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Inferential study on lifetime performance index with the generalized inverted exponential model under progressive first-failure censoring

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Abstract Lifetime performance assessment is widely used in quality control of the manufacturing industry. This paper focuses on the progressively first-failure-censored data coming from the generalized inverted exponential distribution. We present the maximum likelihood (ML) estimate and the Bayesian estimate for the lifetime performance index (C_L) for a given lower specification level L. The results are used to develop non-Bayesian and Bayesian inferences to determine whether the product performance meets the required level. A Monte Carlo simulation and two real data examples are discussed for illustration purposes.

Keywords Generalized inverted exponential distribution; Maximum likelihood estimate; Bayesian inference; Monte Carlo simulation

AMS 2010 subject classifications 62N02

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1. Introduction

Process capability analysis is a set of tools utilized to assess how well a given process meets the specified limits. In manufacturing, process capability is evaluated using product survival lifetime, as a longer lifetime signifies higher product quality. [1] proposed using the lifetime performance index C_L for assessing the lifetime performance of electronic components, where L is the lower specification limit. It is known that failure time data follows a non-normal distribution, such as Exponential, Gamma, or Weibull distributions. Several references provide an overview of lifetime data models, including [2], [3], [4], [6], [7] and [8] . Recently, [9] proposed a generalization of the censored normal regression based on the extended normal distribution with two additional shape parameters as a and b. It provides flexible estimation in cases symmetric-asymmetric data and especially in case non-normal asymmetric data. [10] adopted two parameters generalized half-logistic lifetime model in the generalized Type I hybrid censoring scheme.

This article specifically considers the Generalized Inverted Exponential Distribution (GIED) that is a generalization of the Inverted Exponential Distribution (IED), developed to address limitations in modeling more complex failure behaviors in reliability and survival analysis. While the exponential distribution is commonly used due to its mathematical simplicity and assumption of a constant failure rate, this assumption often does not hold in real-world scenarios, especially for systems like electronic devices or mechanical components, where the failure rate tends to decrease over time. The Inverted Exponential Distribution (IED) is better suited for such cases, as it naturally captures a decreasing failure rate. [11] provided the estimation of the parameters and reliability function for IED under maximum likelihood method under complete data. The parameters as well as the risk function of IED were further estimated with Bayes method in [12]. However, the IED is limited when data exhibits more complex

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failure patterns , such as an inverted bathtub-shaped hazard rate, where the failure rate initially increases and then decreases. To address this, [13] introduced GIED by adding a shape parameter β, which allows for more flexibility in modeling different types of failure rates. Depending on the value of β, the GIED can accommodate increasing, decreasing, or even non-monotonic failure rates, making it more versatile in capturing various real-world lifetime behaviors. This generalization makes the GIED a powerful tool for modeling complex lifetime data across different industries and applications. [14] derived statistical and reliability properties of GIED by fitting GIED to some sets of reliability data and showed that GIED can be used effectively, in place of other existing models to fit lifetime data. For more details on applications of the GIED, (see [15, 16, 17, 18]).

In life-testing experiments, it can be difficult for experimenters to observe the lifetimes of all products placed on the test due to time or other resource restrictions. Hence, the procedure of censoring is usually used in survival analysis. There are many types of censorship schemes, but the most common schemes are Type-I and Type-II censoring. In Type I censoring, all items n are put on a test and the test is terminated after a pre-specified test time. In Type II censoring, the test is terminated exactly when a pre-specified $m(< n)$ units fail. A drawback of Type-I and Type-II censoring schemes is that they do not allow the removal of products before the experiment is complete. [19] proposed progressive censoring that allows removals of units from the life test at various stages during the experiment. Although progressive censoring can improve the experimental efficiency, the duration of the test is still too long when products have a long life expectancy. It was Johnson [20] who introduced the concept of the first failure censoring scheme. [21] combined the concepts of first-failure censoring and progressive censoring to develop a new life-test plan called the progressive first-failure censoring scheme. This scheme can be described as follows: suppose that n independent batches with k units within each batch are put on a test and the progressive censoring scheme $\mathbf{R} = (R_1, R_2, \cdots, R_m)$ is pre-fixed; when the fist failure $x^{R_1}_{1:m:n:k}$ occurs, R_1 random selected batches, as well as the batch including the first failure, are removed; when the second failure $x^{R_2}_{2:m:n:k}$ occurs, R_2 random selected batches, as well as the batch including the second failure, are removed, and so on; finally, when the m^{th} failure $x^{R_m}_{m:m:n:k}$ occurs, the remaining batches are removed, and the test is terminated. Then, the observed failure times $x^{R_1}_{1:m:n:k} < x^{R_2}_{2:m:n:k} < \cdots < x^{R_m}_{m:m:n:k}$ are called a progressive first failure censoring order statistics with pre-specified constants m and $\mathbf{R} = (R_1, R_2, \cdots, R_m)$.

The statistical analysis of the lifetime performance index has been extensively explored in the literature, considering various censoring schemes and distributions. For instance, [22] derived the ML estimate of C_L by employing data transformation techniques under the Pareto distribution with a Type-II censored sample and then they further utilized the ML estimate to develop a hypothesis testing procedure. Additionally, [23] discussed statistical inference for C_L on the basis of the progressive first-failure censored data from the Weibull distribution with known shape parameters. [24] obtained different estimators of C_L under the Compound Rayleigh distribution with progressively first-failure censored samples. Recently, [25] investigated both Bayesian and non-Bayesian analysis of C_L under the Pareto distribution with progressively first-failure censored samples. In this paper, we focus on the progressively first-failure censoring scheme. We apply data transformation techniques to the generalized inverted exponential distribution to obtain the ML estimate of C_L and additionally, the Bayesian estimate of C_L. These estimates are then used to investigate non-Bayesian and Bayesian inference for C_L, enabling us to determine whether the product performance meets the required level.

The paper is structured as follows. Section 2 introduces the lifetime performance index C_L and the conforming rate. Section 3 focuses on the development of ML estimates. In Section 4, we present the Bayes estimates for C_L. Section 5 discusses interval estimation, including the lower confidence interval and lower credible bound. Section 6 is dedicated to hypothesis testing. To illustrate the proposed methods, Section 7 provides two practical examples. In Section 8, we conduct Monte Carlo simulations to examine the performance of the ML estimate and Bayes estimate. Finally, the paper concludes in Section 9.

2. Lifetime Performance Index

Let X be the lifetime of products. The lifetime performance index C_L of X, denoted by C_{L_X} is defined as [1]

$$C_{L_X} = \frac{\mu_X - L_X}{\sigma_X}, \tag{1}$$

where μ_X and σ_X are, respectively, the mean and standard deviation of lifetime products, and L_X is a lower specification limit. The lifetime is generally required to exceed L_X unit times to be both financially profitable and satisfying customers. Note that L_X is determined according to the customer's requirements and it specifies the minimum acceptable lifetime of a product.

Suppose that the lifetime of products, X may be modeled by a GIED with shape and scale parameters β and λ respectively and denoted by GIED (β, λ). Then probability density function (pdf) of X is given by

$$f_X(x;\beta,\lambda) = \frac{\beta\lambda}{x^2} e^{-\frac{\lambda}{x}}(1 - e^{-\frac{\lambda}{x}})^{\beta-1}, \;\; x > 0, \; \beta > 0, \; \lambda > 0. \tag{2}$$

The corresponding cumulative distribution function (cdf) is given by

$$F_X(x;\beta,\lambda) = 1 - (1 - e^{-\frac{\lambda}{x}})^{\beta}, \;\; x > 0, \; \beta > 0, \; \lambda > 0. \tag{3}$$

Since we cannot obtain the mean and variance of GIED in closed form, we use the data transformation technique defining a new variable Y as $Y = -\ln(1 - e^{-\frac{\lambda}{x}})$ for a given scale parameter λ. The distribution of Y becomes an exponential distribution with one parameter β. The pdf and cdf of Y respectively are,

$$f_Y(y;\beta) = \beta e^{-\beta y}, \;\; y > 0, \; \beta > 0, \tag{4}$$

and

$$F_Y(y;\beta) = 1 - e^{-\beta y}, \;\; y > 0, \; \beta > 0. \tag{5}$$

The mean and the variance of y are $\frac{1}{\beta}$ and $\frac{1}{\beta^2}$, respectively. Then the corresponding lifetime performance index C_{L_Y} for Y is

$$C_{L_Y} = \frac{\mu_Y - L_Y}{\sigma_Y} = \frac{\frac{1}{\beta} - L_Y}{\frac{1}{\beta}} = 1 - \beta L_Y, \tag{6}$$

where $L_Y = -\ln(1 - e^{-\frac{\lambda}{L_X}})$ and L_X be the lower specification limit of X. The failure rate function $r(t)$ is defined by:

$$r(t) = \frac{f_Y(t)}{1 - F_Y(t)} = \beta, \;\; \beta > 0. \tag{7}$$

From equations (6) and (7), we can see that C_{L_Y} is decreasing in β while the failure rate is increasing in β. This property motivates that C_{L_Y} reasonably and accurately represents the lifetime performance of products.

If the lifetime of a product X, transformed as $Y = -\ln(1 - e^{-\frac{\lambda}{X}})$, exceeds the lower specification limit L_Y, then the product is considered as a conforming product. Therefore, the conforming rate, which is the ratio of conforming products, can be defined based on the random variable Y as follows:

$$P_r = P(Y \geq L_Y) = \int_{L_Y}^{\infty} \beta e^{-\beta Y} dy = e^{-\beta L_Y} = e^{(C_{L_Y} - 1)}, \;\; -\infty < C_{L_Y} < 1. \tag{8}$$

The conforming rate and C_{L_Y} are in a strictly increasing relationship. The values of C_{L_Y} and the corresponding conforming rates P_r are listed in Table (1).

3. Maximum likelihood estimation

The maximum likelihood (ML) estimation method is based on empirical information provided by the data. In this section, we obtain the ML estimate of the C_{L_Y} of the GIED(β, λ) model based on progressively first-failure censored samples.

Table 1. The lifetime performance index C_{L_Y} versus the conforming rate P_r

C_{L_Y}	P_r	C_{L_Y}	P_r	C_{L_Y}	P_r	C_{L_Y}	P_r
$-\infty$	0.00000	-3.00	0.01832	0.15	0.42741	0.60	0.67032
-9.00	0.00005	-2.50	0.03020	0.20	0.44933	0.65	0.70469
-8.00	0.00012	-2.00	0.04979	0.25	0.47237	0.70	0.74082
-7.00	0.00034	-1.50	0.08208	0.30	0.49659	0.75	0.77880
-6.00	0.00091	-1.00	0.13534	0.35	0.52205	0.80	0.81873
-5.00	0.00248	-0.50	0.22313	0.40	0.54881	0.85	0.86071
-4.50	0.00409	0.00	0.36788	0.45	0.57695	0.90	0.90484
-4.00	0.00674	0.05	0.38674	0.50	0.60653	0.95	0.95123
-3.50	0.01111	0.10	0.40657	0.55	0.63763	1.00	1.00000

We consider the case that parameter λ is given (known) and β is unknown. Let $x^{\mathbf{R}}_{i:m:n:k}$, $i = 1, 2, \cdots, m$ be the progressive first-failure censored sample from a GIED(β, λ) with pdf and cdf given by equations (2) and (3), respectively.

