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The Odd Generalized Rayleigh Reciprocal Weibull Family of Distributions with Applications

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Abstract We introduced a novel family of models in this paper, which we named the odd-generalized Rayleigh reciprocal Weibull-G (OGR-RW-G) family. This family is noteworthy because it applies the T-X model construction technique to the generalized Rayleigh reciprocal Weibull model, addressing the inflexibility limits associated with traditional models and allowing one to use any baseline distribution. We examine some valuable statistical inferences from the OGR-RW-G, including its probability density function (pdf) represented in a linear fashion, its order statistics' pdf, moments, residual life functions and Rényi entropy. Additionally, the hazard rate functions (hrfs) and pdfs of a few particular models are determined to have analytical shapes. The OGR-RW-G model parameters are determined by the widely recognized maximum likelihood estimation (MLE) technique. We also perform a simulation exercise to evaluate the performance of the MLEs. Ultimately, the utility of the OGR-RW-G family is demonstrated by using the odd generalized Rayleigh reciprocal Weibull Burr-XII (OGR-RW-BXII) example of the OGR-RW-G to three distinct datasets. In actuality, the four parameter OGR-RW-BXII outperforms the four parameter non-nested models and some nested models that are presented.

Keywords Probability weighted moment; residual life function; reverse residual life function; generalized Rayleigh Reciprocal Weibull; simulations; maximum likelihood estimation; key risk indicators.

AMS 2010 subject classifications 62E99, 60E05

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1. Introduction

In actuarial science, probability models play a vital role in evaluating and managing financial risks related to insurance and pension programs. These models are frequently used to estimate the expected value of future claims, potential losses, and the likelihood of adverse events. They are also applicable in analyzing events such as insurance claims, fatalities, and policy cancellations. However, traditional probability distributions often fall short in accurately representing extreme events or outliers, which occur more frequently in actuarial science and financial markets. Lighter-tailed distributions, such as the normal, Weibull, and exponential distributions, may underestimate the risks associated with these extreme occurrences.

In contrast, heavy-tailed probability distributions are designed to assign greater probability to extreme values or outliers, making them better suited for modeling such phenomena. Unlike light-tailed distributions, where the probability of extreme values diminishes rapidly, heavy-tailed distributions provide a more realistic representation of data with long tails. For instance, the Pareto distribution, known for its long right tail, is widely used to model wealth and income distributions. The Student's t-distribution, with its heavier tails, is effective for hypothesis testing and handling data with outliers. The Levy distribution, which has even heavier tails than the normal distribution, finds applications in financial mathematics and physics. Similarly, the Cauchy distribution,

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characterized by infinite variance, is often used to model phenomena with extended tails, such as network response times and earthquakes.

These heavy-tailed distributions are indispensable in fields like actuarial science, finance, economics, physics, and engineering, as they provide a more accurate reflection of real-world scenarios where extreme events occur more frequently than predicted by lighter-tailed models. Over the past few decades, significant research has been devoted to the development and enhancement of heavy-tailed distributions. Many of these distributions are constructed by introducing additional parameters to a parent cumulative density function (cdf), creating new families of distributions with thicker and longer tails. Researchers have also worked on modifying classical distributions such as the normal, Weibull, and exponential models to enhance their analytical flexibility and better capture the characteristics of heavy-tailed data [1, 2, 3, 7, 10, 11, 12, 15, 20, 23, 22].

The Odd Generalized Rayleigh Reciprocal Weibull-G (OGR-RW-G), a unique probability-based reciprocal Weibull distribution, constructed via the T-X technique [3], is offered in order to provide an adequate explanation of risk exposure under the reinsurance revenues data set. The cdf and probability distribution function (pdf) of the Generalized Rayleigh Reciprocal Weibull (GR-RW) are given by

$$F_{GR-RW}(u) = 1 - \exp\left\{-\left[\frac{e^{-2\delta_1 u^{-\delta_2}}}{(1-e^{-2\delta_1 u^{-\delta_2}})^2}\right]\right\},$$

and

$$\begin{aligned} f_{GR-RW}(u) &= \frac{2\delta_1\delta_2 u^{-(\delta_2-1)}\exp\left\{-\left[\frac{e^{-2\delta_1 u^{-\delta_2}}}{(1-e^{-2\delta_1 u^{-\delta_2}})^2}\right]\right\}}{(1-e^{-2\delta_1 u^{-\delta_2}})^3} \\ &\times \; [1-e^{-2\delta_1 u^{-\delta_2}}(1-e^{-2\delta_1 u^{-\delta_2}})], \end{aligned}$$

respectively, for $u > 0, \delta_1 > 0$, and $\delta_2 > 0$.
Now, the cdf and pdf of the Odd Generalized Rayleigh Reciprocal Weibull-G (OGR-RW-G) distribution are given by

$$\begin{aligned} F_{OGR-RW-G}(y) &= \int_0^{\nabla} f_{GR-RW}(u)du \\ &= 1 - \exp\left\{-\left[\frac{e^{-2\delta_1\nabla^{-\delta_2}}}{(1-e^{-2\delta_1\nabla^{-\delta_2}})^2}\right]\right\}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} f_{OGR-RW-G}(y) &= \frac{2\delta_1\delta_2\nabla^{-(\delta_2-1)}\nabla'\exp\left\{-\left[\frac{e^{-2\delta_1\nabla^{-\delta_2}}}{(1-e^{-2\delta_1\nabla^{-\delta_2}})^2}\right]\right\}}{(1-e^{-2\delta_1\nabla^{-\delta_2}})^3} \\ &\times \; [1-e^{-2\delta_1\nabla^{-\delta_2}}(1-e^{-2\delta_1\nabla^{-\delta_2}})], \end{aligned} \tag{2}$$

respectively, where δ_1, δ_2 are non-negative scale and shape parameters respectively, $y \in \mathbb{R}$, $\nabla = \frac{G(y;\underline{\omega})}{\bar{G}(y;\underline{\omega})}, \nabla' = \frac{g(y;\underline{\omega})}{\bar{G}^2(y;\underline{\omega})}$, $G(y;\underline{\omega})$ is a baseline cdf, $\bar{G}(y;\underline{\omega})$ is a survival function and $\underline{\omega}$ is a vector of parameters. The corresponding hazard rate function (hrf) is given by

$$\begin{aligned} h(y) &= \frac{2\delta_1\delta_2\nabla^{-(\delta_2-1)}g(y;\underline{\omega})\exp\left\{-\left[\frac{e^{-2\delta_1\nabla^{-\delta_2}}}{(1-e^{-2\delta_1\nabla^{-\delta_2}})^2}\right]\right\}}{\bar{G}^2(y;\underline{\omega})(1-e^{-2\delta_1\nabla^{-\delta_2}})^3} \\ &\times \; \exp\left\{\frac{e^{-2\delta_1\nabla^{-\delta_2}}}{(1-e^{-2\delta_1\nabla^{-\delta_2}})^2}\right\}\left[1-e^{-2\delta_1\nabla^{-\delta_2}}\left(1-e^{-2\delta_1\nabla^{-\delta_2}}\right)\right]. \end{aligned}$$

Using the generalized binomial expansion and the power series, the pdf in equation (2) can be expressed as

$$f_{OGR-RW-G}(y) = \sum_{a,b,c=0}^{\infty} \chi_\tau^* g_\tau^*(y;\underline{\omega}), \tag{3}$$

where

$$\chi_\tau^* = \sum_{d,e=0}^{\infty} \frac{(-1)^{a+b+c+d+e}[2\delta_1(a+b+c+1)]^d}{(-(d\delta_2+\delta_2)+h)a!d!} \times \binom{2a+b+2}{b}\binom{1}{c}\binom{d\delta_2+\delta_2+e-4}{e}, \tag{4}$$

and

$$g_\tau^*(y;\underline{\omega}) = \tau g(y;\underline{\omega})G^{\tau-1}(y;\underline{\omega}), \tag{5}$$

is an exponentiated-G (Exp-G) distribution with power parameter $\tau = -(d\delta_2+\delta_2)+h$. Similarly, the pdf of the k^{th} order statistic of the OGR-RW-G family of distributions can be written as

$$f_{k:m}(y) = \sum_{a,b,c=0}^{\infty} \phi_\tau^* g_\tau^*(y;\underline{\omega}), \tag{6}$$

where

$$\phi_\tau^* = \frac{m!}{(k-1)!(m-k)!}\sum_{l,p,d,e=0}^{\infty} \frac{(-1)^{l+p+a+b+c+d+e}(p+1)^a[2\delta_1(a+b+c+1)]^d}{(-(d\delta_2+\delta_2)+h)(p+1)^{-a}a!d!} \times \binom{m-k}{l}\binom{k+l-1}{p}\binom{2a+b+2}{b}\binom{1}{c}\binom{d\delta_2+\delta_2+e-4}{e}. \tag{7}$$

Thus, the pdf of the order statistic of the OGR-RW-G can be written as a linear combination of Exp-G with power parameter τ.

2. Special Cases

The OGR-RW-G family of distributions' special instances are discussed in this section. Examples of log-logistic, exponential, Fŕetchet and Burr-XII distributions for the baseline distribution function are given. The pdf and hrf plots are also shown.

