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Optimal Search for a Lost Moving Target Whose Truncated Markov Chain with Continuous Time

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Abstract The main contribution of this paper center a round searching for a lost target which moves among finite number of cells according to truncated Markov chain with continuous time , the searcher distributes the effort among the states and the purpose is to minimize the search effort and the probability of undetection at the same time.

Keywords Optimal search plan, Truncated Markov chain with continuous time, probability of undetection.

AMS 2010 subject classifications 37A50, 60k30.

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1. Introduction

The study of search plans for lost targets that are either stationary or which move according to any random motion is very important and has recently various application such electrical power lines, missing boats and submarines and search for a randomly located goal at two disjoint separated region, see [4–8]. The aim of this work is to maximize the probability of detecting the target where the search effort takes a fixed value, see [8, 11, 13]. Effort may be time, energy, looks, or whatever is appropriate for the given search. When the target is assumed to be in one of several states and a searcher must divided his effort among the states with the object of finding the target, the optimal distribution of effort under a variety of assumption is studied, see [5] and [13]. The search problems with a stationary or moving target among a finite number of states according to random motion are studied, see [8, 9, 11, 12, 14, 16]. Our purpose here is to obtain the search plan which made the probability of undetection of the target moving according to truncated markov chain with continuous time and the search effort of detecting the target are minimum, where the effort is unrestricted and states are not identical, also to find stability of search.

The study of truncation on Markov chain and stochastic processes has applications in some different field see [17–19]. It may happen sometimes that the space of a Markov chain at a given time is not suitable for data according to some reasons so that this partial will be truncated. Sometimes in some applications, we face a problem that need to be solved. For example, when studying the search for a hidden moving randomly which follows a finite states Markov chain within a city that has been divided into a certain number of cells and for circumstances a cell or group of cells was removed during that searching, this mean that the state space of the chain must be truncated, hence the significance of studying the Markov chain with truncated space in this paper. Markov chain play a great role in the study of Economic, Social, Physical and other different sciences. As we may have to truncated a umber of states of Markov chain for the same reasons.

Markov chain has many application as statistical models of real world process, searching for a lost target which moves randomly among a finite number of cells, see [1, 2]. Also, queues or lines of customers arriving at an airport

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is one of the famous applications on Markov chain see [3]. The authors in [3] delete some state in infinite Markov chain where the transition probabilities tend to zero. Also, for its application studying search plans for lost target with truncated states.

2. Problem Formulation

A target is assumed to be in one several states not necessarily identical states and a state is truncated from states at time interval k at $[t, t+h]$ and the searcher must distribute his effort among the states in such a manner as to minimize the probability of undetection, let the number of states be M, the state r is truncated from states at time interval k in $[t, t+h]$, the target occupies one state during the time intervals n in $[t, t+h]$. Before truncated, the probabilities that the target exists in state j at time interval i in $[t, t+h]$ is p_{ij}. We suppose the rate position of the effort in state is constant at any state and change for other state such as $[\phi_{1j} \neq \phi_{2j}]$ where ϕ_{ij} the allocation of search effort.

During the time interval j^{th} in $[t, t+h]$, the effort is given by $L_i(\phi, \phi')$ will be distributed among the states. The allocation of search effort is ϕ_{ij}, where $i = 1, 2, \cdots, k-1$, $j = 1, 2, \cdots, m$, which gives the effort to put into state j at time interval i in $[t, t+h]$ we call $\phi = [\phi_{ij}]$ of a search plan. After truncated, the probabilities that the target exists in state j at time interval i^{th} in $[t, t+h]$ is p'_{ij}, the effort is given by $L_i(\phi, \phi']$ which will be distributed among the states. The allocation of search effort is ϕ'_{ij}, where, $i = k, k+1, \cdots, N$, $j = 1, 2, \cdots, m-1$, $j \neq r$ which gives the effort to put into state j at time interval i in $[t, t+h]$ we call $\phi' = [\phi'_{ij}]$ a search plan. Before truncated let $b(i, j, \phi_{ij})$ be the conditional probability of detecting the target at time $i = 1, 2, \cdots, k-1$, $j = 1, 2, \cdots, m$ with ϕ_{ij} amount of effort given the target is located in state j which is defined by the detection function. After truncated let $b' = (i, j, \phi'_{ij})$ be the conditional probability of detecting the target at time i, $i = k, k+1, \cdots, N$ with ϕ'_{ij} amount of effort given the target is located in state j which is defined by the detection function. There is an implicit assumption here that the probability of detection in state j at time interval i in $[t, t+h]$ depends only on the total amount of effort applied and not on the way the effort is applied. We assume that the searches at continuous time are independent, and the motion of the target is independent of its searcher action. The probability of undetection of the target over the whole time is given by:

$$\lim_{N\to\infty} H(\phi, \phi') = \lim_{N\to\infty} \prod_{i=1}^{k-1} \sum_{j=1}^{m} p_{ij}(1 - b(i, j, \phi_{ij})) \prod_{i=k}^{N} \sum_{\substack{j=1,\\ j\neq r}}^{m-1} p'_{ij}(1 - b'(i, j, \phi'_{ij})_{j\neq r}),$$

where $p'_{ij, j\neq r} = \frac{p_{ij}}{1-p_{ir}}$.

We assume that the search strategy ϕ is Borel function such that $\phi : J \times (0, \infty) \to (0, \infty)$ (see [13]).

The probability of not detecting the target at the interval i in time $[t, t+h]$ is independent and we will truncate state (r)

$$\lim_{N\to\infty} \prod_{i=1}^{k-1} \sum_{\substack{j=1,\\ i\neq j}}^{m} p_{ij}(t) \exp\left[\int_{t+(k-1)h}^{t+kh} \phi(J, S)\, dt \right]$$

$$\prod_{i=k}^{N} \sum_{\substack{J=1, J\neq r,\\ i\neq J}}^{m-1} p'_{ij}(t) \exp\left[\int_{t+(k-1)h}^{t+kh} \phi'(J, S)\, dt \right],$$

where $P_{iJ}(t)$ is the probability of the target before truncated at $i = 1, 2, \cdots, k$, $j = 1, 2, \cdots, m$ of t and $k(J)$ is a factor due to the search in cell J and the dimensions of it and the interval $[t, t+h]$. After truncated $p'_{ij}(t)$ is the probability of the target at $i = k, k+1, \cdots, N$ $j = 1, 2, \cdots, m-1$ of t and $k'(J)$ is the factor due to search in cell J after truncate is the factor $[t, t+h]$,

$$H(\phi,\phi') = \prod_{i=1}^{k-1} \sum_{\substack{J=1,\\ i\neq j}}^{m} p_{ij}(t)\ \exp\left[-k(J) \int_{t+(k-1)h}^{t+kh} \phi(J,S)\, dt\right]$$

$$\prod_{L=K}^{N} \sum_{\substack{J=1,\\ J=r}}^{m} p'_{ij}(t)\ \exp\left[-k(J) \int_{t+(k-1)h}^{t+kh} \phi'(J,S)\, dt\right]$$

$$L(\phi_{iJ},\phi'_{iJ}) = \sum_{j=1}^{k-1} \sum_{J=1}^{m} \phi_{iJ} + \sum_{i=k}^{N} \sum_{\substack{J=1,\\ J\neq r}}^{m-1} \phi'_{ij},$$

$$L_i(\phi_{iJ},\phi'_{iJ}) = \sum_{\substack{J=1,\\ 1\leq i\leq k-1}}^{m} \phi_{iJ} + \sum_{\substack{J=1, J\neq r,\\ k\leq i\leq N}}^{m-1} \phi'_{iJ},$$

$$\sum_{J=1}^{m} p_{ij} = 1, \sum_{\substack{J=1,\\ j\neq r}}^{m-1} p_{iJ} = 1,$$

where $p'_{iJ} = \frac{p_{iJ,J=r}}{1-p_{ir}}$.