The joint PDF of progressively first-failure censored order statistics is

$$f_{1,2,\cdots,m}(x^{R_1}_{1:m:n:k}, x^{R_2}_{2:m:n:k}, \cdots, x^{R_m}_{m:m:n:k}) = Ck^m \prod_{i=1}^{m} f(x^{R_i}_{i:m:n:k})(1 - F(x^{R_i}_{i:m:n:k}))^{k(R_i+1)-1}, \tag{9}$$

where $0 < x^{R_1}_{1:m:n:k} < x^{R_2}_{2:m:n:k} < \cdots < x^{R_m}_{m:m:n:k} < \infty$ and $C = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$.

Using the transformation $Y = -\ln(1 - e^{-\frac{\lambda}{X}})$, the order statistics $Y^{R_1}_{1:m:n:k} < Y^{R_2}_{2:m:n:k} < \cdots < Y^{R_m}_{m:m:n:k}$ will be the corresponding progressive first-failure censored data from the one-parameter exponential distribution with pdf and cdf given by equations (4) and (5), respectively. Substituting equations (4), (5), and censoring scheme $\mathbf{R} = (R_1, R_2, \cdots, R_m)$ into equation (9), the likelihood function is then given by

$$L(\beta|y_1, y_2, \cdots, y_m) = Ck^m \beta^m e^{-\beta k \sum_{i=1}^{m}(R_i+1)y_i}. \tag{10}$$

Taking the logarithm of equation (10)

$$log(L(\beta|Y_1, Y_2, \cdots, Y_m) = logC + mlogk + mlog\beta - \beta k \sum_{i=1}^{m}(R_i + 1)y_i. \tag{11}$$

Then

$$\frac{\partial log(L(\beta|Y_1, Y_2, \cdots, Y_m)}{\partial \beta} = \frac{m}{\beta} - k \sum_{i=1}^{m}(R_i + 1)y_i = 0. \tag{12}$$

So the MLE of β is

$$\hat{\beta} = \frac{m}{T}, \tag{13}$$

where $T = k \sum_{i=1}^{m}(R_i + 1)Y^{R}_{i:m:n:k}$. By the invariant property of the ML estimates [26], the ML estimate for C_{L_Y} denoted by $\hat{C}^{ML}_{L_Y}$ is obtained from equation (6),

$$\hat{C}^{ML}_{L_Y} = 1 - \frac{m}{T} L_Y. \tag{14}$$

It can be noticed that $2\beta T$ follows the chi-square distribution with $2m$ degrees of freedom ([21]) , denoted by χ^2_{2m}. Hence, the expectation of $\hat{C}^{ML}_{L_Y}$ can be derived as

$$E(\hat{C}_{L_Y}^{ML}) = 1 - \frac{m\beta L_Y}{m-1}. \tag{15}$$

But $E(\hat{C}_{L_Y}^{ML}) \neq C_{L_Y}$, where $C_{L_Y} = 1 - \beta L_Y$. Hence, the MLE $\hat{C}_{L_Y}^{ML}$ is not an unbiased estimator of C_{L_Y}. However, when $m \to \infty$, $E(\hat{C}_{L_Y}^{ML}) \to C_{L_Y}$, so $\hat{C}_{L_Y}^{ML}$ is asymptotically unbiased estimator. Furthermore, we can also show that $\hat{C}_{L_Y}^{ML}$ is consistent.

4. Bayesian Estimation

In Bayesian analysis, the prior distribution plays a defining role in Bayesian inference and represents the information about an uncertain parameter. A common approach to constructing a prior density is to use the conjugate priors ([27]). A conjugate prior is a prior distribution that, when combined with the likelihood function, results in a posterior distribution that belongs to the same family as the prior. The Gamma distribution is the conjugate prior for the parameter β of the Exponential family, which is widely used in lifetime data modeling because it leads to a posterior that is also Gamma distributed, avoiding the need for complex numerical integration and Bayesian inference process. In our study, under the assumption that shape parameter β is unknown, we consider a conjugate gamma prior density for β as

$$\pi(\beta) = \frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-b\beta}, \;\; \beta > 0, \tag{16}$$

where a and b are the hyper parameters that reflect prior beliefs.

We will get the posterior PDF of the parameter β, using equations (10) and (16) as,

$$\pi^*(\beta|y_1, y_2, \cdots, y_m) = \frac{L(\beta|y_1, y_2, \cdots, y_m)\pi(\beta)}{\int_{-\infty}^{\infty} L(\beta|y_1, y_2, \cdots, y_m)\pi(\beta)d\beta} \tag{17}$$

$$= \frac{Ck^m\beta^m e^{-\beta t}\frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-\frac{b}{\beta}}}{ck^m\frac{b^a}{\Gamma(a)}\frac{\Gamma(a+m)}{(b+t)^{a+m}}}$$

$$= \frac{\beta^{a+m-1}e^{-\beta(b+t)}(b+t)^{a+m}}{\Gamma(a+m)}$$
$$= A\beta^{a+m-1}e^{-\beta(b+t)},$$

where $t = k\sum_{i=1}^{m}(R_i+1)y_i$ and $A = \frac{(b+t)^{a+m}}{\Gamma(a+m)}$ is normalizing constant. Thus, $(\beta|y_1, y_2, \cdots, y_m)$ has Gamma distribution with parameters $(a+m)$ and $(b+t)$.

Under the squared error loss function, $l(\hat{\beta}, \beta) = (\hat{\beta} - \beta)^2$, the Bayes estimate of the parameter β is the posterior mean $\frac{a+m}{b+t}$. Therefore, we obtain the Bayes estimator of lifetime performance index C_{L_Y} denoted by $\hat{C}_{L_Y}^{BSE}$ as

$$\hat{C}_{L_Y}^{BSE} = E^{\pi^*}(C_{L_Y}|\mathbf{y}) = 1 - \frac{a+m}{b+t}L_Y. \tag{18}$$

5. Interval Estimation

In the non-Bayesian approach, given the specified significance level α, the $100(1-\alpha)\%$ one-sided confidence interval for C_{L_Y} can be obtained by using the ML estimate given by (14). Using the pivotal quantity $2\beta T$, where $2\beta T \sim \chi^2_{2m}$ and χ^2_{2m} representing the lower $(1-\alpha)$ percentile of chi-squared distribution, we have

$$P(2\beta T \leq \chi^2_{2m}(1-\alpha)) = 1-\alpha$$

$$P\left(1-\beta L_Y \geq 1-\frac{L_Y\chi^2_{2m}(1-\alpha)}{2T}\right) = 1-\alpha$$

$$P\left(C_{L_Y} \geq 1-\frac{(1-\hat{C}_{L_Y})\chi^2_{2m}(1-\alpha)}{2m}\right) = 1-\alpha, \tag{19}$$

from (19), then

$$C_{L_Y} \geq 1-\frac{(1-\hat{C}_{L_Y})\chi^2_{2m}(1-\alpha)}{2m} \tag{20}$$

is the level $100(1-\alpha)\%$ one-sided confidence interval from C_{L_Y}. Thus, the $100(1-\alpha)\%$ lower confidence bound for C_{L_Y} can be written as

$$LB_{ML} = 1-\frac{(1-\hat{C}_{L_Y})\chi^2_{2m}(1-\alpha)}{2m}. \tag{21}$$

where $\hat{C}_{L_Y}$ and α denote the ML estimate of C_{L_Y} and the specified significance level respectively.

In the Bayesian approach, we obtain the lower credible bound of C_{L_Y}. Using the fact $\beta \sim \Gamma(a+m, b+t)$ stated in the equation (17), let $T^* = (b+t)$, then we can show that $2\beta T^* \sim \chi^2_{2(a+m)}(1-\alpha)$ as follows.

Let $Z = 2\beta T^*$, then $\beta = \frac{Z}{2T^*}$, we have

$$f_Z(z) = f_{\beta[y_1,y_2\ldots,y_m]}\left(\frac{Z}{2T^*}\right) \times \left(\frac{1}{2T^*}\right)$$

$$= \frac{1}{\Gamma(a+m)2^{(a+m)}} Z^{a+m-1} e^{\frac{-Z}{2}}$$

$$= \frac{1}{2^{\frac{2(a+m)}{2}}\Gamma(\frac{2(a+m)}{2})} Z^{\frac{2(a+m)}{2}-1} e^{-\frac{Z}{2}}.$$

Analogously, $2\beta T^* \sim \chi^2_{2(a+m)}(1-\alpha)$.

We can now obtain a confidence interval of C_{L_Y} by considering the distribution of $2\beta T^*$ as follows.

$$P\left(2\beta T^* \leq \chi^2_{2(a+m)}(1-\alpha)\right) = 1-\alpha$$

$$P\left(1-\beta L_Y \geq 1-\frac{L_Y\chi^2_{2(a+m)}(1-\alpha)}{2T^*}\right) = 1-\alpha$$

$$P\left(1-\beta L_Y \geq 1-\frac{(1-\hat{C}^{BSE}_{L_Y})}{2(a+m)}\chi^2_{2(a+m)}(1-\alpha)\right) = 1-\alpha$$

$$P\left(C_{L_Y} \geq 1-\frac{(1-\hat{C}^{BSE}_{L_Y})}{2(a+m)}\chi^2_{2(a+m)}(1-\alpha)\right) = 1-\alpha.$$

Then we can obtain that $100(1-\alpha)\%$ on sided credible interval of C_{L_Y} as

$$LB_{Bayes} = 1-\frac{(1-\hat{C}^{BSE}_{L_Y})}{2(a+m)}\chi^2_{2(a+m)}(1-\alpha). \tag{22}$$

6. Hypothesis Testing

In this section, we construct statistical tests for evaluating whether the lifetime performance index of products meets the required level via the Bayesian and the non-Bayesian approaches. Assuming that the required index value of lifetime performance is greater than the target value c^*, thus, the null and alternative hypothesis can be stated as

$$H_0 : C_{L_Y} \leq c^* \quad vs \quad H_a : C_{L_Y} > c^*.$$

In the non-Bayesian approach, let $\hat{C}_{L_Y}$ be the test statistic, the rejection region can be expressed as $\{\hat{C}_{L_Y} > c_0\}$ where c_0 is the critical value. Given the specified significance level α, the critical value c_0 can be derived as follows by using the fact $2\beta T \sim \chi^2_{2m}$.

$$P(\hat{C}_{L_Y} > c_0 | C_{L_Y} = c^*) = \alpha$$

$$P\left(1 - \frac{mL_Y}{T} > c_0 \middle| C_{L_Y} = c^*\right) = \alpha$$

$$P\left(2\beta T > \frac{2\beta m L_Y}{1 - c_0} \middle| \beta = \frac{1 - c^*}{L_Y}\right) = \alpha$$

$$P\left(2\beta T \leq \frac{2m(1 - c^*)}{1 - c_0}\right) = 1 - \alpha. \tag{23}$$

Hence, $\chi^2_{2m}(1 - \alpha)$ function represents the lower $(1 - \alpha)$ percentile of χ^2_{2m}, that is

$$\frac{2m(1 - c^*)}{1 - c_0} = \chi^2_{2m}(1 - \alpha) \tag{24}$$

Thus, the following critical value can be derived as

$$c_0 = 1 - \frac{2m(1 - c^*)}{\chi^2_{2m}(1 - \alpha)}. \tag{25}$$

The power of a statistical test is the probability of correctly rejecting a false null hypothesis. Thus, the power of the statistical test $H_0 : C_{L_Y} \leq c^* \quad vs \quad H_a : C_{L_Y} > c^*$ can be derived as follows:

Under progressively first-failure censoring, we get a size α test with the rejection region $\{\hat{C}_{L_Y} | \hat{C}_{L_Y} > 1 - \frac{2m(1-c^*)}{\chi^2_{2m}(1-\alpha)}\}$, for sample size n $(m \leq n)$. The power $P(c_1)$ of the test at this point $C_L = c_1 (> c^*)$ is computed as

$$P(c_1) = P\left(\hat{C}_{L_Y} > 1 - \frac{2m(1 - c^*)}{\chi^2_{2m}(1 - \alpha)} \middle| C_{L_Y} = c_1\right)$$

$$= P\left(1 - \frac{mL_Y}{T} > 1 - \frac{2m(1 - c^*)}{\chi^2_{2m}(1 - \alpha)} \middle| \beta = \frac{1 - c_1}{L_Y}\right)$$

$$= P\left(2\beta T > \frac{\beta L_Y \chi^2_{2m}(1 - \alpha)}{1 - c^*} \middle| \beta = \frac{1 - c_1}{L_Y}\right)$$

$$= P\left(2\beta T > \frac{(1 - c_1)\chi^2_{2m}(1 - \alpha)}{1 - c^*}\right) \tag{26}$$

Following [25], in Bayesian approach for testing $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$ where $\Theta_0 \bigcup \Theta_1 = \Theta$ (the space of the parameter θ), and $\Theta_0 \cap \Theta_1 = \emptyset$, the empty set, the actions of interest are a_0 and a_1, where a_i denotes acceptance of $H_i,\ i = 0, 1$. When the loss function is '$0 - K_i$' defined by

$$L(\theta, a_i) = \begin{cases} 0 & \text{if } \theta \in \Theta_i \\ K_i & \text{if } \theta \in \Theta_j\ (i \neq j). \end{cases} \tag{27}$$

Hence, on the basis of the posterior distribution of C_{L_Y} in (17), the Bayesian rejection region for testing $H_0 : C_{L_Y} \leq c$ versus $H_1 : C_{L_Y} > c$ is

$$C^* = \left\{ (y_1, \cdots, y_m) : \int_c^1 \pi^*(C_{L_Y}|y_1, \cdots, y_m) dC_{L_Y} > \frac{K_1}{K_0 + K_1} \right\}$$

$$= \left\{ (y_1, \cdots, y_m) : \int_0^{\frac{(1-c)}{L_Y}} D u^{a+m-1} exp(-u(b+t)) du > \frac{K_1}{K_0 + K_1} \right\}$$

$$= \left\{ (y_1, \cdots, y_m) : \int_0^{\frac{(1-c)}{L_Y}} g(u) du > \frac{K_1}{K_0 + K_1} \right\},$$

where D is the normalizing constant and $g(u)$ is the PDF of the gamma distribution with parameters $a + m$ and $b + t$. Thus, the Bayesian rejection region C^* is reduced to

$$C^* = \left\{ (y_1, \cdots, y_m) : \frac{1-c}{L_Y} > \Gamma_{a+m,b+t}\left(\frac{K_1}{K_0 + K_1} \right) \right\} \tag{28}$$

where $\Gamma_{\eta,\xi}(\gamma)$ stands for the γth quantile of the gamma distribution with parameters η and ξ. Note that for any positive integer a, the Bayesian rejection region C^* in (28) can be rewritten as

$$C^* = \left\{ (y_1, \cdots, y_m) : 2(b+t)\left(\frac{1-c}{L_Y} \right) > \chi^2_{2(m+a)}\left(\frac{K_1}{K_0 + K_1} \right) \right\}. \tag{29}$$

7. Illustrative examples

Example 7.1: We use a lifetime data set on the survival times (in days) of guinea pigs injected with different doses of tubercle bacillia shown in Table 2, which was originally discussed by [28] and also given in [18].

Table 2. Survival times (in days) of guinea pigs injected with different doses of tubercle bacilli

12	15	22	24	24	32	32	33	34	38	38	43
44	48	52	53	54	54	55	56	57	58	58	59
60	60	60	60	61	62	63	65	65	67	68	70
70	72	73	75	76	76	81	83	84	85	87	91
95	96	98	99	109	110	121	127	129	131	143	146
146	175	175	211	233	258	258	263	297	341	341	376

In order to estimate the scale parameter λ in the GIED(β, λ), Gini statistic due to [5] is suggested. The Gini statistic is defined as,

$$G_m = \frac{\sum_{i=1}^{m-1} iD_{i+1}}{(m-1)\sum_{i=1}^{m} D_i}, \tag{30}$$

where $D_1 = NY^{\mathbf{R}}_{1:m:n:k}$ and $D_i = k(n - \sum_{j=1}^{i-1}(R_j+1))(Y^{\mathbf{R}}_{i:m:n:k} - Y^{\mathbf{R}}_{i-1:m:n:k})$ for $i = 2, \cdots, m$ while $Y^{\mathbf{R}}_{i:m:n:k} = -\ln(1 - e^{-\frac{\lambda}{X^{\mathbf{R}}_{i:m:n:k}}})$, $i = 1, \cdots, m$. For $m \geq 20$, $(12(m-1))^{\frac{1}{2}}(G_m - 0.5)$ tends to the standard normal distribution $N(0,1)$. Hence for large m, the p-value can be given as follows.

$$\text{p-value} = P\{|Z| > |(12(m-1))^{\frac{1}{2}}(g_m - 0.5)|\},$$

where g_m is the observed value of G_m and Z has an approximation of $N(0,1)$. When m is small, we can use the exact distribution of G_m which has been studied by [5] and is symmetric about 0.5. For $m = 3, \cdots, 20$, percentiles of the Gini statistics are given on page 352 in [5].

For the data set in Table 2, the Gini statistics $G_{15} = \frac{\sum_{i=1}^{15-1} iD_{i+1}}{(15-1)\sum_{i=1}^{m} D_i}$, $i = 1, 2, ..., 15$. Hence, the p-value=$P\{|Z| > |(12(15-1))^{\frac{1}{2}}(g_{15} - 0.5)|\}$, where g_{15} is the observed value of G_{15} and Z has an approximation of $N(0,1)$. The values of p-value and the various values of λ are shown in Table 3. Based on Table 3, it appears that $\lambda = 55.48$ is very close to the optimum value, while p-value = 0.9999 is the maximum value. So, we can assume that the parameter λ is known. Using $\lambda = 55.48$, the observed exponential first-failure censored sample obtained with transformation $Y^{\mathbf{R}}_{i:m:n:k} = -\ln(1 - e^{-\frac{55.48}{X^{\mathbf{R}}_{i:m:n:k}}})$, where $n = 36$, $m = 15$, $k = 2$ and $\mathbf{R} = (3,0,1,1,0,0,0,0,1,2,3,3,3,2,2)$, are reported in Table 4.

Table 3. Values of p-value for various values of λ in Example 7.1

λ	p-value	λ	p-value	λ	p-value	λ	p-value	λ	p-value
55.15	0.9888	55.30	0.9938	55.45	0.9988	55.60	0.9961	55.75	0.9911
55.16	0.9891	55.31	0.9942	55.46	0.9992	55.61	0.9958	55.76	0.9908
55.17	0.9895	55.32	0.9945	55.47	0.9995	55.62	0.9955	55.77	0.9905
55.18	0.9898	55.33	0.9948	**55.48**	**0.9999**	55.63	0.9951	55.78	0.9901
55.19	0.9901	55.34	0.9952	55.49	0.9998	55.64	0.9948	55.79	0.9898
55.20	0.9905	55.35	0.9955	55.50	0.9995	55.65	0.9945	55.80	0.9895
55.21	0.9908	55.36	0.9958	55.51	0.9991	55.66	0.9941	55.81	0.9891
55.22	0.9911	55.37	0.9962	55.52	0.9988	55.67	0.9938	55.82	0.9888
55.23	0.9915	55.38	0.9965	55.53	0.9985	55.68	0.9935	55.83	0.9885
55.24	0.9918	55.39	0.9968	55.54	0.9981	55.69	0.9931	55.84	0.9881
55.25	0.9921	55.40	0.9972	55.55	0.9978	55.70	0.9928	55.85	0.9878
55.26	0.9925	55.41	0.9975	55.56	0.9975	55.71	0.9925	55.86	0.9875
55.27	0.9928	55.42	0.9978	55.57	0.9971	55.72	0.9921	55.87	0.9871
55.28	0.9932	55.43	0.9982	55.58	0.9968	55.73	0.9918	55.88	0.9868
55.29	0.9935	55.44	0.9985	55.59	0.9965	55.74	0.9915	55.89	0.9865

Following [18], the lower lifetime limit, L_X is assumed to be 20 then transformed limit $L_Y = -\ln(1 - e^{-\frac{55.48}{20}}) = 0.06444$. We also assume $(a,b) = (2,3)$. The lower confidence bound is calculated as $LB_{ML} = 0.96957$ and the lower credible bound is calculated as $LB_{Bayes} = 0.96827$ for C_{L_Y} from equations (21) and (22) respectively at the level of significance $\alpha = 0.05$.

For $L_X = 20$, the conforming rate P_r of products is assumed to exceed 70%. From Table (1), the C_{L_Y} will exceed 0.65. Therefore, We take the performance index value $c^* = 0.65$. Then, testing the null hypothesis $H_0 : C_L \leq 0.65$ against the alternative $H_a : C_L > 0.65$ is interested. Since $\hat{C}_{L_Y} = 0.97914 > 1 - \frac{2m(1-c^*)}{\chi^2_{2m}(1-\alpha)} = 0.8629291$ and $\frac{2(b+w)(1-c^*)}{L_Y} = 306.3202 > \chi^2_{2(m+a)}(0.5) = 33.33571$ under both the non-Bayesian and Bayesian approaches, the

Table 4. The generated progressively first-failure-censored sample for Example 7.1

i	1	2	3	4	5	6	7	8
$X^r_{i:m:n:k}$	12	15	24	32	38	48	53	58
R_i	3	0	1	1	0	0	0	0
$Y^r_{i:m:n:k}$	0.00987	0.02507	0.10436	0.19434	0.26427	0.37804	0.43242	0.48486

i	9	10	11	12	13	14	15
$X^r_{i:m:n:k}$	60	61	87	98	110	143	258
R_i	1	2	3	3	3	2	2
$Y^r_{i:m:n:k}$	0.50528	0.51537	0.75185	0.83869	0.92606	1.13454	1.64253

null hypothesis is rejected. Thus, based on both methods, the lifetime performance index of products meets the required level.

Example 7.2: The data for this application was originally discussed by [29] and also given in [30]. This data represents the strength measured in GPa (giga-Pascals) for single-carbon fibers and impregnated 1000-carbon fiber tows.

Table 5. The strength measured in GPa (giga-Pascals) for single carbon fibers and impregnated 1000-carbon fiber tows

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958	1.966	1.997
2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270
2.272	2.274	2.301	2.301	2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.490
2.511	2.514	2.535	2.554	2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684
2.697	2.726	2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585			

We estimate the scale parameter by the maximum p-value method. For the data set in Table 5, the estimated value of λ is calculated by using the concept of Gini statistic with maximum $p-$value method as we explained in Example 7.1. The optimum values that maximize $p-$value as given in Table 6 is $\lambda = 10.94$ with $p-$ value $= 0.9983$. Hence, the parameter λ is known.