2.1. *Odd-Generalized Rayleigh Reciprocal Weibull-Log-Logistic*

Suppose we have a log-logistic baseline distribution with cdf and pdf given by $G(y;\alpha) = 1-(1+y^\alpha)^{-1}$ and $g(y;\alpha) = \alpha y^{\alpha-1}(1+y^\alpha)^{-2}$ respectively for $y,\alpha > 0$. Now, the cdf and pdf of the Odd-Generalized Rayleigh Reciprocal Weibull-Log-Logistic (OGR-RW-LLoG) distribution are given by

$$F_{OGR-RW-LLoG}(y) = 1-\exp\left\{-\left[\frac{e^{-2\delta_1\nabla_1^{-\delta_2}}}{(1-e^{-2\delta_1\nabla_1^{-\delta_2}})^2}\right]\right\}, \tag{8}$$

and

$$f_{OGR-RW-LLoG}(y) = \frac{2\delta_1\delta_2\nabla_1^{-(\delta_2+1)}\nabla_1' e^{-2\delta_1\nabla_1^{-\delta_2}}(1+e^{-2\delta_1\nabla_1^{-\delta_2}})}{(1-e^{-2\delta_1\nabla_1^{-\delta_2}})^3} \times \exp\left\{-\left[\frac{e^{-2\delta_1\nabla_1^{-\delta_2}}}{(1-e^{-2\delta_1\nabla_1^{-\delta_2}})^2}\right]\right\}, \tag{9}$$

respectively, for $\alpha, \delta_1, \delta_2 > 0$, $\nabla_1 = \frac{1-(1+y^{\alpha})^{-1}}{(1+y^{\alpha})^{-1}}$ and $\nabla_1' = \frac{\alpha y^{\alpha-1}(1+y^{\alpha})^{-2}}{(1+y^{\alpha})^{-2}}$.

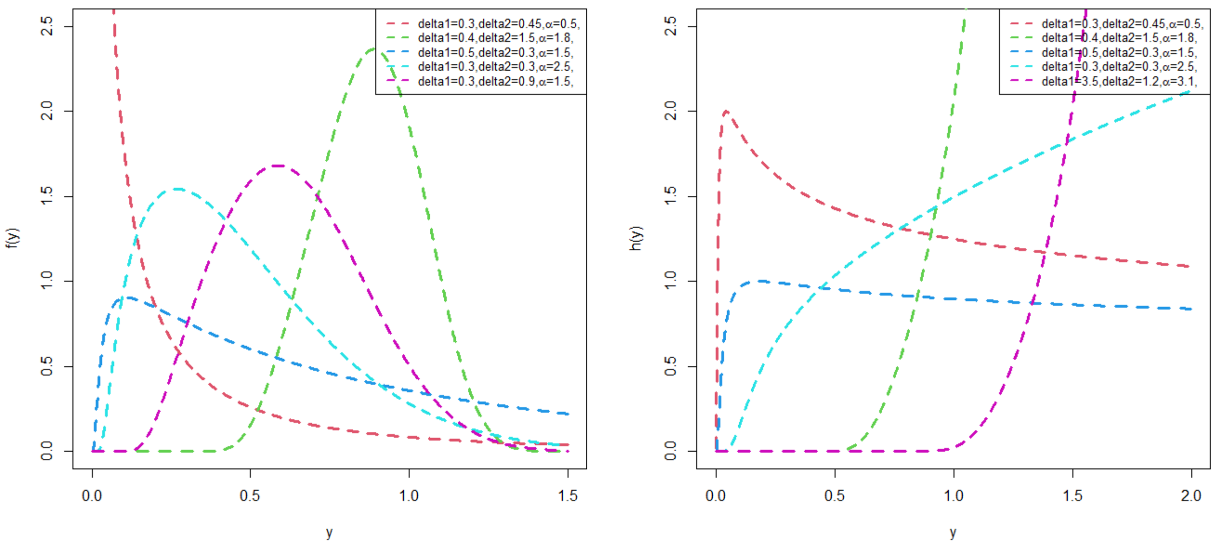

Figure 1. The pdf and hrf plots of the OGR-RW-LLoG distribution

2.2. *Odd-Generalized Rayleigh Reciprocal Weibull - Exponential*

By taking an exponential baseline distribution with cdf and pdf given by $G(y;\alpha) = 1 - e^{-\alpha y}$ and $g(y;\alpha) = \alpha e^{-\alpha y}$ respectively for $y, \alpha > 0$. Now, the cdf and pdf of the Odd-Generalized Rayleigh Reciprocal Weibull-Exponential (OGR-RW-E) distribution are given by

$$F_{OGR-RW-E}(y) = 1 - \exp\left\{-\left[\frac{e^{-2\delta_1 \nabla_2^{-\delta_2}}}{(1-e^{-2\delta_1 \nabla_2^{-\delta_2}})^2}\right]\right\}, \tag{10}$$

and

$$f_{OGR-RW-E}(y) = \frac{2\delta_1\delta_2 \nabla_2^{-(\delta_2+1)} \nabla_2' e^{-2\delta_1 \nabla_2^{-\delta_2}}(1+e^{-2\delta_1 \nabla_2^{-\delta_2}})}{(1-e^{-2\delta_1 \nabla_2^{-\delta_2}})^3} \times \exp\left\{-\left[\frac{e^{-2\delta_1 \nabla_2^{-\delta_2}}}{(1-e^{-2\delta_1 \nabla_2^{-\delta_2}})^2}\right]\right\}, \tag{11}$$

respectively, for $\alpha, \delta_1, \delta_2 > 0$, $\nabla_2 = \frac{1-e^{-\alpha y}}{e^{-\alpha y}}$ and $\nabla_2' = \frac{\alpha e^{-\alpha y}}{e^{-2\alpha y}}$.

2.3. *Odd-Generalized Rayleigh Reciprocal Weibull-Fŕetchet*

Suppose our baseline distribution is Fŕetchet with cdf and pdf given by $G(y;\lambda,\alpha) = e^{-\lambda y^{-\alpha}}$ and $g(y;\lambda,\alpha) = \lambda\alpha y^{-(\alpha+1)} e^{-\lambda y^{-\alpha}}$ respectively for $y, \lambda, \alpha > 0$. Now, the cdf and pdf of the Odd-Generalized Rayleigh Reciprocal Weibull-Fŕetchet (OGR-RW-Fr) distribution are given by

$$F_{OGR-RW-Fr}(y) = 1 - \exp\left\{-\left[\frac{e^{-2\delta_1 \nabla_3^{-\delta_2}}}{(1-e^{-2\delta_1 \nabla_3^{-\delta_2}})^2}\right]\right\}, \tag{12}$$

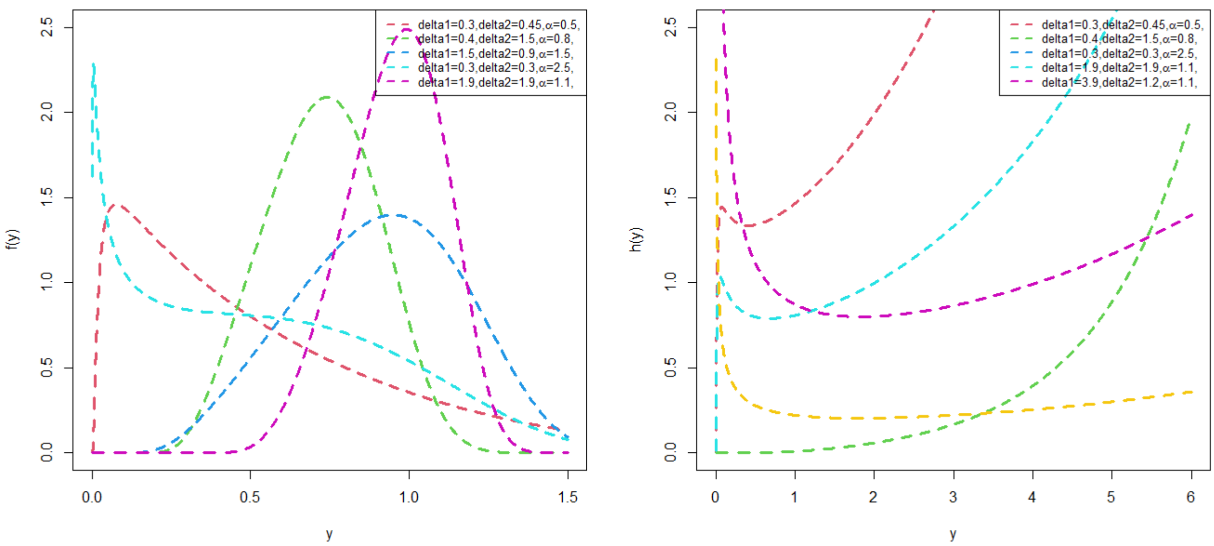

Figure 2. The pdf and hrf plots of the OGR-RW-E distribution

and

$$f_{OGR-RW-Fr}(y) = \frac{2\delta_1\delta_2\nabla_3^{-(\delta_2+1)}\nabla_3' e^{-2\delta_1\nabla_3^{-\delta_2}}(1+e^{-2\delta_1\nabla_3^{-\delta_2}})}{(1-e^{-2\delta_1\nabla_3^{-\delta_2}})^3} \times \exp\left\{-\left[\frac{e^{-2\delta_1\nabla_3^{-\delta_2}}}{(1-e^{-2\delta_1\nabla_3^{-\delta_2}})^2}\right]\right\}, \tag{13}$$

respectively, for $\lambda, \alpha, \delta_1, \delta_2 > 0$, $\nabla_3 = \frac{e^{-\lambda y^{-\alpha}}}{1-e^{-\lambda y^{-alpha}}}$ and $\nabla_3' = \frac{\lambda\alpha y^{-(\alpha+1)}e^{-\lambda y^{-\alpha}}}{(1-e^{-\lambda y^{-\alpha}})^2}$.

