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Numerical Solution of the Lotka-Volterra Stochastic Differential Equation

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Abstract This paper presents the modeling of the stochastic differential equation of Lotka-Volterra and introduces the application of two numerical methods to approximately obtain the solution to this stochastic model. The methods used to solve the stochastic differential equation are the Euler-Maruyama method and the Milstein method. Additionally, a methodology will be presented to obtain the parameters of the predator-prey model equation based on empirically obtained data from observations conducted over a fixed period of time.

Keywords Stochastic differential equations, Lotka-Volterra model, Euler-Maruyama method, Milstein method.

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I. Introduction

One of the most significant challenges in fields such as science and industry lies in studying the behavior of a quantity underlying systems governed by random factors. The concept of an ”underlying quantity” refers to an object whose value is known in the present but is subject to changes in the future. Common examples include the count of cancer cells, stock prices, values of minerals and oil, among others. These systems are generally represented by differential equations which, when affected by random disturbances like system volatility, become stochastic differential equations. Since it is difficult to find explicit solutions for certain stochastic differential equations, deterministic approaches are adapted to the stochastic context.

A scalar stochastic differential equation (SDE) is presented in the following form:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \quad X_0 = x_0, \tag{1}$$

where $t \in [0, T]$, and a, b are scalar functions. The unknown is the process X_t, while the coefficients a and b are the trend coefficient and diffusion coefficient, respectively. The diffusion coefficient is accompanied by a Wiener process.

It is common to rewrite the equation in its integral form:

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s. \tag{2}$$

The first term is a Riemann integral, while the second is a stochastic Itô integral.

The first integral of (2) is a Riemann integral, while the second is a stochastic Itô integral. The solution X_t is a random variable for each t.

Processes of the form (2) are called Itô processes, and just as in the deterministic case, in order for them to have a solution, the coefficients must meet some conditions.

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I.1. Existence and uniqueness theorem for stochastic differential equations

Given the following conditions:

1. **Continuity Condition:** The coefficients in the stochastic differential equation (1) must be measurable and continuous functions t and x for all t in an interval $I = [0, T]$ and for almost all $x \in \mathbb{R}$.
2. **Linearity Condition:** The SDE must be linear in x in the sense that $a(t, x)$ and $b(t, x)$ must be linear functions in x. This means that for every fixed t, $a(t, x)$ and $b(t, x)$ must be linear in x.
3. **Lipschitz Condition:** There are constants L and K such that:
   - For $a(t, x)$: $|a(t, x) - a(t, y)| \leq L|x - y|$
   - For $b(t, x)$: $|b(t, x) - b(t, y)| \leq K|x - y|$
4. **Integrability Condition:** The integral of $b(t, x)$ squared with respect to x must be bounded at $[0, T]$:

$$\int_0^T |b(x,t)|^2 dx < \infty$$

Then, under these conditions, there exists a unique solution X_t for the SDE (1) in the interval $I = [0, T]$ in a suitable probability space [1, 2].

It is important to note that specific conditions may vary depending on the context and the nature of the SDE. This theorem provides a solid foundation for the existence and uniqueness of solutions for SDE under certain conditions of regularity and boundedness of the coefficients.

Definition 1: A numerical method is said to have strong order of convergence equal to r if there exists a constant C such that:

$$E\{X_n - X(t)\} \leq C\Delta t^r \tag{3}$$

For any choice $t = n\Delta t \in [0, T]$ and Δt small enough [3, 5].

Definition 2: A numerical method is said to have weak order of convergence equal to r if there exists a constant C for every function p such that:

$$|E\{p(X_n)\} - E\{p(X(t))\}| \leq C\Delta t^r \tag{4}$$

For any choice $t = n\Delta t \in [0, T]$ and Δt small enough [4, 5].

II. CONTENT

II.1. Euler-Maruyama Method:

The Euler-Maruyama method is a numerical method used to approximate the solution of stochastic differential equations (SDE) of the form:

$$dX(t) = f(X(t), t)dt + g(X(t), t)dW(t) \tag{5}$$

Where:

- $X(t)$ is the stochastic process we are trying to approximate.
- $f(X(t), t)$ is the deterministic derivative of X at time t.
- $g(X(t), t)$ is the stochastic derivative of X at time t.
- $dW(t)$ is a stochastic differential following a Wiener process (Brownian process), which is a source of stochastic noise.