Since $\lambda = 10.94$, for given $n = 23$, $k = 3$, $m = 20$ and $R = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1)$, the observed exponential first-failure censored sample obtained with transformation $Y^{\mathbf{R}}_{i:m:n:k} = -\ln(1 - e^{-\frac{10.94}{X^{\mathbf{R}}_{i:m:n:k}}})$, are reported in Table 7.

Following [30], the lower lifetime limit, L_X is assumed to be 2.8, then transformed limit $L_Y = -\ln(1 - e^{-\frac{10.94}{2.8}}) = 0.02030$. We also assume $(a,b) = (50,2)$. The lower confidence bound is calculated as $LB_{ML} = 0.27254$ and the lower credible bound is calculated as $LB_{Bayes} = 0.38524$ for C_{L_Y} from equations (21) and (22) respectively at the level of significance $\alpha = 0.05$. When $L_X = 2.80$ and the conforming rate P_r of products is assumed to exceed 90%. From Table 1, the C_{L_Y} will exceed 0.90. Therefore, the performance index value is supposed to be $c^* = 0.90$. Then, we test the null hypothesis $H_0 : C_L \leq 0.90$ against the alternative $H_a : C_L > 0.90$. Since $\hat{C}_{L_Y} = 0.48648 < 1 - \frac{2m(1-c^*)}{\chi^2_{2m}(1-\alpha)} = 0.92941$ and $\frac{2(b+w)(1-c^*)}{L_Y} = 26.71241 < \chi^2_{2(m+a)}(0.5) = 135.3339$ under both the non-Bayesian and Bayesian approaches, the null hypothesis is not rejected. Thus, based on both methods, the lifetime performance index of products does not meet the required level.

Table 6. Values of p-value for various values of λ in Example 7.2

λ	p-value	λ	p-value	λ	p-value	λ	p-value	λ	p-value
10.55	0.8444	10.70	0.9036	10.85	0.9629	11.00	0.9781	11.15	0.9198
10.56	0.8483	10.71	0.9076	10.86	0.9669	11.01	0.9742	11.16	0.9160
10.57	0.8522	10.72	0.9115	10.87	0.9708	11.02	0.9703	11.17	0.9121
10.58	0.8562	10.73	0.9155	10.88	0.9747	11.03	0.9664	11.18	0.9083
10.59	0.8601	10.74	0.9194	10.89	0.9787	11.04	0.9625	11.19	0.9044
10.60	0.8641	10.75	0.9234	10.90	0.9826	11.05	0.9586	11.20	0.9006
10.61	0.8680	10.76	0.9274	10.91	0.9866	11.06	0.9547	11.21	0.8967
10.62	0.8720	10.77	0.9313	10.92	0.9905	11.07	0.9508	11.22	0.8929
10.63	0.8759	10.78	0.9353	10.93	0.9944	11.08	0.9469	11.23	0.8891
10.64	0.8799	10.79	0.9392	**10.94**	**0.9983**	11.09	0.9430	11.24	0.8853
10.65	0.8838	10.80	0.9432	10.95	0.9977	11.10	0.9392	11.25	0.8815
10.66	0.8878	10.81	0.9471	10.96	0.9938	11.11	0.9353	11.26	0.8776
10.67	0.8918	10.82	0.9511	10.97	0.9899	11.12	0.9314	11.27	0.8738
10.68	0.8957	10.83	0.9550	10.98	0.9860	11.13	0.9275	11.28	0.8700
10.69	0.8997	10.84	0.9590	10.99	0.9820	11.14	0.9237	11.29	0.8662

Table 7. The generated progressively first-failure-censored sample for Example 2

i	1	2	3	4	5	6	7	8	9	10
$X^r_{i:m:n:k}$	1.8030	1.9440	1.9970	2.0060	2.0270	2.0550	2.1790	2.2240	2.2700	2.2720
R_i	0	0	0	0	0	0	0	0	0	0
$Y^r_{i:m:n:k}$	0.00232	0.00360	0.00419	0.00429	0.00454	0.00489	0.00662	0.00733	0.00811	0.00814
i	11	12	13	14	15	16	17	18	19	20
$X^r_{i:m:n:k}$	2.4260	2.5110	2.5868	2.6970	2.8210	2.8480	2.9540	3.0120	3.0900	3.5850
R_i	0	0	0	0	0	0	0	1	1	1
$Y^r_{i:m:n:k}$	0.01107	0.01290	0.01467	0.01746	0.02091	0.02170	0.02495	0.02682	0.02943	0.04844

8. Simulation study

In this section, we perform a Monte Carlo simulation study to examine and compare the performance of Bayes estimates with ML estimates for different combinations of sample sizes and first-failure censoring schemes. Several different combinations of progressive censoring scheme $\mathbf{R}$, sample size n, and the number of observed failures before termination m are labeled in Table 8. The performances of the lower confidence interval and the lower credible intervals are compared by using the average empirical confidence level $\overline{(1-\alpha)}$ and the corresponding sample mean square error (SMSE). Also, we take the Bayesian approach to test the null hypothesis $H_0 : C_L \leq c^*$ versus the alternative hypothesis $H_a : C_L > c^*$ by calculating the estimated risk of Bayesian rejection region based on equation (27) with "0-1" loss function; In contrast, we calculate the power function for non-Bayesian approach. All simulation results are reported in Tables 9-20.

Algorithm 1 shows the steps of generating the progressively first failure censored sample, then we follow Algorithm 2 to calculate the average empirical confidence level $\overline{(1-\alpha)}$ based on a $100(1-\alpha)\%$ one-sided credible interval and a $100(1-\alpha)\%$ one-sided confidence interval of the lifetime performance index C_{L_Y}, and the corresponding sample mean square error (SMSE) denoted by B.SMSE for Bayes estimate and M.SMSE for ML estimate respectively. The results of Monte Carlo simulation are reported in Tables 9-12 for different censoring schemes and different values of prior parameters where $\lambda = 4.9, 2, (a, b) = (3.3, 2.88), (0.5, 0.7), C_{L_Y} = 0.8, 0.9, L_X = 2.5, 1.4$ and $k = 3, 5$ for the significance level $\alpha = 0.05$.

Table 8. Different choices of the progressive first-failure-censoring schemes

n	m	R	CS	n	m	R	CS	n	m	R	CS
10	5	$(3, 0^*3, 2)$	[1]	20	10	$(0^*9, 10)$	[15]			$(1, 0^*22, 2^*2)$	[29]
		(1^*5)	[2]			(1^*10)	[16]			$(0^*10, 1^*5, 0^*10)$	[30]
		$(2, 1^*3, 0)$	[3]			$(1^*4, 0^*4, 3^*2)$	[17]		30	(0^*30)	[31]
	10	(0^*10)	[4]		15	$(0^*10, (1^*5)$	[18]	50	15	$(35, 0^*14)$	[32]
15	5	$(7, 3, 0^*3)$	[5]			$((1^*3, 0^*10, (1^*2)$	[19]			$(0^*14, 35)$	[33]
		$(4, 0^*2, 2, 4)$	[6]			$(3, 0^*13, 2)$	[20]			$(0^*7, 35, 0^*7)$	[34]
		$(0, 1, 2, 3, 4)$	[7]		20	(0^*20)	[21]		30	$((2, 0, 0)^*10)$	[35]
	10	$(3, 0^*8, 2)$	[8]	30	10	$(20, 0^*9)$	[22]			$(1^*20, 0^*10)$	[36]
		$((1, 0)^*5)$	[9]			$((5^*3, 0^*6, 5)$	[23]			$(20, 0^*29)$	[37]
		$(5, 0^*9)$	[10]			$(0^*4, 10^*2, 0^*4)$	[24]		40	$(10, 0^*39)$	[38]
	15	(0^*15)	[11]		20	$(5^*2, 0^*18)$	[25]			$(0^*39, 10)$	[39]
20	5	$(3, 0^*3, 12)$	[12]			$(10, 0^*19)$	[26]			$(0^*15, 1^*10, 0^*15)$	[40]
		$(0^*2, 15, 0^*2)$	[13]			$((0, 1)^*10)$	[27]		50	(0^*50)	[41]
		(3^*5)	[14]		25	$(2^*2, 0^*22, 1)$	[28]				

Algorithm 1 Generate the progressively first failure censored sample

Step 1: Generate a random sample of size m from a standard uniform distribution, denoted by $W_1, \cdots, W_m$.

Step 2: Set $V_i = W_i^{\frac{1}{a_i}}$ for $i = 1, \cdots, m$ where $a_i = i + \sum_{j=m-i+1}^{m} R_j$.

Step 3: Set $U_{i:m:n}^R = 1 - \prod_{j=m-i+1}^{m} V_j$ for $i = 1, \cdots, m$. Thus $U_{1:m:n}^R, \cdots, U_{m:m:n}^R$ is a progressively Type-II censored sample from the standard uniform distribution.

Step 4: set $Z_{1:m:n:k}^R = 1 - (1 - U_{1:m:n}^R)^{\frac{1}{k}}$ for $i = 1, \cdots, m$. The data set $Z_{1:m:n:k}^R, \cdots, Z_{m:m:n:k}^R$ is a progressively first failure censored sample from the standard uniform distribution.

Step 5: Finally, set $X_{i:m:n:k}^R = \frac{-\lambda}{ln[1-(1-Z^R)^{\frac{1}{\beta}}]}$ with $i = 1, \cdots, m$ and hence, the required progressively first failure censored sample is $X_{1:m:n:k}^R, \cdots, X_{m:m:n:k}^R$.

From the confidence interval simulation results, the following points can be drawn:

- All of the average empirical confidence level $\overline{(1-\alpha)}$ both the Bayesian and Maximum Likelihood (ML) estimates convergent to the target confidence level $(1-\alpha) = 0.95$. However, for small sample sizes $(n = 10)$, the results tend to be slightly higher than 0.95.
- The simulation results demonstrate very small SMSEs, with both the Bayesian and ML estimates performing equivalently well. The Bayesian SMSE (B.SMSE) and ML SMSE (M.SMSE) values are minimal, ranging from 0.000040 to 0.00057, in relation to the average empirical confidence level $(1-\alpha)$.
- As the sample size n increases, the SMSE values decreases slightly overall . For moderate sample sizes $(n = 15, 20, 30)$, as the effective sample proportion $\frac{m}{n}$ increases, the Bayesian estimate demonstrates slightly superior performance compared to the ML estimate, showing relatively smaller B.SMSE values.
- When n and m are fixed, no significant difference in B.SMSE and M.SMSE are observed under different censoring schemes.
- The estimated vales of the average empirical confidence level and the computed SMSE display good stability for $k = 3$ and $k = 5$, across various prior parameter settings (λ, a, b, L_X).

Also, we follow Algorithm 3 to compute the estimated risks of the Bayesian rejection region C^*, given by the equation (28) under $0-1$ loss function. The results of Monte Carlo simulation are reported in Tables 13-16 for different censoring schemes and different values of prior parameters where $\lambda = 1, 0.5, (a, b) = (0.8, 0.5), (2, 2), L_X = 1.7, 0.8$ and $k = 3, 5$.

