STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **13**, May 2025, pp 2011–2026.
Published online in International Academic Press (www.IAPress.org)

Analysis of uncertainty in the Leontief model by interval arithmetic

Benhari Mohamed Amine*, Kaicer Mohammed

Laboratory of analysis, geometry and applications, Ibn-Tofail University, Morocco

Abstract This paper presents an innovative strategy to enhance the precision of economic projections through the integration of interval arithmetic into the Leontief model. We emphasise the utilisation of the Gauss-Seidel method for solving linear systems with interval coefficients. In this paper, we present a method that use the Gauss-Seidel approach to effectively solve linear systems consisting of interval coefficients. This technique enhances traditional methods by incorporating potential value intervals, in addition to exact numerical values. The result is a more precise reflection of uncertainty and a more accurate calculation of solution intervals for economic variables. We have implemented this approach in the Moroccan economic context and the Washington state context using the Gauss-Seidel method to solve linear systems with interval coefficients. Based on real economic data, we have demonstrated how this technique can have a positive impact on the accuracy of output and sensitivity evaluations in the Leontief model.

Keywords System Of Interval Linear Equations, Uncertainty, Economic Modeling, Leontief Model

DOI: 10.19139/soic-2310-5070-2279

1. Introduction

In economic modelling and forecasting, accounting for uncertainty and change is essential to ensure informed and resilient decision-making. Traditional economic models, while often robust, can underestimate the impact of the uncertainty inherent in the input data, leaving considerable margins of error in forecasts. It is in this context that the application of interval arithmetic in the Leontief model emerged as an innovative strategy for estimating production levels and assessing the sensitivity of the results. The Leontief model [1], also known as the **input-output** model, is an important analytical tool for describing the complex interactions between different sectors of the economy. It is based on the assumption that production in one sector depends on demand from other sectors, creating a network of economic interdependence. This model makes it possible to quantify the way in which changes in demand in one sector are propagated throughout the economy.

Briefly, the I-O model can be expressed as X-AX=Z, where X is a vector representing sectoral output, A is a matrix of technical input-output coefficients and Z is a vector of final demand. Solving equation $(I-A)X=Z$ gives the Leontief matrix $(I-A)$. If the final demand is known, the quantity of goods required to satisfy that demand can be determined [4]. However, the technical coefficients of the Leontief matrix are unknown and must be estimated, thus introducing uncertainty. Several sources of uncertainty can be identified in I-O models, such as the source data and the assumptions inherent in the I-O analysis (linearity, proportionality, distribution, aggregation).

*Correspondence to: Benhari Mohamed Amine (Email: mohamedamine.benhari@uit.ac.ma). Laboratory of analysis, geometry and applications, Faculty of Sciences, Ibn-Tofail University, Kenitra, Morocco.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

Uncertainty in I-O models can be modeled using two general approaches :

A probabilistic approach assumes a probability distribution of all uncertainties associated with the coefficients of the Leontief matrix. It allows probability distributions to be used to represent the variability of uncertain values.

The unknown but bounded approach is a method used in uncertainty analysis. It is often applied when there are uncertainties in the model parameters, but instead of specifying a probability distribution for these uncertainties, upper and lower bounds are simply defined for each parameter.

A variety of analytical and computational techniques are used to examine the impact of uncertain inputs, including :

Sensitivity analysis [3], which includes methods for calculating the impact of parameter changes on model predictions. This can be done univariately (one parameter at a time) or multivariate (several parameters simultaneously).

Uncertainty propagation [2] is a technique for calculating the uncertainty in model outputs caused by input uncertainties.

Other deterministic and probabilistic techniques, such as one-to-one analysis, conjoint analysis, parametric analysis, and Monte Carlo simulation, emphasize the importance of sensitivity analysis and uncertainty, with particular emphasis on taking into account multiple parameters as possible consequences of simultaneous changes. The results are significantly different and more representative of the real situation.
Interval arithmetic [9] offers a fundamentally different approach to modeling and quantifying uncertainty. Instead of providing a single peak, it generates a range of possible values, capturing the diversity of scenarios resulting from natural fluctuations and uncertainties in the data. This approach is particularly important in the context of the Leontief model, which aims to describe the complex economic relationships between different sectors of an economy. By incorporating interval arithmetic into the model, potential changes in output levels can be better understood and predicted, and the way in which uncertainty affects these estimates can be analyzed.
The main objective of this paper is to explore the application of interval arithmetic to the Leontief model to estimate production levels and perform a thorough sensitivity analysis. First, it proposes the use of interval arithmetic as an advanced methodological framework for modeling uncertainty, thereby improving the reliability of economic forecasts in the Leontief model. Secondly, he highlights the efficiency of the Gauss-Seidel solution method for linear systems with interval coefficients, allowing solutions to be calculated in interval form.

Two empirical applications in the Moroccan and American economic contexts are also presented, demonstrating the potential of this method to improve the accuracy of economic projections and sensitivity analyses. Finally, a comparative analysis with traditional approaches highlights better management of uncertainty and a significant reduction in margins of error.

Ultimately, this article aims to show that integrating interval arithmetic into economic models can transform the way we approach economic forecasts, thereby meeting the growing challenges of accuracy and robustness in an uncertain environment.

2. Interval arithmetic

2.1. Generalized intervals

Let $I\mathbb{R} = \{\widehat{a} = [a_1; a_2] : a_1 \leqslant a_2 and a_1, a_2 \in \mathbb{R}\}$ be the set of all proper intervals and $\overline{I\mathbb{R}} = \{\widehat{a} = [a_1; a_2] : a_1 > a_2 and a_1, a_2 \in \mathbb{R}\}$ be the set of all improper intervals on the real line R. If $a_1 = a_2 = a$, then $\widehat{a} = [a, a] = a$ is a real number (or a degenerate interval). We shall use the terms "interval" and "interval number" interchangeably. The mid-point and width(or half-width) of an interval number $\widehat{a} = [a_1, a_2]$ are defined as $m(\widehat{a}) = \dfrac{a_1 + a_2}{2}$ and $w(\widehat{a}) = \dfrac{a_2 - a_1}{2}$. We denote the set of generalized intervals (proper and improper) by :

$$K\mathbb{R} = I\mathbb{R} \cup \overline{I\mathbb{R}} = \{[a_1; a_2] : a_1, a_2 \in \mathbb{R}\}$$