The Euler-Maruyama method is based on a time discretization and is used to approximate the solution $X(t)$ at discrete points in time. $t_n = n\Delta t$, where Δt is the step size of time. The numerical approximation of $X(t)$ at time

t_n is denoted as X_n:

$$X_{n+1} = X_n + f(X_n, t_n)\Delta t + g(X_n, t_n)\Delta W_n \tag{6}$$

Where:

- X_{n+1} is the approximation of $X(t_{n+1})$ at time t_{n+1}.
- X_n is the approximation of $X(t_n)$ at time t_n.
- Δt is the time step size.
- $f(X_n, t_n)$ is the value of f at time t_n and X_n.
- $g(X_n, t_n)$ is the value of g at time t_n and X_n.
- ΔW_n is a Wiener increment following a normal distribution with zero mean and variance Δt.

The Euler-Maruyama method is a first-order method, which means that its approximation error is proportional to $\sqrt{\Delta t}$. Therefore, to achieve higher accuracy, the time step size Δt is typically reduced.

It is important to mention that, due to the presence of stochastic noise in SDEs, the numerical solutions generated by the Euler-Maruyama method are stochastic approximations of the true solutions. Therefore, multiple simulations are required to estimate probabilistic statistics of interest.

II.2. Milstein Method

The Milstein method is commonly used to solve SDEs involving stochastic processes based on Brownian motion, such as geometric Brownian motion or the Wiener process. This method is an extension of the Euler-Maruyama method, which is a simpler numerical technique for approximating solutions of SDEs but tends to have larger errors, especially in SDEs with non-linear coefficients.

The Milstein method is based on a second-order Taylor series expansion and takes into account additional terms due to the stochastic nature of the equation. Essentially, the Milstein method improves the accuracy of the Euler-Maruyama method by considering higher-order terms in the Taylor series expansion.

Consider the SDE:

$$dX_t = a(X_t)dt + b(X_t)dW_t, \tag{3}$$

with initial condition $X_0 = x_0$, τ_j, τ_{j+1} consecutive points of the discretization.

Itô's equation establishes that for a twice differentiable function we can write:

$$\theta(X_s) = \theta(X_t) + \int_{t_j}^{s} \theta'(X_u)a(X_u) + \frac{1}{2}\theta''(X_u)b(X_u)^2\, du + \int_{t_j}^{s} \theta'(X_u)b(X_u)dW_u. \tag{4}$$

And when applying Itô's formula to the expressions $a(X_s)$ and $b(X_s)$, which are the coefficients of the SDE, we obtain [6, 7]:

$$\begin{aligned} X_{t_{j+1}} = X_{t_j} + \int_{t_j}^{t_{j+1}} a(X_{t_j})\, du + \int_{t_j}^{s} \left(a'(X_u)a(X_u) + \frac{1}{2}a''(X_u)b(X_u)^2\right) du + \int_{t_j}^{s} a'(X_u)b(X_u)\, dW_u\, ds \\ + \int_{t_j}^{t_{j+1}} b(X_{t_j})\, du + \int_{t_j}^{s} \left(b'(X_u)a(X_u) + \frac{1}{2}b''(X_u)b(X_u)^2\right) du + \int_{t_j}^{s} b'(X_u)b(X_u)\, dW_u\, dW_s. \end{aligned}$$

If you want to obtain a method of strong convergence order equal to 1, you can ignore the double integrals that are of the type dW_s and $dsds$. Then, you get:

$$\begin{aligned} X_{t_{j+1}} &\approx X_{t_j} + \int_{t_j}^{t_{j+1}} \left(a(X_{t_j})\, ds + \int_{t_j}^{t_{j+1}} b(X_{t_j})\, ds + \int_{t_j}^{s} b'(X_u)b(X_u)\, dW_u\right) dW_s \\ &\approx X_{t_j} + a(X_{t_j})\Delta t + b(X_{t_j})\Delta W_{j+1} + \int_{t_j}^{t_{j+1}} \int_{t_j}^{s} b'(X_u)b(X_u)\, dW_u\, dW_s. \end{aligned}$$

