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Function Representation in Hilbert Spaces Using Haar Wavelet Series

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Abstract This work explores the application of integral transforms using Scale and Haar wavelet functions to numerically represent a function $f(t)$. It is based on defining a vector space where any function can be represented as a linear combination of orthogonal basis functions. In this case, the Haar wavelet transform is used, employing Haar functions generated from Scale functions. First, the fundamental mathematical concepts such as Hilbert spaces and orthogonality, necessary for understanding the Haar wavelet transform, are presented. Then, the construction of the Scale and Haar wavelet functions and the process for determining the coefficients for function representation are detailed. The methodology is applied to the function $f(t) = t^2$ over the interval $t \in [-3, 3]$, showing how to calculate the series coefficients for different resolution levels. As the resolution level increases, the approximation of $f(t)$ improves significantly. Furthermore, the representation of the function $f(t) = \sin(t)$ over the interval $t \in [-6, 6]$ using the Haar wavelet series is presented.

Keywords Hilbert Space, Haar Wavelet, Kernel, Inner Product, Cauchy Sequence.

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1. Introduction

This work aims to demonstrate a specific application of integral transforms, using Scale and Haar wavelet functions as kernels [1, 2]. The primary objective is to determine the numerical series of a function $f(t)$ that best represents it. To achieve this, we need to define the mathematical foundation underlying this concept.

We will define the vector space on which we work, where knowing a basis allows us to represent any other element in the space as a linear combination of the basis elements. In our case, these elements will be functions, for which we will impose orthogonality to simplify the necessary calculations to determine the coefficients of the linear combination. Many orthogonal functions are used in literature for representing arbitrary functions, with the most common being trigonometric sines and cosines, Legendre polynomials, Hermite polynomials, etc. [3, 4, 5].

The Haar wavelet transform is a relatively recent technique where the orthogonal functions consist of a set of Haar functions generated by a combination of Scale functions [6]. This work will detail how to construct these orthogonal functions and how to determine the coefficients necessary for representing an arbitrary function as a linear combination.

In Section 2, we will define the mathematical concepts underlying this application. Section 3 defines the Scale and Haar wavelet functions, showing detailed calculations for determining the coefficients. Finally, Section 4 presents the application to a function $f(t)$.

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2. Mathematical Preliminaries

This section defines several necessary concepts, such as Hilbert spaces [7, 8], which provide the most immediate generalization to infinite-dimensional spaces from finite-dimensional Euclidean spaces, preserving geometric notions such as orthogonality. These foundations are essential to better understand the Haar wavelet transform as a tool for function representation:

Definition 2.1. A vector space V over the field $\mathbb{K}$ is an inner product space if there exists a mapping $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{K}$ satisfying the following axioms:

i. $\langle x|x\rangle \geq 0\, \forall x \in V$; $\langle x|x\rangle = 0 \Leftrightarrow x = 0$.
ii. $\langle x|y\rangle = \overline{\langle y|x\rangle}\, \forall x, y \in V$.
iii. $\langle \alpha x|y\rangle = \alpha\langle x|y\rangle\, \forall x, y \in V, \alpha \in \mathbb{K}$.
iv. $\langle x + y|z\rangle = \langle x|z\rangle + \langle y|z\rangle\, \forall x, y, z \in V$.

Every inner product space $(V, \langle\cdot|\cdot\rangle)$ is a normed space if we define:

$$\|x\| = \sqrt{\langle x|x\rangle}. \tag{1}$$

Now, every normed space $(V, \|\cdot\|)$ gives rise to a metric space (V, d) (and thus to a topological space) where:

$$d(x, y) = \|x - y\|. \tag{2}$$

In this work, we are interested in the vector space $L^2([a, b])$ of Lebesgue-measurable functions on the real interval $[a, b]$ with integrable square [9]:

$$L^2([a,b]) = \left\{ x : [a,b] \to \mathbb{K} \,|\, x \text{ is Lebesgue measurable}, \int_a^b |x(t)|^2\, dt < \infty \right\},$$

where pointwise algebraic operations are defined, and the inner product is defined by:

$$\langle x|y\rangle = \int_a^b x(t)\overline{y(t)}\, dt, \quad \forall x, y \in L^2([a, b]). \tag{3}$$

Definition 2.2. An inner product space $(X, \langle\cdot|\cdot\rangle)$ is called a Hilbert space if the metric space (X, d) is complete, where d is defined as in 2 and the norm is given by 1.

