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Central Metric Dimension of Rooted Product Graph

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Abstract The Central metric dimension is a type of metric dimension on graph. Some special graphs for which the central metric dimension have been found include path graph, cycle graph, complete graph, and complete bipartite graph. The aim of this study is to determine the central metric dimension of rooted product graph. Let G be a connected graph of order n and $\mathcal{H}$ is a sequence of n rooted graphs $H_1, H_2, H_3, \ldots, H_n$. The rooted product graph G and $\mathcal{H}$ denoted by $G \circ \mathcal{H}$. In this paper, we determine the central metric dimension of rooted product graph $G \circ \mathcal{H}$, which denoted by $dim_{cen}(G \circ \mathcal{H})$. The results obtained for $G \circ \mathcal{H}$, where $\mathcal{H}$ is a sequence of rooted graphs that all have the same radius and the rooted vertex is the central vertex. For $\mathcal{H}$ is a sequence of rooted cycle graph, the cycle with the largest radius has an impact on the central set, while the central metric dimension is affected by the central set of $G \circ \mathcal{H}$. For $\mathcal{H}$ is a sequence of rooted complete graph, the central set is affected by the central set of a graph G, while the central metric dimension is affected by the central set of the graph G.

Keywords radius, central set, central resolving set, rooted product graph, central metric dimension

Mathematics Subject Classification 2020: 05C12, 05C75, 05C38

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1. Introduction

Graph theory, as a field of study in mathematics, is experiencing rapid development. Mathematicians have discovered various new concepts and ideas, one of which is eccentricity in graphs. In Chartrand dan Lesniak (2000), the eccentricity of a vertex is introduced as the distance of that vertex to the furthest vertex on the graph. Then the largest eccentricity among the vertices on the graph is called the *diameter* and the smallest eccentricity among the vertices on the graph is called the *radius*. Thus, the central vertex is the vertex that has the smallest eccentricity in the graph, and the set of all central vertices is called the central set [1]. Different variants of eccentricity have been studied, ranging from average eccentricity for unweighted graphs [2, 3], to that for weighted graphs [4].

Another concept that has also been studied extensively is the concept of metric dimensions in graphs. According to Chartrand [1], a mathematician named Slater introduced the concept of a resolving set for a connected graph G under the term *locating set* where he referred to a minimum resolving set as a *reference set* for G and called the cardinality of a minimum resolving set the *location number* of G. On the other hand, Harary and Melter [5] were the first to use the term *metric dimension*, rather than location number. A subset W of G is called a resolving set

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if every vertex in G is uniquely determined by its distances to the vertices of W. The resolving set with minimum cardinality is called a basis and the cardinality of the basis is called the metric dimension of the graph [6, 5].

In the development of metric dimensions, several mathematicians have developed metric dimensions into several types. Sebo and Tannier introduced the concept of strong metric dimensions [7], Okamoto developed the metric dimension into local metric dimensions [8], Jannesari and Omoomi developed the concept of the neighbourhood metric dimension [3]. Further development of the concept of metric dimensions have also been carried out by combining the concept of resolving sets with other concepts. Brigham [9] introducing the combination of a resolving set with a dominating set called the resolving dominating set. The resolving dominating set with minimal cardinality is called the *dominant metric dimension*. Henning and Oellermann [10] combined the resolving set and the dominating set to form the *locating dominating number*. The combination of eccentricity through the central set with other graph invariants was independently introduced by Listiana et. al. [11]. Here, the ideas of the local metric dimension and the central set were combined to form the *central local metric dimension* of a graph. Susilowati et. al. [12] further extended the combination of resolving set and central set as the *central resolving set*, whose minimum cardinality is called the *central basis*. Surprisingly, none of these papers addressed the rooted product graph, first introduced in 1978 by Godsil and McKay [13]. Our aim in this paper is to cover that gap for the rooted product graphs for the central metric dimension.

This research presents the central metric dimension concept. The concept of the central metric dimension has a significant role in optimization, namely to determine strategic places as public service places. The meaning of strategic here is that the place has a relatively close distance to all places in the area, while being able to detect the condition of other places precisely.

To support this research, an initial concept is required as presented below.

Definition 1.1. [12] A vertex v is called a central vertex of graph G if vertex v has the same eccentricity as the radius of the graph G that is $e(v) = rad(G)$.
Definition 1.2. [12] Given a connected graph G. The central set of a graph G is the set whose elements are all central vertices and is denoted by $C(G)$.
Definition 1.3. [12] Let G be a connected graph. The Finite set $W \subseteq V(G)$ with $W \neq \emptyset$ is called the central of graph G if W is a resolving set and contains the central set. The central resolving set that has minimum cardinality is called the central basis and the cardinality of the central basis of G is called the *central metric dimension*, denoted as $dim_{cen}(G)$.
Lemma 1.4. [1] Let G be a connected graph of order $n \geq 2$, then

(i) $dim(G) = 1$ if only if $G = P_n$.

(ii) $dim(G) = n - 1$ if only if $G = K_n$.

(iii) For $n \geq 4$, $dim(G) = n - 2$ if only if $G = K_{r,s}$; $(r; s \geq 1)$,
$G = K_r + K_s$, $(r \geq 1; s \geq 2)$, or $G = K_r + (K_1 \cup K_s)$,$(r; s \geq 1)$.

(iv) For $n \geq 3$, $dim(C_n) = 2$.

Definition 1.5. [13] Let G be a connected graph of order n, $\mathcal{H}$ is a sequence of n rooted graphs $H_1, H_2, H_3, \ldots, H_n$ and $o_i \in V(H_i)$ is the rooted vertex of the graph H_i. Rooted product graph of graph G and sequence $\mathcal{H}$ denoted $G \circ \mathcal{H}$ is a graph obtained by identifying H_i to the i-th of the G.