The first three addends are well known from the Euler-Maruyama method. The fourth addend can be approximated by:

$$\int_{t_j}^{t_{j+1}} \int_{t_j}^{s} b'(X_u)b(X_u)\, dW_u\, dW_s \approx b'(X_{t_j})b(X_{t_j}) \int_{t_j}^{t_{j+1}} \int_{t_j}^{s} dW_u\, dW_s. \tag{5}$$

The right side integral is:

$$\int_{t_j}^{t_{j+1}} \int_{t_j}^{s} dW_u\, dW_s = \frac{1}{2}\big((\Delta W_{j+1})^2 - \Delta t\big). \tag{6}$$

Substituting into the previous approximation, the Milstein method is finally obtained:

$$X_0 = x_0$$

$$X_{j+1} = X_j + a(X_j)\Delta t + b(X_j)\Delta W_{j+1} + \frac{1}{2}b'(X_j)b(X_j)\big((\Delta W_{j+1})^2 - \Delta t\big). \tag{7}$$

III. APPLICATION

The following table has a sample representation of thousands of individuals of lynx (predators) and rabbits (prey) prepared by the Hudson Bay company between the years 1900 and 1920 in a forest in northern Canada.

Year	Rabbits	Lynxes	Year	Rabbits	Lynxes
1900	30.0	4.0	1911	40.3	8.0
1901	47.2	6.1	1912	57.0	12.3
1902	70.2	9.8	1913	76.6	19.5
1903	77.4	35.2	1914	52.3	45.7
1904	36.3	59.4	1915	19.5	51.1
1905	20.6	41.7	1916	11.2	29.7
1906	18.1	19.0	1917	7.6	15.8
1907	21.4	13.0	1918	14.6	9.7
1908	22.0	8.3	1919	16.2	10.1
1909	25.4	9.1	1920	24.7	8.6
1910	27.1	7.4	1921		

Table 1. Samples of lynx and rabbits in thousands of individuals.

The deterministic Lotka-Volterra model is given by:

$$\frac{dX}{dt} = \alpha X - \beta XY, \quad \frac{dY}{dt} = \delta XY - \gamma Y \tag{8}$$

Where:

- $\frac{dX}{dt}$ is the rate of change of the prey population with respect to time.
- α is the intrinsic growth rate of the prey.
- β is the predation rate of the prey by the predators.
- X is the prey population.
- Y is the predator population.
- $\frac{dY}{dt}$ is the rate of change of the predator population with respect to time.
- δ is the growth rate of the predators due to consumed prey.
- γ is the intrinsic mortality rate of the predators.

The stochastic equations for the stochastic Lotka-Volterra model are given by [8]:

$$dX_t = (\alpha X_t - \beta X_t Y_t)dt + \alpha X_t dW_{1,t}, \quad dY_t = (\delta X_t Y_t - \gamma Y_t)dt + \delta X_t Y_t dW_{2,t} \tag{9}$$

Where the parameters are the same as those in the deterministic model, and $dW_{1,t}$ and $dW_{2,t}$ are independently generated randomly sampled Wiener processes that introduce uncertainty into the system dynamics.

The Milstein method is an extension of the Euler-Maruyama method that provides a better approximation of solutions in stochastic equations by including additional terms to account for stochastic variability.

To derive the Milstein discretization for the stochastic Lotka-Volterra system, we apply Itô's lemma to each equation in Equation (9).

Applying Itô's formula to the prey equation in Equation (9):

$$\begin{aligned} dX_t &= (\alpha X_t - \beta X_t Y_t)dt + \sigma_1 X_t dW_{1,t} \\ d(X_t)^2 &= 2X_t dX_t + (dX_t)^2 \\ &= 2X_t(\alpha X_t - \beta X_t Y_t)dt + 2X_t\sigma_1 X_t dW_{1,t} + \sigma_1^2 X_t^2 dt. \end{aligned}$$

Thus, the Milstein discretization for X_t is:

$$X_{t+\Delta t} = X_t + (\alpha X_t - \beta X_t Y_t)\Delta t + \sigma_1 X_t \Delta W_{1,t} + \frac{1}{2}\sigma_1^2 X_t((\Delta W_{1,t})^2 - \Delta t). \tag{9.1}$$