Based on the estimated risk simulation results, the following points can be drawn:

Algorithm 2 The Monte Carlo simulation algorithm for the confidence level

Step 1: Given C_{L_Y}, $\mathbf{R}$, L_X, α, and$k$, where $C_L < 1$.
Step 2: The value of β is calculated by the equation $C_{L_Y} = 1 - \beta L, C_{L_Y} < 1$.
Step 3: (a) The generation of data $X^R_{1:m:n}, X^R_{2:m:n}, \ldots, X^R_{m:m:n}$ from **Algorithm 1** is a random sample from the GIED with parameters α and β with PDF as equation (3).
(b) The random sample obtained by the transformation $Y^R_{i:m:n} = ln\left[1 - e^{-\lambda / X^R_{i:m:n}}\right], i = 1, ..., m$, follows an exponential distribution ith the PDF given in equation (4).
(c) The value of LB_{ML} is calculated by Equation (21).
(d) The value of LB_{Bayes} is calculated by Equation (22).
Step 4: (a) The step 3 is repeated 100 times.
(b) The confidence level $(1-\alpha)$ for one-sided credible interval is estimated by,

$$(1 - \hat{\alpha}) = \frac{\text{total Count B}}{100}$$

(c) The confidence level $(1-\alpha)$ for one-sided confidence interval is estimated by,

$$(1 - \hat{\alpha}) = \frac{\text{total Count M}}{100}$$

Step 5: (a) Repeat step (3) and step (4) 1000 times, then we can get the 1000 estimations of confidence level as follows.

$$(1-\hat{\alpha})_1, (1-\hat{\alpha})_2, ..., (1-\hat{\alpha})_{1000}$$

(b) The average empirical confidence level is

$$\overline{1-\alpha} = \frac{1}{1000} \sum_{i=1}^{1000} (1-\hat{\alpha})_i \text{ for } i = 1, ..., 1000$$

(c) The sample mean square error (SMSE) of $(1-\hat{\alpha})_1, (1-\hat{\alpha})_2, ..., (1-\hat{\alpha})_{1000}$

$$SMSE = \frac{1}{1000} \sum_{i=1}^{1000} [(1-\hat{\alpha})_i - (1-\alpha)]^2$$

Algorithm 3 The estimated risks of the Bayesian rejection region

Step 1: The generation of data $X^R_{1:m:n}, X^R_{2:m:n}, \ldots, X^R_{m:m:n}$ from **Algorithm 1** is a random sample from the GIED with parameters α and β with PDF as equation (3).
Step 2: we test the null hypothesis $H_0 : C_{L_Y} \leq c^* \quad vs \quad H_a : C_{L_Y} > c^*$ using the Bayesian approach and from Equation (28) with '0-1' loss function. Based on the simulated progressively first-failure-censored sample in Step 1, we recorded the estimated risks of the Bayesian rejection region resulting from such hypothesis testing when c=0.2 and 0.5.

- As the effective sample proportion $\frac{m}{n}$ increases, the estimated risks of the Bayesian rejection region C^* decrease. This reduction in risk becomes particularly noticeable with larger sample sizes n.
- As n and m are fixed, there is no significant difference between the estimated risks of the Bayesian rejection region under the different censoring schemes.

- Although the estimated risks of the Bayesian rejection region vary with different choices of prior parameters $(\lambda, \beta, a, b, L_X)$, the overall patterns remain consistent, indicating the robustness of Bayesian testing.

Finally, we follow Algorithm 4 to the power of the tests by the equation (26) when $c_1 = 0(0.1)0.9$ and $\alpha = 0.05$. The results of Monte Carlo simulation are reported in Tables 17-20 for different censoring schemes and different values of prior parameters where $\lambda = 6, 2.75, c^* = 0.3, L_X = 1.5, 1.25$ and $k = 3, 5$.

Algorithm 4 The Monte Carlo simulation algorithm for Power

Step 1: Given $C_{L_Y}, \mathbf{R}, L_X, \alpha$, and$k$, where $C_L < 1$.
Step 2: The value of β is calculated by the equation $C_{L_Y} = 1 - \beta L, C_{L_Y} < 1$.
Step 3: (a) The generation of data $X^R_{1:m:n}, X^R_{2:m:n}, \ldots, X^R_{m:m:n}$ from **Algorithm 1** is a random sample from the GIED with parameters α and β with PDF as equation (3).
(b) The random sample obtained by the transformation $Y^R_{i:m:n} = ln\left[1 - e^{-\lambda/X^R_{i:m:n}}\right], i = 1, ..., m$, follows an exponential distribution ith the PDF given in equation (4).
(d) The value of LB_{ML} is calculated by Equation (21).
if $C_{L_Y} \geq LB_{ML}$ then count $M = 1$, else count $M = 0$.
Step 4: (a) The step 3 is repeated 10000 times.
(b) The power is estimated by,

$$\frac{\text{total Count B}}{10000}$$

Based on Monte Carlo power simulation results, the following points can be drawn:

- As we expect, the power of the test approaches α when $c_1 = c^*$.
- There is no significant difference in powers for different censoring schemes when m and n are fixed.
- The power of the test becomes larger as c_1 increases for fixed m and n.
- When the effective sample proportion $\frac{m}{n}$ increases, the power of the test decreases if $c_1 \leq c^*$ and increases if $c_1 > c^*$ for fixed c_1.

9. Concluding remarks

The lifetime performance index C_L is utilized to evaluate the quality of products. In this paper, given a progressively first-failure censored sample from a generalized inverted exponential distribution, the ML estimate and the Bayes estimate of C_L are obtained. Then we turn attention to the inferential analysis based on the estimates of C_L from both the Bayesian and the non-Bayesian approaches. In the Bayesian approach, we construct $100(1-\alpha)\%$ lower credible interval and use "0-1" loss function for hypothesis testing $H_0 : C_L \leq c^*$ versus the alternative hypothesis $H_a : C_L > c^*$. The results of Monte Carlo simulation and real data examples confirm that these statistical inferences on C_L can be used for assessing whether the product performance meets customer expectations or requirements.

The data transformation techniques based on a given scale parameter simplify the computational process but limit the generalization to some rea-world data that emphasizes the variation of scale parameter. Additionally, it seems that the estimated risks associated with the Bayesian rejection appear to be sensitive to the choice of prior parameters, making it essential for researchers to carefully select parameters to ensure accurate and reliable results.

In future research, we plan to extend our study by applying the proposed method to progressive censoring schemes, including the hybrid censoring scheme (HCS), which combines features of both Type I and Type II censoring. We are also interested in exploring other lifetime models in our future work.

Table 9. Simulation results of $\overline{1-\alpha}$ and SMSE under $k=3,\ \lambda=4.9,\ (a,b)=(3.3,2.88), C_{L_Y}=0.80, L_X=2.5$ and $\alpha=0.05$

n	m	R	Bayes	MLE	B.SMSE	M.SMSE	n	m	R	Bayes	MLE	B.SMSE	M.SMSE
10	5	[1]	0.9652	0.9496	0.00055	0.00048	30	10	[22]	0.9560	0.9490	0.00047	0.00047
		[2]	0.9653	0.9495	0.00056	0.00046			[23]	0.9566	0.9495	0.00045	0.00047
		[3]	0.9653	0.9496	0.00056	0.00047			[24]	0.9563	0.9490	0.00045	0.00048
	10	[4]	0.9563	0.9490	0.00045	0.00048		20	[25]	0.9510	0.9492	0.00043	0.00046
15	5	[5]	0.9652	0.9494	0.00056	0.00049			[26]	0.9505	0.9487	0.00041	0.00044
		[6]	0.9652	0.9496	0.00056	0.00048			[27]	0.9510	0.9492	0.00045	0.00047
		[7]	0.9653	0.9495	0.00056	0.00046		25	[28]	0.9502	0.9496	0.00044	0.00045
	10	[8]	0.9568	0.9496	0.00046	0.00048			[29]	0.9501	0.9495	0.00045	0.00045
		[9]	0.9562	0.9491	0.00046	0.00048			[30]	0.9500	0.9495	0.00046	0.00047
		[10]	0.9565	0.9494	0.00046	0.00047		30	[31]	0.9500	0.9502	0.00045	0.00045
	15	[11]	0.9539	0.9506	0.00045	0.00047	50	15	[32]	0.9526	0.9491	0.00043	0.00045
20	5	[12]	0.9652	0.9494	0.00056	0.00049			[33]	0.9532	0.9498	0.00044	0.00047
		[13]	0.9653	0.9496	0.00057	0.00047			[34]	0.9539	0.9506	0.00045	0.00047
		[14]	0.9653	0.9497	0.00056	0.00048		30	[35]	0.9500	0.9502	0.00045	0.00045
	10	[15]	0.9567	0.9495	0.00046	0.00048			[36]	0.9498	0.9501	0.00045	0.00045
		[16]	0.9563	0.9492	0.00046	0.00048			[37]	0.9493	0.9494	0.00044	0.00044
		[17]	0.9564	0.9493	0.00045	0.00047		40	[38]	0.9484	0.9495	0.00045	0.00044
	15	[18]	0.9538	0.9504	0.00045	0.00047			[39]	0.9488	0.9499	0.00045	0.00044
		[19]	0.9538	0.9505	0.00045	0.00047			[40]	0.9492	0.9503	0.00044	0.00043
		[20]	0.9540	0.9507	0.00046	0.00048		50	[41]	0.9486	0.9502	0.00048	0.00046
	20	[21]	0.9511	0.9493	0.00045	0.00047							

Table 10. Simulation results of $\overline{1-\alpha}$ and SMSE under $k=5,\ \lambda=4.9,\ (a,b)=(3.3,2.88), C_{L_Y}=0.80, L_X=2.5$ and $\alpha=0.05$

n	m	R	Bayes	MLE	B.SMSE	M.SMSE	n	m	R	Bayes	MLE	B.SMSE	M.SMSE
10	5	[1]	0.9652	0.9494	0.00056	0.00049	30	10	[22]	0.9569	0.9496	0.00045	0.00045
		[2]	0.9652	0.9496	0.00055	0.00048			[23]	0.9562	0.9491	0.00044	0.00045
		[3]	0.9651	0.9495	0.00056	0.00048			[24]	0.9563	0.9492	0.00046	0.00048
	10	[4]	0.9563	0.9492	0.00046	0.00048		20	[25]	0.9506	0.9488	0.00040	0.00043
15	5	[5]	0.9654	0.9495	0.00057	0.00049			[26]	0.9506	0.9487	0.00043	0.00046
		[6]	0.9652	0.9492	0.00057	0.00051			[27]	0.9509	0.9491	0.00044	0.00045
		[7]	0.9652	0.9496	0.00056	0.00047		25	[28]	0.9502	0.9497	0.00043	0.00044
	10	[8]	0.9565	0.9494	0.00046	0.00047			[29]	0.9503	0.9498	0.00044	0.00044
		[9]	0.9566	0.9495	0.00046	0.00048			[30]	0.9500	0.9494	0.00046	0.00046
		[10]	0.9563	0.9491	0.00047	0.00048		30	[31]	0.9499	0.9501	0.00046	0.00046
	15	[11]	0.9538	0.9505	0.00045	0.00047	50	15	[32]	0.9541	0.9504	0.00046	0.00047
20	5	[12]	0.9652	0.9493	0.00057	0.00049			[33]	0.9533	0.9498	0.00044	0.00047
		[13]	0.9653	0.9495	0.00056	0.00046			[34]	0.9538	0.9505	0.00045	0.00047
		[14]	0.9652	0.9493	0.00056	0.00049		30	[35]	0.9502	0.9504	0.00044	0.00044
	10	[15]	0.9564	0.9493	0.00046	0.00048			[36]	0.9501	0.9503	0.00045	0.00044
		[16]	0.9567	0.9495	0.00046	0.00048			[37]	0.9496	0.9497	0.00050	0.00050
		[17]	0.9566	0.9494	0.00046	0.00048		40	[38]	0.9483	0.9494	0.00045	0.00045
	15	[18]	0.9539	0.9506	0.00044	0.00046			[39]	0.9488	0.9499	0.00046	0.00045
		[19]	0.9540	0.9506	0.00046	0.00048			[40]	0.9489	0.9500	0.00045	0.00044
		[20]	0.9534	0.9500	0.00047	0.00049		50	[41]	0.9484	0.9499	0.00047	0.00046
	20	[21]	0.9509	0.9491	0.00044	0.00046							