The rooted product graph $G \circ \mathcal{H}$ is obtained as follows. Let G be a connected graph of order n, $\mathcal{H}$ is a sequence of n rooted graphs $H_1, H_2, H_3, \ldots, H_n$ and $o_i \in V(H_i)$ is the rooted vertex of the graph H_i. The vertex set of $G \circ \mathcal{H}$, $V(G \circ \mathcal{H}) = \{(a, v_{ij}) | a \in V(G), v_{ij} \in V(H_i), i = 1, 2, 3, \ldots, n, j = 1, 2, 3, \ldots, |V(H_i)|\}$ and $(a, v_{ij})(b, v_{kl}) \in E(G \circ \mathcal{H})$, with $a = b$ if only if $v_{ij}v_{kl} \in E(H_i)$ or $a \neq b$ if only if $ab \in E(G)$. In other words $ab \in E(G)$ if and only if $v_{ij} = o_i, v_{kl} = o_k, i \neq k$.

Based on above definition, Godsil and McKay [13] have presented the adjacency matrix of the rooted product graph and found the polynomials of the matrix. Research on root product graphs has been conducted by Rosyida et.al for the case of rooted graph sequences of isomorphic graphs, which are called comb products [14, 15].

2. Central Resolving Set of Rooted Product Graph

As a first step in this research, we determine the central set of the special graphs used in this research. The central set of some special graphs can be seen in the next observations.

a. The central set of the cycle graph is the vertex set of the cycle graph.

b. The central set of the complete graph is the vertex set of the complete graph.

c. The central vertex of the path graph P_n is the $(\frac{n+1}{2})$-th vertex for n odd and $(\frac{n}{2})$-th, $(\frac{n-2}{2})$-th vertices for n even.

d. The central set of the complete bipartite graph is the vertex set of complete bipartite graph.

Based on the definitions of several metric dimension concepts that have been discovered, the following general conclusions can be drawn.

a. The metric dimension of graph less than or equal to its central metric dimension.

b. The local metric dimension of graph less than or equal to its metric dimension.

c. The central local metric dimension of graph less than or equal to its central metric dimension.

The following describes the central set of the rooted product graph, followed by the central metric dimension of the rooted product graph.

Theorem 2.1. Let G be a connected graph of order n, $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$ is a sequence of rooted graphs, and $o_i \in V(H_i)$ is a rooted vertex of graph H_i. Suppose for every i imply o_i is a central vertex of graph H_i. If $rad(H_i) = rad(H_j), \forall i, j$ then central set of graph $G \circ \mathcal{H}$ is $C(G \circ \mathcal{H}) = \{(a_i, o_i) | a \in C(G)\}$.

Proof. Let $V(G) = \{a_i | i = 1, 2, 3, \ldots, n\}$, $V(H_i) = \{v_{ij} | j = 1, 2, 3, \ldots, |V(H_i)|\}$ and $o_i = v_{i1}$ is a rooted vertex and central vertex of graph H_i for every $i = 1, 2, 3, \ldots, n$. Let $C(G)$ is a central set of graphs G then for every $a \in C(G)$ imply $e(a_i) \geq e(b)$ for every $b \in V(G)$. Because for every i imply $o_i \in C(H_i)$, then $rad(H_i) = e(o_i) \leq e(v_{ij}), \forall v_{ij} \in V(H_i)$. Take any $a_i \in V(G)$, then it imply $e(a_i) \leq e(b), \forall b \in V(G)$. Moreover, since $o_i = v_{i1}$ is a central vertex of graph H_i, then for $(a_i, o_i) \in V(G \circ \mathcal{H}), e((a_i, o_i)) = e(a_i) + e(o_i) \leq e(a_i) + e(v_{ij}) \leq e(a_i, v_ij), \forall v_{ij} \in V(H_i)$. On the other side $e((a_i, o_i)) = e(a_i) + e(o_i) \leq e(b) + e(o_k) \leq e(b, v_{kj}), \forall v_{kj} \in V(H_k)$. So $e((a_i, o_i)) \leq e(b, v_{kj}), \forall (b, v_{kj}) \in C(G \circ \mathcal{H})$ and $C(G \circ \mathcal{H}) = \{(a_i, o_i) | a_i \in C(G)\}$. ■

Corollary 2.2. Let G be a connected graph of order n, $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$ is a sequence of rooted graphs, and $o_i \in V(H_i)$ is a rooted vertex of H_i. Suppose for every i, o_i is a central vertex of H_i. If $rad(H_i) = rad(H_j), \forall i, j$ then $rad(G \circ \mathcal{H}) = rad(G) + rad(H_i)$.

Proof. Let $V(G) = \{a_i | i = 1, 2, 3, \ldots, n\}$ and $V(H_i) = \{v_{ij} | j = 1, 2, 3, \ldots, |V(H_i)|\}$ and $o_i = v_{i1}$ is a rooted vertex and central vertex of graph H_i, for every $i = 1, 2, 3, \ldots, n$. Let $C(G)$ be a central set of a graph G. By Theorem 2.1, $C(G \circ \mathcal{H}) = \{(a_i, o_i) | a_i \in C(G)\}$, then for $\forall a_i \in C(G)$. Therefore $rad(G \circ \mathcal{H}) = e(a_i, o_i) = e(a_i) + e(o_i) = rad(G) + rad(H_i)$. ■

In Figure 1. given graph G and a sequence of rooted graph $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$, the central set of G is $C(G) = \{a_1\}$ and the central set of H_i is $C(H_i) = \{v_{i1}\}, \forall i$.

