	1.49325	0.00675	0.00005
400	1.29293	0.00707	0.00005	1.49475	0.00525	0.00003
800	1.29596	0.00404	0.00002	1.49803	0.001965	0.00000
n	$\hat{\mu}$	Abs. Bias	MSE	$\hat{\theta}$	Abs. Bias	MSE
200	1.38192	0.01808	0.00033	0.63685	0.03685	0.00136
400	1.38848	0.01152	0.00013	0.62428	0.02428	0.00059
800	1.39224	0.00776	0.00006	0.62195	0.02195	0.00048

Table 3. Recurrence times of infections of 38 kidney patients

Patient	T_1	T_2	Patient	T_1	T_2
1	8	16	20	15	108
2	23	13	21	152	562
3	22	28	22	402	24
4	447	318	23	13	66
5	30	12	24	39	46
6	24	245	25	12	40
7	7	9	26	113	201
8	511	30	27	132	156
9	53	196	28	34	30
10	15	154	29	2	25
11	17	333	30	130	26
12	141	8	31	27	58
13	96	38	32	5	43
14	149	70	33	152	30
15	536	25	34	190	5
16	17	4	35	119	8
17	185	177	36	54	16
18	292	114	37	6	78
19	22	159	38	63	8

We have plotted the boxplot and scaled TTT plot for X_1 and X_2 and is given in figure 6.1.1 and 6.1.2 respectively. According to [55] a scaled TTT transform plot provides an idea regarding the empirical hazard function. It is stated that the scaled TTT transform is concave (convex) if the hazard rate is increasing (decreasing). From the figure it is clear that both X_1 and X_2 has a decreasing hazard function. It is evident that q-BVGED, which has decreasing hazard function for the marginals, provides better fit than the BVGED model which has only constant hazard function for the marginals.

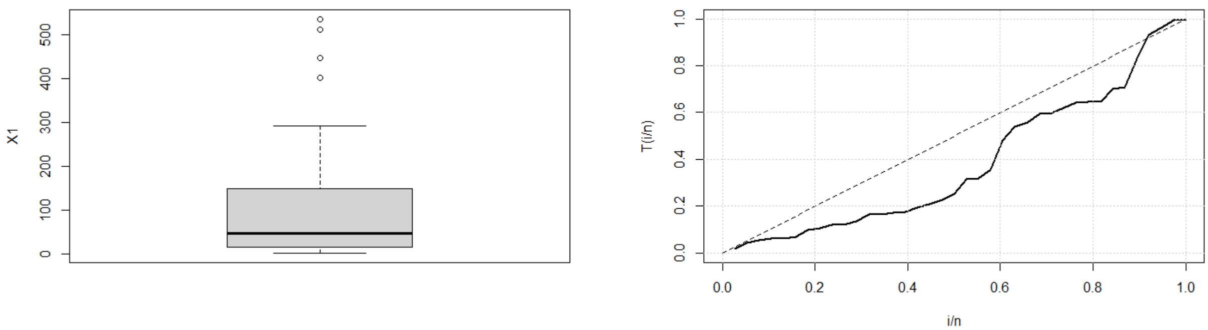

figure 6.1.1 Box plot and scaled TTT transform for X1

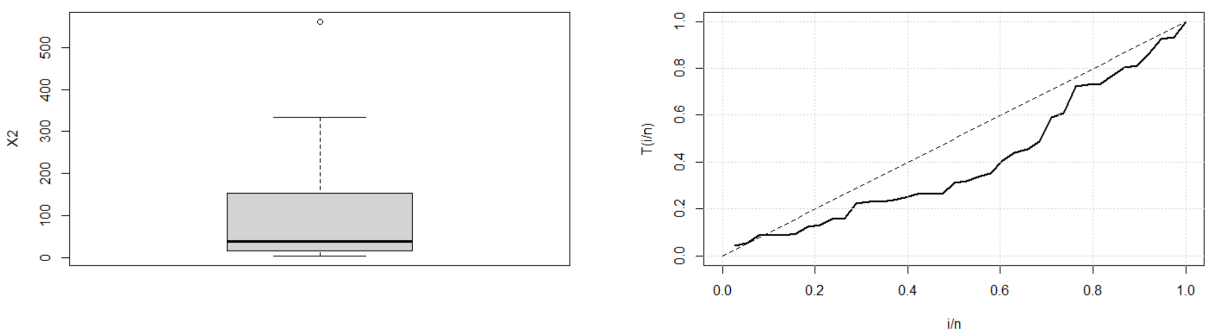

figure 6.1.2 Box plot and scaled TTT transform for X2

We fitted the data for q-BVGED. The maximum likelihood estimates, log likelihood value and the AIC, BIC, and HQIC values are shown in table 4. We have compared the results with the bivariate Gumbel's exponential distribution (BVGED), bivariate distribution with weighted exponential marginals proposed by [56](BWE) and bivariate generalized exponential distribution introduced by [57](BGE) we can see that the q-BVGED is a better fit to the data with low AIC, BIC and HQIC values.

We have used the bivariate version of Kolmogrov-Smirnov test given in [58] for testing the goodness of fit. The values of the K.S. statistic are $D_1 = 0.1503$, $D_2 = 0.1287$, $D_3 = 0.0494$, $D_4 = 0.0261$ and $D_5 = 0.02402$ and thus D^*=0.1503 $(\max(D_1, D_2, D_3, D_4, D_5))$. The above value is less than the value 0.2103 at 25^th percentile thus we can conclude that q-BVGED is appropriate for the given data.