Table 11. Simulation results of $\overline{1-\alpha}$ and SMSE under $k=3,\ \lambda=2,\ (a,b)=(0.5,0.7), C_{L_Y}=0.9, L_X=1.4$ and $\alpha=0.05$

n	m	R	Bayes	MLE	B.SMSE	M.SMSE	n	m	R	Bayes	MLE	B.SMSE	M.SMSE
10	5	[1]	0.9614	0.9496	0.00048	0.00048	30	10	[22]	0.9574	0.9490	0.00048	0.00047
		[2]	0.9614	0.9495	0.00049	0.00046			[23]	0.9580	0.9495	0.00046	0.00047
		[3]	0.9614	0.9496	0.00048	0.00047			[24]	0.9577	0.9490	0.00045	0.00048
	10	[4]	0.9577	0.9490	0.00045	0.00048		20	[25]	0.9550	0.9492	0.00043	0.00046
15	5	[5]	0.9614	0.9494	0.00049	0.00049			[26]	0.9544	0.9487	0.00041	0.00044
		[6]	0.9614	0.9496	0.00049	0.00048			[27]	0.9551	0.9492	0.00044	0.00047
		[7]	0.9614	0.9495	0.00049	0.00046		25	[28]	0.9550	0.9496	0.00043	0.00045
	10	[8]	0.9581	0.9496	0.00046	0.00048			[29]	0.9549	0.9495	0.00043	0.00045
		[9]	0.9577	0.9491	0.00046	0.00048			[30]	0.9549	0.9495	0.00044	0.00047
		[10]	0.9579	0.9494	0.00046	0.00047		30	[31]	0.9548	0.9502	0.00044	0.00045
	15	[11]	0.9571	0.9506	0.00046	0.00047	50	15	[32]	0.9557	0.9491	0.00044	0.00045
20	5	[12]	0.9614	0.9494	0.00049	0.00049			[33]	0.9564	0.9498	0.00045	0.00047
		[13]	0.9614	0.9496	0.00050	0.00047			[34]	0.9571	0.9506	0.00046	0.00047
		[14]	0.9615	0.9497	0.00049	0.00048		30	[35]	0.9548	0.9502	0.00043	0.00045
	10	[15]	0.9580	0.9495	0.00046	0.00048			[36]	0.9546	0.9501	0.00043	0.00045
		[16]	0.9577	0.9492	0.00046	0.00048			[37]	0.9541	0.9494	0.00043	0.00044
		[17]	0.9578	0.9493	0.00046	0.00047		40	[38]	0.9535	0.9495	0.00042	0.00044
	15	[18]	0.9570	0.9504	0.00045	0.00047			[39]	0.9539	0.9499	0.00042	0.00044
		[19]	0.9570	0.9505	0.00046	0.00047			[40]	0.9544	0.9503	0.00043	0.00043
		[20]	0.9572	0.9507	0.00047	0.00048		50	[41]	0.9538	0.9502	0.00045	0.00046
	20	[21]	0.9551	0.9493	0.00044	0.00047							

Table 12. Simulation results of $\overline{1-\alpha}$ and SMSE under $k=5,\ \lambda=2,\ (a,b)=(0.5,0.7), C_{L_Y}=0.9, L_X=1.4$ and $\alpha=0.05$

n	m	R	Bayes	MLE	B.SMSE	M.SMSE	n	m	R	Bayes	MLE	B.SMSE	M.SMSE
10	5	[1]	0.9614	0.9494	0.00049	0.00049	30	10	[22]	0.9584	0.9496	0.00047	0.00045
		[2]	0.9614	0.9496	0.00048	0.00048			[23]	0.9576	0.9491	0.00045	0.00045
		[3]	0.9613	0.9495	0.00049	0.00048			[24]	0.9577	0.9492	0.00046	0.00048
	10	[4]	0.9577	0.9492	0.00046	0.00048		20	[25]	0.9545	0.9488	0.00040	0.00043
15	5	[5]	0.9615	0.9495	0.00050	0.00049			[26]	0.9547	0.9487	0.00041	0.00046
		[6]	0.9613	0.9492	0.00050	0.00051			[27]	0.9549	0.9491	0.00043	0.00045
		[7]	0.9614	0.9496	0.00048	0.00047		25	[28]	0.9548	0.9497	0.00041	0.00044
	10	[8]	0.9579	0.9494	0.00047	0.00047			[29]	0.9550	0.9498	0.00042	0.00044
		[9]	0.9580	0.9495	0.00047	0.00048			[30]	0.9548	0.9494	0.00044	0.00046
		[10]	0.9577	0.9491	0.00047	0.00048		30	[31]	0.9546	0.9501	0.00043	0.00046
	15	[11]	0.9570	0.9505	0.00046	0.00047	50	15	[32]	0.9568	0.9504	0.00046	0.00047
20	5	[12]	0.9614	0.9493	0.00049	0.00049			[33]	0.9565	0.9498	0.00045	0.00047
		[13]	0.9614	0.9495	0.00049	0.00046			[34]	0.9570	0.9505	0.00046	0.00047
		[14]	0.9614	0.9493	0.00049	0.00049		30	[35]	0.9550	0.9504	0.00042	0.00044
	10	[15]	0.9578	0.9493	0.00047	0.00048			[36]	0.9549	0.9503	0.00042	0.00044
		[16]	0.9580	0.9495	0.00046	0.00048			[37]	0.9546	0.9497	0.00046	0.00050
		[17]	0.9579	0.9494	0.00046	0.00048		40	[38]	0.9536	0.9494	0.00042	0.00045
	15	[18]	0.9572	0.9506	0.00046	0.00046			[39]	0.9539	0.9499	0.00043	0.00045
		[19]	0.9572	0.9506	0.00047	0.00048			[40]	0.9541	0.9500	0.00043	0.00044
		[20]	0.9566	0.9500	0.00047	0.00049		50	[41]	0.9536	0.9499	0.00044	0.00046
	20	[21]	0.9549	0.9491	0.00044	0.00046							

Table 13. The estimated risks of the Bayesian rejection region under $k = 3,\ \ \lambda = 1,\ \ \beta = 1(0.1)1.8, L_X = 1.7, (a, b) = (0.8, 0.5), c = 0.5$ and $K_1 = K_2 = 1$

n	m		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
10	5	[1]	0.18517	0.14937	0.12126	0.09909	0.08151	0.06750	0.05626	0.04718	0.03980
		[2]	0.18575	0.14984	0.12163	0.09939	0.08175	0.06769	0.05641	0.04730	0.03990
		[3]	0.18491	0.14915	0.12107	0.09894	0.08139	0.06740	0.05617	0.04711	0.03975
	10	[4]	0.12155	0.08489	0.05948	0.04190	0.02970	0.02122	0.01527	0.01108	0.00811
15	5	[5]	0.18641	0.15041	0.12214	0.09983	0.08214	0.06803	0.05671	0.04757	0.04014
		[6]	0.18614	0.15015	0.12189	0.09961	0.08194	0.06786	0.05656	0.04743	0.04002
		[7]	0.18634	0.15032	0.12203	0.09972	0.08203	0.06793	0.05662	0.04748	0.04006
	10	[8]	0.12252	0.08573	0.06018	0.04247	0.03016	0.02158	0.01556	0.01131	0.00829
		[9]	0.12309	0.08618	0.06052	0.04273	0.03036	0.02173	0.01567	0.01139	0.00835
		[10]	0.12252	0.08572	0.06017	0.04246	0.03016	0.02158	0.01556	0.01131	0.00829
	15	[11]	0.08406	0.05156	0.03159	0.01941	0.01200	0.00747	0.00469	0.00298	0.00191
20	5	[12]	0.18552	0.14959	0.12139	0.09916	0.08155	0.06751	0.05625	0.04716	0.03977
		[13]	0.18555	0.14967	0.12150	0.09929	0.08168	0.06764	0.05638	0.04728	0.03989
		[14]	0.18488	0.14914	0.12108	0.09897	0.08144	0.06747	0.05625	0.04719	0.03983
	10	[15]	0.12157	0.08513	0.05984	0.04230	0.03011	0.02159	0.01561	0.01137	0.00835
		[16]	0.12243	0.08581	0.06036	0.04269	0.03039	0.02179	0.01575	0.01147	0.00843
		[17]	0.12219	0.08565	0.06026	0.04262	0.03035	0.02177	0.01573	0.01146	0.00842
	15	[18]	0.08432	0.05173	0.03169	0.01945	0.01200	0.00746	0.00467	0.00295	0.00188
		[19]	0.08352	0.05112	0.03124	0.01913	0.01178	0.00730	0.00456	0.00287	0.00183
		[20]	0.08312	0.05078	0.03096	0.01893	0.01163	0.00719	0.00449	0.00282	0.00179
	20	[21]	0.05667	0.03037	0.01618	0.00862	0.00461	0.00248	0.00135	0.00074	0.00041
30	10	[22]	0.12098	0.08440	0.05909	0.04160	0.02948	0.02105	0.01515	0.01099	0.00804
		[23]	0.12020	0.08388	0.05875	0.04138	0.02934	0.02096	0.01510	0.01096	0.00802
		[24]	0.12001	0.08363	0.05849	0.04113	0.02911	0.02076	0.01493	0.01082	0.00790
	20	[25]	0.05612	0.02998	0.01592	0.00845	0.00451	0.00242	0.00131	0.00072	0.00040
		[26]	0.05626	0.02999	0.01588	0.00840	0.00447	0.00239	0.00129	0.00070	0.00039
		[27]	0.05593	0.02981	0.01578	0.00835	0.00444	0.00237	0.00128	0.00070	0.00039
	25	[28]	0.03905	0.01818	0.00834	0.00381	0.00174	0.00080	0.00037	0.00017	0.00008
		[29]	0.03891	0.01810	0.00830	0.00379	0.00173	0.00080	0.00037	0.00017	0.00008
		[30]	0.03893	0.01816	0.00836	0.00383	0.00176	0.00081	0.00038	0.00018	0.00009
	30	[31]	0.02731	0.01121	0.00451	0.00180	0.00072	0.00029	0.00012	0.00005	0.00002
50	15	[32]	0.08028	0.04874	0.02957	0.01800	0.01102	0.00681	0.00424	0.00267	0.00170
		[33]	0.08110	0.04940	0.03007	0.01837	0.01129	0.00699	0.00437	0.00275	0.00176
		[34]	0.08075	0.04922	0.02999	0.01833	0.01128	0.00699	0.00437	0.00276	0.00176
	30	[35]	0.02716	0.01114	0.00448	0.00179	0.00071	0.00029	0.00012	0.00005	0.00002
		[36]	0.02696	0.01104	0.00444	0.00177	0.00071	0.00028	0.00012	0.00005	0.00002
		[37]	0.02728	0.01119	0.00451	0.00181	0.00073	0.00029	0.00012	0.00005	0.00002
	40	[38]	0.01416	0.00453	0.00141	0.00043	0.00013	0.00004	0.00001	0.00000	0.00000
		[39]	0.01409	0.00453	0.00141	0.00044	0.00014	0.00004	0.00001	0.00000	0.00000
		[40]	0.01411	0.00451	0.00141	0.00043	0.00013	0.00004	0.00001	0.00000	0.00000
	50	[41]	0.00724	0.00178	0.00042	0.00009	0.00002	0.00001	0.00000	0.00000	0.00000