Recall that a metric space (X, d) is said to be complete if every Cauchy sequence in X converges to an element of X.

In a metric space (X, d), the distance δ from an element $x \in X$ to a non-empty subset $M \subset X$ is defined as:

$$\delta = \inf_{y \in M} d(x, y). \tag{4}$$

In our work, X will be a certain function space, and M will be a subset of X consisting of well-behaved functions, such as the scale and Haar wavelet functions. It is important to determine if there exists a unique $\tilde{y} \in M$ that minimizes the distance to $x \in X$. This constitutes the well-known problem of existence and uniqueness of the best approximation to x from M [10, 11].

Definition 2.3. A subset $M \subset X$ of a vector space is said to be convex if for any $x, y \in M$ and any $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in M$.

Theorem 2.4. *Every complete, non-empty convex subset M of an inner product space X contains an element with minimum norm.*

Theorem 2.5. *If X is an inner product space and M is a complete, non-empty convex subset of X, then for every $x \in X$, there exists a unique best approximation to x from M, that is, a unique $\tilde{y} \in M$ such that*

$$\delta = \inf_{y \in M} \|x - y\| = \|x - \tilde{y}\|.$$

From the previous theorem, if M is a subspace of X with an orthogonal basis $\{v_i\}_{i=1}^{n}$, then the best approximation to x from M is given by:

$$\tilde{y} = \sum_{k=1}^{n} c_k v_k, \tag{5}$$

where $c_k = \langle x | v_k \rangle$.

Definition 2.6. Let $(X, \langle \cdot | \cdot \rangle)$ be an inner product space. Given $M \subset X$, the set

$$M^{\perp} = \{z \in X : z \perp M\} = \{z \in X : \langle z | x \rangle = 0 \, \forall x \in M\}$$

is called the annihilator of M.

In the case where X is a Hilbert space and M is a closed subspace of X, $M^{\perp}$ is called the orthogonal complement of M.

Definition 2.7. A vector space X is said to be the direct sum of its subspaces Y and Z, denoted $X = Y \oplus Z$, if every $x \in X$ can be uniquely written as the sum of an element $y \in Y$ and another $z \in Z$. In this case, Y and Z are said to be complementary subspaces of X.

Theorem 2.8. *Let M be a closed subspace of a Hilbert space H. Then $H = M \oplus M^{\perp}$. More precisely:*

1. *Every $x \in H$ has a unique decomposition $x = Px + Qx$ as the sum of $Px \in M$ and $Qx \in M^{\perp}$.*
2. *Px and Qx are the best approximations to $x \in H$ from M and $M^{\perp}$, respectively.*
3. *$\|x\|^2 = \|Px\|^2 + \|Qx\|^2$.*

3. Scale and Haar Wavelet Functions

In this section, we define the base function from the set $B \subset X$ that allows us to determine the best approximation of $f \in X$, known as the **Scale Function**, where X is the set of functions defined as:

$$X = \left\{ f : [a, b] \to \mathbb{R} \,|\, f \in L^2([a, b]) \right\}.$$

Definition 3.1. The Scale function is defined on the set $\Omega = [0, 1]$ as follows:

$$\phi(t) = \begin{cases} 1 & 0 \leq t < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Now, for $f \in X$, we can determine a unique best approximation from $M \subset X$, meaning there exists a unique $\tilde{f} \in M$ such that:

$$f(t) \approx \tilde{f}(t) = \sum_{k \in I} c_k \phi_k(t), \tag{7}$$

where $\phi_k(t) = \phi(t - k)$ with $k \in I$, an index set, generates the subspace $M = \text{span}\{\phi_k(t)\}$. We define V_0 as the closure of M, that is,