The set of generalized intervals $K\mathbb{R}$ is a group with respect to addition and multiplication operations of zero free intervals, while maintaining the inclusion monotonicity.
The "dual" is an important monadic operator proposed by kaucher that reverses the end-points of the intervals in $K\mathbb{R}$. For $\widehat{a} = [a_1, a_2] \in K\mathbb{R}$, its dual is defined by $dual(\widehat{a}) = dual([a_1, a_2]) = [a_2, a_1]$. The opposite of an interval $\widehat{a} = [a_1, a_2]$ is $opp([a_1, a_2]) = [-a_1, -a_2]$ which is the additive inverse of $[a_1, a_2]$ and $\left[\dfrac{1}{a_1}, \dfrac{1}{a_2}\right]$ is the multiplicative inverse of $[a_1, a_2]$, provided $0 \notin [a_1, a_2]$.
That is, $\widehat{a} + (-dual(\widehat{a})) = [0, 0]$ and $\widehat{a} \times \dfrac{1}{dual(\widehat{a})} = [1, 1]$.
Ganesan and Veeramani [5] proposed new interval arithmetic on $\mathbb{IR}$. We extend these arithmetic operations to the set of generalized interval numbers $K\mathbb{R}$ and incorporating the concept of dual.
For $\widehat{a} = [a_1; a_2]$, $\widehat{b} = [b_1; b_2] \in K\mathbb{R}$ and for $* \in \{+; -; \times; \div\}$ we define :

$$\widehat{a} * \widehat{b} = [m(\widehat{a}) * m(\widehat{b}) - k; m(\widehat{a}) * m(\widehat{b}) + k] \text{ and } k = min\left\{(m(\widehat{a}) * m(\widehat{b}) - \alpha; \beta - m(\widehat{a}) * m(\widehat{b}))\right\}$$

α and β are the end points of the interval $\widehat{a}$ et $\widehat{b}$.

2.2. Algebraic solution of an interval linear system

Definition: an interval matrix [6] $\widehat{A}$ is a matrix whose elements are intervals.
Let $\widehat{A}$ be an interval matrix of order $n \times n$.
To solve a linear system involving interval matrices, we try to find the smallest interval vector containing the set of vectors $\widehat{X}$ in a way that there exists a elementary matrix $A \in \widehat{A}$ and $B \in \widehat{B}$ and we have the equality $Ax = B$.
When solving linear systems with interval coefficients, the choice of method depends on various factors, including the nature of the system, the complexity of the interval, and the modeling objectives. However, depending on the characteristics of the system, some common methods can be considered, such as :
Cholesky decomposition method [12]: the application of the method to linear systems with interval coefficients presents certain specific constraints and conditions. It requires a positive definite symmetric matrix and during the decomposition process, uncertainty can propagate at each step. These uncertainties affect the final solution.
Cramer's method [11]: the application of Cramer's method to linear systems with interval coefficients can be complex due to the calculation of determinants, which involves delicate calculations, particularly on high-order interval-coefficient matrices.
However, iterative methods can provide better numerical stability than some direct methods, allowing the management of large systems. Nevertheless, it is important to note that the choice of method depends on the specific characteristics of the Leontief model, the nature and size of the interval coefficient matrix, the required convergence tolerance, and other considerations.
After testing different approaches on the test cases to determine which is best suited to the problem, the Gauss-Seidel method is chosen.

Solving the $\widehat{A}$ $\widehat{X}$=$\widehat{B}$ system using the Gauss-Seidel method: the method of **Gauss-Seidel** in $K\mathbb{R}$ is a method for solving linear systems $\widehat{A}\widehat{X} = \widehat{B}$, that $\widehat{A}$ is a matrix $n \times n$ with interval coefficients and $\widehat{X}$, $\widehat{B}$ are vectors with interval coefficients.
The Gauss-Seidel method consists of decomposing $\widehat{A}$ into the form $\widehat{A} = \widehat{D} - \widehat{E} - \widehat{F}$, that $\widehat{D}$ is a diagonal matrix, $-\widehat{E}$ is a lower triangular matrix, and $-\widehat{F}$ is an upper triangular matrix.

The system $\widehat{A}\widehat{X} = \widehat{B}$ is equivalent to:
If the matrix $\widehat{D} - \widehat{E}$ is invertible then:

$$\widehat{A}\widehat{X} = \widehat{B} \Leftrightarrow (\widehat{D} - \widehat{E})\widehat{X} - \widehat{F}\widehat{X} = \widehat{B} \Leftrightarrow \widehat{X} = (\widehat{D} - \widehat{E})^{-1}\widehat{F}\widehat{X} + (\widehat{D} - \widehat{E})^{-1}\widehat{B}$$

A sequence of vectors $\widehat{X}^k$ will be calculated directly by solving the following system with an arbitrary $\widehat{X}^0$:

$$\widehat{X}^{k+1} = (\widehat{D} - \widehat{E})^{-1}\widehat{F}\widehat{X}^K + (\widehat{D} - \widehat{E})^{-1}\widehat{B} \text{ with } k \in \mathbb{N}$$

Concerning the stopping condition of the Gauss-Seidel method is determined according to one of the following circumstances:
- Maximum number of iterations.
- Absolute convergence : $\widehat{X^{n+1}} - \widehat{X^n} = \widehat{\epsilon} \approx \widehat{0}$, with $\widehat{\epsilon}$ is a vector whose coefficients are amplitude intervals tending to 0 and $\widehat{0} = \begin{pmatrix} [0,0] \\ [0,0] \\ \vdots \\ [0,0] \end{pmatrix}$.

We stop the calculation when one of the conditions is verified.

Gauss-Seidel algorithm using generalized interval numbers: let $\widehat{A}$ be a square matrix and $\widehat{B}$ a vector with n-order interval coefficients.
The resolution of this linear system with interval coefficients consists in following the following steps :
Step 1: decompose $\widehat{A}$ as $\widehat{D} - \widehat{E} - \widehat{F}$.
Step 2: check if matrix $(\widehat{D} - \widehat{E})$ is invertible and calculate $(\widehat{D} - \widehat{E})^{-1}$.
Step 3: set $\widehat{X}^0 = \begin{pmatrix} [0,0] \\ [0,0] \\ \vdots \\ [0,0] \end{pmatrix}$ and $\widehat{Y}$=$(\widehat{D} - \widehat{E})^{-1}\widehat{B}$.
Step 4: do a recursive calculation in $\widehat{X}^{n+1} = (\widehat{D} - \widehat{E})^{-1}\widehat{F}\widehat{X}^n + \widehat{Y}$ with $n \in \mathbb{N}^*$.
After each iteration, we check if $\widehat{X}^{n+1} - \widehat{X}^n \approx \widehat{0}$.

2.3. Gauss-Seidel complexity for systems with interval coefficients

The complexity of the Gauss-Seidel method, when adapted to linear systems with interval coefficients, depends on several factors linked to the nature of the calculations on the intervals. Unlike the traditional method applied to linear systems with real coefficients, where the calculations are direct, interval arithmetic requires additional operations to manage the bounds of the intervals and the dependencies between them. This increases the complexity significantly.
For systems with interval coefficients, each elementary operation becomes more expensive because of the arithmetic of the intervals. In particular :

Calculations on intervals: arithmetic operations on intervals (addition, subtraction, multiplication) require the processing of lower and upper bounds, doubling the number of elementary operations.