Table 14. The estimated risks of the Bayesian rejection region under $k=5,\ \lambda=1,\ \beta=1(0.1)1.8, L_X=1.7$, (a,b)=(0.8,0.5),c=0.5 and $\mathrm{K}_1=K_2=1$

n	m		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
10	5	[1]	0.18512	0.14933	0.12123	0.09907	0.08150	0.06749	0.05624	0.04717	0.03979
		[2]	0.18566	0.14976	0.12156	0.09933	0.08170	0.06765	0.05637	0.04727	0.03987
		[3]	0.18486	0.14912	0.12105	0.09892	0.08138	0.06739	0.05617	0.04710	0.03974
	10	[4]	0.12157	0.08491	0.05950	0.04191	0.02972	0.02122	0.01528	0.01109	0.00811
15	5	[5]	0.18635	0.15035	0.12207	0.09977	0.08209	0.06798	0.05667	0.04753	0.04010
		[6]	0.18614	0.15016	0.12190	0.09962	0.08195	0.06787	0.05657	0.04744	0.04003
		[7]	0.18635	0.15032	0.12203	0.09972	0.08204	0.06794	0.05662	0.04749	0.04006
	10	[8]	0.12253	0.08574	0.06019	0.04247	0.03017	0.02158	0.01556	0.01131	0.00829
		[9]	0.12310	0.08618	0.06053	0.04273	0.03036	0.02173	0.01567	0.01139	0.00835
		[10]	0.12247	0.08569	0.06015	0.04244	0.03015	0.02157	0.01555	0.01131	0.00828
	15	[11]	0.08411	0.05159	0.03161	0.01943	0.01201	0.00748	0.00470	0.00298	0.00191
20	5	[12]	0.18544	0.14952	0.12134	0.09912	0.08151	0.06748	0.05622	0.04714	0.03976
		[13]	0.18556	0.14967	0.12150	0.09929	0.08169	0.06765	0.05638	0.04728	0.03989
		[14]	0.18476	0.14903	0.12099	0.09889	0.08138	0.06741	0.05620	0.04715	0.03979
	10	[15]	0.12165	0.08518	0.05988	0.04233	0.03013	0.02161	0.01562	0.01138	0.00836
		[16]	0.12245	0.08583	0.06037	0.04270	0.03040	0.02180	0.01576	0.01148	0.00843
		[17]	0.12218	0.08565	0.06026	0.04262	0.03035	0.02177	0.01573	0.01146	0.00842
	15	[18]	0.08431	0.05173	0.03168	0.01945	0.01200	0.00746	0.00467	0.00295	0.00188
		[19]	0.08353	0.05112	0.03124	0.01913	0.01178	0.00730	0.00456	0.00287	0.00183
		[20]	0.08310	0.05076	0.03095	0.01892	0.01162	0.00719	0.00448	0.00282	0.00179
	20	[21]	0.05667	0.03038	0.01619	0.00863	0.00461	0.00249	0.00135	0.00074	0.00041
30	10	[22]	0.12100	0.08443	0.05911	0.04161	0.02949	0.02106	0.01516	0.01100	0.00805
		[23]	0.12013	0.08383	0.05871	0.04135	0.02932	0.02095	0.01509	0.01095	0.00802
		[24]	0.11996	0.08359	0.05845	0.04109	0.02909	0.02074	0.01491	0.01080	0.00789
	20	[25]	0.05620	0.03002	0.01594	0.00847	0.00451	0.00243	0.00131	0.00072	0.00040
		[26]	0.05633	0.03003	0.01590	0.00841	0.00447	0.00239	0.00129	0.00070	0.00039
		[27]	0.05592	0.02981	0.01578	0.00835	0.00444	0.00237	0.00128	0.00070	0.00039
	25	[28]	0.03909	0.01821	0.00836	0.00382	0.00175	0.00080	0.00037	0.00018	0.00008
		[29]	0.03890	0.01810	0.00830	0.00379	0.00173	0.00080	0.00037	0.00017	0.00008
		[30]	0.03896	0.01817	0.00837	0.00383	0.00176	0.00081	0.00038	0.00018	0.00009
	30	[31]	0.02731	0.01121	0.00451	0.00180	0.00072	0.00029	0.00012	0.00005	0.00002
50	15	[32]	0.08029	0.04874	0.02956	0.01799	0.01102	0.00680	0.00424	0.00267	0.00169
		[33]	0.08110	0.04940	0.03007	0.01837	0.01129	0.00699	0.00437	0.00275	0.00176
		[34]	0.08070	0.04919	0.02997	0.01832	0.01127	0.00698	0.00437	0.00276	0.00176
	30	[35]	0.02718	0.01115	0.00449	0.00179	0.00071	0.00029	0.00012	0.00005	0.00002
		[36]	0.02695	0.01104	0.00443	0.00177	0.00071	0.00028	0.00012	0.00005	0.00002
		[37]	0.02734	0.01122	0.00453	0.00182	0.00073	0.00030	0.00012	0.00005	0.00002
	40	[38]	0.01418	0.00454	0.00141	0.00043	0.00013	0.00004	0.00001	0.00000	0.00000
		[39]	0.01409	0.00453	0.00141	0.00044	0.00014	0.00004	0.00001	0.00000	0.00000
		[40]	0.01411	0.00451	0.00140	0.00043	0.00013	0.00004	0.00001	0.00000	0.00000
	50	[41]	0.00726	0.00179	0.00042	0.00010	0.00002	0.00001	0.00000	0.00000	0.00000

Table 15. The estimated risks of the Bayesian rejection region under $k = 3,\ \ \lambda = 0.5,\ \ \beta = 2(0.1)2.8, L_X = 0.8, (a, b) = (2, 2), c = 0.2$ and $K_1 = K_2 = 1$

n	m		2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8
10	5	[1]	0.21330	0.19707	0.18256	0.16955	0.15786	0.14733	0.13782	0.12923	0.12143
		[2]	0.21381	0.19753	0.18298	0.16993	0.15821	0.14766	0.13813	0.12950	0.12168
		[3]	0.21313	0.19691	0.18241	0.16941	0.15773	0.14721	0.13772	0.12912	0.12133
	10	[4]	0.10076	0.08542	0.07263	0.06194	0.05298	0.04547	0.03915	0.03381	0.02930
15	5	[5]	0.21442	0.19811	0.18352	0.17044	0.15869	0.14811	0.13856	0.12991	0.12207
		[6]	0.21429	0.19798	0.18340	0.17033	0.15858	0.14801	0.13846	0.12982	0.12198
		[7]	0.21442	0.19810	0.18351	0.17043	0.15868	0.14809	0.13854	0.12989	0.12205
	10	[8]	0.10154	0.08614	0.07328	0.06253	0.05351	0.04595	0.03957	0.03420	0.02964
		[9]	0.10197	0.08652	0.07361	0.06282	0.05377	0.04617	0.03977	0.03437	0.02979
		[10]	0.10152	0.08611	0.07326	0.06250	0.05350	0.04593	0.03956	0.03418	0.02963
	15	[11]	0.05295	0.04140	0.03245	0.02550	0.02011	0.01591	0.01263	0.01006	0.00805
20	5	[12]	0.21361	0.19734	0.18279	0.16975	0.15803	0.14749	0.13796	0.12935	0.12154
		[13]	0.21374	0.19748	0.18293	0.16989	0.15818	0.14763	0.13811	0.12949	0.12167
		[14]	0.21322	0.19700	0.18250	0.16950	0.15782	0.14730	0.13781	0.12921	0.12142
	10	[15]	0.10098	0.08570	0.07293	0.06226	0.05331	0.04580	0.03946	0.03412	0.02959
		[16]	0.10157	0.08622	0.07339	0.06266	0.05366	0.04610	0.03973	0.03435	0.02979
		[17]	0.10138	0.08606	0.07326	0.06255	0.05357	0.04602	0.03966	0.03429	0.02974
	15	[18]	0.05306	0.04148	0.03250	0.02554	0.02012	0.01591	0.01262	0.01005	0.00803
		[19]	0.05252	0.04102	0.03211	0.02520	0.01984	0.01568	0.01243	0.00989	0.00790
		[20]	0.05220	0.04074	0.03186	0.02499	0.01966	0.01552	0.01230	0.00978	0.00781
	20	[21]	0.02754	0.01982	0.01429	0.01032	0.00748	0.00544	0.00397	0.00291	0.00214
30	10	[22]	0.10041	0.08510	0.07234	0.06168	0.05275	0.04526	0.03896	0.03365	0.02916
		[23]	0.09977	0.08457	0.07189	0.06130	0.05243	0.04499	0.03874	0.03346	0.02899
		[24]	0.09961	0.08439	0.07170	0.06110	0.05224	0.04481	0.03856	0.03329	0.02883
	20	[25]	0.02724	0.01957	0.01409	0.01017	0.00736	0.00535	0.00390	0.00286	0.00210
		[26]	0.02724	0.01955	0.01406	0.01013	0.00732	0.00531	0.00387	0.00283	0.00208
		[27]	0.02705	0.01941	0.01396	0.01006	0.00727	0.00527	0.00384	0.00281	0.00206
	25	[28]	0.01463	0.00965	0.00637	0.00421	0.00279	0.00186	0.00124	0.00083	0.00056
		[29]	0.01454	0.00959	0.00632	0.00418	0.00277	0.00184	0.00123	0.00082	0.00056
		[30]	0.01460	0.00964	0.00637	0.00422	0.00280	0.00186	0.00125	0.00084	0.00056
	30	[31]	0.00801	0.00488	0.00297	0.00181	0.00111	0.00068	0.00042	0.00026	0.00016
50	15	[32]	0.05038	0.03923	0.03063	0.02398	0.01884	0.01486	0.01176	0.00934	0.00745
		[33]	0.05097	0.03974	0.03107	0.02436	0.01916	0.01513	0.01199	0.00953	0.00761
		[34]	0.05074	0.03958	0.03095	0.02427	0.01910	0.01508	0.01195	0.00951	0.00759
	30	[35]	0.00797	0.00485	0.00295	0.00180	0.00110	0.00067	0.00042	0.00026	0.00016
		[36]	0.00789	0.00480	0.00292	0.00178	0.00109	0.00067	0.00041	0.00025	0.00016
		[37]	0.00803	0.00489	0.00299	0.00182	0.00112	0.00069	0.00043	0.00026	0.00017
	40	[38]	0.00259	0.00135	0.00070	0.00037	0.00019	0.00010	0.00005	0.00003	0.00002
		[39]	0.00258	0.00135	0.00070	0.00037	0.00019	0.00010	0.00005	0.00003	0.00002
		[40]	0.00257	0.00134	0.00070	0.00036	0.00019	0.00010	0.00005	0.00003	0.00002
	50	[41]	0.00081	0.00036	0.00016	0.00007	0.00003	0.00001	0.00001	0.00000	0.00000