$$H(\phi_{iJ},\phi'_{iJ}) = \prod_{i=1}^{k-1} \left[p_{i1}\, e^{-k(1) \int_{t+(i-1)h}^{t+ih} \phi_{iJ} dt} + \cdots + p_{it}\, e^{-k(t) \int_{t+(i-1)h}^{t+ih} \phi_{it} dt} + \cdots + p_{im}\, e^{-k(m) \int_{t+(i-1)h}^{t+ih} \phi_{im} dt} \right]$$

$$\prod_{i=k}^{N} \left[p'_{i1}(t)\, e^{-k'(1) \int_{t+(i-1)h}^{t+ih} \phi'_{it} dt} + \cdots + p'_{it}\, e^{-k'(t) \int_{t+(i-1)h}^{t+ih} \phi'_{it} dt} + \cdots + p'_{im-1} \right.$$

$$\left. e^{-k'(m-1) \int_{t+(i-1)h}^{t+ih} \phi'_{im-1} dt} \right]$$

$$= \left[p_{11}\, e^{-k(1) \int_{t}^{t+h} \phi_{11} dt} + \cdots + p_{1t}\, e^{-k(t) \int_{t}^{t+h} \phi_{1t} dt} + \cdots + p_{1m}\, e^{-k(m) \int_{t}^{t+h} \phi_{1m} dt} \right]$$

$$\left[p_{q1}\, e^{-k(q) \int_{t+(q-1)h}^{t+qh} \phi_{q1} dt} + \cdots + p_{qt}\, e^{-k(t) \int_{t+(q-1)h}^{t+qh} \phi_{qt} dt} + \cdots + p_{qm}\, e^{-k(m) \int_{t+(q-1)h}^{t+qh} \phi_{qm} dt} \right]$$

$$\left[p_{k-11}\, e^{-k(1) \int_{t+(k-2)h}^{t+(k-1)h} \phi_{k-11} dt} + \cdots + p_{k-1t}\, e^{-k(t) \int_{t+(k-2)h}^{t+(k-1)h} \phi_{k-1t} dt} + \cdots + p_{k-1m} \right.$$

$$\left. e^{-k(m) \int_{t+(k-2)h}^{t+(k-1)h} \phi_{k-1m} dt} \right] \left[p'_{k1}\, e^{-k'(1) \int_{t+(k-1)h}^{t+kh} \phi'_{k1} dt} + \cdots + p'_{kt}\, e^{-k'(t) \int_{t+(k-1)h}^{t+kh} \phi'_{kt} dt} + \cdots \right.$$

$$+p'_{km-1}\, e^{-k'(m-1)\int_{t+(k-1)h}^{t+kh}\phi'_{km-1}dt}\Bigg]\Bigg[p'_{q'1}\, e^{-k'(1)\int_{t+(q'-1)h}^{t+q'h}\phi'_{q'1}dt}+\cdots$$

$$+p'_{q't}\, e^{-k'(t')\int_{t+(q'-1)h}^{t+q'h}\phi'_{q't}dt}+\cdots+p'_{q'm-1}\, e^{-k'(m-1)\int_{t+(q'-1)h}^{t+q'h}\phi'_{q'm-1}dt}\Bigg]$$

$$\Bigg[p'_{q'N1}\, e^{-k'(1)\int_{t+(N-1)h}^{t+Nh}\phi'_{N1}dt}+\cdots+p'_{Nt}\, e^{-k'(t)\int_{t+(N-1)h}^{t+Nh}\phi'_{Nt}dt}+\cdots$$

$$+p'_{q'Nm-1}\, e^{-k'(m-1)\int_{t+(N-1)h}^{t+Nh}\phi'_{Nm-1}dt}\Bigg],$$

$$\frac{\partial}{\partial\,\phi qt}\left[p_{qt}\, e^{-k(t)\int_{t+(q-1)h}^{t+qh}\phi_{qt}dt}\right]=\frac{\partial}{\partial\,\phi qt}\left[p_{qt}\, e^{-k(t)\phi_{qt}t\Big|_{t+(q-1)h}^{t+qh}}\right]$$

$$=\frac{\partial}{\partial\,\phi qt}\left[p_{qt}\, e^{-k(t)\phi_{qt}[t+qh-t-qh+h]}\right],$$

$$\frac{\partial}{\partial\,\phi qt}\left[p_{qt}\, e^{-k(t)\phi_{qt}h}\right]=-k(t)p_{qt}\,h\,e^{-k(t)\phi_{qt}h},$$

$$\frac{\partial}{\partial\,\phi q't}\left[p'_{q't}\, e^{-k'(t)\int_{t+(q'-1)h}^{t+q'h}\phi_{q't}dt}\right]=-k'(t)p'_{q't}\,h\,e^{-k'(t)\phi'_{qt}h},$$