Propagation of dependencies: dependencies between intervals need to be tracked, to avoid overextending the bounds. This requires additional checks.
Management of uncertain coefficients: when a coefficient is modified during an iteration, it affects other interval coefficients in the system, increasing the calculations required to stabilize the bounds.

Some factors and criteria influence convergence and minimize computation time and complexity :
A well-structured matrix: convergence and minimized complexity are guaranteed if the matrix is strictly diagonal

dominant or symmetric positive definite or a hollow matrix.
Interval coefficients with a small amplitude: wider intervals increase the propagation of uncertainties, requiring a greater number of iterations to achieve convergence, which implies more computation time.
However, complementary approaches, such as interval reduction before each iteration, can reduce complexity.

The complexity of the Gauss-Seidel method on a linear system with interval coefficients, although increased compared with the classical version, remains suitable for economic models of reasonable size thanks to its iterative structure. However, for very large systems or ill-conditioned matrices, it may be necessary to adopt optimizations or alternative methods.

2.4. Gauss-Seidel limits for linear systems with interval coefficients

The first thing to note is the computational complexity. Although the method is effective for matrices of moderate size, solving large systems with coefficients in intervals quickly becomes costly in terms of computing time. Convergence is not guaranteed: the Gauss-Seidel method requires the system matrix to have certain properties to guarantee convergence. With interval coefficients, these properties are not necessarily verified for all possible realizations, complicating the convergence analysis.
An ill-conditioned system with large interval coefficients can lead to an amplification of the uncertainties, widening the bounds of the solutions and making the results less meaningful. These limitations underline the importance of carefully choosing the parameters of the system, optimally structuring the data, and, if necessary, considering the use of complementary or hybrid methods to overcome these obstacles. Therefore, Solutions to these problems include:
- Reformulating the problem as an optimization task aimed at minimizing the width of the intervals while guaranteeing viable solutions.
- Minimising the size of the system.
- Combining with other solution methods.

2.5. Application and comparison of results

We consider the system $\widehat{A}\widehat{X} = \widehat{B}$ with interval coefficient :

$$\widehat{A} = \begin{pmatrix} [3.7; 4.3] & [-1.5; -0.5] & [0; 0] \\ [-1.5; -0.5] & [3.7; 4.3] & [-1.5; -0.5] \\ [0; 0] & [-1.5; -0.5] & [3.7; 4.3] \end{pmatrix} \text{ and } \widehat{B} = \begin{pmatrix} [-14; 0] \\ [-9; 0] \\ [-3; 0] \end{pmatrix}$$

By using the Gauss-Seidel algorithm and generalized interval numbers, we find :

$$\widehat{X} = \begin{pmatrix} [-4.03; 0.86] \\ [-2.73; 1.32] \\ [-1.12; 0.65] \end{pmatrix}$$

Comparaison of results :

Definition: let $V = \begin{pmatrix} \widehat{x_0} \\ \widehat{x_1} \\ \vdots \\ \widehat{x_n} \end{pmatrix}$ be a vector with interval coefficients and $\widehat{x_i} = [a_i; b_i]$ with $i \in \mathbb{N}$.

The total amplitude width of the vector V is the value T_v which equals the sum of the interval widths of all its interval coefficients, such that:

$$T_v = \sum_{i=0}^{n} |b_i - a_i|$$

The aim of this numerical optimisation method is to compare different methods of solving linear systems with interval coefficients and to determine the optimum solution with minimum uncertainty.

Table 1. Comparison between resolution methods

Results	Methods	Width of the vector
$\widehat{X} = \begin{pmatrix} [-4.03; 0.86] \\ [-2.73; 1.32] \\ [-1.12; 0.65] \end{pmatrix}$	Using the Gauss-Seidel method	$T = 10.71$
$\widehat{X} = \begin{pmatrix} [-4.96; 0] \\ [-4.51; 0] \\ [-2.2; 0] \end{pmatrix}$	Using the inverse matrix proposition [7]	$T = 11.67$
$\widehat{X} = \begin{pmatrix} [-6.38; 0] \\ [-6.40; 1.32] \\ [-3.40; 0] \end{pmatrix}$	The result find by **Ning and al** [8] using Gauss elimination	$T = 17.5$
$\widehat{X} = \begin{pmatrix} [-6.38; 1.12] \\ [-6.40; 1.54] \\ [-3.40; 1.40] \end{pmatrix}$	The result find by **Ning and al** using the technique of **Hansen** [8]	$T = 20.02$
$\widehat{X} = \begin{pmatrix} [-4.76; 0.31] \\ [-3.78; 0] \\ [-1.80; 0.08] \end{pmatrix}$	Using the Cramer's method [11]	$T = 10.73$
$\widehat{X} = \begin{pmatrix} [-4.12; 0.96] \\ [-2.99; 1.63] \\ [-1.37; 0.096] \end{pmatrix}$	Using the Jacobi method	$T = 12.03$
$\widehat{X} = \begin{pmatrix} [-5.77; 1.44] \\ [-4.81; 1.48] \\ [-2.79; 1.20] \end{pmatrix}$	Using interval hull method [13]	$T = 17.52$
$\widehat{X} = \begin{pmatrix} [-4.53; 0] \\ [-3.9; 0] \\ [-1.76; 0] \end{pmatrix}$	using the Choleski decomposition[12]	$T = 10.19$

To assess the efficiency and convergence of the various methods applied to linear systems with interval coefficients, a detailed analysis was carried out, including the inverse matrix method, Gauss elimination, Cramer's method, Jacobi's method, interval hull method, technique of Hansen and the Cholesky decomposition, in addition to the Gauss-Seidel method. Of these, the Cholesky decomposition has shown the best accuracy and stability, with convergence guaranteed for symmetric and positive definite matrices, making it particularly suitable for well-conditioned systems. The Gauss-Seidel method also demonstrated high performance, with rapid convergence in many cases, even for large matrices that are not necessarily well-conditioned, which is a major advantage.

The Gauss-Seidel method generally converges if the matrix is strictly diagonal dominant or symmetric and positive definite, although the latter condition is not always necessary in the context of interval coefficients. Its adaptability to large matrices comes from its iterative structure, which reduces memory requirements and computations compared with direct methods such as Cramer or matrix inversion, which are often costly for large systems. This property makes Gauss-Seidel particularly effective for low-cost applications where matrix sizes can be large.