$$V_0 = \overline{\text{span}\{\phi_k(t)\}}. \tag{8}$$

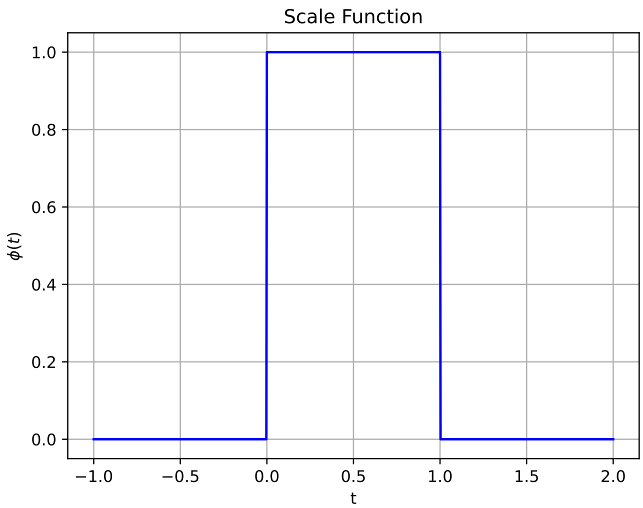

Figure 1. Scale Function

It is easy to see that the subspace V_0 is orthonormal; therefore, for $c_k = \langle f|\phi_k\rangle$ defined as in the inner product 3, we have:

$$c_k = \int_k^{k+1} f(t)\phi_k(t)\,dt. \tag{9}$$

If we want to improve the representation of the function $f \in X$, we need to construct a basis with higher resolution. This involves generating a subspace of Scale functions, called daughter functions, from the Scale function 6 by scaling and translating:

$$\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k), \tag{10}$$

where the first subscript $j \in \mathbb{N}$ represents the resolution level, and $k \in I$ denotes the translations. Thus, the representation of $f \in X$ will be given by:

$$\tilde{f}(t) = \sum_{k\in I} c_{j,k}\phi_{j,k}(t), \tag{11}$$

where

$$c_{j,k} = \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} f(t)\phi_{j,k}(t)\,dt. \tag{12}$$

Now, we define the orthonormal subspace of functions

$$V_j = \overline{\text{span}\{\phi_{j,k}(t)\}} \tag{13}$$

noting that $V_i < V_j$ for all $i < j$.

Definition 3.2. The Haar wavelet function is defined on the set $\Omega = [0, 1]$ as follows:

$$\psi(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}, \\ -1 & \frac{1}{2} \le t < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

It can be observed that $\psi(t)$ is orthogonal to $\phi(t)$ on the interval $t \in [0, 1]$, i.e.,

$$\langle \phi(t) | \psi(t) \rangle = \int_0^1 \phi(t)\psi(t)\, dt = 0. \tag{15}$$

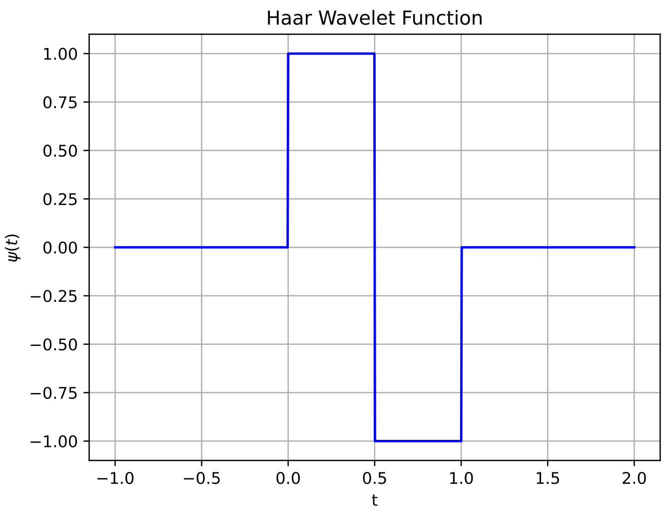

Figure 2. Haar Wavelet Function

With this orthogonality, we can define the orthogonal complement of M as the set of all functions that can be obtained through linear combinations of translations of the mother wavelet function:

$$M^{\perp} = \text{span}\{\psi_k(t)\}, \tag{16}$$

where $\psi_k(t) = \psi(t - k)$. We define the space W_0 as the closure of $M^{\perp}$, that is,

$$W_0 = \overline{\text{span}\{\psi_k(t)\}}. \tag{17}$$

For a function $g \in V_1$, its representation can be decomposed as the sum of Scale and wavelet functions of equal resolution:

$$g(t) = \sum_{k \in I} c_{0,k}\phi_{0,k}(t) + \sum_{k \in I} d_{0,k}\psi_{0,k}(t), \tag{18}$$

or, equivalently, $V_1 = V_0 \oplus W_0$. In general, we have:

$$V_{j+1} = V_j \oplus W_j, \tag{19}$$

where W_j is the closure of the space generated by $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$. From 19, once the Scale function is selected, the choice of the wavelet function is not arbitrary.

Finally, the representation of $f \in X$ over the space V_{j+1} will be given by:

$$f(t) = \sum_{k \in I} c_{j,k}\phi_{j,k}(t) + \sum_{k \in I} d_{j,k}\psi_{j,k}(t). \tag{20}$$

4. Results and Simulations

This section presents the calculations and conditions required to obtain the best representation of the function $f(t)$ over a predefined interval. We start by analyzing the function $f(t) = t^2$ over the interval $t \in [-3, 3]$ to simplify the calculation of the series coefficients.

Before calculating these coefficients, we verify that $f \in X$, i.e., $f \in L^2([-3, 3])$, which means:

$$\int_{-3}^{3} f^2(t)\, dt < +\infty.$$

This integral yields:

$$\int_{-3}^{3} t^4\, dt = \frac{486}{5} < +\infty.$$

Thus, we can guarantee that a unique best approximation of f exists over the space V_j.

Using equation 20, we select the resolution level j. For $j = 0$, we have:

$$f(t) \approx \sum_k c_{0,k}\phi_{0,k}(t) + \sum_k d_{0,k}\psi_{0,k}(t),$$

where $c_{0,k} = \langle f|\phi_{0,k}\rangle$ and $d_{0,k} = \langle f|\psi_{0,k}\rangle$.

Expanding the expressions for $c_{0,k}$ and $d_{0,k}$, we get:

$$c_{0,k} = \int_{k}^{k+1} t^2\, dt = \frac{3k^2 + 3k + 1}{3},$$

for all $k \in \mathbb{Z}$ within the range $k = -3$ to $k = 2$, as defined by the interval of approximation. Similarly, the coefficients for the Haar wavelet functions are given by:

$$d_{0,k} = \int_{k}^{k+0.5} t^2\, dt - \int_{k+0.5}^{k+1} t^2\, dt = \frac{2k-1}{4}.$$

To improve the approximation, we increase the resolution level. For $j = 1$, we calculate twice the coefficients for both $c_{0,k}$ and $d_{0,k}$, now denoted $c_{1,k}$ and $d_{1,k}$, respectively. They are given by:

$$c_{1,k} = \int_{k/2}^{(k+1)/2} t^2\, dt = \frac{3k^2 + 3k + 1}{12\sqrt{2}},$$

and

$$d_{1,k} = \int_{k/2}^{(k+0.5)/2} t^2\, dt - \int_{(k+0.5)/2}^{(k+1)/2} t^2\, dt = -\frac{2k+1}{16\sqrt{2}}.$$

As shown, the approximation of $f(t) = t^2$ improves significantly with an increased resolution level. At a high enough resolution, such as $j = 4$, the functions become indistinguishable from the original function, as shown in Figure 5.