2.4. Odd-Generalized Rayleigh Reciprocal Weibull-Burr-XII

Suppose our baseline distribution is a Burr-XII distribution with cdf and pdf given by $G(y) = 1-(1+y^\beta)^{-\alpha}$ and $g(y) = \alpha\beta y^{\beta-1}(1+x^\beta)^{-(\alpha+1)}$ respectively for $y, \alpha, \beta > 0$. Now, the cdf and pdf of the Odd-Generalized Rayleigh Reciprocal Weibull-Burr-XII (OGR-RW-BXII) distribution are given by

$$F_{OGR-RW-BXII}(y) = 1-\exp\left\{-\left[\frac{e^{-2\delta_1\nabla_4^{-\delta_2}}}{(1-e^{-2\delta_1\nabla_4^{-\delta_2}})^2}\right]\right\}, \tag{14}$$

and

$$f_{OGR-RW-BXII}(y) = \frac{2\delta_1\delta_2\nabla_4^{-(\delta_2+1)}\nabla_4' e^{-2\delta_1\nabla_4^{-\delta_2}}(1+e^{-2\delta_1\nabla_4^{-\delta_2}})}{(1-e^{-2\delta_1\nabla_4^{-\delta_2}})^3} \times \exp\left\{-\left[\frac{e^{-2\delta_1\nabla_4^{-\delta_2}}}{(1-e^{-2\delta_1\nabla_4^{-\delta_2}})^2}\right]\right\}, \tag{15}$$

respectively, for $\alpha, \beta, \delta_1, \delta_2 > 0$, $\nabla_4 = \frac{1-(1+y^\beta)^{-\alpha}}{(1+y^\beta)^{-\alpha}}$ and $\nabla_4' = \frac{\alpha\beta y^{\beta-1}(1+x^\beta)^{-(\alpha+1)}}{(1+y^\beta)^{-2\alpha}}$.

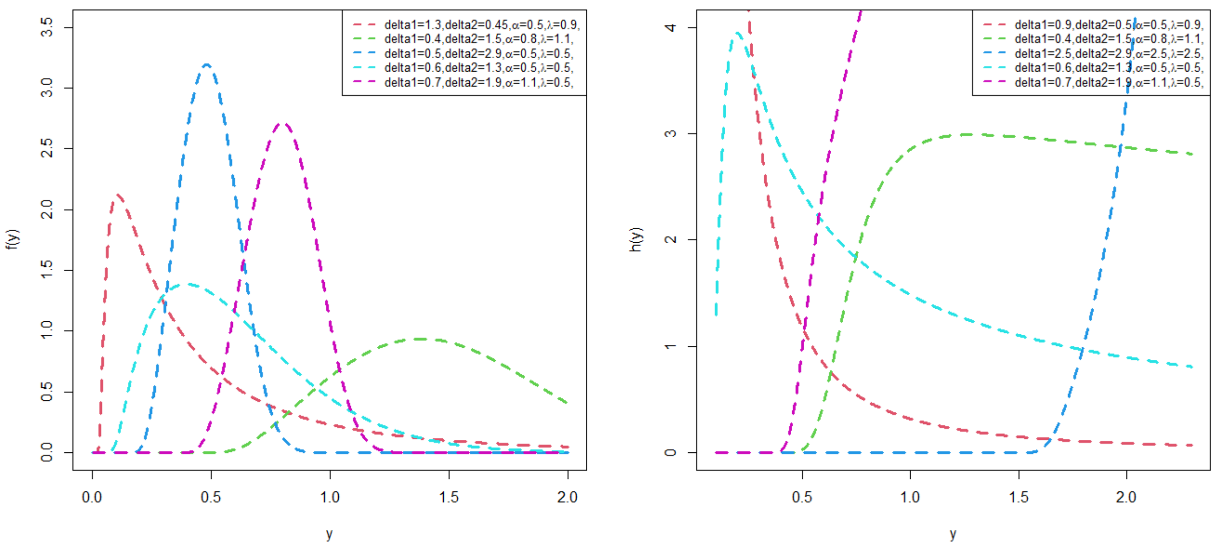

Figure 3. The pdf and hrf plots of the OGR-RW-Fr distribution

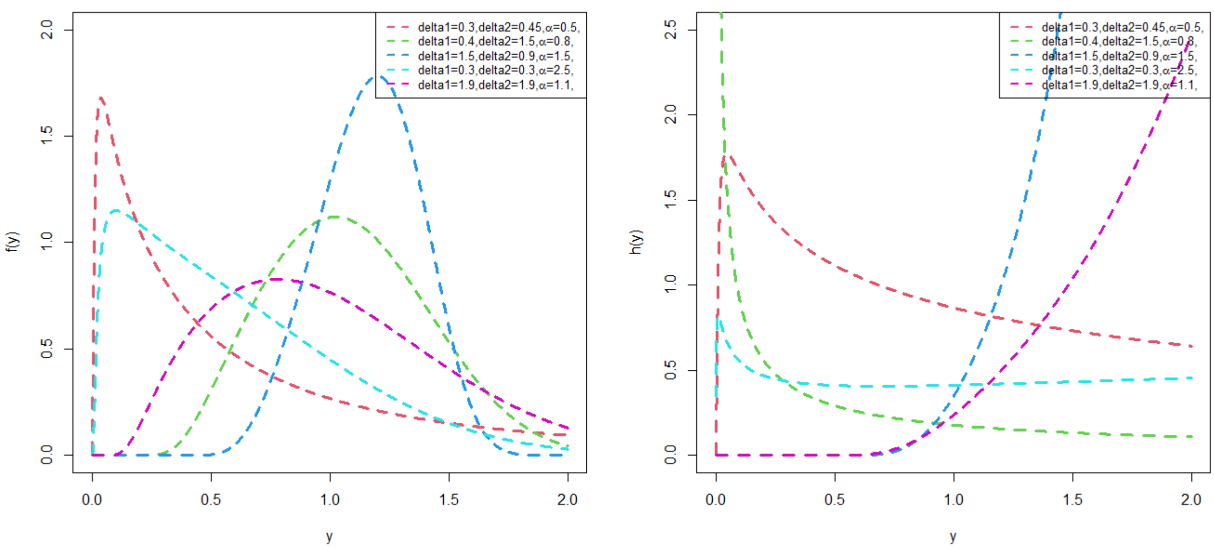

Figure 4. The pdf and hrf plots of the OGR-RW-BXII distribution

3. Statistical Properties

Assessing the statistical properties of a model is essential for evaluating its performance, reliability, and applicability in both scientific and practical contexts. Key properties, such as measures of central tendency, variability, goodness-of-fit, and error metrics, offer valuable insights into the model's ability to represent the underlying data or processes it is designed to simulate. These evaluations help identify potential biases, measure prediction accuracy, and detect issues like overfitting or underfitting. Moreover, statistical analysis facilitates comparisons between models, aiding in the selection of the most suitable one for a specific task. Understanding these properties ensures that the model's outputs are interpretable, reproducible, and valid for informed decision-making or further research. In this section, we outline some of the key statistical properties.

The r^{th} moment of $Y \sim$ OGR-RW-G , say μ_r' follows from equation (3) as

$$\mu_r' = E(Y^r) = \sum_{a,b,c=0}^{\infty} \chi_\tau^* E(Y_\tau^r). \tag{16}$$

The mean and variance can be easily derived from equation (16). The moment generating function (mgf) of the OGR-RW-G distribution can be derived from equation (3) and is given by

$$M_Y(t) = \sum_{a,b,c=0}^{\infty} \chi_\tau^* M_\tau(t), \tag{17}$$

where $M_\tau(t)$ is the MGF of Y_τ. Hence, $M_Y(t)$ can be determined from the MGF of Exp-G.

The r^{th} incomplete moment can be obtained as follows;

$$I_r(t) = \int_0^t y^r f_{OGR-RW-G}(y)dy = \sum_{a,b,c=0}^{\infty} \chi_\tau^* \int_0^t y^r g_\tau^*(y; \underline{\omega})dy. \tag{18}$$

The probability weighted moment (PWM) of a random variable $Y \sim$ OGR-RW-G, say $(\varrho_{r,j})$ is given by

$$\begin{aligned} \varrho_{r,j} &= E(Y^r F^j(Y)) = \int_{-\infty}^{\infty} y^r F_{OGR-RW-G}^j(y; \delta, \gamma, \beta, \kappa) f_{OGR-RW-G}(y; \delta, \gamma, \beta, \kappa)dy. \\ &= \sum_{a,b,c=0}^{\infty} Q_\tau^* \int_{-\infty}^{\infty} y^r g_{i+m+1}^*(y; \kappa)dy = \sum_{a,b,c=0}^{\infty} Q_\tau^* E[Y_\tau^r], \end{aligned} \tag{19}$$

where

$$\begin{aligned} Q_\tau^* &= \sum_{p,d,e=0}^{\infty} \frac{(-1)^{p+a+b+c+d+e}(p+1)^a[2\delta_1(a+b+c+1)]^d}{(-(d\delta_2+\delta_2)+h)(p+1)^{-a}a!d!} \\ &\times \binom{j}{p}\binom{2a+b+2}{b}\binom{1}{c}\binom{d\delta_2+\delta_2+e-4}{e}, \end{aligned} \tag{20}$$

$E(Y_\tau^r)$ is the r^{th} moment of Exponentiated-G (Exp-G) density with power parameter $\tau = -(d\delta_2 + \delta_2) + h$ and $g_{i+m+1}^*(y; \kappa)$ is given in equation (5).