$$\frac{\partial H(\phi,\phi')}{\partial\phi_{qt}}=-k(t)p_{qt}\,h\,e^{-k(t)\phi_{qt}h}\prod_{i=1}^{k-1}\sum_{J=1}^{m}p_{iJ}\,e^{-k(J)\int_{t+(i-1)h}^{t+ih}\phi_{iJ}dt}\prod_{i=k}^{N}\sum_{\substack{J=1,\\ J\neq r}}^{m-1}p'_{iJ}\,e^{-k(J)\int_{t+(i-1)h}^{t+ih}\phi'_{iJ}dt},$$

$$\frac{\partial H(\phi,\phi')}{\partial\phi'_{q't}}=-k'(t)p'_{q't}\,e^{-k'(t)\phi'_{q't}h}\prod_{i=k}^{N}\sum_{\substack{J=1,\\ J\neq r}}^{m-1}p'_{iJ}\,e^{-k'(J)\int_{t+(i-1)h}^{t+ih}\phi'_{iJ}(t)dt}\prod_{i=1}^{k-1}\sum_{J=1}^{m}p_{iJ}\,e^{-k(J)\int_{t+(i-1)h}^{t+ih}\phi_{iJ}(t)dt},$$

$$g_i(\phi_{iJ},\phi'_{iJ})=L_i(\phi_{iJ},\phi'_{iJ})-\varepsilon_i\leq 0,$$

$$g_i(\phi_{iJ},\phi'_{iJ})=\left[\sum_{\substack{J=1,\\ 1\leq i\leq k-1}}^{m}\phi_{iJ}(t)+\sum_{\substack{J=1,\\ k\leq i\leq N}}^{m-1}\phi'_{iJ}(t)-\varepsilon_i(\phi_{iJ},\phi'_{(iJ)}\right],$$

$$g_i(\phi_{iJ},\phi'_{iJ}) = [\phi_{i1}+\cdots+\phi_{im}] + [\phi'_{11}+\cdots+\phi'_{im-1}], \qquad J \neq r,$$

$$\frac{\partial g_i(\phi_{iJ},\phi'_{iJ})}{\partial \phi_{qt}} = 1, \quad 1 \leq q \leq k-1,$$

$$\frac{\partial g_i(\phi_{iJ},\phi'_{iJ})}{\partial \phi_{q't}} = 1, \quad k \leq q' \leq N,$$

$$\frac{\partial H(\phi_{iJ},\phi'_{iJ})}{\partial \phi_{qt}} + \sum_{i=1}^{k-1} u_i(\phi,\phi')\,\partial g_i(\phi_{iJ},\phi'_{iJ}) = 0,$$

$$-k(t)p_{qt}h\,e^{-k(t)\phi_{qt}h} \prod_{i=1}^{k-1}\sum_{J=1}^{m} p_{iJ}\,e^{-k(J)\int\limits_{t+(i-1)h}^{t+ih}\phi_{iJ}(t)dt} \prod_{i=k}^{N}\sum_{\substack{J=1\\ j\neq r}}^{m-1} p'_{iJ}\,e^{-k'(J)\int\limits_{t+(i-1)h}^{t+ih}\phi'_{iJ}(t)dt}$$
$$+u_q(\phi_{ij}(t),\phi'_{iJ}(t)) = 0, \tag{1}$$

$$-k'(t)p'_{qt'}h\,e^{-k'(t')\phi_{q't}h} \prod_{i=k}^{N}\sum_{\substack{J=1\\ J\neq r}}^{m-1} p'_{iJ}\,e^{-k'(J)\int\limits_{t+(i-1)h}^{t+ih}\phi'_{iJ}(t)dt} \prod_{i=k}^{k-1}\sum_{J=1}^{m} p_{iJ}\,e^{-k(J)\int\limits_{t+(i-1)h}^{t+ih}\phi_{iJ}(t)dt}$$
$$+u_{q'}(\phi_{ij}(t),\phi'_{iJ}(t)) = 0. \tag{2}$$