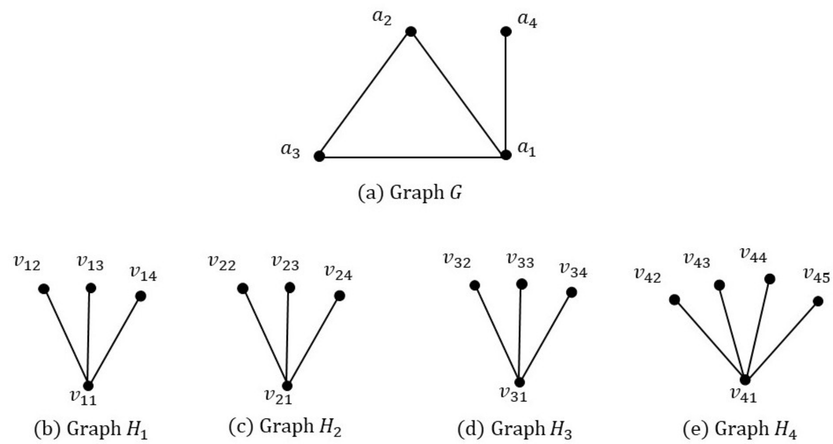

Figure 1. Graph G and sequence of rooted graph $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$

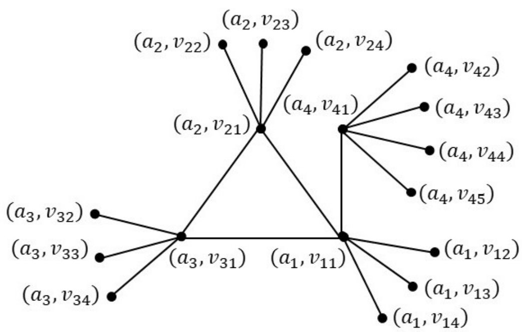

Figure 2. Graph $G \circ \mathcal{H}$

Figure 2 is rooted product graph $G \circ \mathcal{H}$ obtained from graph G and sequence $\mathcal{H}$ described in Figure 1. The rooted vertex is the central vertex of H_i for every i and thus, $rad(H_i) = rad(H_j) = 1, \forall i, j$. This condition satisfies Theorem 2.1, where $C(G \circ \mathcal{H}) = \{(a_1, v_{11})\}$ with $a_1 \in C(G)$.

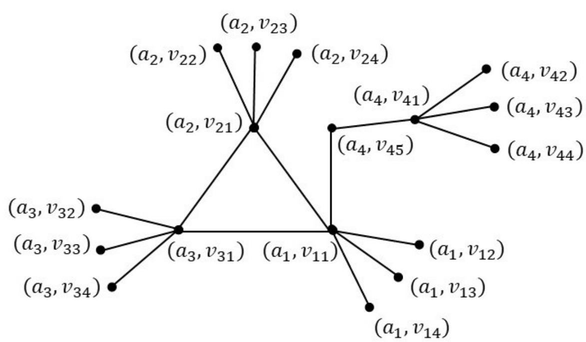

Figure 3. Graph $G \circ \mathcal{H}$

Figure 3 is an example of a rooted product graph $G \circ \mathcal{H}$ obtained from the graph on Figure 1, that doesn't fulfill the conditions of Theorem 2.1. Here, the rooted vertex of H_i isn't the central vertex of H_i, for i and $rad(H_i) = rad(H_j) = 1, \forall i, j$. Hence, $C(G \circ \mathcal{H}) = \{(a_1, v_{11}), (a_4, v_{45})\}$ with $a_4 \notin C(G)$.

Observation 2.3. Let G be a connected graph of order n, $\mathcal{H} = H_i, H_2, H_3, \ldots, H_n$ is a sequence of rooted graphs, and $o_i \in V(H_i)$ is a rooted vertex of graph H_i. If there are two distinct vertices $(a_i, v_{ij}), (a_i, v_{ik}) \in V(G \circ \mathcal{H})$ so that $d((a_i, v_{ij}), (a_i, v_{i1})) = d((a_i, v_{ik}), (a_i, v_{i1}))$ then $r((a_i, v_{ij})|V(G \circ \mathcal{H}) \backslash H_i') = r((a_i, v_{ik})|V(G \circ \mathcal{H}) \backslash H_i')$, where

$$H_i' = \{(a_i, v_{ij})|j = 2, 3, \ldots, |V(H_i')|\}.$$

The following presents the central set of rooted product graph $G \circ \mathcal{H}$, for case $G \approx K_n$ and $\mathcal{H}$ is a sequence of rooted cycle graph.

Lemma 2.4. Let G be a connected graph of order n, $\mathcal{H} = H_i, H_2, H_3, \ldots, H_n$ is a sequence of rooted graphs, and $o_i \in V(H_i)$ is a rooted vertex of graph H_i. Let $K_n' = \{(a_i, o_i)|a \in V(G), i = 1, 2, 3, \ldots, n\}$. If H_l for any l with the largest radius, then

$$C(G \circ \mathcal{H}) = \begin{cases} \{(a_i, o_l)\}, & \text{if } H_l \text{ exaclty one} \\ K_n', & \text{others.} \end{cases}$$

Proof. Let $G \approx K_n$, $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$ is a sequence of rooted cycle graphs and for every i imply $o_i \in V(H_i)$ is a rooted vertex of graph H_i. Let $K_n' = \{(a, o_i)|a \in V(G), i = 1, 2, 3, \ldots, n\}$.

1. Case 1: Suppose there is exactly one l, so that H_l is cycle with the largest radius, as a result $e(a_i, o_l) = rad(H_l)$. While $e(a_i, o_j) = 1 + rad(H_l)$ for every $j \neq l$. Then $C(G \circ \mathcal{H}) = \{(a_i, o_l)\}$.
2. Case 2: For another cases, suppose there are H_k and H_l two cycles with the largest radius $rad(H_k) = rad(H_l)$, so $e(a_i, o_i) = 1 + rad(H_k)$ for $i \neq k$ and $e(a_i, o_k) = 1 + rad(H_l)$. So that $e(a_i, o_i) = 1 + rad(H_l)$ for every $a_i \in K_n$. Then $C(G \circ \mathcal{H}) = K_n' = \{(a_i, o_i)|a \in V(G), i = 1, 2, 3, \ldots, n\}$. ■

Corollary 2.5. Let G be a connected graph of order n, $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$ is a sequence of rooted complete graphs or a sequence of rooted star graph with the rooted vertex is the central vertex, then $|C(G \circ \mathcal{H})| = |C(G)|$.

Proof. Let G be a connected graph of order n. Because $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$ is a sequence of the rooted complete graphs or a sequence of the rooted star graph with the rooted vertex being the central vertex, then Theorem 2.1 is satisfied such that $C(G \circ \mathcal{H}) = \{(a_i, o_i)|a_i \in C(G)\}$, then $|C(G \circ \mathcal{H})| = |C(G)|$. ■

Theorem 2.6. Let G be a connected graph of order n, $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$ is a sequence of rooted cycle graphs, then $dim_{cen}(G \circ \mathcal{H}) = |C(G \circ \mathcal{H})| + \sum_{i=1}^{n}(dim(H_i) - 1)$.