We have also performed the Kolmogrov-Smirnov(K-S) test for the marginal distributions. The K-S distance between the empirical distribution function and fitted distribution function of X_1 and X_2 with their corresponding p values (in bracket) are 0.1127 (0.7191) and 0.11079 (0.7394) respectively. Thus we can conclude that q-BVGED fits the marginal distribution of the data well.

A likelihood ratio test can be used to compare the fit of a distribution having additional parameters with some of its sub-models with respect to a particular data set. Here we use likelihood ratio test to evaluate the performance of q-BVGED with respect to BVGED.

We have, under the null hypothesis,

$$\lambda = -2\ln\left(\frac{\textit{likelihood under the null hypothesis}}{\textit{likelihood in the whole parameter space}}\right) \sim \chi^2_{(d)} \tag{50}$$

Table 4. Parameter estimates, Log likelihood, AIC,BIC and HQIC values for the recurrence time data

Distribution	Estimates	Log Likelihood	AIC	BIC	HQIC
q-BVGED	$\hat{q}$=1.41517	-422.95	853.9	860.4503	856.2306
	$\hat{\alpha}$=0.0146				
	$\hat{\mu}$=0.01698				
	$\hat{\theta}$=0.00007				
BVGED	$\hat{\alpha}$=0.00895	-426.8473	859.6946	864.6074	861.4425
	$\hat{\mu}$=0.01092				
	$\hat{\theta}$=2.43x10^{-11}				
BWE	$\hat{\lambda}_1$=0.0.00897	-426.6662	859.3324	864.2452	861.0803
	$\hat{\lambda}_2$=0.01096				
	$\hat{\lambda}_{12}$=3.5868				
BGE	$\hat{\alpha}$=1.1409	-425.3271	856.6542	861.567	858.4021
	$\hat{\lambda}_1$=0.00938				
	$\hat{\lambda}_2$=0.01155				

where, $\chi^2_{(d)}$ follows a chi-square distribution having d degrees of freedom, d denotes the number of additional parameters in the extended model. Applying this result and using standard statistical tables, critical values for the test statistic can be obtained.
Here our null hypothesis is,
$H_0 : q \to 1$ against $H_1 : q \neq 1$, and our test statistic gives the value,

$$\lambda = -2\ln(-426.8473 + 422.95) = 7.7946 \tag{51}$$

which follows a chi-square distribution with 1 degrees of freedom. The likelihood ratio test yields a p-value given by 0.00524. Thus we reject the null hypothesis and we can conclude that the q-BVGED model is better compared to the BVGED model for the data.

6.2. *Finance Data*

Here we consider a financial data set to demonstrate the applicability of q-BVGED in financial domain. The data set consists of the daily adjusted closing stock prices of Apple (AAPL) and Microsoft (MSFT) for the time period 1 February 2024 to 1 February 2025. The observations are scaled (divided by 1000) to ensure numerical stability. The data was extracted from https://finance.yahoo.com/.
We fitted the data for q-BVGED. The maximum likelihood estimates, log likelihood value and the AIC, BIC, and HQIC values are shown in table 5. We have compared the results with the bivariate Gumbel's exponential distribution (BVGED), bivariate distribution with weighted exponential marginals proposed by [56](BWE) and bivariate generalized exponential distribution introduced by [57](BGE) we can see that the q-BVGED is a better fit to the data with low AIC, BIC and HQIC values.
The likelihood ratio test has also been carried out and the test yields a p-value of 0. Thus we reject the null hypothesis and we can conclude that the q-BVGED model is better compared to the BVGED model for the data.
We have used the bivariate version of Kolmogrov-Smirnov test given in [58] for testing the goodness of fit. The values of the K.S. statistic are $D_1 = 0.0705$, $D_2 = 0.004$, $D_3 = 0.0026$, $D_4 = 0.0462$ and $D_5 = 0.0027$ and thus D^*= 0.0705 $(\max(D_1, D_2, D_3, D_4, D_5))$. The above value is less than the value 0.0792 at $25^t h$ percentile thus we can conclude that q-BVGED is appropriate for the given data.

Table 5. Parameter estimates, Log likelihood, AIC,BIC and HQIC values for the stock price data

Distribution	Estimates	Log Likelihood	AIC	BIC	HQIC
q-BVGED	$\hat{q}$=1.3232	-4623.591	9255.182	9269.284	9260.857
	$\hat{\alpha}$=0.04047				
	$\hat{\mu}$=0.02340396				
	$\hat{\theta}$=0.0447				
BVGED	$\hat{\alpha}$=0.0101	-5779.489	11564.98	11575.55	11569.23
	$\hat{\mu}$=0.0101				
	$\hat{\theta}$= 0.5				
BWE	$\hat{\lambda}_1$=0.0026	-13455.46	26916.92	26927.5	26921.18
	$\hat{\lambda}_2$=0.0047				
	$\hat{\lambda}_{12}$=0.0146				
BGE	$\hat{\alpha}$=3.9703	-13087.88	26181.76	26192.34	26186.02
	$\hat{\lambda}_1$=0.0043				
	$\hat{\lambda}_2$=0.01155				

7. Conclusion

In this paper we have proposed a generalization of bivariate Gumbel's exponential distribution whose marginals are q-exponential distributions. We derived various properties of the proposed distribution. The parameters were estimated using the method of MLE and was illustrated using two real data sets and we observed that for both the data sets q-BVGED model provides a better fit than the bivariate Gumbel's exponential model.

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