Now, applying Itô's formula to the predator equation in Equation (9):

$$\begin{aligned} dY_t &= (\delta X_t Y_t - \gamma Y_t)dt + \sigma_2 X_t Y_t dW_{2,t} \\ d(Y_t)^2 &= 2Y_t dY_t + (dY_t)^2 \\ &= 2Y_t(\delta X_t Y_t - \gamma Y_t)dt + 2Y_t\sigma_2 X_t Y_t dW_{2,t} + \sigma_2^2 X_t^2 Y_t^2 dt. \end{aligned}$$

Thus, the Milstein discretization for Y_t is:

$$Y_{t+\Delta t} = Y_t + (\delta X_t Y_t - \gamma Y_t)\Delta t + \sigma_2 X_t Y_t \Delta W_{2,t} + \frac{1}{2}\sigma_2^2 X_t^2 Y_t((\Delta W_{2,t})^2 - \Delta t). \tag{9.2}$$

The cross terms such as $\frac{\partial b_Y}{\partial X}$ in the stochastic equation for Y_t involve derivatives of the diffusion term with respect to X. Since the Wiener processes $dW_{1,t}$ and $dW_{2,t}$ are assumed to be independent, their mixed differentials (e.g., $dW_{1,t}dW_{2,t}$) have expectation zero and do not contribute to the main approximation order in the Milstein scheme. Additionally, the absence of explicit mixed derivatives in the diffusion term $\sigma_2 X_t Y_t$ means that such contributions are negligible at the order of accuracy considered here.

By excluding these terms, we maintain a computationally efficient scheme while preserving the dominant stochastic effects relevant to the system dynamics. For higher-order approximations, one could investigate extensions incorporating such dependencies, but they are typically negligible in standard applications of the Milstein method.

By applying equation (7) to equation (9), we obtain the Milstein method for the stochastic Lotka-Volterra model [9]:

$$\begin{aligned} X_{t+\Delta t} &= X_t + (\alpha X_t - \beta X_t Y_t)\Delta t + \sigma_1 X_t \Delta W_{1,t} + \frac{1}{2}\sigma_1^2 X_t((\Delta W_{1,t})^2 - \Delta t) \\ Y_{t+\Delta t} &= Y_t + (\delta X_t Y_t - \gamma Y_t)\Delta t + \sigma_2 X_t Y_t \Delta W_{2,t} + \frac{1}{2}\sigma_2^2 X_t^2 Y_t((\Delta W_{2,t})^2 - \Delta t) \end{aligned} \tag{10}$$

To obtain the parameters $\alpha, \beta, \gamma, \delta$ and the data in Table (1), the following analysis is used.

The deterministic Lotka-Volterra model is given by equation (8); From this, the following relation is derived:

$$\frac{d}{dt}\ln X(t) = \alpha - \beta Y(t) \tag{11}$$

And the average value of $Y(t)$ over the interval $[0, T]$ is given by:

$$\frac{1}{T}\int_0^T Y(t)\,dt \tag{12}$$

By substituting Equation (11) into Equation (12), we obtain:

$$\frac{1}{T}\int_0^T \frac{1}{\beta}\left(\alpha - \frac{d}{dt}\ln X(t)\right)\,dt = \frac{\alpha}{\beta}. \tag{13}$$

In an analogous way, it is proven that the average value of $X(t)$ is given by:

$$X(t) = \frac{\gamma}{\delta}. \tag{14}$$

IV. RESULTS

To estimate the parameters $\alpha, \beta, \gamma, \delta$, we employ the Maximum Likelihood Estimation (MLE) method, leveraging the time-series data of prey and predator populations.

Likelihood Formulation

The stochastic version of the Lotka-Volterra model introduces diffusion terms:

$$\begin{aligned} dX_t &= (\alpha X_t - \beta X_t Y_t)dt + \sigma_1 X_t dW_{1,t}, \\ dY_t &= (\delta X_t Y_t - \gamma Y_t)dt + \sigma_2 X_t Y_t dW_{2,t}, \end{aligned}$$

where $W_{1,t}$ and $W_{2,t}$ are independent Wiener processes, and σ_1 and σ_2 represent diffusion intensities.