Proof. Let G be a connected graph of order n, $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$ is a sequence of rooted cycle graphs and $o_i \in V(H_i)$ is rooted vertex of graph H_i. Let $V(G) = \{a_i|i = 1, 2, \ldots, n\}$ and $C(G)$ be a central set of a graph G. Let $V(H_i) = \{v_{ij}|j = 1, 2, \ldots, |V(H_i)|\}$ and $o_i = v_{i1}$ be a rooted vertex of $\mathcal{H}_1, H_2, H_3, \ldots, H_n$. Because H_i is the cycle graph, then every vertex of H_i has similar property as root vertex. We consider two cases:

Case 1: There is exactly one H_l with the largest radius. Choose $W = (a_l, o_l) \cup \{(a_i, x)|(a_i, x)(a_i, o_i) \in E(G \circ \mathcal{H}), o_i \in V(H_i)\}$, then $|W| = 1 + n$. By Lemma 2.4, $\{(a_l, o_l)\} = C(G \circ \mathcal{H}) \subseteq W$, so that $W = |C(G \circ \mathcal{H})| + n$, and by Lemma 1.4 iv $W = |C(G \circ \mathcal{H})| + \sum_{i=1}^{n}(dim(H_i) - 1)$. Take any two distinct vertices $x, y \in V(G \circ \mathcal{H})$ so that there are three possibilities,

a. $x = (a_l, v_{lj}), y = (a_l, v_{lk})$ with $j \neq k \neq 1$, so that $d((a_l, v_{lj}), (a_l, x)) \neq d((a_l, v_{lk}), (a_l, x))$. Since $(a_l, x) \in W$, then $r(x|W) \neq r(y|W)$.

b. $x = (a_i, v_{i1}), y = (a_l, v_{l1})$ with $i \neq l$, so that $d((a_i, v_{i1}), (a_l, x)) \neq d((a_l, v_{l1}), (a_l, x))$. Since $(a_l, x) \in W$, then $r(x|W) \neq r(y|W)$.

c. $x = (a_i, v_{ij}), y = a_l, v_{lk})$ with $i \neq l$, so that $d((a_i, v_{ij}), (a_i, x)) \neq d((a_l, v_{lk}), (a_i, x))$. Since $(a_i, x) \in W$, then $r(x|W) \neq r(y|W)$.

Hence, W is a central resolving set of $G \circ \mathcal{H}$. we prove that W is a central resolving set with minimal cardinality.

Take any set $S < W$, with $C(G \circ \mathcal{H}) \subset S$, such that $(a_l, o_l) \in S$. As a result, there exists i such that for every j it holds $(a_i, v_{ij}) \notin S, i \neq l$. By Observation 2.3, there are two distinct vertices (a_i, v_{ij}) and (a_i, v_{ik}), where $(a_i, v_{ij})(a_i, v_{i1}) \in E(G \circ \mathcal{H})$ and $(a_i, v_{ik})(a_i, v_{i1}) \in E(G \circ \mathcal{H})$ so that $r((a_i, v_{ij})|S) = r((a_i, v_{ik})|S)$. So W is a central basis of graph $G \circ \mathcal{H}$.

Case 2: There are H_k and H_l with the largest radius. Choose $W = \{(a_i, o_i)|(a_i, o_i) \in V(G)\} \cup \{(a_i, x)|(a_i, x)(a_i, o_i) \in E(G \circ \mathcal{H}), o_i \in V(H_i)\}$, so that $|W| = n + n$. By Lemma 2.4, $\{(a_i, o_i)|(a_i, o_i) \in V(G)\} = C(G \circ \mathcal{H}) \subseteq W$ so that $W = |C(G \circ \mathcal{H})| + n$, and by Lemma 1.4 iv $W = |C(G \circ \mathcal{H}| + \sum_{i=1}^{n}(dim(H_i) - 1)$. Take any two distinct vertices $x, y \in V(G \circ \mathcal{H})$ so there are three possibilities,

a. $x = (a_l, v_{lj}), y = (a_l, v_{lk})$ with $j \neq k \neq 1$, so that $d((a_l, v_{lj}), (a_l, x)) \neq d((a_l, v_{lk}), (a_l, x))$, implying $r(x|W) \neq r(y|W)$.

b. $x = (a_i, v_{i1}), y = (a_l, v_{l1})$ with $i \neq l$, so that $d((a_i, v_{i1}), (a_l, x)) \neq d((a_l, v_{l1}), (a_l, x))$, implying $r(x|W) \neq r(y|W)$.

c. $x = (a_i, v_{ij}), y = (a_l, v_{lk})$ with $i \neq l$, so that $d((a_i, v_{ij}), (a_i, x)) \neq d((a_l, v_{lk}), (a_i, x))$. Since $(a_i, x) \in W, r(x|W) \neq r(y|W)$.

So W is a central resolving set of $G \circ \mathcal{H}$. Next, we prove that W is a central resolving set with minimal cardinality. Take any set $S < W$, with $C(G \circ \mathcal{H}) \subset S$, satisfying $(a_l, o_l) \in S$. As a result, there exists i such that for every j, it follows that $(a_i, v_{ij}) \in S, i \neq l$. By Observation 2.3, there are two distinct (a_i, v_{ij}) and (a_i, v_{ik}), where $(a_i, v_{ij})(a_i, v_{i1}) \in E(G \circ \mathcal{H})$ and $(a_i, v_{ik})(a_i, v_{i1}) \in E(G \circ \mathcal{H})$ so that $r((a_i, v_{ij})|S) = r((a_i, v_{ik})|S)$. Therefore, W is a central basis of graph $G \circ \mathcal{H}$. ∎

The following is an example of a case that satisfies Theorem 2.6. Suppose G is a complete graph of order 4 and $\mathcal{H}$ is a rooted cycle graph $\mathcal{H} = H_1, H_2, H_3, H_4$ as in Figure 4 and rooted product graph $G \circ \mathcal{H}$ is shown in Figure 5. Here, there is exactly one H_4 with the largest radius. Suppose the basis of H_1, H_2, H_3, H_4 sequential are B_1, B_2, B_3, B_4, respectively. It is easy to see from the figures that $C(G) = \{a_1, a_2, a_3, a_4\}, B_1 = \{v_{11}, v_{12}\}, B_2 = \{v_{21}, v_{22}\}, B_3 = \{v_{31}, v_{32}\}$, and $B_4 = \{v_{41}, v_{42}\}$.