The others methods, while effective in some cases, have shown convergence limits for ill-conditioned systems or matrices with strong interval dependence. Cramer's method, although accurate in simple cases, has a high computational complexity (calculating the determinant), which limits its applicability for large systems. Finally, the inverse method, despite its theoretical simplicity, has demonstrated increased instability in the face of uncertainties, making its solutions less reliable.

In brief, the Cholesky decomposition offers the best accuracy for well-conditioned matrices. At the same time, Gauss-Seidel stands out for its robustness and its ability to adapt to large matrices and less stringent conditions, making it a particularly practical tool in complex economic analyses. These observations reinforce the importance of assessing the specific characteristics of the system before selecting a method, to ensure both convergence and accuracy.

3. Integrating interval arithmetic into Leontief's model: Approach and Applications

In a global economic context marked by growing uncertainty, mathematical models play a central role in analysis and decision-making. However, the accuracy of the results obtained is highly dependent on the quality and reliability of the data used. The integration of interval arithmetic is emerging as a powerful approach for explicitly incorporating this uncertainty into calculations, by replacing the precise values of coefficients with intervals representing their variability.
Among the economic models where interval arithmetic can be successfully applied, the Leontief model stands out. This model, which is based on a matrix of technical coefficients describing the interdependent relationships between an economy's sectors, is particularly sensitive to variations in the parameters. Using intervals makes it possible to better manage the uncertainties associated with imprecise economic data, providing ranges of results for in-depth sensitivity analysis. The benefits include a more realistic assessment of the robustness of predictions and better identification of critical sectors.
Another relevant example is the gravity model applied to international trade, which examines trade flows between two countries as a function of their economic masses and the distance separating them. The arithmetic of intervals in this context makes it possible to manage the imprecision of data linked to economic distances, logistical costs, or variations in national GDPs. This improves the assessment of factors influencing trade and enables various uncertainty scenarios to be tested.
In addition, applications can be envisaged in economic forecasting models, such as general equilibrium models or models of the propagation of shocks in supply chains. These frameworks benefit from the ability of intervals to encapsulate uncertainty, facilitate more realistic simulations, and guide decision-makers in complex economic environments.
In the following sections, we will apply our approach to the Leontief model.

3.1. Leontief's model

The input-output equilibrium model, or Leontief model [1], is a mathematical representation of intersectoral relations within an economy. It describes how the output of each sector is used to satisfy both final consumption needs and the demands of the other sectors of the economy.
The model is written in the form :

$$(I - A)X = D$$

With:

I: is the identity matrix.
A: is the matrix of technical coefficients, representing the quantity of inputs required to produce one unit of output for each sector.

X: is the vector of sectoral production levels.
D: is the final demand vector.

This model is based on the assumption of a linear relationship between inputs and outputs, which allows for the resolution of associated linear systems to ascertain the requisite production levels.
In the case of interval arithmetic, this model is suitable for managing the uncertainties inherent in economic data. The technical coefficients a_{ij} of the matrix A are replaced by intervals, representing their variability or uncertainty. Similarly, the components of the vectors X and D are replaced by intervals.
The model becomes in the form $(\widehat{I} - \widehat{A})\widehat{X} = \widehat{D}$, the solution of this system offers a range of possible solutions, reflecting the uncertainty of the data and allowing a more robust sensitivity analysis.
Leontief's model is based on a set of assumptions that must be adapted in the case of interval arithmetic [table 2].

Table 2. Assumptions of the Leontief model with interval arithmetic

Hypothesis	**Description in the classical case**	**Adaptation in case of intervals**
Constant proportionality	The inputs required to produce one unit of each good are constant, regardless of the production level.	The technical coefficients become $\widehat{a_{ij}}$ intervals to represent uncertainty or variability in input requirements.
Sector independence	Each sector depends solely on the other sectors via fixed technical coefficients.	Sector dependencies are expressed in terms of ranges of values, allowing more flexible relationships to be modeled.
Technology stability	The technology used to produce goods remains unchanged over the period.	Technological stability is represented by intervals, allowing for potential variations in technologies or production methods.
Fixed external demand	Final demand Y is given and constant.	Final demand becomes a vector of intervals, reflecting variations in demand.
No substitution	Inputs from one sector cannot be replaced by inputs from another sector.	This assumption is retained in the interval case, but intervals offer flexibility for modeling potential variations in utilization ratios.
No excess capacity	Each sector produces exactly what is needed.	This assumption is retained, but the uncertainties in total production are calculated to take account of possible variations in the final demand vector.

3.2. Applying Gauss-Seidel algorithm using generalized interval numbers in the Leontief model to estimate the level of production : The case of Morocco 2013

The Moroccan economy counts 12 economic regions. Considering this regional macroeconomic identity of Morocco, the components of the gross regional product are solely made up of sectoral variables such as gross output, added value, household final consumption, and employment [10].

These different values can be put into a table, called the (IIOM-MOR) [table 2]. The rows of the [table 2] indicate the distribution of production among the different regions in millions of (MAD), while the columns determine the consumption of each region. We have also included the regional demand for imported products (the last row) which has been estimated by considering the structure of demand by need, and international exports by region (last column).

It is possible to add up the figures in each column, thus obtaining the cost of production for each region. On the other hand, the sum of the corresponding line gives the total consumption of each region (the values are in millions of MAD) [table 3].

Table 3. Interregional commerce in Morocco, 2013 (million MAD)[7]

•	R1	R2	R3	R4	R5	R6	R7	R8	R9	R10	R11	R12	EXP
R1	69980	2187	3537	3911	1218	12832	2171	1224	1479	282	565	414	20098
R2	2681	54731	3813	2401	995	7686	1414	1335	1068	225	583	409	8447
R3	5956	5656	81361	6755	2346	18203	2860	2768	1956	408	904	636	9157
R4	7778	3291	6460	111369	2753	37929	4284	2065	2505	625	1117	686	17403
R5	2018	1189	2394	2673	39855	18271	4089	997	2038	270	537	359	15242
R6	34756	18362	27080	52858	19104	215240	35012	10375	16944	3220	4212	2862	120080
R7	3899	2319	3308	5759	4330	25670	85581	1774	4588	832	1360	839	10513
R8	1056	810	1655	1131	693	4768	1080	23678	835	113	287	198	1466
R9	2974	2088	2540	3767	2187	12059	5128	1421	55014	1923	1732	983	5838
R10	295	175	257	376	188	1094	430	131	943	10547	342	152	1742
R11	438	269	365	437	209	2729	497	179	540	201	14457	314	2847
R12	80	63	89	79	43	236	80	38	96	23	90	3730	2609
IMP	48842	26748	37534	47534	21206	160187	41340	12627	23625	3651	5759	2579	0000

Table 4. Total consumption and production of each region of Morocco, 2013 (in millions of MAD)

•	Total consumption	Total production
R1	180753	119898
R2	117888	85785
R3	170393	138966
R4	239050	198265
R5	95127	89932
R6	516904	560105
R7	183966	150772
R8	58612	37770
R9	111631	97654
R10	22320	16672
R11	31945	23482
R12	14161	7256

To determine the production level of each region, we must solve the equation $X - AX = Z$.

$$X - AX = Z \Leftrightarrow (I - A)X = Z \Leftrightarrow X = (I - A)^{-1}Z \ (1)$$

We define :

X : The production level of the regions.
A : The technical coefficient matrix (to determine the technical coefficients of production, we simply divide the inputs of each region by its total production).
AX : Domestic consumption (between the 12 regions).
Z : Vector export.
I : The identity matrix.