Now, for $f(t) = \sin(t)$ over the interval $t \in [-6, 6]$, its Haar wavelet series representation at resolution level $j = 2$ is given by:

$$\sin(t) \approx \sum_k c_{2,k}\phi(4t - k) + \sum_k d_{2,k}\psi(4t - k).$$

Finally, in Figure 7, we present the representation of a piecewise function:

$$f(t) = \begin{cases} t^2 & -6 \le t < 0, \\ 36 - 6t & 0 \le t < 6, \\ 0 & \text{otherwise}, \end{cases}$$

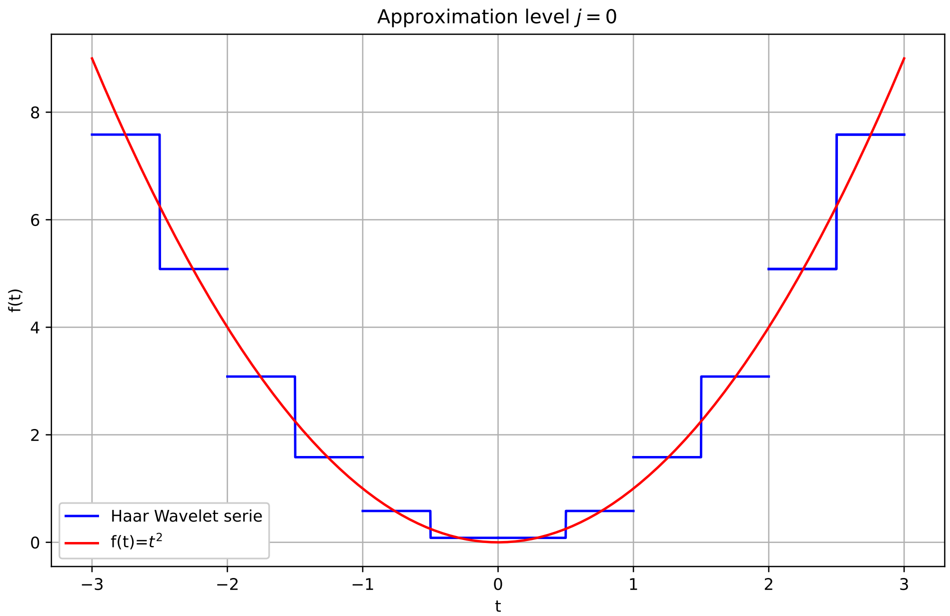

Figure 3. Haar Wavelet Series Representation with $j = 0$

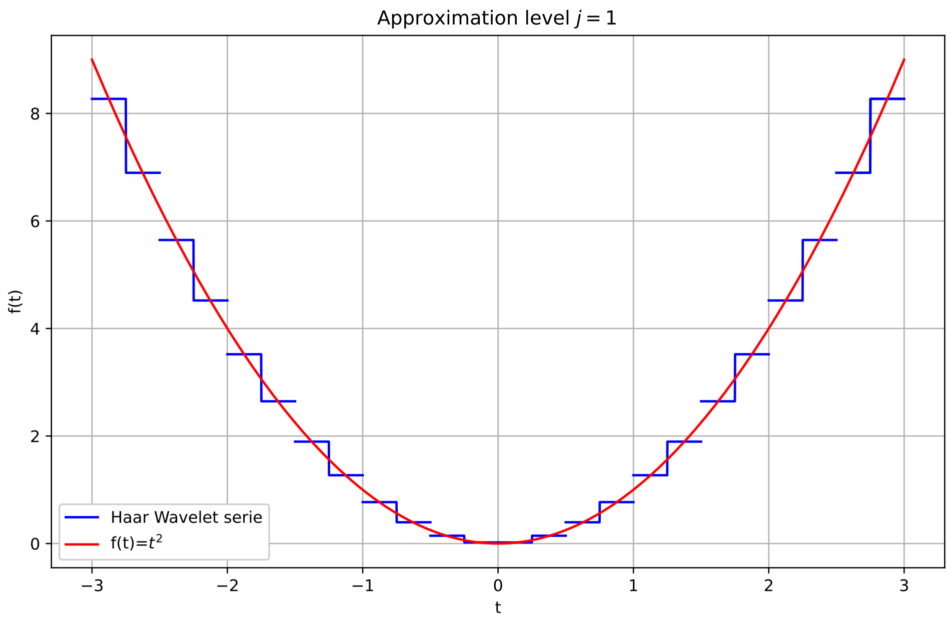

Figure 4. Haar Wavelet Series Representation with $j = 1$

using both Fourier series and Haar wavelet series. It is evident that in the Fourier series representation of discontinuous functions, the Gibbs phenomenon appears as overshoots near the discontinuities. In contrast,