Assuming small time intervals Δt, the transition probabilities are approximated as Gaussian distributions:

$$P(X_{t+\Delta t}, Y_{t+\Delta t} | X_t, Y_t) \sim \mathcal{N}(\mu, \Sigma),$$

where:

$$\begin{aligned} \mu_X &= X_t + (\alpha X_t - \beta X_t Y_t)\Delta t, \\ \mu_Y &= Y_t + (\delta X_t Y_t - \gamma Y_t)\Delta t, \\ \Sigma &= \text{diag}(\sigma_1^2 X_t^2 \Delta t, \sigma_2^2 X_t^2 Y_t^2 \Delta t). \end{aligned}$$

The log-likelihood function for n data points is:

$$\log \mathcal{L}(\alpha, \beta, \gamma, \delta) = -\frac{1}{2}\sum_{i=1}^{n-1}\left[\log|\Sigma| + (\mathbf{z}_{i+1} - \mu)^T \Sigma^{-1}(\mathbf{z}_{i+1} - \mu)\right],$$

where $\mathbf{z}_i = [X_i, Y_i]^T$.

The parameters $\alpha, \beta, \gamma, \delta$ are obtained by maximizing the log-likelihood function using numerical optimization techniques. The estimated values and their 95% confidence intervals are summarized below:

Parameter	Estimate	CI Lower	CI Upper
α	0.368	-0.693	1.429
β	0.015	-0.023	0.054
γ	0.799	-1.159	2.758
δ	0.028	-0.637	0.693

Table 2. Estimated parameters for the Lotka-Volterra model with confidence intervals.

By using the parameters found for α, β, γ, and δ, we have:

$$\begin{aligned} dX_t &= (0.368X_t - 0.015Y_t)dt + 0.368X_t dW_{1,t}, \\ dY_t &= (0.028X_tY_t - 0.799Y_t)dt + 0.028X_tY_t dW_{2,t}. \end{aligned} \tag{15}$$

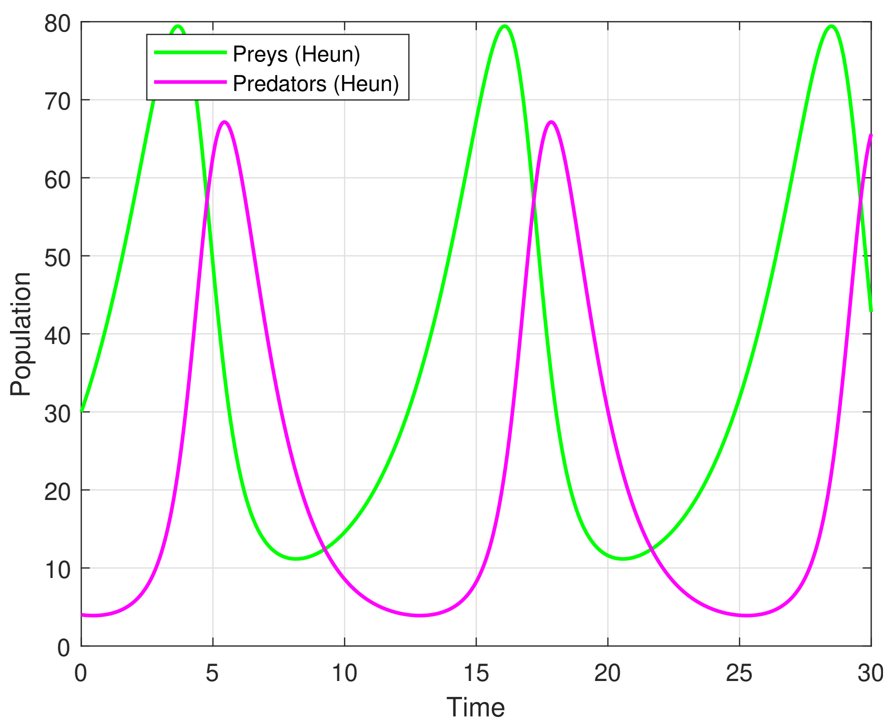


Figure 1. Solution of equation (8) using the Heun's method with the parameters found in the previous section. The green curve represents the rabbit population, and the pink one represents the lynx population.