With :

$$A = \begin{pmatrix} 0,5837 & 0,0255 & 0,0254 & 0,0197 & 0,0135 & 0,0229 & 0,0144 & 0,0324 & 0,0151 & 0,0169 & 0,0241 & 0,0570 \\ 0,0224 & 0,6380 & 0,0274 & 0,0121 & 0,0111 & 0,0137 & 0,0094 & 0,0353 & 0,0109 & 0,01350 & 0,0248 & 0,0563 \\ 0,0497 & 0,0660 & 0,5855 & 0,0341 & 0,0261 & 0,0325 & 0,0190 & 0,0733 & 0,0200 & 0,0245 & 0,03851 & 0,0876 \\ 0,0649 & 0,0384 & 0,0465 & 0,5617 & 0,0306 & 0,0677 & 0,0284 & 0,0547 & 0,0256 & 0,0375 & 0,0476 & 0,0945 \\ 0,0168 & 0,0139 & 0,0172 & 0,0135 & 0,4432 & 0,0326 & 0,0271 & 0,0264 & 0,0209 & 0,0162 & 0,0229 & 0,0495 \\ 0,2899 & 0,2140 & 0,1949 & 0,2666 & 0,2124 & 0,3843 & 0,2322 & 0,2747 & 0,1735 & 0,1932 & 0,1794 & 0,3944 \\ 0,0325 & 0,0270 & 0,0238 & 0,0290 & 0,0481 & 0,0458 & 0,5676 & 0,0469 & 0,0470 & 0,0499 & 0,0579 & 0,1156 \\ 0,0088 & 0,0094 & 0,0119 & 0,0057 & 0,0077 & 0,0085 & 0,0071 & 0,6269 & 0,0085 & 0,0068 & 0,0122 & 0,0273 \\ 0,0248 & 0,0243 & 0,0183 & 0,0190 & 0,0243 & 0,0215 & 0,0340 & 0,0376 & 0,5633 & 0,1153 & 0,0737 & 0,1354 \\ 0,0025 & 0,0020 & 0,0019 & 0,0019 & 0,0020 & 0,0019 & 0,0028 & 0,0035 & 0,0096 & 0,6327 & 0,0145 & 0,0209 \\ 0,0036 & 0,0031 & 0,0026 & 0,0022 & 0,0023 & 0,0049 & 0,0033 & 0,0047 & 0,0055 & 0,0120 & 0,6156 & 0,0433 \\ 0,0006 & 0,0007 & 0,0006 & 0,0004 & 0,0005 & 0,0004 & 0,0005 & 0,0010 & 0,0009 & 0,0014 & 0,0038 & 0,5140 \end{pmatrix}$$

And

$$Z = \begin{pmatrix} 20098 \\ 8447 \\ 9157 \\ 17403 \\ 15242 \\ 120080 \\ 10513 \\ 1466 \\ 5838 \\ 2847 \\ 1742 \\ 2609 \end{pmatrix}$$

From (1), we find :

$$X = \begin{pmatrix} 119899.7890 \\ 85794.3109 \\ 138968.1226 \\ 198267.7031 \\ 89933.4731 \\ 560115.8013 \\ 150776.0223 \\ 37772.3468 \\ 97655.3334 \\ 16675.4799 \\ 23480.7273 \\ 7254.9865 \end{pmatrix}$$

We notice that the **[R6]** (Casablanca-Settat) region has the highest level of production of the whole country with an amount of 560115.8013 millions MAD, on the other hand the **[R12]** region (Dakhla-Oued Ed-Dahab) has the lowest level of production of the whole country with an amount of 7254.9865 millions MAD.

3.3. The advantages of changing a fixed Leontief coefficient per interval that contains it

A country's growth rate is a measure of the change in gross domestic product (GDP) or other economic indicators over a given period, generally expressed as a percentage. It is a key indicator for assessing the health of a country's economy and its ability to create wealth and improve living standards. A country's growth rate can be influenced by many factors, such as investment, consumption, exports, government policies, political stability, technological innovation, and so on.

The assumption of a fixed Leontief coefficient means that the production process in each industry or sector is constant, indicating that the inputs required to produce a unit of output do not change. However, the elements required for production can vary for many reasons, such as technological advances, changes in the skills of the workforce, and the availability of resources. Therefore, using fixed Leontief coefficients could produce incorrect results when assessing the effect of policy changes or external shocks on the economy.

By using interval arithmetic and integrating the growth rate into the Leontief matrix and the export matrix, we replace each coefficient of these two matrices with an interval that surrounds them. We therefore obtain matrices whose intervals reflect the uncertainty associated with each coefficient.

As a result, this change offers a range of potential outcomes that can be used for decision-making purposes. For example, it can make it easier to identify the most unfavorable and the most favorable scenarios, which can help to guide risk management tactics.

This approach takes better account of the uncertainty of technical coefficients, which are difficult to estimate accurately. This approach provides a formal representation of this uncertainty.

These intervals capture possible changes in technology coefficients over time, providing a more realistic model of economic relationships subject to fluctuations. This allows policy decision-makers and analysts to understand the range of changes associated with forecasts.

3.4. Application : Estimated production levels by region in Morocco in 2014

Consider the technical coefficient matrix with an uncertainty of $\pm 2.67\%$ which is the annual growth and decline rate that has been forecasted in Morocco in 2014.