If we put

$$A = \prod_{i-1}^{k-1}\sum_{J=1}^{m} p_{ij}\,e^{-k(J)\int\limits_{t+(i-1)h}^{t+ih}\phi_{iJ}(t)dt},$$

$$B = \prod_{i=k}^{N}\sum_{\substack{J=1,\\ J\neq r}}^{m-1} p'_{ij}\,e^{-k'(J)\int\limits_{t+(i-1)h}^{t+ih}\phi'_{iJ}(t)dt},$$

we can get

$$-k(t)p_{qt}h\,e^{-k(t)\phi_{qt}h}AB + u_q(\phi,\phi') = 0,$$

$$u_q(\phi,\phi') = k(t)p_{qt}\,e^{-k(t)\phi_{qt}h}ABh,$$

$$\ln[u_q(\phi,\phi')] = \ln[k(t)p_{qt}\,e^{-k(t)\phi_{qt}h}AB] - k(t)\phi_{qt}h = \ln\left[\frac{u_q(\phi,\phi')}{k(t)p_{qt}ABh}\right],$$

$$\phi_{qt} = \ln\left[\frac{u_q(\phi,\phi')}{k(t)p_{qt}ABh}\right]^{\frac{-1}{k(t)h}},$$

$$\phi_{qt} = \ln\left[\frac{ABhk(t)p_{qt}}{u_q(\phi,\phi')}\right]^{\frac{1}{k(t)h}}, \quad 1 \le q \le k-1 \tag{3}$$

$$u_q(\phi_{iJ},\phi'_{iJ})g_q(\phi_{iJ},\phi'_{iJ}) = 0,$$

if $u_q(\phi,\phi') \neq 0 \Rightarrow L_q(\phi,\phi') - \varepsilon_q(\phi,\phi') = 0$, $g_q(\phi,\phi') = 0 \Rightarrow L_q(\phi,\phi') = \varepsilon_q(\phi,\phi')$,

$$\sum_{\substack{J=1,\\ 1\le i\le k-1}}^{m} \phi_{iJ}(t) + \sum_{\substack{J=1,J\neq r,\\ 1\le i\le k-1}}^{m-1} \phi_{iJ} = \varepsilon_{q,q'}(\phi,\phi'),$$

$$\varepsilon_q(\phi,\phi') = \sum_{\substack{J=1,\\ 1\le i\le k-1}}^{m} \phi_{qJ}. \tag{4}$$

From Equation (4) into Equation (3) we get

$$\ln\left[\frac{ABhk(1)p_{qt}}{u_q(\phi,\phi')}\right]^{\frac{1}{k(1)h}} + \ln\left[\frac{ABhk(2)p_{qt}}{u_q(\phi,\phi')}\right]^{\frac{1}{k(2)h}} + \cdots + \ln\left[\frac{ABhk(m)p_{qt}}{u_q(\phi,\phi')}\right]^{\frac{1}{k(m)h}}$$
$$= \ln\left[\frac{\prod_{J=1}^{m}(k(J)p_{qJ})^{\frac{1}{k(J)h}} A^{\sum_{J=1}^{m}\frac{1}{k(J)h}} B^{\sum_{J=1}^{m}\frac{1}{k(J)h}} \cdot h^{\sum_{J=1}^{m}\frac{1}{k(J)h}}}{[u_q(\phi,\phi')]^{\sum_{J=1}^{m}\frac{1}{k(J)h}}}\right] = \varepsilon(\phi,\phi') \rightarrow$$

$$\frac{\prod_{J=1}^{m}(k(J)p_{qJ})^{\frac{1}{k(J)h}} A^{\sum_{J=1}^{m}\frac{1}{k(J)h}} B^{\sum_{J=1}^{m}\frac{1}{k(J)h}} \cdot h^{\sum_{J=1}^{m}\frac{1}{k(J)h}}}{[u_q(\phi,\phi')]^{\sum_{J=1}^{m}\frac{1}{k(J)h}}} = e^{\varepsilon_q(\phi,\phi')},$$

$$[[u_q(\phi,\phi')]]^{\left(\sum_{J=1}^{m}\frac{1}{k(J)h}\right)} = \frac{\left[\prod_{J=1}^{m}(k(J)p_{qJ})^{\frac{1}{k(J)h}} A^{\sum_{J=1}^{m}\frac{1}{k(J)h}} B^{\sum_{J=1}^{m}\frac{1}{k(J)h}} \cdot h^{\sum_{J=1}^{m}\frac{1}{k(J)h}}\right]}{e^{\varepsilon_q(\phi,\phi')}},$$