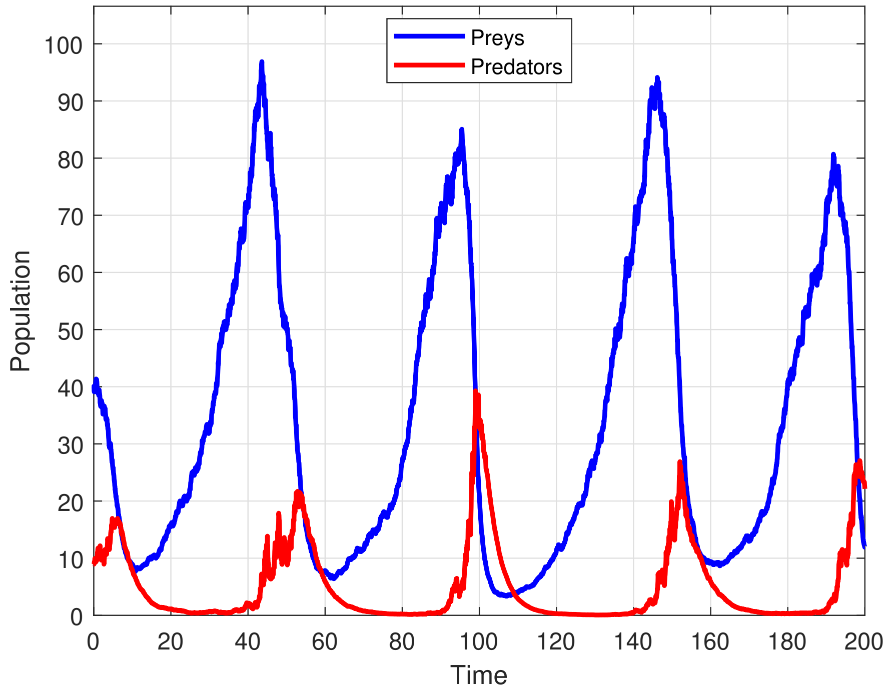


Figure 2. Numerical solution of a trajectory using the Euler-Maruyama method with the parameters found in the previous section. The blue curve represents the rabbit population, and the red one represents the lynx population.