Considering the uncertainty, each element a_i of the technical coefficient matrix will be replaced by an interval in form :
$\left[a_i - (\frac{2.67 a_i}{100}); a_i + (\frac{2.67 a_i}{100})\right]$, it facilitates the identification of best-case and worst-case scenarios by simultaneously considering the uncertainty of growth and decline rates, thus providing the flexibility needed to

adapt to economic changes. The impact of economic development on the interaction between sectors can be realistically simulated based on the growth rate adjustment coefficient.
We find :
From column 1 up to 6

$$\widehat{A} = \begin{pmatrix}
[0.5681, 0.5992] & [0.0248, 0.0262] & [0.0248, 0.0262] & [0.0192, 0.0202] & [0.0132, 0.0139] & [0.0223, 0.0235] \\
[0.0218, 0.0230] & [0.621, 0.6550] & [0.0267, 0.0282] & [0.0117, 0.0124] & [0.0107, 0.0113] & [0.0133, 0.0141] \\
[0.0483, 0.0510] & [0.0641, 0.0676] & [0.5698, 0.6011] & [0.0331, 0.0349] & [0.0253, 0.0268] & [0.0316, 0.0334] \\
[0.0631, 0.0666] & [0.0373, 0.0393] & [0.0452, 0.0477] & [0.5467, 0.5767] & [0.0297, 0.0314] & [0.0659, 0.0695] \\
[0.0163, 0.0172] & [0.0134, 0.0142] & [0.0167, 0.0176] & [0.0131, 0.0138] & [0.4313, 0.4550] & [0.0317, 0.0334] \\
[0.2821, 0.2976] & [0.2083, 0.2197] & [0.1896, 0.2] & [0.2594, 0.2737] & [0.2067, 0.2180] & [0.3740, 0.3945] \\
[0.0316, 0.0334] & [0.0263, 0.0277] & [0.0231, 0.0244] & [0.0282, 0.0298] & [0.0468, 0.0494] & [0.0446, 0.0470] \\
[0.0085, 0.0090] & [0.009, 0.0096] & [0.0115, 0.01227] & [0.0055, 0.0058] & [0.0075, 0.0079] & [0.0082, 0.0087] \\
[0.0241, 0.0254] & [0.0236, 0.0249] & [0.0177, 0.0187] & [0.0184, 0.0195] & [0.0236, 0.0249] & [0.0209, 0.0221] \\
[0.0023, 0.0025] & [0.0019, 0.0020] & [0.0018, 0.0019] & [0.0018, 0.0019] & [0.002, 0.0021] & [0.0019, 0.002] \\
[0.0035, 0.0037] & [0.003, 0.0032] & [0.0025, 0.0026] & [0.0021, 0.0022] & [0.0022, 0.0023] & [0.0047, 0.005] \\
[0.0006, 0.0006] & [0.0007, 0.0007] & [0.0006, 0.0006] & [0.0003, 0.0004] & [0.0004, 0.0004] & [0.0004, 0.0004]
\end{pmatrix}$$

From column 7 up to 12

$$\begin{pmatrix}
[0.0140, 0.0148] & [0.0315, 0.0332] & [0.0147, 0.0155] & [0.0164, 0.0173] & [0.0234, 0.0247] & [0.0555, 0.0586] \\
[0.0091, 0.0097] & [0.0344, 0.0362] & [0.0106, 0.0112] & [0.0131, 0.0138] & [0.0241, 0.0255] & [0.0549, 0.0578] \\
[0.0184, 0.0195] & [0.0713, 0.0752] & [0.0194, 0.0205] & [0.0238, 0.0251] & [0.0374, 0.0395] & [0.0853, 0.0899] \\
[0.0276, 0.0291] & [0.0532, 0.0561] & [0.0249, 0.0263] & [0.0364, 0.0384] & [0.0462, 0.0488] & [0.0920, 0.0970] \\
[0.0263, 0.0278] & [0.0256, 0.0271] & [0.0203, 0.0214] & [0.0157, 0.0166] & [0.0222, 0.0234] & [0.0481, 0.0507] \\
[0.2260, 0.2384] & [0.2673, 0.2820] & [0.1688, 0.1781] & [0.1880, 0.1983] & [0.1745, 0.1841] & [0.3838, 0.4049] \\
[0.5524, 0.5827] & [0.0457, 0.0482] & [0.0457, 0.0482] & [0.0485, 0.0512] & [0.0563, 0.0594] & [0.1125, 0.1186] \\
[0.0069, 0.0073] & [0.6101, 0.6436] & [0.0083, 0.0087] & [0.0065, 0.0069] & [0.0118, 0.0125] & [0.0265, 0.0280] \\
[0.0331, 0.0349] & [0.0366, 0.0386] & [0.5483, 0.5783] & [0.1122, 0.1184] & [0.0717, 0.0757] & [0.1318, 0.1390] \\
[0.0027, 0.0029] & [0.0033, 0.0035] & [0.0093, 0.0099] & [0.6158, 0.6495] & [0.0141, 0.0149] & [0.0203, 0.0215] \\
[0.0032, 0.0033] & [0.0046, 0.0048] & [0.0053, 0.0056] & [0.0117, 0.0123] & [0.5991, 0.6320] & [0.0421, 0.0444] \\
[0.0005, 0.0005] & [0.0009, 0.0010] & [0.0009, 0.001] & [0.0013, 0.0014] & [0.003, 0.003] & [0.5, 0.5277]
\end{pmatrix}$$

And as for the export vector, we replace each element z_i of the export vector by an interval in the form $\left[z_i - (\frac{2.67a_i}{100}); z_i + (\frac{2.67a_i}{100})\right]$
We find :

$$\begin{pmatrix}
[19561.38, 20634.62] \\
[8221.465, 8672.535] \\
[8912.508, 9401.492] \\
[16938.34, 17867.66] \\
[14835.04, 15648.96] \\
[116873.9, 123286.1] \\
[10232.3, 10793.7] \\
[1426.858, 1505.142] \\
[5682.125, 5993.875] \\
[1695.489, 1788.511] \\
[2770.985, 2923.015] \\
[2539.34, 2678.66]
\end{pmatrix}$$

To determine the production level of each region taking into account the proposed uncertainty, one must solve the equation : $\widehat{X} - \widehat{A}\widehat{X} = \widehat{Z}$

$$\widehat{X} - \widehat{A}\widehat{X} = \widehat{Z} \Leftrightarrow (\widehat{I} - \widehat{A})\widehat{X} = \widehat{Z}$$

By applying the Gauss-Seidel algorithm using generalized interval numbers, we find :

$$\widehat{X} = \begin{pmatrix} [100724.66, 146235.25] \\ [69600.76, 108560.79] \\ [111465.35, 177764.24] \\ [161471.77, 249678.72] \\ [75476.16, 109852.86] \\ [472073.36, 681750.30] \\ [121584.75, 191787.33] \\ [29724.70, 49253.83] \\ [78148.25, 125185.95] \\ [13548.07, 21070.34] \\ [19372.85, 29176.96] \\ [6548.03, 8150.17] \end{pmatrix}$$