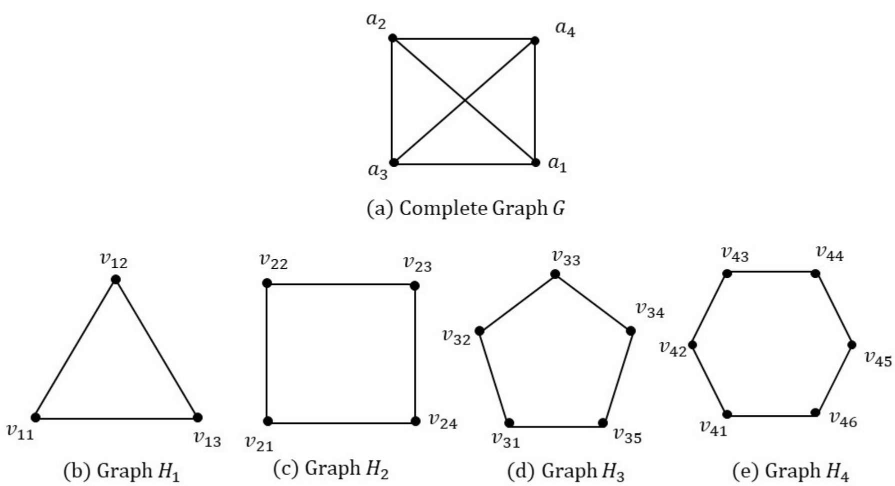

Figure 4. Graph G and Sequence $\mathcal{H}$

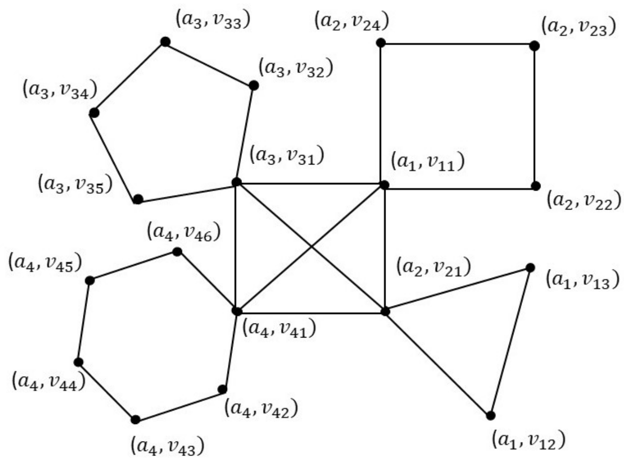

Figure 5. Graph $G \circ \mathcal{H}$

Figure 5 is an example that shows the central set of $G \circ \mathcal{H}$ is $C(G \circ \mathcal{H}) = \{(a_1, v_{11}), (a_2, v_{21}), (a_3, v_{31}), (a_4, v_{41})\}$. Graph H_4 is a cycle graph with the largest radius of $\mathcal{H}$, and so we choose $W = \{(a_4, v_{41})\} \cup \{(a_1, v_{12}), (a_2, v_{22}), (a_3, v_{32}), (a_4, v_{42})\}$. As a result $C(G \circ \mathcal{H}) \subseteq W$. Since the set W involves the basis of all graphs in sequence $\mathcal{H}$, then it can be easily shown that W is a basis containing the central set of $G \circ \mathcal{H}$ and $|W| = |C(G \circ \mathcal{H})| + \sum_{i=1}^{4}(dim(H_i) - 1) = 1 + 4 = 5$.

Theorem 2.7. Let G be a connected graph of order n, $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$a sequence of rooted complete graph, then $dim_{cen}(G \circ \mathcal{H}) = |C(G)| + \displaystyle\sum_{i=1}^{n}(dim(H_i) - 1)$.

Proof. Let G be a connected graph of order n, $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$ a sequence of rooted complete graph and $o_i \in V(H_i)$ is a rooted vertex of H_i. Let $V(G) = \{a_i | i = 1, 2, \ldots, n\}$ and $C(G)$ is a central set of graph G. Let $V(H_i) = \{v_{ij} | j = 1, 2, \ldots, |V(H_i)|\}$ with $o_i = v_{i1}$. Because H_i is complete graph, then every vertex of H_i has similar property as root vertex. Let B_i be a basis of H_i for every i, by Lemma 1.4 (ii), $B_i = \{v_{ij} | j = 1, 3, 4, 5 \ldots, |V(H_i)|\}$ for $i = 1, 2, \ldots, n$. In other words $dim(H_i) = |V(H_i)| - 1$, for every i. Choose $W = \{(a_i, v_{i1}) | (a_i, v_{i1}) \in C(G \circ \mathcal{H})\} \cup \{(a_i, v_{il}) | a_i \in V(G), v_{il} \in B_i, v_{il} \neq v_{i1}\}$. As a result W contains $C(G \circ \mathcal{H})$ and by Corollary 2.5 $|C(G \circ \mathcal{H})| = |C(G)|$, so that $|W| = |C(G \circ \mathcal{H})| + \sum_{i=1}^{n}(dim(H_i) - 1) = |C(G)| + \sum_{i=1}^{n}(dim(H_i) - 1)$. Next, we prove that W is a resolving set of graph $G \circ \mathcal{H}$. Take any two distinct vertices $x, y \in V(G \circ \mathcal{H}) \backslash W$, we consider the three possibilities below.