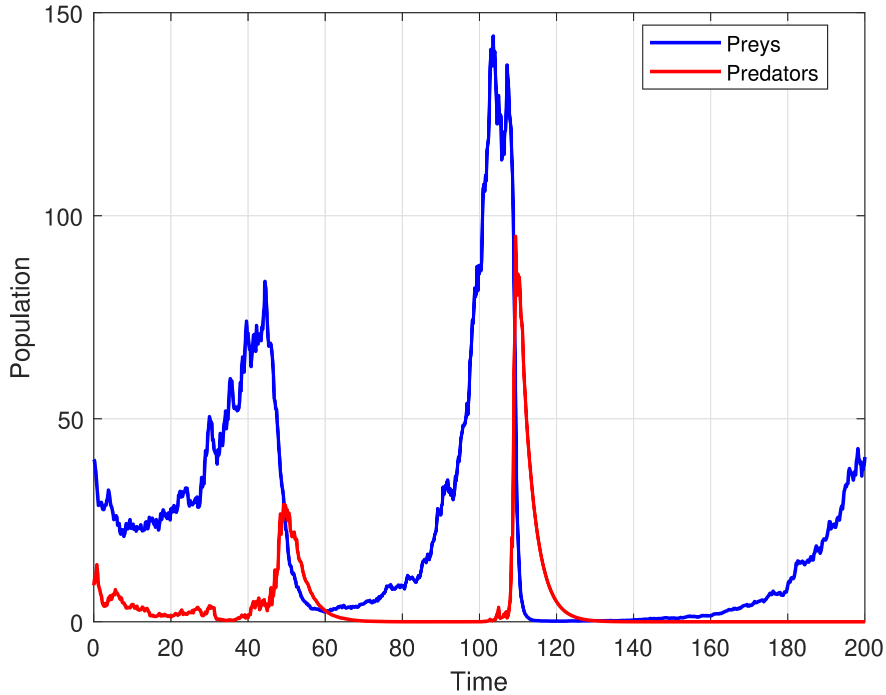
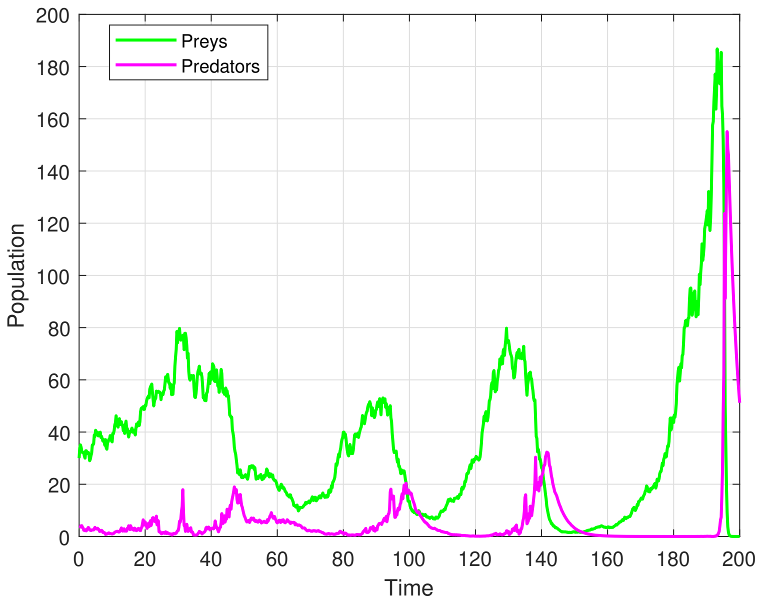
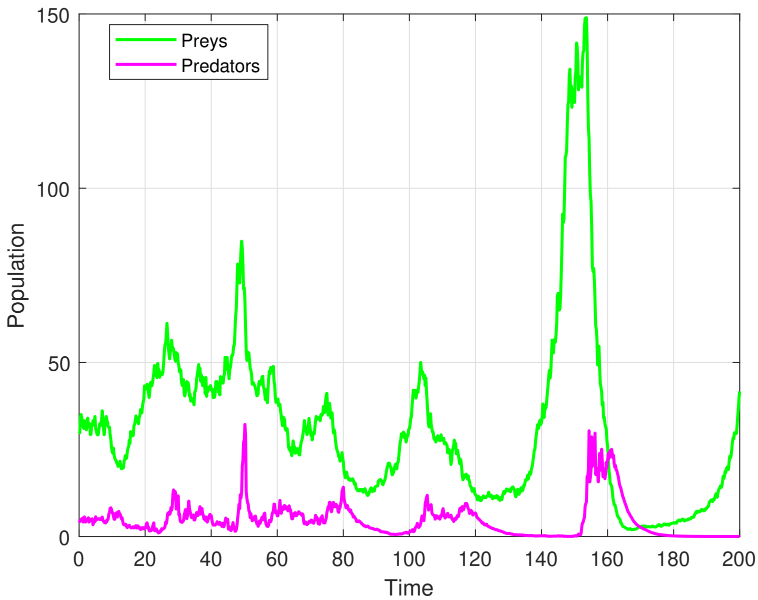


Figure 3. Numerical solution of a trajectory using the Euler-Maruyama method with the parameters found in the previous section. The blue curve represents the rabbit population, and the red one represents the lynx population.



(a) Milstein



(b) Milstein

Figure 4. Numerical solution of different trajectories using the Milstein method with the parameters found in the previous section. The green curve represents the rabbit population, and the pink one represents the lynx population.

V. CONCLUSIONS

- From the simulations, it can be observed that even if the model is considered stochastic, the system's dynamics remain consistent. This means that when the lynx population increases, the rabbit population decreases, and when the prey population decreases, the predator population increases.

- The numerical solution of the stochastic Lotka-Volterra model using the Euler-Maruyama method provides valuable insights into the dynamics of prey and predator populations in an uncertain environment. Through this approach, we can appreciate how stochastic fluctuations can influence interactions between species, even though overall trends remain consistent with the deterministic model. This underscores the importance of considering randomness in ecological systems and how it can impact population stability over time. Additionally, the use of the Euler-Maruyama method offers an effective tool for addressing complex biological systems and examining their behavior under variable and realistic conditions. This research contributes to our understanding of ecology and population biology, helping to better predict how species interactions may respond in fluctuating natural environments.

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