If n is an integer value greater than 1 $(n > 1)$ and $y > t$, then the r^{th} moment of residual life of the OGR-RW-G family of distributions is given by;

$$\vartheta_r^*(t) = \frac{1}{\bar{F}(y; \underline{\omega})} \sum_{a,b,c=0}^{\infty} \chi_\tau^* \sum_{s=0}^{\infty} \binom{n}{s} (-t)^s \int_t^{\infty} y^{n-s} g_\tau^*(y; \underline{\omega})dy, \tag{21}$$

where $g_\tau^*(y; \underline{\omega})$ is pdf of an Exp-G distribution with power parameter τ given in equation (5), $\bar{G}(y; \underline{\omega})$ is the survival function and the coefficients of χ_τ^* are given in equation (4).

Now, if n is an integer value greater than 1 $(n > 1)$ and $y < t$, then the r^{th} moment of reverse residual life of the OGR-RW-G family of distributions is given by;

$$\theta_r^*(t) = \frac{1}{F(y; \underline{\omega})} \sum_{a,b,c=0}^{\infty} \chi_\tau^* \sum_{s=0}^{\infty} \binom{n}{s} (t)^{n-s} (-1)^s \int_0^t y^s g_\tau^*(y; \underline{\omega})dy, \tag{22}$$

where $g^*_\tau(y;\underline{\omega})$ is a pdf of an Exp-G distribution with power parameter (τ) given in equation (5) and the coefficients of χ^*_τ are given in equation (4).

Let $X \sim$ OGR-RW-G$(\delta_{1_1}, \delta_{2_1}, \underline{\omega}_1)$ and $Y \sim$ OGR-RW-G$(\delta_{1_2}, \delta_{2_2}, \underline{\omega}_2)$ respectively, the reliability model of the OGR-RW-G family of distributions is given by

$$\begin{aligned} R &= \int_{x=0}^{\infty} \frac{2\delta_{1_1}\delta_{2_1}\nabla_1^{-(\delta_{2_1}+1)}\nabla_1' e^{-2\delta_{1_1}\nabla_1^{-\delta_{2_1}}}(1+e^{-2\delta_{1_1}\nabla_1^{-\delta_{2_1}}})}{(1-e^{-2\delta_{1_1}\nabla_1^{-\delta_{2_1}}})^3} \\ &\times \exp\left\{-\left[\frac{e^{-2\delta_{1_1}\nabla_1^{-\delta_{2_1}}}}{(1-e^{-2\delta_{1_1}\nabla_1^{-\delta_{2_1}}})^2}\right]\right\} \\ &\times \left(1-\exp\left\{-\left[\frac{e^{-2\delta_{1_2}\nabla_2^{-\delta_{2_2}}}}{(1-e^{-2\delta_{1_2}\nabla_2^{-\delta_{2_2}}})^2}\right]\right\}\right) dx, \end{aligned} \tag{23}$$

where $\nabla_1 = \frac{G(y;\underline{\omega}_1)}{\bar{G}(y;\underline{\omega}_1)}$, $\nabla_1' = \frac{g(y;\underline{\omega}_1)}{\bar{G}^2(y;\underline{\omega}_1)}$ and $\nabla_2 = \frac{G(y;\underline{\omega}_2)}{\bar{G}(y;\underline{\omega}_2)}$.

The Rényi Entropy for the OGR-RW-G family of distribution given by;

$$\begin{aligned} I_R(\nu) &= \frac{1}{1-\nu}\log\Bigg[(2\delta_1\delta_2)^\nu \sum_{a,b,c,d,e=0}^{\infty} \frac{(-1)^{a+b+c+d+e}\nu^a[2\delta_1(a+b+c+1)]^d}{a!d!} \\ &\times \binom{2a+b+3\nu-1}{b}\binom{\nu}{c}\binom{d\delta_2+\delta_2\nu-\nu+e-1}{e} \\ &\times \int_0^\infty g^\nu(x;\zeta)G^\tau(y;\underline{\omega})dy\Bigg], \end{aligned} \tag{24}$$

where $\tau = -(d\delta_2+\delta_2)+h$, $\nu \neq 1$ and $\nu > 0$.

4. Key Risk Indicators

In this section, we analyze five Key Risk Indicators (KRIs) for our new model: Value-at-Risk (VaR), Tail-Value-at-Risk (TVaR), Conditional-Value-at-Risk (CVaR), Tail Variance (TV), and Tail Mean-Variance (TMV). These metrics are critical for quantifying and managing financial and operational risks, particularly in environments characterized by extreme events or significant uncertainties. They provide valuable insights into potential losses under adverse conditions by focusing on the tail of the distribution, where the most severe outcomes are concentrated.

These indicators not only assess the likelihood of losses but also capture their severity and variability, making them indispensable tools for stress testing, regulatory compliance, and developing effective risk mitigation strategies. By incorporating these measures, organizations can enhance their preparedness for low-probability, high-impact events and make informed decisions to ensure stability and resilience in the face of uncertainty. Let $Y \sim$ OGR-RW-G denote a loss random variable.

4.1. VAR Indicator

The VAR of Y at the $100\alpha\%$ confidence level, say VAR$(Y;\alpha)$ is given by

$$\text{VAR}(Y;\alpha) = G^{-1}\left(\frac{\psi(\alpha)}{1+\psi(\alpha)}\right), \tag{25}$$

for $\psi(\alpha) = \left[\frac{\log\left\{\frac{-\log(1-\alpha)}{1-\log(1-\alpha)}\right\}}{-2\delta_1}\right]^{-\delta_2^{-1}}$.

4.2. TVAR Indicator

The TVAR of Y at the $100\alpha\%$ confidence level, say TVAR$(Y;\alpha)$ is given by

$$\begin{aligned}\text{TVAR}(Y;\alpha) &= E[Y|Y>\text{VAR}(Y;\alpha)] = (1-\alpha)^{-1}\int_{\text{VAR}(Y;\alpha)}^{\infty} y f_{\text{OGR-RW-G}}(y)dy \\ &= (1-\alpha)^{-1}\sum_{a,b,c=0}^{\infty}\chi_\tau^{*}\int_{\text{VAR}(Y;\alpha)}^{\infty} y g_\tau^{*}(y;\underline{\omega})dy, \end{aligned} \tag{26}$$

where χ_τ^* is given in equation (4), $g_\tau^*(y;\underline{\omega})$ is given in equation (5) and $\tau=-(d\delta_2+\delta_2)+h$ is the power parameter.

4.3. CVAR Indicator

The CVAR of Y at the $100\alpha\%$ confidence level, say CVAR$(Y;\alpha)$ is given by

$$\begin{aligned}\text{CVAR}(Y;\alpha) &= E[Y|Y<\text{VAR}(Y;\alpha)] = \frac{1}{\alpha}\int_{-\infty}^{\text{VAR}(Y;\alpha)} y f_{\text{OGR-RW-G}}(y)dy \\ &= \frac{1}{\alpha}\sum_{a,b,c=0}^{\infty}\chi_\tau^{*}\int_{-\infty}^{\text{VAR}(Y;\alpha)} y g_\tau^{*}(y;\underline{\omega})dy, \end{aligned} \tag{27}$$

where χ_τ^* is given in equation (4), $g_\tau^*(y;\underline{\omega})$ is given in equation (5) and $\tau=-(d\delta_2+\delta_2)+h$ is the power parameter.

4.4. TV Risk Indicator

The TV risk indicator of Y at the $100\alpha\%$ confidence level, say TV$(Y;\alpha)$ is given by

$$\begin{aligned}\text{TV}(Y;\alpha) &= E[Y^2|Y>\text{VAR}(Y;\alpha)] - \Big(\text{TVAR}(Y;\alpha)\Big)^2 \\ &= (1-\alpha)^{-1}\int_{\text{VAR}(Y;\alpha)}^{\infty} y^2 f_{\text{OGR-RW-G}}(y)dy - \Big(\text{TVAR}(Y;\alpha)\Big)^2 \\ &= (1-\alpha)^{-1}\sum_{a,b,c=0}^{\infty}\chi_\tau^{*}\int_{\text{VAR}(Y;\alpha)}^{\infty} y^2 g_\tau^{*}(y;\underline{\omega})dy - \Big(\text{TVAR}(Y;\alpha)\Big)^2, \end{aligned} \tag{28}$$

where χ_τ^* is given in equation (4), $g_\tau^*(y;\underline{\omega})$ is given in equation (5) and $\tau=-(d\delta_2+\delta_2)+h$ is the power parameter.

4.5. TMV Risk Indicator

The TMV risk indicator of Y at the $100\alpha\%$ confidence level, say TMV$(Y;\alpha)$ is given by

$$\begin{aligned}\text{TMV}(Y;\alpha) &= E[(Y-\text{TVAR}(Y;\alpha))^2|Y>\text{VAR}(Y;\alpha)] \\ &= (1-\alpha)^{-1}\int_{\text{VAR}(Y;\alpha)}^{\infty} \left(y-\text{TVAR}(Y;\alpha)\right)^2 f_{\text{OGR-RW-G}}(y)dy \\ &= (1-\alpha)^{-1}\sum_{a,b,c=0}^{\infty}\chi_\tau^{*}\int_{\text{VAR}(Y;\alpha)}^{\infty} g_\tau^{*}(y;\underline{\omega})\left(y-\text{TVAR}(Y;\alpha)\right)^2 dy, \end{aligned} \tag{29}$$

where χ_τ^* is given in equation (4), $g_\tau^*(y;\underline{\omega})$ is given in equation (5) and $\tau=-(d\delta_2+\delta_2)+h$ is the power parameter.