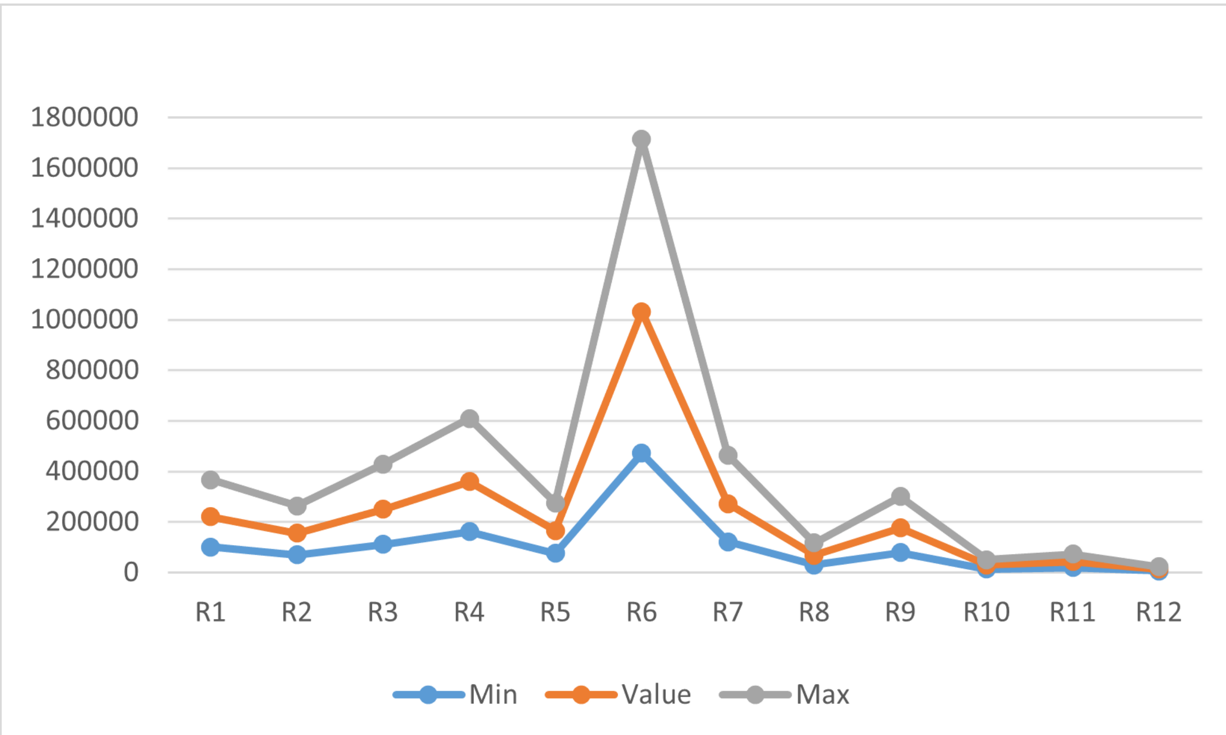

Figure 1. Estimation of the production level of each region of Morocco

Figure [1] provides a visual illustration of the impact of using interval arithmetic on Leontief production level estimates. The resulting production intervals highlight the range of possible values, underlining the inherent variability of the input data and the complex interactions between economic sectors. This visual representation enhances understanding of the uncertainties associated with forecasting, while providing insight into the advantages of the interval approach over conventional methods based on point values.

The results obtained confirm that this method can effectively evaluate optimistic and pessimistic economic scenarios while identifying sectors that are sensitive to variations in the parameters. Although the study focused on the Moroccan economy, the methodology easily adapts to other countries and economic contexts, particularly those sharing similar structures. However, the effectiveness of this approach depends on the rigorous structuring of the data and careful estimation of the interval amplitudes, as an incorrect transformation of the real coefficients into interval coefficients can compromise the reliability and significance of the results.

3.5. Application 2: Economic impact of increased exports for Washington State 1987

This application uses the direct purchase coefficients estimated for the State of Washington in 1987 with an uncertainty of 10%, as presented in [Table 5] (the technology interval matrix from the table in Chase et al., 1993)[14]. The objective is to estimate the total economic impact of an economic shock caused by an increase in exports of natural resources worth between 30 and 40 million, manufacturing products worth between 40 and 45 million, commercial services and personal consumption worth between 50 and 60 million, and personal consumption worth between 100 and 110 million.

Table 5. 1987 Washington State Input-Output Study : Direct Purchase Coefficient Table with ± 10 % uncertainty

Sectors	**Natural resources**	**Manufacturing**	**Trade and services**	**Personal consumption**
Natural resources	[0.0947, 0.115]	[0.0385, 0.04707]	[0.00258,0.00316]	[0.00274, 0.00336]
Manufacturing	[0.0743, 0.0910]	[0.097, 0.1196]	[0.0525, 0.0642]	[0.0289, 0.03534]
Trade and services	[0.0780, 0.0953]	[0.0917, 0.1121]	[0.1828, 0.2235]	[0.3199, 0.3911]
Personal consumption	[0.5627, 0.6879]	[0.3103, 0.3794]	[0.5495, 0.6717]	[0.0718, 0.0878]

We put:

$$\widehat{A} = \begin{pmatrix} [0.0947, 0.115] & [0.0385, 0.04707] & [0.00258, 0.00316] & [0.00274, 0.00336] \\ [0.0743, 0.0910] & [0.097, 0.1196] & [0.0525, 0.0642] & [0.0289, 0.03534] \\ [0.0780, 0.0953] & [0.0917, 0.1121] & [0.1828, 0.2235] & [0.3199, 0.3911] \\ [0.5627, 0.6879] & [0.3103, 0.3794] & [0.5495, 0.6717] & [0.0718, 0.0878] \end{pmatrix} \text{ and } \widehat{B} = \begin{pmatrix} [30; 40] \\ [40; 45] \\ [50; 60] \\ [100, 110] \end{pmatrix}$$

The aim is to assess the total economic impact of a forecast increase in the financial sectors, taking account of the uncertainties associated with the coefficients. We apply the Leontief model and the Gauss-Seidel method to the resolution of linear systems with interior coefficients applied to the modelled system $(\widehat{I} - \widehat{A})\widehat{X} = \widehat{B}$.
We find :

$$\widehat{X} = \begin{pmatrix} [30.18, 43.61] \\ [32.47, 47.51] \\ [10.91, 65.40] \\ [12.59, 84.86] \end{pmatrix}$$

The results of the total economic impact, expressed as intervals, show that the natural resources sector would contribute between 30.19 and 43.61 million dollars, while the manufacturing sector would generate between 32.47 and 47.51 million dollars. The trade and services sector shows a more uncertain contribution, varying between 10.91 and 65.40 million dollars, while the personal consumption sector shows a wide range, between 12.59 and 84.86 million dollars. In economic terms, these results highlight the industries that are most sensitive to uncertainties and where efforts to improve data accuracy could have the greatest impact on improving the reliability of global forecasts.