$$[[u_q(\phi,\phi')]]^{\left(\sum_{J=1}^{m}\frac{1}{k(J)h}\right)\left[1\Big/\left(\sum_{J=1}^{m}\frac{1}{k(J)h}\right)\right]} =$$
$$\left[\frac{\left[\prod_{J=1}^{m}(k(J)p_{qJ})^{\frac{1}{k(J)h}} A^{\sum_{J=1}^{m}\frac{1}{k(J)h}} B^{\sum_{J=1}^{m}\frac{1}{k(J)h}} \cdot h^{\sum_{J=1}^{m}\frac{1}{k(J)h}}\right]}{e^{\varepsilon_q(\phi,\phi')}}\right]^{\left[1\Big/\left(\sum_{J=1}^{m}\frac{1}{k(J)h}\right)\right]},$$

$$u_q(\phi,\phi') = \left[\frac{\prod_{J=1}^{m}(k(J)p_{qJ})^{\frac{1}{k(J)h}} A^{\sum_{J=1}^{m}\frac{1}{k(J)h}} B^{\sum_{J=1}^{m}\frac{1}{k(J)h}} \cdot h^{\sum_{J=1}^{m}\frac{1}{k(J)h}}}{e^{\varepsilon_q(\phi,\phi')}}\right]^{\left[1\Big/\left(\sum_{J=1}^{m}\frac{1}{k(J)h}\right)\right]}. \tag{5}$$

From Equation (5) into Equation (3) we get

$$\phi_{qt} = \ln \left[\frac{ABhk(t)p_{qt}\left[e^{\varepsilon_q(\phi,\phi')}\right]^{\left[1/\sum_{J=1}^{m}\frac{1}{k(J)h}\right]\frac{1}{k(t)h}}}{\prod_{J=1}^{m}[k(J)p_{qpJ}]^{\frac{1}{k(J)h}}A^{\sum_{J=1}^{m}\frac{1}{k(J)h}}B^{\sum_{J=1}^{m}\frac{1}{k(J)h}}h^{\sum_{J=1}^{m}\frac{1}{k(J)h}}} \right],$$

$$\phi_{qt} = \frac{1}{k(t)h}\ln\frac{\left[k(t)p_{qt}\left[e^{\varepsilon_q(\phi,\phi')}\right]^{1/\sum_{J=1}^{m}\frac{1}{k(J)h}}\right]}{\left[\prod_{J=1}^{m}[k(J)p_{qJ}]^{\frac{1}{k(J)h}}\right]^{1/\sum_{J=1}^{m}\frac{1}{k(J)h}}}$$

$$= \frac{1}{k(t)h}\left[\ln k(t) + \ln p_{qt} + \varepsilon_q(\phi,\phi')\left[1\Big/\sum_{J=1}^{m}\frac{1}{k(J)h}\right] - \left[\ln\prod_{J=1}^{m}k(J)p_{qJ}\right]^{\frac{1}{k(J)h}}\right]\left[1\Big/\sum_{J=1}^{m}\frac{1}{k(J)h}\right],$$

$$\phi_{qt} = \frac{1}{k(t)h}\left[\ln\left[\frac{k(t)p_{qt}}{\left[\left[\prod_{J=1}^{m}(k(J)p_{qJ})^{1/k(J)h}\right]^{\left[1/\sum_{J=1}^{m}\frac{1}{k(J)h}\right]}\right]}\right] + \varepsilon_q(\phi,\phi')\frac{1}{\sum_{J=1}^{m}\frac{1}{k(J)h}}\right],$$

where $1 \le q \le k-1$, $J = 1,2,\cdots,m$, before truncated.