5. Maximum Likelihood Estimation

Let $X \sim$OGR-RW-G$(\delta_1, \delta_2, \underline{\omega})$ and $\Delta = (\delta_1, \delta_2, \underline{\omega})^T$ the vector of model parameters, then the log-likelihood function $\ell_n(\Delta) = \ell_n$ based on a random sample of size n from the OGR-RW-G family of distributions is given by

$$\begin{aligned}\ell_n &= n\log(2\delta_1\delta_2) - (\delta_2+1)\sum_{i=1}^{n}\log(\nabla) + \sum_{i=1}^{n}\log(\nabla') - 2\sum_{i=1}^{n}\log(\delta_1\nabla^{-\delta_2}) \\ &+ \sum_{i=1}^{n}\log(1+e^{-2\delta_1\nabla^{-\delta_2}}) - 3\sum_{i=1}^{n}\log(1-e^{-2\delta_1\nabla^{-\delta_2}}) - \sum_{i=1}^{n}\left[\frac{e^{-2\delta_1\nabla^{-\delta_2}}}{1-e^{-2\delta_1\nabla^{-\delta_2}}}\right].\end{aligned}$$

The score equations for the OGR-RW-G model are

$$\begin{aligned}\frac{\partial\ell_n}{\partial\delta_1} &= \frac{n}{\delta_1} - 2\sum_{i=1}^{n}\nabla^{-\delta_2} - 2\sum_{i=1}^{n}\frac{\nabla^{-\delta_2}e^{-2\delta_1\nabla^{-\delta_2}}}{1+e^{-2\delta_1\nabla^{-\delta_2}}} \\ &- 6\sum_{i=1}^{n}\frac{\nabla^{-\delta_2}e^{-2\delta_1\nabla^{-\delta_2}}}{1-e^{-2\delta_1\nabla^{-\delta_2}}} + 2\sum_{i=1}^{n}\frac{\nabla^{-\delta_2}e^{-2\delta_1\nabla^{-\delta_2}}}{(1-e^{-2\delta_1\nabla^{-\delta_2}})^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell_n}{\partial\delta_2} &= \frac{n}{\delta_2} - \sum_{i=1}^{n}\log(\nabla) - 2\sum_{i=1}^{n}\nabla^{-\delta_2} - 2\sum_{i=1}^{n}\delta_1\nabla^{-\delta_2}\log(\nabla) \\ &- 2\sum_{i=1}^{n}\frac{\delta_1\nabla^{-\delta_2}e^{-2\delta_1\nabla^{-\delta_2}}\log(\nabla)}{1+e^{-2\delta_1\nabla^{-\delta_2}}} - 6\sum_{i=1}^{n}\frac{\delta_1\nabla^{-\delta_2}e^{-2\delta_1\nabla^{-\delta_2}}\log(\nabla)}{1-e^{-2\delta_1\nabla^{-\delta_2}}} \\ &+ 2\sum_{i=1}^{n}\frac{\delta_1\nabla^{-\delta_2}e^{-2\delta_1\nabla^{-\delta_2}}\log(\nabla)}{(1-e^{-2\delta_1\nabla^{-\delta_2}})^2},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ell_n}{\partial\underline{\omega}_k} &= -(\delta_2+1)\sum_{i=1}^{n}\partial(\log(\nabla))/\partial\underline{\omega}_k + \sum_{i=1}^{n}\partial(\log(\nabla'))/\partial\underline{\omega}_k \\ &- 2\sum_{i=1}^{n}\partial(\log(\delta_1\nabla^{-\delta_2}))/\partial\underline{\omega}_k + \sum_{i=1}^{n}\partial(\log(1+e^{-2\delta_1\nabla^{-\delta_2}}))/\partial\underline{\omega}_k \\ &- 3\sum_{i=1}^{n}\partial(\log(1-e^{-2\delta_1\nabla^{-\delta_2}}))/\partial\underline{\omega}_k - \sum_{i=1}^{n}\partial\left(\left[\frac{e^{-2\delta_1\nabla^{-\delta_2}}}{1-e^{-2\delta_1\nabla^{-\delta_2}}}\right]\right)/\partial\underline{\omega}_k,\end{aligned}$$

where $\nabla = \frac{G(y;\underline{\omega})}{\bar{G}(y;\underline{\omega})}$ and $\nabla' = \frac{g(y;\underline{\omega})}{\bar{G}^2(y;\underline{\omega})}$. Therefore the estimates of the parameters are obtained by solving the non-linear systems of equation $\hat{\mathbf{\Delta}} = (\frac{\partial\ell_n}{\partial\delta_1}, \frac{\partial\ell_n}{\partial\delta_2}, \frac{\partial\ell_n}{\partial\underline{\omega}_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson technique [17, 18], via statistical softwares such as MATHEMATICA, MAPLE, Ox and R.

6. Simulation Study

In this section, a simulation exercise was conducted using the R programming language (stats4 package) to assess the consistency of the maximum likelihood estimators (MLEs). The OGR-RW-BXII distribution was used to generate 3000 samples with varying sizes ($n = 100, 200, 400, 800, 1000, 1600, 1800$ and 2000) through iterative simulations. For each of the 3000 replications, the MLEs were computed. The average bias (ABias) and root mean square error (RMSE) were calculated for the estimated parameters (refer to [14, 16] for the mathematical formulations of ABias and RMSE).

It is expected that as the sample size increases, both the ABias and RMSE will decrease toward zero, indicating good model performance. As shown in Tables 1 and 2, the ABias and RMSE for all parameter estimates consistently decline toward zero with increasing sample size. This demonstrates that the MLEs are consistent and produce reliable estimates for the model parameters, confirming the robustness of the model.

Table 1. **OGR-RW-BXII Simulation Results 1**

		$(1.0, 1.0, 1.0, 1.0)$			$(1.0, 1.0, 1.5, 1.0)$		
parameter	Sample Size	Mean	RMSE	ABias	Mean	RMSE	A.Bias
δ_1	100	1.7963	2.8691	0.7963	1.6712	3.6361	0.6712
	200	1.4372	1.8024	0.4372	1.4566	2.5401	0.4566
	400	1.2227	1.2856	0.2227	1.4837	2.4789	0.4837
	800	1.0722	0.4964	0.0722	1.2589	1.2322	0.2589
	1000	1.0573	0.4668	0.0573	1.2223	1.0633	0.2223
	1600	1.0253	0.2943	0.0253	1.1974	0.8863	0.1974
	1800	1.0361	0.2612	0.0361	1.1951	0.8478	0.1951
	2000	1.0299	0.2653	0.0299	1.2350	0.8681	0.2350
δ_2	100	0.8811	0.3679	-0.1189	0.8047	-0.6712	0.1953
	200	0.9220	0.3131	-0.0780	0.8411	-0.6154	0.1589
	400	0.9705	0.2395	-0.0295	0.9005	-0.5531	0.0995
	800	0.9854	0.1493	-0.0146	0.9184	-0.5060	0.0816
	1000	0.9883	0.1331	-0.0117	0.9421	-0.4821	0.0579
	1600	0.9971	0.0899	-0.0029	0.9630	-0.4316	0.0370
	1800	0.9951	0.0656	-0.0049	0.9496	-0.4344	0.0504
	2000	0.9955	0.0777	-0.0045	0.9672	-0.4264	0.0328
α	100	1.3614	1.1162	0.3614	1.7869	1.7529	0.2869
	200	1.1819	0.7627	0.1819	1.5973	1.1795	0.0973
	400	1.0768	0.4658	0.0768	1.5642	0.9855	0.0642
	800	1.0310	0.2701	0.0310	1.4986	-0.5367	0.0014
	1000	1.0172	0.2479	0.0172	1.4739	-0.4672	0.0261
	1600	1.0086	0.1903	0.0086	1.4857	-0.3765	0.0143
	1800	1.0188	0.1706	0.0188	1.4975	-0.3615	0.0025
	2000	1.0142	0.1638	0.0142	1.5107	0.3539	0.0107
β	100	1.4381	1.5242	0.4381	3.2173	4.3280	2.2173
	200	1.3329	1.3022	0.3329	2.7617	3.5730	1.7617
	400	1.1425	0.7157	0.1425	2.2086	2.6175	1.2086
	800	1.0455	0.3623	0.0455	1.8826	1.9930	0.8826
	1000	1.0409	0.3122	0.0409	1.7821	1.8426	0.7821
	1600	1.0094	0.1643	0.0094	1.5556	1.4604	0.5556
	1800	0.9994	0.0927	-0.0006	1.5595	1.4105	0.5595
	2000	1.0040	0.1327	0.0040	1.5074	1.3515	0.5074

7. Application

This section demonstrates the utility of the OGR-RW-G family by applying the OGR-RW-BXII distribution to fit three distinct datasets. The model's performance is assessed using various goodness-of-fit (GoF) statistics, including -2 log-likelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramér-von Mises (W^*), Anderson-Darling (A^*), and the Kolmogorov-Smirnov (K-S) statistic along with its p-value. Among the models considered, the one with the lowest values for these statistics and the highest K-S p-value is deemed the most optimal.