a. $x = (a_i, v_{ik})$ and $y = (a_i, v_{il})$, where $(a_i, v_{i1}) \in C(G \circ \mathcal{H})$. As a result, there is exactly one vertex $(a_i, v_{i2}) \notin W$, so that $r((a_i, v_{i2}) | W) \neq r((a_i, v_{ij}) | W), \forall j$. Then $r(x|W) \neq r(y|W)$.

b. $x = (a_i, v_{ik})$ and $y = (a_i, v_{il})$, where $(a_i, v_{i1}) \notin C(G \circ \mathcal{H})$. As a result, there are exactly two vertices $(a_i, v_{i1}), (a_i, v_{i2}) \in W$, so that $r((a_i, v_{i1}) | W) \neq r((a_i, v_{i2}) | W)$ and $(x|W) \neq r(y|W)$.

c. $x = (a_i, v_{ik})$ and $y = (a_j, v_{jl})$, with $i \neq j$, so that $d((a_i, v_{ik}), (a_i, v_{i3})) \neq d((a_j, v_{jl}), (a_i, v_{i3}))$. Since $(a_i, v_{i3}) \in W$, then $r((a_i, v_{ik}) | W) \neq r((a_j, v_{jl}) | W)$.

Hence W is a resolving set of the graph $G \circ \mathcal{H}$. Moreover, since W contains central set of $G \circ \mathcal{H}$ then W is a central resolving set of the graph $G \circ \mathcal{H}$. Next, we prove that W is a central resolving set with minimum cardinality.

Take any set $S \subseteq V(G \circ \mathcal{H})$ with $|S| < |W|$ and $C(G \circ \mathcal{H}) \subseteq S$. By Theorem 2.1 $C(G \circ \mathcal{H}) = \{(a, o_i) | a \in C(G)\}$. Since $|S| < |W|$, there are two vertices $(a_i, v_{ik}), (a_i, v_{il}) \in V(G \circ \mathcal{H})$, with $k \neq l \neq 1$, so that $d((a_i, v_{ik}), (a_i, v_{i1})) = d((a_i, v_{il}), (a_i, v_{i1}))$. By Observation 2.3, $r((a_i, v_{ik}) | W) = r((a_i, v_{il}) | W)$. Therefore, S is

not central resolving set and W is a central basis of $G \circ \mathcal{H}$. Then $dim_{cen}(G \circ \mathcal{H}) = |C(G)| + \sum_{i=1}^{n}(dim(H_i) - 1)$.
■

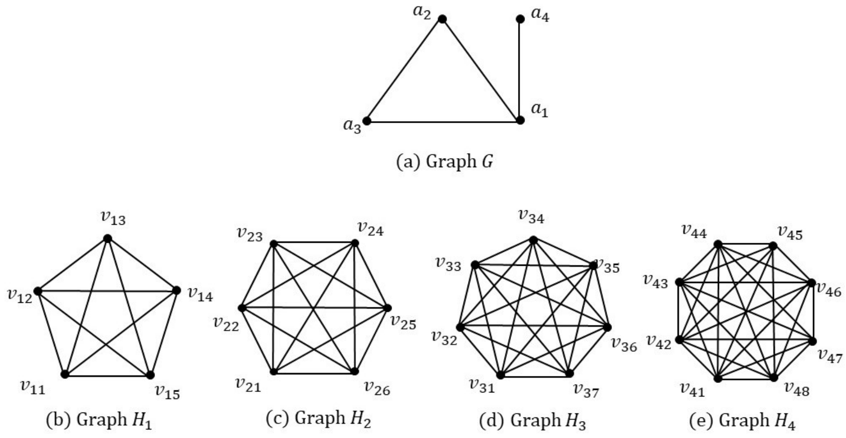

Figure 6. Graph G and sequence $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$

The following is an example of a case that satisfies Theorem 2.7. Let G be a connected graph of order n as in Figure 6 and $\mathcal{H}$ a sequence of rooted complete graphs $\mathcal{H} = H_1, H_2, H_3, H_4$. The rooted product graph $G \circ \mathcal{H}$ presented in Figure 7 is obtained from G and $\mathcal{H}$. Suppose the basis of graph H_1, H_2, H_3, H_4 sequentially are B_1, B_2, B_3, B_4. It is easy to see from the figure that $C(G) = \{a_1\}, B_1 = \{v_{1j}|j = 1, 2, \ldots, 4\}, B_2 = \{v_{2j}|j = 1, 2, \ldots, 5\}, B_3 = \{v_{3j}|j = 1, 2, \ldots, 6\}, B_4 = \{v_{4j}|j = 1, 2, \ldots, 7\}$.

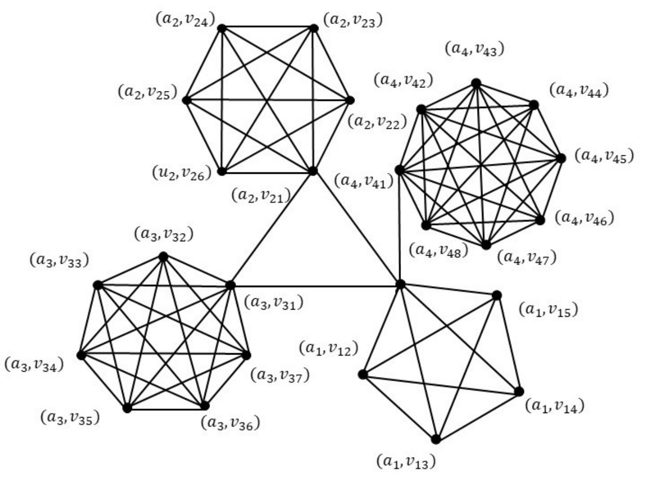

Figure 7. Graph $G \circ \mathcal{H}$

Figure 7 shows that the central set of $G \circ \mathcal{H}$ is $C(G \circ \mathcal{H}) = \{(a_1, v_{11})\}$. Notice that $W = \{(a_1, v_{11})\} \cup \{(a_1, x)|x \in B_1 \backslash \{v_{11}\}\} \cup \{(a_2, x)|x \in B_2 \backslash \{v_{21}\}\} \cup \{(a_3, x)|x \in B_3 \backslash \{v_{31}\}\} \cup \{(a_4, x)|x \in B_4 \backslash \{v_{41}\}\}$. It follows that $C(G \circ \mathcal{H}) \subseteq W$. In such a case, set W involves the basis of all graphs in sequence $\mathcal{H}$.Thus, it can be easily shown that W is a basis containing the central set of $G \circ \mathcal{H}$ and $|W| = |C(G \circ \mathcal{H})| + |B_1 \cup B_2 \cup B_3 \cup B_4| = |C(G \circ \mathcal{H})| + \sum_{i=1}^{4}(dim(H_i) - 1) = 20$.