After truncated, by the same way we can get $\phi'_{q't}$

$$-k'(t)p_{q't}h'e^{-k'(t)\phi_{q't}h}AB + uq'(\phi,\phi') = 0,$$

$$uq'(\phi,\phi') = k'(t)p'_{q't}he^{-k'(t)\phi_{q't}h}AB,$$

by applying the same way we get

$$\phi'_{q't} = \frac{1}{k'(t)h}\left[\ln\left[\frac{k'(t)p'_{q't}}{\left[\left[\prod_{\substack{J=1,\\J\neq r}}^{m-1}k'(J)p'_{q'J}\right]^{1/k'(J)h}\right]^{\left[1/\sum_{\substack{J=1,\\J\neq r}}^{m-1}\frac{1}{k'(J)h}\right]}}\right] + [\varepsilon'_q(\phi,\phi')\frac{1}{\sum_{J=1}^{m}\frac{1}{k'(J)h}}\right],$$

where $k \le q \le N$, $J = 1,2,\cdots,m-1$, $J \neq r$, then

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$$\lim_{N\to\infty}\prod_{i=1}^{k-1}\sum_{\substack{J=1,\\ J\neq r}}^{m} p_{iJ}\exp\left[-k(J)\int\limits_{t+(k-1)h}^{t+kh}\frac{1}{k(J)h}\left[\ln\left[\frac{k(J)p_{iJ}}{\prod\limits_{J=1}^{m}k(J)p_{iJ}^{(1/k(J)h}}\right]^{\left[\sum\limits_{J=1}^{m}1/k(J)h\right]}\right.\right.$$

$$\left.\left.+[\varepsilon_{q'}(\phi,\phi')]\frac{1}{\sum\limits_{J=1}^{m}\frac{1}{k(J)h}}\right]dt$$

$$\prod_{i=k}^{N}\sum_{\substack{J=1,\\ J\neq r}}^{m-1} p'_{iJ}\exp\left[-k'(J)\int\limits_{t+(k-1)h}^{t+kh}\frac{1}{k'(J)h}\left[\ln\left[\frac{k'(J)p'_{iJ}}{\left[\left[\prod\limits_{\substack{J=1,\\ j\neq r}}^{m-1}k'(J)p'_{iJ}\right]^{1/k'(J)h\left[\sum\limits_{\substack{J=1,\\ j\neq r}}^{m-1}1/k'(J)h\right]}\right]}\right]^{\frac{1}{\sum\limits_{\substack{J=1,\\ J\neq r}}^{m-1}\frac{1}{k'(J)h}}}\right.\right.$$

$$\left.\left.+[\varepsilon_{q'}(\phi,\phi')]\frac{1}{\sum\limits_{\substack{J=1,\\ J\neq r}}^{m-1}\frac{1}{k'(J)h}}\right]dt.$$

$$\lim_{N\to\infty}\left[\left[\prod_{i=1}^{k-1}\sum_{\substack{J=1,\\ J\neq r}}^{m-1} p_{ij}\exp\left[-k(J)\int\limits_{t+(k-1)h}^{t+kh}\frac{1}{k(J)h}\ln\left[\frac{k(J)p_{iJ}}{\left[\left[\prod\limits_{J=1}^{n}k(j)p_{iJ}\right]^{1/k(J)h}\right]^{\left[1\Big/\sum\limits_{J=1}^{m}\frac{1}{k(J)h}\right]}}\right]\right.\right.\right.$$

$$\left.\left.\left.+\varepsilon_{q}(\phi,\phi')\frac{1}{\sum\limits_{J=1}^{m}\frac{1}{k(J)h}}\right]\right]dt$$

$$\left[\prod_{i=k}^{N}\sum_{J=1}^{m-1} p'_{iJ}\ \exp\left[-k'(J)\int_{t+(k-1)h}^{t+kh}\frac{1}{k(J)h}\ln\left[\frac{k'(J)p'_{iJ}}{\left[\left[\prod_{J=1}^{m-1}k'(J)p'_{iJ}\right]^{1/k'(J)h}\right]^{\left[1\Big/\sum_{\substack{J=1,\\ J\neq r}}^{m-1}\frac{1}{k(J)h}\right]}}\right]\right.\right.$$
$$\left.\left.+\varepsilon_{q'}(\phi,\phi')\left[\frac{1}{\sum_{\substack{J=1,\\ J\equiv r}}\frac{1}{k'(J)h}}\right]\right]\right]\ dt.$$

3. Conclusions

There exist some cases of truncation like truncated some states from discrete state space or continuous state space with discrete parameter or continuous parameter. This is future study.

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