Table 2. **OGR-RW-BXII Simulation Results 2**

		$(1.0, 1.0, 1.2, 1.0)$			$(1.1, 1.1, 1.5, 1.1)$		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	A.Bias
δ_1	100	1.7641	3.6072	0.7641	1.7645	3.8650	0.6645
	200	1.4134	2.1130	0.4134	1.6397	3.3009	0.5397
	400	1.3403	1.7863	0.3403	1.4861	2.1106	0.3861
	800	1.1283	0.8190	0.1283	1.3035	1.1731	0.2035
	1000	1.1013	0.7398	0.1013	1.2955	1.1402	0.1955
	1600	1.0610	0.5193	0.0610	1.2596	0.9110	0.1596
	1800	1.0758	0.5158	0.0758	1.2742	0.8951	0.1742
	2000	1.0604	0.4850	0.0604	1.2892	0.8778	0.1892
δ_2	100	0.8343	-0.5182	0.1657	0.8910	-0.7265	0.2090
	200	0.8870	-0.4353	0.1130	0.9053	-0.6524	0.1947
	400	0.9564	-0.3752	0.0436	0.9910	-0.5801	0.1090
	800	0.9871	-0.2943	0.0129	1.0033	-0.5247	0.0967
	1000	0.9811	-0.2868	0.0189	1.0377	-0.5043	0.0623
	1600	0.9958	-0.2161	0.0042	1.0556	-0.4461	0.0444
	1800	0.9901	-0.2229	0.0099	1.0350	-0.4307	0.0650
	2000	0.9947	-0.2047	0.0053	1.0539	-0.4291	0.0461
α	100	1.5397	1.4386	0.3397	1.7967	1.7967	0.2967
	200	1.3428	0.8876	0.1428	1.6365	1.3629	0.1365
	400	1.2814	0.6875	0.0814	1.5326	0.8775	0.0326
	800	1.2194	0.3536	0.0194	1.4795	-0.4979	0.0205
	1000	1.2055	0.3209	0.0055	1.4651	-0.4630	0.0349
	1600	1.2046	0.2473	0.0046	1.4745	-0.3710	0.0255
	1800	1.2168	0.2280	0.0168	1.4935	-0.3534	0.0065
	2000	1.2095	0.2184	0.0095	1.4940	-0.3400	0.0060
β	100	1.9868	2.3237	0.9868	3.4431	4.6703	2.3431
	200	1.6594	1.6981	0.6594	2.8542	3.5304	1.7542
	400	1.3901	1.2132	0.3901	2.2719	2.5523	1.1719
	800	1.1884	0.7732	0.1884	1.9650	1.9745	0.8650
	1000	1.1968	0.7513	0.1968	1.8443	1.7809	0.7443
	1600	1.0879	0.4855	0.0879	1.6265	1.4069	0.5265
	1800	1.0964	0.5083	0.0964	1.6240	1.3590	0.5240
	2000	1.0772	0.4546	0.0772	1.5792	1.3033	0.4792

The OGR-RW-BXII distribution is compared to its nested models as well as five competing non-nested five-parameter models: the Beta Generalized Lindley (BGL) distribution [19], Generalized Gompertz-Poisson (GGP) distribution [8], Beta Odd Lindley-Exponential (BOLE) distribution [5], Weibull Lomax (WL) distribution [21], and Marshall-Olkin-Gompertz-Weibull (MO-Gom-W) distribution [6].

Parameter estimates, along with their standard errors (in parentheses), for the selected datasets are provided in Tables 3 and 7, while the corresponding GoF statistics are summarized in Tables 4 and 6. Figures 5 and 9 illustrate the fitted density, probability plots, and Kaplan-Meier (KM) curves for the OGR-RW-BXII model. Additionally, Figures 6 and 10 present the empirical cumulative distribution function (ECDF), estimated hazard rate function (HRF), and total-time-on-test (TTT) plots for the OGR-RW-BXII model.

These analyses highlight the flexibility and effectiveness of the OGR-RW-BXII model in accurately fitting diverse datasets.

7.1. Repair Lifetimes of an Airborne Transceiver

The dataset represents the duration of active repairs measured in hours for an airborne communication transceiver. These data points are sourced from the studies conducted by Raj S. Chhikara and J Leroy Folks [4] and Victor Leiva et.al. [9]. The variance-covariance matrix for OGR-RW-BXII model on Repair lifetimes of an airborne transceiver

Table 3. **Fitted models parameter estimates for the repair lifetimes of an airborne transceiver dataset**

Model	**Estimates and standard errors**			
	δ_1	δ_2	α	β
OGR-RW-BXII	2.2552×10^{-4}	0.7492	2.6700×10^{-5}	1.0424
	(3.7497×10^{-7})	(8.5481×10^{-10})	(2.3730×10^{-6})	(3.3238×10^{-11})
OGR-RW-BXII$(1,\delta_2,\alpha,\beta)$	-	14.7030	1.0104	2.8863
		(4.1329×10^{-6})	(0.0048)	(0.0029)
OGR-RW-BXII$(\delta_1,1,\alpha,\beta)$	3.0031×10^{-5}	-	5.3642×10^{-5}	0.7338
	(1.5652×10^{-6})		(8.7628×10^{-7})	(3.1085×10^{-11})
	α	θ	a	b
BGL	1.1741×10^{-1}	1.3021×10^{-7}	0.0408	8.0996
	(0.0010)	(1.8480×10^{-6})	(6.0198×10^{-3})	(2.9912×10^{-5})
	θ	α	β	γ
GGP	1.5838×10^{-4}	0.8065	0.2266	3.9440×10^{-9}
	(1.0223)	(0.2288)	(0.0668)	(0.0180)
	λ	θ	a	b
BOLE	80.209	8.0481×10^{-5}	0.9317	40.511
	(1.2470×10^{-11})	(1.2289×10^{-5})	(1.0488×10^{-9})	(2.4410×10^{-11})
	a	b	α	β
WL	2.0041×10^{2}	1.1500	0.1738	3.7634
	(0.0017)	(0.2283)	(0.0086)	(3.6149)
	δ	θ	λ	γ
MO-GOm-W	0.0273	0.0122	1.1551	0.0403
	(0.0747)	(0.0342)	(0.1734)	(0.0632)
			a	b
G			0.2585	0.9323
			(0.0615)	(0.1701)
				λ
W				0.5649
				(0.0500)

dataset is given by

$$\begin{bmatrix} 1.4060\times10^{-13} & 3.2053\times10^{-16} & -8.8983\times10^{-13} & -1.2463\times10^{-17} \\ 3.2053\times10^{-16} & 7.3069\times10^{-19} & -2.0284\times10^{-15} & -2.8412\times10^{-20} \\ -8.8982\times10^{-13} & -2.0285\times10^{-15} & 5.6312\times10^{-12} & 7.8875\times10^{-17} \\ -1.2463\times10^{-17} & -2.8412\times10^{-20} & 7.8875\times10^{-17} & 1.1048\times10^{-21} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta_1 \in [2.2552\times10^{-4} \pm 7.3495\times10^{-7}]$, $\delta_2 \in [0.7492 \pm 1.6754\times10^{-6}]$, $\alpha \in [2.6700\times10^{-5} \pm 4.6511\times10^{-6}]$ and $\beta \in [1.0424 \pm 6.5147\times10^{-11}]$. Table 4 shows that the OGR-RW-BXII model provides the overall best fit on the repair lifetimes of an airborne transceiver dataset as compared to nested and non-nested models presented in the table.

Table 4. **Fitted models GoF statistics for the repair lifetimes of an airborne transceiver dataset**

	GoF Statistics							
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
OGR-RW-BXII	199.3641	207.3641	208.3397	214.6786	0.0460	0.2841	0.0928	0.8235
OGR-RW-BXII$(1, \delta_2, \alpha, \beta)$	204.769	471.4273	471.9988	476.9133	0.0527	0.3351	0.7875	2.2×10^{-16}
OGR-RW-BXII$(\delta_1, 1, \alpha, \beta)$	199.8261	205.8261	206.3975	211.3120	0.0501	0.3234	0.1043	0.6989
BGL	539.3437	547.6217	548.5973	554.9363	0.0742	0.4838	0.9561	2.2×10^{-16}
GGP	210.8166	218.8167	219.7923	226.1313	0.1441	1.0013	0.1334	0.3856
BOLE	209.8717	217.8717	218.8473	225.1862	0.1434	0.9950	0.1455	0.2842
WL	204.2995	212.2995	213.2751	219.6140	0.0803	0.5158	0.1157	0.5692
MO-GOm-W	204.0075	212.0075	212.9831	219.3221	0.0683	0.4692	0.1251	0.4675
G	209.8619	213.8619	214.1410	217.5192	0.1069	0.6729	0.1603	0.1877
W	241.9077	243.9077	243.9986	245.7364	0.0978	0.6719	0.4479	1.9×10^{-8}

Figures 5 and 6 show that the OGR-RW-BXII model can accommodate the extremely tailed data sets, KM

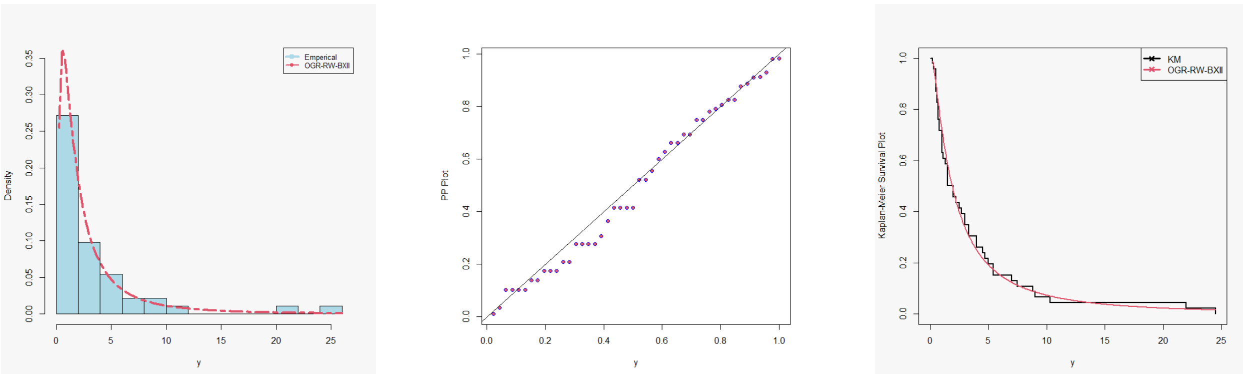

Figure 5. Fitted density, probability plot and KM survival plots of the OGR-RW-BXII model for repair lifetimes of an airborne transceiver data