3.6. Improving Sensitivity Analysis by Interval Arithmetic in Economic Models

Sensitivity analysis is a crucial step in the study of complex economic systems. It consists of assessing the extent to which variations in the input parameters influence the model's results. This approach is essential for identifying critical parameters, improving the robustness of forecasts, and increasing confidence in the results obtained. In the context of this research, interval arithmetic offers an innovative approach to sensitivity analysis, making it possible to quantify the uncertainty inherent in economic data and modeling assumptions.
Unlike traditional methods, which use fixed values for the parameters, the interval arithmetic approach represents these parameters by intervals that frame them. This makes it possible to take account of the uncertainties associated with coefficient estimates, which are often unavoidable in economic models. Sensitivity analysis in this context is carried out by observing the impact of variations in one or more input intervals on the bounds of the calculated results.
To demonstrate the effectiveness of this approach, we have applied sensitivity analysis on the Leontief model extended to interval arithmetic by going through the following steps:

Definition of Input Parameters: Technical coefficients and final demand levels were defined as intervals based on observed economic data.
Variation of Intervals: We modified the bounds of these intervals to simulate different levels of uncertainty, progressively increasing the margins.
Calculation of the results: The Gauss-Seidel method adapted to the intervals was used to solve the associated linear systems, allowing the bounds of the solutions obtained to be determined.
Identification of Critical Parameters: A comparative analysis was used to identify the parameters whose variations have the greatest influence on the results, thereby shedding light on the elements of the model that are most sensitive to uncertainty.

3.6.1. Results and recommendations: When the coefficients of the input parameters (such as technical coefficients or final demand levels) have extensive ranges, the bounds of the solutions obtained deviate considerably. This reflects a strong propagation of uncertainty in the model, making the results less meaningful for decision-making. For example, a model with input coefficients defined by wide intervals may produce economic projections with extensive margins, limiting their relevance. On the other hand, well-calibrated intervals, reflecting reasonable estimates of uncertainties, lead to more accurate and usable results. A moderate amplitude ensures that the model remains sensitive to variations in the parameters while limiting excess uncertainty.
Recommendations:
The intervals must be chosen to reflect the real uncertainties of the economic data without exaggerating them.
A preliminary analysis of the data can help to define the optimal margins for the coefficients, taking into account the acceptable limits for the amplitude of the intervals.
Testing different amplitudes to assess their impact on the results helps to select intervals that produce meaningful and reliable projections.

4. Conclusion

In conclusion, the integration of interval arithmetic within the Leontief model represents a significant advance in economic analysis and decision-making in uncertain environments. The inherent advantages of this approach over traditional methods are indisputable, opening up new prospects for a better understanding and management of uncertainties in the complex context of economic interactions. The use of interval arithmetic provides a more realistic understanding of the variability of production and sensitivity results, taking into account the diversity of possible scenarios resulting from the inherent uncertainty of the input data. The intervals obtained offer a more realistic range of values, more accurately reflecting potential fluctuations in the estimates. This gives decision-makers and economic planners a more complete picture of the implications of variations in economic parameters and relationships. Sensitivity analysis also becomes more relevant and informative thanks to the interval approach. By identifying more precisely the parameters with the greatest impact on results. In short, the application of interval arithmetic to the Leontief model represents a significant step forward in economic modeling and uncertainty management.

This approach takes better account of the realities of a complex and uncertain world, making forecasts more reliable and decisions more robust. Its potential to contribute to more resilient and informed economic planning cannot be underestimated, opening up new perspectives for economic analysis and strategic decision-making.

This study opens up several promising avenues of research. The first is to integrate probabilistic techniques with interval arithmetic, making it possible to combine bounded uncertainty with probability distributions to better model complex systems. Secondly, the development of hybrid algorithms, combining iterative methods such as Gauss-Seidel with optimization or machine learning approaches, could improve both accuracy and computational efficiency. Finally, the application of interval arithmetic to dynamic models, including temporal interactions between economic sectors, could improve our understanding of economic impacts in an evolving context. These prospects will enable us to push back the current limits and make this approach even more relevant to economic analysis and public policy.

REFERENCES

1. Wassily Leontief, *Input-output analysis*, Input-output economics, 2nd ed, oxford university press, pp 19–40, 1986.
2. M. Granger, and M. Henrion, *Uncertainty*, Cambridge University Press, 1990.
3. V. Beletskyy, A. Chemeris and E. Nowatskaja, *Sensitivity analysis of a Leontief model*, SAMO 2001, Third International Symposium on Sensitivity Analysis of Model Output, Madrid, 2001.
4. M. Lenzen, *A generalized Input-Output multiplier Calculus for Australia*, Economic System Research, Vol. 13 No. 1, 2001.
5. K. Ganesan, P. Veeramani, *On arithmetic operations of interval numbers*, International Journal of Uncertainty, Fuzziness and Knowledge - Based Systems, vol. 13, No. 6, pp. 619–631, 2005.
6. R.E.Moore, R. B. Kearfott and M.J.Cloud, *Introduction to Interval Analysis*, SIAM, 2009.
7. T. nirmala, D. datta, H. s. kushwaha, and K. ganesan, *Inverse interval matrix : a new approach*, Book title. Applied mathematical sciences, vol. 5, no. 13, 607 – 624, 2011.
8. S. Ning, and R. B. Kearfott, *A comparison of some methods for solving linear interval equations*, SIAM Journal of Numerical Analysis, vol. 34, no. 4, pp. 1289–1305, 1997.
9. E. R. Hansen, *Bounding the solution of interval linear equations*, SIAM Journal of Numerical Analysis, Vol. 29, no. 5, pp. 1493–1503, 1992.
10. E.A Haddad, F. El-hattab, A. Ait ali, *A practitioner's guide for building the interregional input-output system for Morocco 2013*, august 2017.
11. M.A Benhari, and M. Kaicer, *Using interval arithmetic on the Leontief model*, Intelligent Systems for Sustainable Development (AI2SD'2020), vol. 1, Springer, pp. 1079—1085, 2022.
12. M.A Benhari, and M. Kaicer, *Resolution of linear systems using interval arithmetic and Cholesky decomposition*, Mathematics and Statistics, vol. 11, no. 5, pp. 840–844, 2023.
13. N. Karkar, K.Benmohamed and B. Arres, *Solving Linear Systems Using Interval Arithmetic Approach*, International Journal of Science and Engineering Investigations, vol. 1, no. 1, pp. 29–33, 2012.
14. M. Dehghani, and Madiseh, *Inner and outer estimations of the generalized solution sets and an application in economic*, Journal of Mathematical Modeling, vol. 8, no. 4, pp. 345–361, 2020.