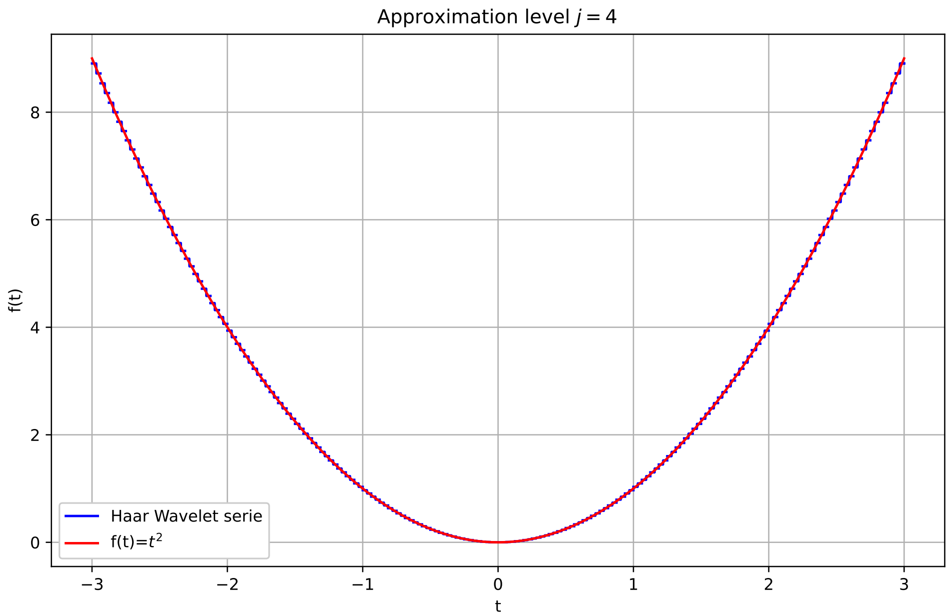

Figure 5. Haar Wavelet Series Representation with $j = 4$

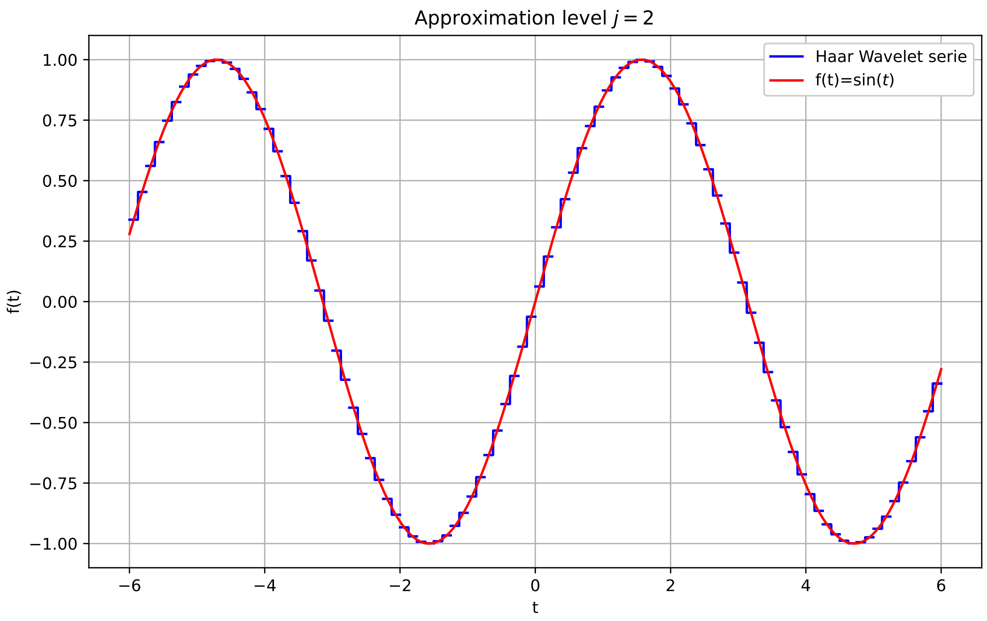

Figure 6. Haar Wavelet Series Representation for $f(t) = \sin(t)$

Haar wavelets, due to their time and frequency localization properties, allow a more accurate representation of discontinuous functions, avoiding the overshoots typical of Fourier series.