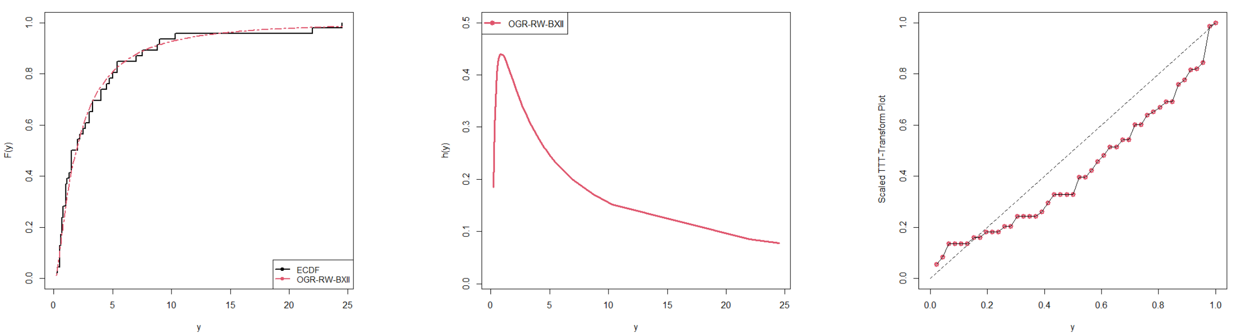

Figure 6. ECDF, estimated hazard rate plot and TTT of the OGR-RW-BXII model for repair lifetimes of an airborne transceiver data

and ECDF curves are close to the empirical data exhibiting better performance of our novel model. Furthermore, the fitted hrf exhibits an upside-down bathtub shape which is supported by the TTT plot produced by the dataset.

7.2. Growth Hormone Dataset

The second dataset comprises the estimated duration from the administration of growth hormone medication until the children reached the desired age [13].

Table 5. **Fitted models parameter estimates for the growth hormone dataset**

Model	**Estimates and standard errors**			
	δ_1	δ_2	α	β
OGR-RW-BXII	2.1510×10^{-4}	1.5690	3.2190×10^{-4}	13.3160
	(6.4410×10^{-4})	(2.0940×10^{-6})	(6.1250×10^{-4})	(6.9100×10^{-8})
OGR-RW-BXII$(1, \delta_2, \alpha, \beta)$	-	0.9760	0.0460	14.0380
		(0.1450)	(0.0040)	(1.4080×10^{-5})
OGR-RW-BXII$(\delta_1, 1, \alpha, \beta)$	0.0670	-	0.1110	0.4910
	(0.1070)		(0.1720)	(0.0850)
	α	θ	a	b
BG-L	0.1151	1.6105×10^{-7}	0.0391	8.0101
	(9.6102×10^{-3})	(1.9382×10^{-6})	(6.5904×10^{-3})	(3.4570×10^{-5})
	θ	α	β	γ
GG-P	5.2833×10^{-9}	1.4484	0.1400	0.1300
	(0.0101)	(0. 4664)	(0.0631)	(0.0590)
	λ	θ	a	b
BOL-E	24.2867	0.0025	4.1427	12.8370
	(0.004)	(4.988×10^{-4})	(0.9550)	(0.0024)
	a	b	α	β
W-L	546.4500	4.4581	0.0664	0.8486
	(6.6900×10^{-5})	(3.2384)	(0.0166)	(2.2971)
	δ	θ	λ	γ
MO-GOm-W	0.0370	2.1544×10^{-5}	0.3410	5.3427
	(0.0107)	(5.9010×10^{-6})	(0.0186)	(7.1244×10^{-4})
			a	b
G			0.7861	4.1711
			(0.1923)	(0.9599)
				λ
W				0.4836
				(0.0513)

The variance-covariance matrix for OGR-RW-BXII model on growth hormone data set set is given by

$$\begin{bmatrix} 1.406\times10^{-13} & 3.205\times10^{-16} & -8.898\times10^{-13} & -1.246\times0^{-17} \\ 3.205\times10^{-16} & 7.307\times10^{-19} & -2.028\times10^{-15} & -2.841\times10^{-20} \\ -8.898\times10^{-13} & -2.028\times10^{-15} & 5.631\times10^{-12} & 7.888\times10^{-17} \\ -1.246\times10^{-17} & -2.841\times10^{-20} & 7.888\times10^{-17} & 1.105\times10^{-21} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta_1 \in [2.151\times10^{-4} \pm 7.349\times10^{-07}]$, $\delta_2 \in [1.569 \pm 1.675\times10^{-09}]$, $\alpha \in [3.219\times10^{-4} \pm 4.651\times10^{-06}]$ and $\beta \in [13.316 \pm 6.515\times10^{-11}]$. Table 6 shows that the OGR-RW-BXII model provides the overall best fit on the growth hormone dataset as compared to the nested and non-nested models under consideration. Figures 9 and 10 that the OGR-RW-BXII model can accommodate the extremely tailed data sets, the KM and ECDF are aligning closely with the model thus exhibiting better performance for our model. Additionally, the estimated hrf exhibits the TTT plot produced by the dataset.

Table 6. **Fitted models GoF statistics for the Growth hormone data set**

	Statistics							
Model	$-2\log L$	AIC	$CAIC$	BIC	W^*	A^*	K-S	p-value
OGR-RW-BXII	154.4357	162.4358	163.7691	168.6572	0.0376	0.2584	0.0822	0.9721
OGR-RW-BXII$(1, \delta_2, \alpha, \beta)$	156.2151	386.5832	387.3574	391.2493	0.0329	0.2332	0.8055	2.2×10^{-16}
OGR-RW-BXII$(\delta_1, 1, \alpha, \beta)$	209.3691	215.3691	216.1433	220.0351	0.0610	0.4173	0.4160	1.095×10^{-5}
BG-L	471.1036	479.1123	480.4456	485.3337	0.0627	0.4285	0.9582	2.2×10^{-16}
GG-P	169.0884	177.0885	178.4218	183.3099	0.1986	1.2386	0.1702	0.2625
BOL-E	160.2746	168.2746	169.6079	174.496	0.1019	0.6609	0.1251	0.6438
W-L	161.0264	169.0264	170.3597	175.2478	0.1098	0.7068	0.1244	0.6508
MO-GOm-W	161.5606	169.5607	170.8940	175.782	0.1134	0.7346	0.1037	0.846
G	160.2165	164.2165	164.5915	167.3272	0.0816	0.5885	0.8671	2.2×10^{-16}
W	258.6514	260.6514	260.7727	262.2068	0.0771	0.5138	0.7650	2.2×10^{-16}

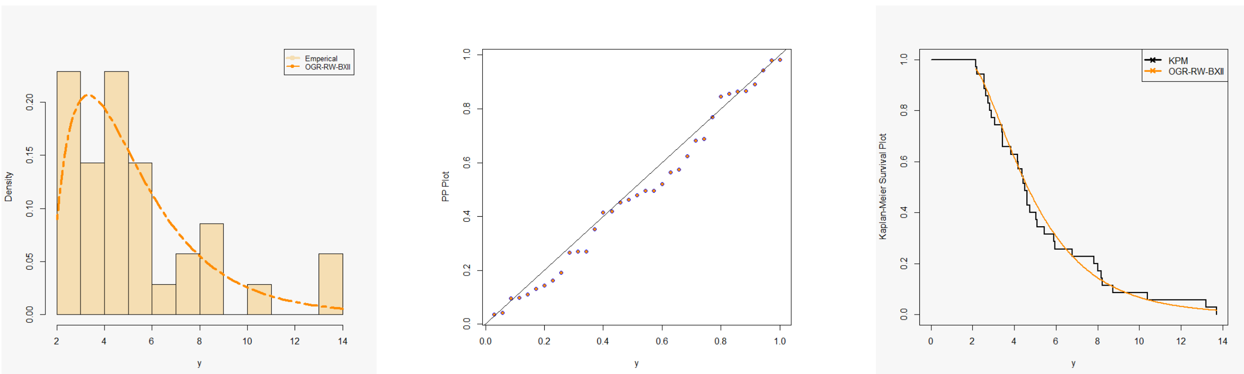

Figure 7. Fitted density, probability plots and KM survival plots of the OGR-RW-BXII model for the growth hormone data