Theorem 2.8. Let G be a connected graph of order n, $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$ is a sequence of rooted star graph, and $o_i \in V(H_i)$ is a rooted vertex of graph H_i, then

$$dim_{cen}(G \circ \mathcal{H}) = \begin{cases} |C(G)| + \sum_{i=1}^{n} dim(H_i), & \text{if } o_i \text{ is not basis element of } H_i \text{ for every } i, \\ |C(G)| + \sum_{i=1}^{n} (dim(H_i) - 1), & \text{if } o_i \text{ is basis element of } H_i \text{ for every } i. \end{cases}$$

Proof. Let G is a connected graph of order n, $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$ is a sequence of rooted star graph, and $o_i \in V(H_i)$ is a rooted vertex of graph H_i. Let $V(G) = \{a_i | i = 1, 2, \ldots, n\}$ and $C(G)$ is a central set of graph G. Suppose for every i occur $V(H_i) = \{v_{ij} | j = 1, 2, \ldots, |V(H_i)|\}$, with v_{i1} is a vertex of degree $|V(H_i)| - 1$. Let B_i be a basis of H_i for every i, by Lemma 1.4 (iii), $dim(H_i) = |V(H_i)| - 2, B_i = \{v_{ij} | j = 3, \ldots, |V(H_i)|\}$ for $i = 1, 2, \ldots, n$. Two cases are considered.

1. The rooted vertex o_i is not a basis element of H_i for every i, that is $o_i = v_{i1}$, choose $W = \{(a_i, v_{i1}) | (a_i, v_{i1}) \in C(G \circ \mathcal{H})\} \cup \{(a_i, v_{il}) | a_i \in V(G), v_{il} \in B_i, i = 1, 2, 3, \ldots, n\}$. As a result W contains $C(G \circ \mathcal{H})$ and by Corollary 2.5, $|C(G \circ \mathcal{H})| = |C(G)|$, so that $|W| = |C(G \circ \mathcal{H})| + \sum_{i=1}^{n} dim(H_i) = |C(G)| + \sum_{i=1}^{n} dim(H_i)$.

   Take any two distinct vertices $x, y \in V(G \circ \mathcal{H}) \backslash W$, we consider the three possibilities below.

   (a) $x = (a_i, v_{i1})$ and $y = (a_i, v_{j1})$, where $x, y \notin C(G \circ \mathcal{H})$. We get $r((a_i, v_{i2})|W) \neq r((a_i, v_{ij})|W), \forall j$. Then $r(x|W) \neq r(y|W)$.

   (b) $x = (a_i, v_{ik})$ and $y = (a_i, v_{il})$, where $(a_i, v_{i1}) \in C(G \circ \mathcal{H})$. As a result, there is exactly one vertex $(a_i, v_{i2}) \notin W$, so that $r((a_i, v_{i2})|W) \neq r((a_i, v_{ij})|W), \forall j$. Then $r(x|W) \neq r(y|W)$.

   (c) $x = (a_i, v_{ik})$ and $y = (a_i, v_{il})$, where $(a_i, v_{i1}) \notin C(G \circ \mathcal{H})$. As a result, there are exactly two vertices $(a_i, v_{i1}), (a_i, v_{i2}) \notin W$, so that $r((a_i, v_{i1})|W) \neq r((a_i, v_{i2})|W)$ and $(x|W) \neq r(y|W)$.

   We can conclude that W is central resolving set of $G \circ \mathcal{H}$. Take any set $S \subseteq V(G \circ \mathcal{H})$ with $|S| < |W|$ and $C(G \circ \mathcal{H}) \subseteq S$. By Corollary 2.5 $C(G \circ \mathcal{H}) = \{(a, o_i) | a \in C(G)\}$. Since $|S| < |W|$, there are two vertices $(a_i, v_{ik}), (a_i, v_{il}) \in V(G \circ \mathcal{H})$, with $k \neq l \neq 1$, so that $d((a_i, v_{ik}), (a_i, v_{i1})) = d((a_i, v_{il}), (a_i, v_{i1}))$. By Observation 2.3, $r((a_i, v_{ik})|W) = r((a_i, v_{il})|W)$. Therefore, S is not central resolving set and W is a central basis of $G \circ \mathcal{H}$. Then $dim_{cen} G \circ \mathcal{H}) = |C(G)| + \sum_{i=1}^{n} dim(H_i)$.

2. The rooted vertex o_i is basis element H_i for every i, that is $o_i = v_{i2} \neq v_{i1}$, choose $W = \{(a_i, v_{i1}) | (a_i, v_{i1}) \in C(G \circ \mathcal{H})\} \cup \{(a_i, v_{il}) | a_i \in V(G), l = 4, 5, 6, \ldots, |V(H_i)|, i = 1, 2, 3, \ldots, n\}$. As a result W contains $C(G \circ \mathcal{H})$ and by Corollary 2.5, $|C(G \circ \mathcal{H})| = |C(G)|$ and by Lemma 1.4 (iii) $|W| = |C(G)| + \sum_{i=1}^{n} (dim(H_i) - 1)$.

   Take any two distinct vertices $x, y \in V(G \circ \mathcal{H}) \backslash W$, we consider the three possibilities below.

   (a) $x = (a_i, v_{i1})$ and $y = (a_i, v_{j1})$, where $x, y \notin C(G \circ \mathcal{H})$. We get $r((a_i, v_{i2})|W) \neq r((a_i, v_{ij})|W), \forall j$. Then $r(x|W) \neq r(y|W)$.