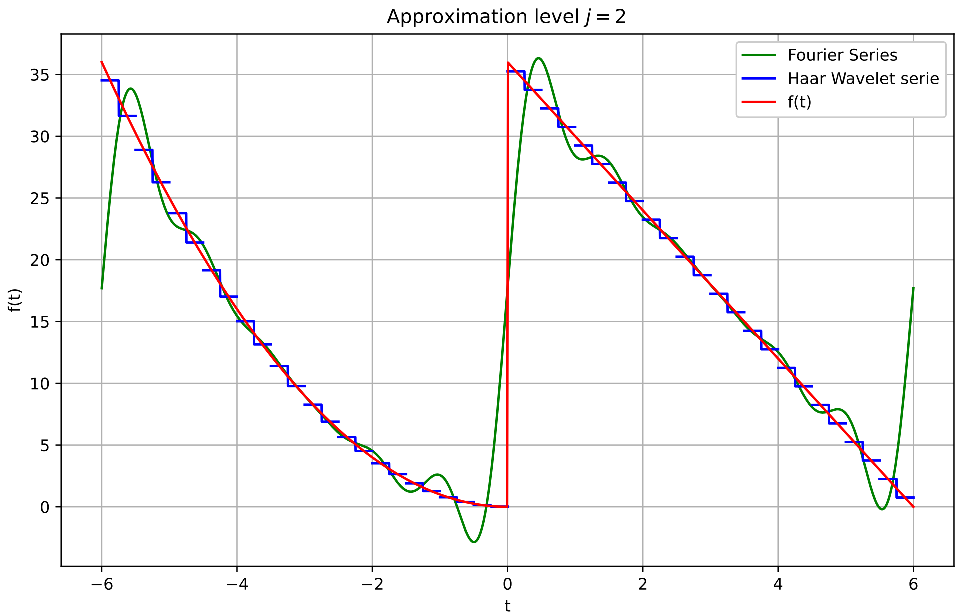

Figure 7. Representation of a Discontinuous Function using Fourier and Haar Wavelet Series

5. Conclusion

The representation of functions using Haar wavelet series has had a significant impact in recent decades due to its simplicity compared to Fourier series representation. Haar wavelet series allow for efficient approximation of functions, especially those with discontinuities, while avoiding the overshoots known as the Gibbs phenomenon, which commonly occur in Fourier series.

In this work, we have presented a detailed construction of the spaces V_{j+1} and the necessary conditions for an arbitrary function $f \in X$ to be approximated as a partial sum of Scale and Haar wavelet functions. We demonstrated the orthogonality calculations of the functions $\phi_k(t)$ and $\psi_k(t)$, and applied this method to functions like $f(t) = t^2$ and $f(t) = \sin(t)$, showing how to calculate the series coefficients at different resolution levels.

By increasing the resolution level from $j = 0$ to $j = 4$, we observed a significant improvement in the approximation of $f(t) = t^2$. This ability to adjust the resolution level makes Haar wavelet series a powerful tool for achieving high accuracy in function approximations. The simplicity of calculating the coefficients is a key aspect that makes Haar wavelet series an accessible and efficient method for numerical representation.

In future research, the application of Haar wavelet transforms could be extended to more complex functions and problems. Additionally, comparing the effectiveness of Haar wavelets with other types of wavelets, such as Daubechies or Morlet, in various fields, including signal processing and image compression, would help further expand the scope and utility of wavelet methods in both theoretical and applied fields.

This work highlights the importance of wavelet methods in areas requiring precise and localized function representations, which can be particularly useful in technological applications where data compression and noise reduction are critical.

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