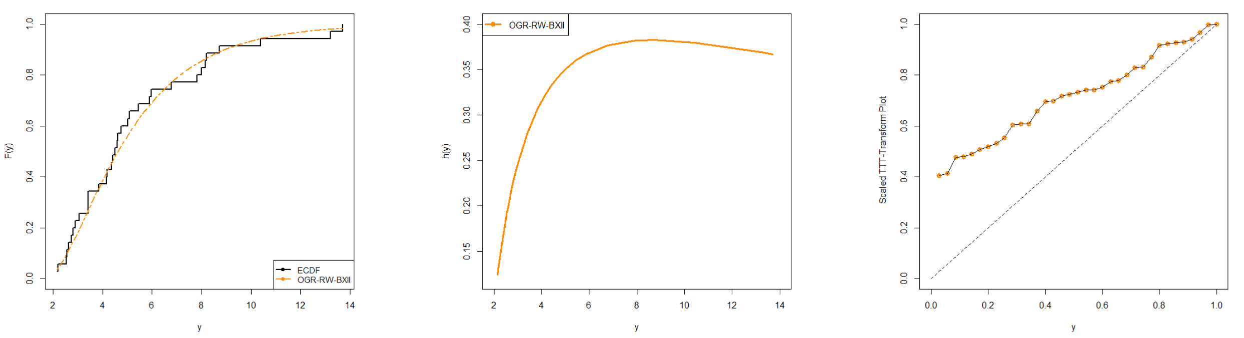

Figure 8. ECDF, estimated hazard rate plot and TTT of the OGR-RW-BXII model for the growth hormone data

7.3. Survival Dataset of Patients Suffering from Acute Myelogeneous Leukaemia

The third dataset comprises of patients suffering from acute myelogeneous leukaemia. The data, that can also be found at library SMIR of the R program (http://cran.r-project.org). The variance-covariance matrix for OGR-RW-

Table 7. **Fitted models parameter estimates for the myelogeneous leukaemia dataset**

Model	**Estimates and standard errors**			
OGR-RW-BXII	δ_1	δ_2	α	β
	1.5252	0.0817	0.0687	57.6924
	(0.2958)	(0.0237)	(0.0215)	(17.6076)
OGR-RW-BXII($1, \delta_2, \alpha, \beta$)	-	0.4183	0.2355	2.4818
		(0.1594)	(0.1780)	(2.5172)
OGR-RW-BXII($\delta_1, 1, \alpha, \beta$)	1.0610	-	0.7025	0.4171
	(1.9050)		(1.0600)	(0.2960)
BG-L	α	θ	a	b
	0.1034	1.6105×10^{-8}	0.0331	8.0997
	(8.378×10^{-3})	(1.409×10^{-6})	(5.766×10^{-3})	(3.129×10^{-5})
GG-P	θ	α	β	γ
	7.142×10^{-5}	0.5478	0.0124	0.0056
	(0.1274)	(0. 1548)	(0.0077)	(0.0073)
BOL-E	λ	θ	a	b
	1.5185	0.0028	0.6204	4.9850
	(2.2114)	(0.0045)	(0.1732)	(0.4522)
W-L	a	b	α	β
	1.2047	0.6017	1.510×10^{3}	1.156×10^{3}
	(0.2141)	(0.0805)	(1.6227×10^{-4})	(2.1186×10^{-6})
MO-GOm-W	δ	θ	λ	γ
	0.1715	0.0106	0.8313	0.0263
	(0.2648)	(0.0181)	(0.2039)	(0.0401)
G			a	b
			0.0168	0.6878
			(0.0050)	(0.1440)
W				λ
				0.2393
				(0.0259)

BXII model on myelogeneous leukaemia dataset set is given by

$$\begin{bmatrix} 8.7490 \times 10^{-2} & -4.7800 \times 10^{-5} & 1.5970 \times 10^{-3} & 0.8730 \\ -4.7800 \times 10^{-5} & 5.6330 \times 10^{-4} & -2.3720 \times 10^{-4} & -0.1958 \\ 1.5970 \times 10^{-3} & -2.3720 \times 10^{-4} & 0.0004 & -0.1737 \\ 0.8732 & -0.1958 & -0.1737 & 310.0262 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta_1 \in [1.5252 \times 10^{-4} \pm 0.5797]$, $\delta_2 \in [0.0817 \pm 0.0465]$, $\alpha \in [0.0687 \times 10^{-4} \pm 0.0421]$ and $\beta \in [57.6924 \pm 34.5108]$. Table 8 shows that the OGR-RW-BXII model provides the overall best fit on the myelogeneous leukaemia dataset as compared to the nested and non-nested models under consideration. Figures 9 and 10 that the OGR-RW-BXII model can accommodate the extremely tailed data sets, the KM and ECDF are aligning closely with the model thus exhibiting better performance for our model. Additionally, the estimated hrf exhibits the TTT plot produced by the dataset.

Table 8. **Fitted models GoF statistics for the myelogeneous leukaemia dataset**

	Statistics							
Model	$-2\log L$	AIC	$CAIC$	BIC	W^*	A^*	K-S	p-value
OGR-RW-BXII	298.9213	306.9213	308.3499	312.9074	0.0705	0.4767	0.1267	0.6644
OGR-RW-BXII$(1,\delta_2,\alpha,\beta)$	304.6768	339.974	340.8016	344.4635	0.1097	0.7076	0.4068	3.62×10^{-5}
OGR-RW-BXII$(\delta_1,1,\alpha,\beta)$	305.3852	311.3852	312.2128	315.8747	0.0862	0.5710	0.1440	0.5008
BG-L	549.7148	557.8677	559.2963	563.8538	0.1276	0.8035	0.9644	2.2×10^{-16}
GG-P	307.1409	315.1409	316.5694	321.1269	0.1030	0.7129	0.1172	0.7556
BOL-E	307.0134	315.0134	316.4420	320.9995	0.0987	0.6849	0.1369	0.5664
W-L	306.4082	376.8067	376.9357	378.3032	0.1001	0.6548	0.6368	4.78×10^{-12}
MO-GOm-W	305.3181	313.3181	314.7467	319.3041	0.0857	0.5889	0.1312	0.6206
G	307.3473	311.3473	311.7473	314.3403	0.1284	0.8533	0.2695	0.0166
W	374.8067	260.6514	260.7727	262.2068	0.0771	0.5138	0.7650	2.2×10^{-16}

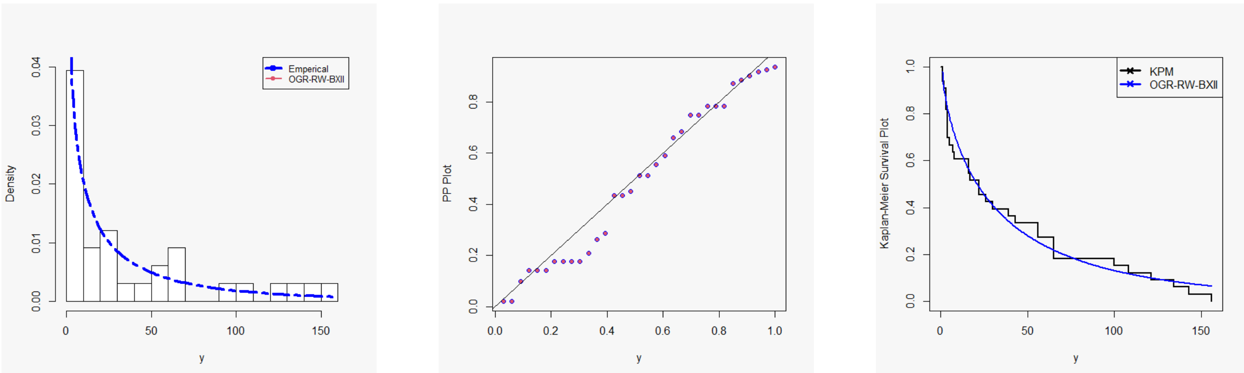

Figure 9. Fitted density, probability plots and KM survival plots of the OGR-RW-BXII model for the myelogeneous leukaemia data

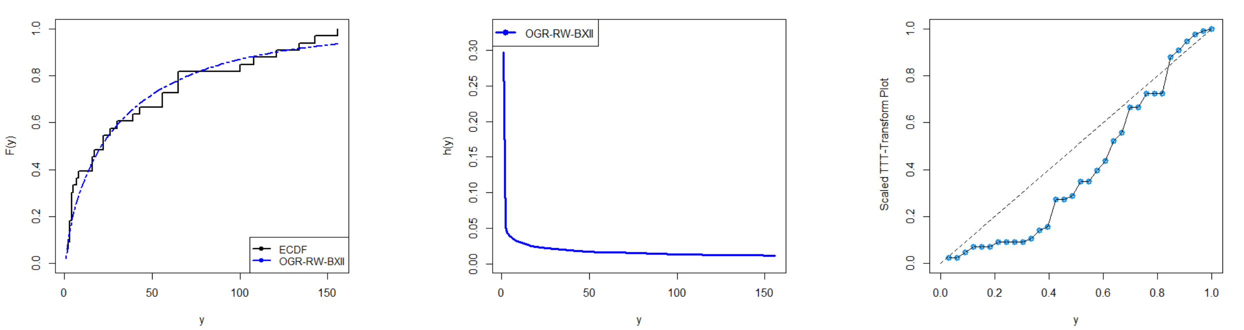

Figure 10. ECDF, estimated hazard rate plot and TTT of the OGR-RW-BXII model for the myelogeneous leukaemia data

8. Concluding Remarks

In this study, we introduced the Odd Generalized Rayleigh Reciprocal Weibull (OGR-RW-G) family of probability distributions, representing a novel and versatile class of probability models. The statistical properties of this newly developed family were thoroughly analyzed, and key characteristics were derived. The maximum likelihood estimation (MLE) method was employed to estimate the model parameters, and a Monte Carlo simulation study was conducted to evaluate the performance of the MLEs.

Additionally, the empirical performance of the OGR-RW-BXII distribution, a member of the OGR-RW-G family, was assessed using real-world datasets. The results demonstrate that the OGR-RW-BXII distribution outperforms its nested distributions as well as other non-nested competing distributions under consideration. These findings underscore the flexibility and effectiveness of the OGR-RW-G family in modeling complex datasets and provide a foundation for its application in various fields requiring robust statistical modeling.

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