   (b) $x = (a_i, v_{ik})$ and $y = (a_i, v_{il})$, where $(a_i, v_{i1}) \in C(G \circ \mathcal{H})$. As a result, there is exactly one vertex $(a_i, v_{i2}) \notin W$, so that $r((a_i, v_{i2})|W) \neq r((a_i, v_{ij})|W), \forall j$. Then $r(x|W) \neq r(y|W)$.

   (c) $x = (a_i, v_{ik})$ and $y = (a_i, v_{il})$, where $(a_i, v_{i1}) \notin C(G \circ \mathcal{H})$. As a result, there are exactly two vertices $(a_i, v_{i1}), (a_i, v_{i2}) \notin W$, so that $r((a_i, v_{i1})|W) \neq r((a_i, v_{i2})|W)$ and $(x|W) \neq r(y|W)$.

   We can conclude that W is central resolving set of $G \circ \mathcal{H}$. Take any set $S \subseteq V(G \circ \mathcal{H})$ with $|S| < |W|$ and $C(G \circ \mathcal{H}) \subseteq S$. By Corollary 2.5 $C(G \circ \mathcal{H}) = \{(a, o_i) | a \in C(G)\}$. Since $|S| < |W|$, there are two vertices $(a_i, v_{ik}), (a_i, v_{il}) \in V(G \circ \mathcal{H})$, with $k \neq l \neq 1$, so that $d((a_i, v_{ik}), (a_i, v_{i1})) = d((a_i, v_{il}), (a_i, v_{i1}))$.

By Observation 2.3, $r((a_i, v_{ik})|W) = r((a_i, v_{il})|W)$. Therefore, S is not central resolving set and W is a central basis of $G \circ \mathcal{H}$. Then $dim_{cen}(G \circ \mathcal{H}) = |C(G)| + \sum_{i=1}^{n}(dim(H_i) - 1)$.■

Based on Theorem 2.6, 2.7 and 2.8, we present the conjecture as below.

Conjecture 2.9. Let G be a connected graph of order n, $\mathcal{H} = H_1, H_2, H_3, \ldots, H_n$ is a sequence of rooted graphs, then

$$dim_{cen}(G \circ \mathcal{H}) = \begin{cases} |C(G \circ \mathcal{H})| + \sum_{i=1}^{n} dim(H_i), & \text{if } o_i \text{ is not basis element of } H_i \text{ for every } i, \\ |C(G \circ \mathcal{H})| + \sum_{i=1}^{n}(dim(H_i) - 1), & \text{if } o_i \text{ is basis element of } H_i \text{ for every } i. \end{cases}$$

3. Conclusion and Suggestions

The results found in this study are the central metric dimensions of rooted product graph $G \circ \mathcal{H}$ with $\mathcal{H}$ as a sequence of complete, cycle, or star graph. The value of the central metric dimensions of a rooted product graph $G \circ \mathcal{H}$ depends on whether the root point of the graph H_i is a central vertex of H_i or is not a central vertex of H_i. From these three results, we present a conjecture for a more general case, namely for $\mathcal{H}$ sequences of any graph. This study provides an opportunity for further study to prove the conjecture. A further interesting thing that can be done is the application of the central metric dimension concept to the problem of developing strategic public service places in urban planning mapping.

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REFERENCES

1. G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, *Resolvability in graphs and the metric dimension of a graph*, Discrete Appl. Math. 105, 99–113, 2000.
2. P. Dankelmann, W. Goddard, and C. S. Swart, *The average eccentricity of a graph and its subgraphs*, Utilitas Mathematica. 65, 41 – 51, 2004.
3. M. Jannesari, and B. Omoomi, *The Metric Dimension of the Lexicographic Product of Graphs*, Discrete Mathematics, 312, 3349-3356, 2012.
4. P. Johnson, and F. J. Osaye, *On edge-weighted mean eccentricity of graphs*, International Journal of Mathematics and Computer Science. 16(4), 1325 - 1339, 2021.
5. F. Harary, and R. A. Melter, *On the metric dimension of a graph*, Ars Combinatoria, 2, 191 -195, 1976.
6. G. Chartrand, and L. Lesniak, *Graph and Digraph, Third Edition*, Chapman & Hall/CRC, Florida, 2000.
7. A. Sebo, and E. Tannier, *On Metric Generator of Graphs*, Mathematics of Operations Research, 29:2, 383-393, 2004.
8. F. Okamoto, L. Crosse, B. Phinezy, and Kalamazoo, *The Lokal Metric Dimension of a Graph*, Mathematica Bohemica, 135:3, 239-255, 2010.
9. R. C. Brigham, G. Chartrand, R. D. Dutton, and P. Zhang, *Resolving Domination in Graphs*, Mathematica Bohemica, 128(1): 25–36, 2003.
10. M. A. Henning, and O. R. Oellermann, *Metric-Locating-Dominating Sets in Graphs*, Ars Combinatoria, Vol. 73, 2004.
11. Y. Listiana, L. Susilowati, S. Slamin, and F. J. Osaye, *A central local metric dimension on acyclic and grid graphs*, AIMS Mathematics, 8(9), 21298 - 21311, 2023.
12. L. Susilowati, U. Maulida, S. Slamin, N. Estuningsih, S. Zahidah, S. Prabhu, *On the Central Resolver set of the Edge Coronation Graphs*, Journal of Discrete Mathematical Sciences and Cryptography, Accepted in Press, 2024.
13. C. D. Godsil, and B. D. Mckay, *A new graph and its spectrum*, Bulletin of the Australian Mathematical Society, 18, 21 – 28, 1978.
14. N. M. Rosyidah, S. Zahidah, U. D. Purwati, and L. Susilowati, *On comb product graphs with respect to the complement metric dimension*, AIP Conference Proceedings 2329, 020006, 2021.

15. B. J. Septory, L. Susilowati, D. Dafik, and M. Venkatachalam, *On the study of rainbow antimagic connection number of comb product of tree and complete bipartite graph*, hms and Applications, 2350098 (18 pages), 2023.