Table 16. The estimated risks of the Bayesian rejection region under $k = 5,\ \ \lambda = 0.5,\ \ \beta = 2(0.1)2.8, L_X = 0.8, (a, b) = (2, 2), c = 0.2$ and $K_1 = K_2 = 1$

n	m		2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8
10	5	[1]	0.213302	0.197071	0.182556	0.169545	0.157856	0.147328	0.137824	0.129225	0.121425
		[2]	0.213807	0.197533	0.182978	0.169933	0.158212	0.147656	0.138127	0.129504	0.121684
		[3]	0.213131	0.196912	0.182409	0.169409	0.157729	0.147211	0.137716	0.129124	0.121331
	10	[4]	0.100756	0.085423	0.072629	0.061936	0.052983	0.045468	0.039145	0.03381	0.029296
15	5	[5]	0.214419	0.198107	0.183519	0.170442	0.158693	0.14811	0.138556	0.129911	0.122069
		[6]	0.214291	0.197983	0.1834	0.170328	0.158583	0.148005	0.138456	0.129815	0.121978
		[7]	0.214419	0.198102	0.183509	0.170428	0.158676	0.148091	0.138536	0.129889	0.122046
	10	[8]	0.101543	0.08614	0.07328	0.062525	0.053514	0.045946	0.039574	0.034195	0.029642
		[9]	0.101969	0.086517	0.073612	0.062817	0.05377	0.04617	0.039771	0.034368	0.029794
		[10]	0.101515	0.086114	0.073256	0.062504	0.053495	0.04593	0.03956	0.034183	0.029632
	15	[11]	0.05295	0.0414	0.032447	0.025501	0.020105	0.015905	0.012627	0.010062	0.008049
20	5	[12]	0.213606	0.197335	0.182787	0.169748	0.158034	0.147485	0.137964	0.129348	0.121535
		[13]	0.213743	0.197477	0.182931	0.169893	0.158179	0.14763	0.138106	0.129488	0.121671
		[14]	0.213223	0.197002	0.182498	0.169498	0.157819	0.147301	0.137806	0.129213	0.121419
	10	[15]	0.100984	0.085698	0.072934	0.062258	0.053311	0.045795	0.039464	0.034118	0.02959
		[16]	0.101572	0.086217	0.07339	0.062658	0.05366	0.046099	0.039729	0.034348	0.02979
		[17]	0.101383	0.086059	0.073258	0.062547	0.053566	0.046019	0.039661	0.03429	0.02974
	15	[18]	0.053063	0.041483	0.032504	0.025537	0.020124	0.015911	0.012624	0.010052	0.008034
		[19]	0.052521	0.04102	0.03211	0.025204	0.019844	0.015676	0.012427	0.009888	0.007897
		[20]	0.052201	0.040737	0.031864	0.024992	0.019662	0.015522	0.012297	0.009778	0.007805
	20	[21]	0.027543	0.019821	0.014288	0.010324	0.007483	0.005442	0.003973	0.002912	0.002144
30	10	[22]	0.10041	0.085103	0.072338	0.061675	0.05275	0.045261	0.038962	0.033649	0.029155
		[23]	0.099772	0.084568	0.071889	0.061297	0.052432	0.044993	0.038736	0.033457	0.028992
		[24]	0.099609	0.084386	0.071698	0.061104	0.052241	0.044808	0.038558	0.033289	0.028834
	20	[25]	0.027237	0.019572	0.014088	0.010166	0.007359	0.005346	0.003898	0.002855	0.0021
		[26]	0.027241	0.019552	0.014056	0.01013	0.007322	0.005312	0.003868	0.002828	0.002077
		[27]	0.027047	0.019411	0.013955	0.010057	0.00727	0.005273	0.00384	0.002808	0.002063
	25	[28]	0.014625	0.009648	0.006368	0.00421	0.002791	0.001856	0.001239	0.000831	0.00056
		[29]	0.014539	0.009585	0.006323	0.004179	0.002769	0.001841	0.001229	0.000824	0.000555
		[30]	0.014603	0.009642	0.006371	0.004217	0.002799	0.001864	0.001246	0.000837	0.000564
	30	[31]	0.008007	0.004877	0.002969	0.00181	0.001106	0.000678	0.000417	0.000258	0.00016
50	15	[32]	0.050381	0.039232	0.030629	0.023984	0.018844	0.014858	0.011759	0.009343	0.007452
		[33]	0.050965	0.039742	0.031069	0.02436	0.019162	0.015126	0.011985	0.009532	0.007611
		[34]	0.050742	0.039579	0.030949	0.024273	0.019099	0.01508	0.011951	0.009508	0.007593
	30	[35]	0.007967	0.004852	0.002954	0.0018	0.0011	0.000674	0.000415	0.000257	0.000159
		[36]	0.007885	0.004799	0.00292	0.001779	0.001087	0.000666	0.00041	0.000253	0.000157
		[37]	0.008025	0.004893	0.002985	0.001824	0.001118	0.000687	0.000425	0.000264	0.000165
	40	[38]	0.002588	0.001348	0.000702	0.000365	0.000191	0.0001	0.000052	0.000028	0.000015
		[39]	0.002583	0.001349	0.000704	0.000368	0.000192	0.000101	0.000053	0.000028	0.000015
		[40]	0.00257	0.00134	0.00070	0.00036	0.00019	0.00010	0.00005	0.00003	0.00002
	50	[41]	0.00081	0.00036	0.00016	0.00007	0.00003	0.00001	0.00001	0.00000	0.00000

Table 17. The power for $c_1 = 0(0.1)0.9$ with $k = 3, \lambda = 6, L_X = 1.5, c^* = 0.3$ and $\alpha = 0.05$

n	m	R	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
10	5	[1]	0.0033	0.0089	0.0222	0.0507	0.1063	0.2177	0.3977	0.6414	0.8728	0.9887
		[2]	0.0033	0.0089	0.0222	0.0507	0.1064	0.2179	0.3978	0.6413	0.8726	0.9887
		[3]	0.0033	0.0089	0.0222	0.0507	0.1063	0.2178	0.3978	0.6414	0.8727	0.9887
	10	[4]	0.0012	0.0047	0.0162	0.0521	0.1398	0.3225	0.5899	0.8578	0.9828	0.9998
15	5	[5]	0.0033	0.0089	0.0222	0.0508	0.1065	0.2180	0.3983	0.6417	0.8730	0.9887
		[6]	0.0033	0.0089	0.0222	0.0508	0.1065	0.2178	0.3980	0.6415	0.8730	0.9887
		[7]	0.0033	0.0089	0.0222	0.0507	0.1064	0.2179	0.3978	0.6413	0.8726	0.9887
	10	[8]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8583	0.9828	0.9998
		[9]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3226	0.5900	0.8580	0.9828	0.9998
		[10]	0.0012	0.0047	0.0162	0.0521	0.1401	0.3227	0.5903	0.8584	0.9828	0.9998
	15	[11]	0.0003	0.0025	0.0119	0.0501	0.1650	0.4017	0.7176	0.9446	0.9975	1.0000
20	5	[12]	0.0033	0.0089	0.0222	0.0508	0.1066	0.2181	0.3983	0.6418	0.8730	0.9887
		[13]	0.0033	0.0089	0.0222	0.0508	0.1064	0.2179	0.3977	0.6412	0.8725	0.9887
		[14]	0.0033	0.0089	0.0222	0.0508	0.1064	0.2177	0.3979	0.6415	0.8730	0.9887
	10	[15]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8582	0.9828	0.9998
		[16]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3226	0.5901	0.8581	0.9828	0.9998
		[17]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8582	0.9828	0.9998
	15	[18]	0.0003	0.0025	0.0119	0.0501	0.1650	0.4016	0.7176	0.9446	0.9975	1.0000
		[19]	0.0003	0.0025	0.0119	0.0501	0.1651	0.4017	0.7177	0.9447	0.9975	1.0000
		[20]	0.0003	0.0025	0.0119	0.0502	0.1651	0.4017	0.7178	0.9449	0.9975	1.0000
	20	[21]	0.0001	0.0015	0.0109	0.0506	0.1862	0.4742	0.8143	0.9784	0.9998	1.0000
30	10	[22]	0.0013	0.0048	0.0162	0.0523	0.1403	0.3232	0.5909	0.8588	0.9829	0.9998
		[23]	0.0012	0.0047	0.0162	0.0522	0.1402	0.3229	0.5905	0.8584	0.9828	0.9998
		[24]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8582	0.9828	0.9998
	20	[25]	0.0001	0.0015	0.0109	0.0507	0.1863	0.4745	0.8144	0.9784	0.9998	1.0000
		[26]	0.0001	0.0015	0.0109	0.0507	0.1865	0.4747	0.8148	0.9785	0.9998	1.0000
		[27]	0.0001	0.0015	0.0109	0.0506	0.1862	0.4743	0.8143	0.9784	0.9998	1.0000
	25	[28]	0.0000	0.0007	0.0092	0.0527	0.2081	0.5472	0.8792	0.9917	1.0000	1.0000
		[29]	0.0000	0.0007	0.0092	0.0527	0.2081	0.5473	0.8792	0.9917	1.0000	1.0000
		[30]	0.0000	0.0007	0.0092	0.0527	0.2080	0.5472	0.8791	0.9917	1.0000	1.0000
	30	[31]	0.0000	0.0008	0.0078	0.0528	0.2306	0.6042	0.9203	0.9975	1.0000	1.0000
50	15	[32]	0.0003	0.0025	0.0118	0.0498	0.1656	0.4034	0.7177	0.9450	0.9975	1.0000
		[33]	0.0003	0.0025	0.0119	0.0501	0.1654	0.4020	0.7180	0.9450	0.9975	1.0000
		[34]	0.0003	0.0025	0.0119	0.0501	0.1650	0.4017	0.7176	0.9446	0.9975	1.0000
	30	[35]	0.0000	0.0008	0.0078	0.0528	0.2306	0.6044	0.9203	0.9975	1.0000	1.0000
		[36]	0.0000	0.0008	0.0078	0.0528	0.2306	0.6044	0.9203	0.9975	1.0000	1.0000
		[37]	0.0000	0.0007	0.0079	0.0529	0.2310	0.6049	0.9205	0.9975	1.0000	1.0000
	40	[38]	0.0001	0.0004	0.0057	0.0523	0.2730	0.7060	0.9656	0.9997	1.0000	1.0000
		[39]	0.0001	0.0004	0.0057	0.0523	0.2730	0.7059	0.9656	0.9997	1.0000	1.0000
		[40]	0.0001	0.0004	0.0057	0.0523	0.2730	0.7060	0.9656	0.9997	1.0000	1.0000
	50	[41]	0.0000	0.0001	0.0045	0.0514	0.3126	0.7824	0.9841	1.0000	1.0000	1.0000

Table 18. The power for $c_1 = 0(0.1)0.9$ with $k = 5, \lambda = 6, L_X = 1.5, c^* = 0.3$ and $\alpha = 0.05$

n	m	R	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
10	5	[1]	0.0033	0.0089	0.0222	0.0507	0.1063	0.2177	0.3977	0.6414	0.8728	0.9887
		[2]	0.0033	0.0089	0.0222	0.0507	0.1064	0.2179	0.3978	0.6413	0.8726	0.9887
		[3]	0.0033	0.0089	0.0222	0.0507	0.1063	0.2178	0.3978	0.6414	0.8727	0.9887
	10	[4]	0.0012	0.0047	0.0162	0.0521	0.1398	0.3225	0.5899	0.8578	0.9828	0.9998
15	5	[5]	0.0033	0.0089	0.0222	0.0508	0.1065	0.2180	0.3983	0.6417	0.8730	0.9887
		[6]	0.0033	0.0089	0.0222	0.0508	0.1065	0.2178	0.3980	0.6415	0.8730	0.9887
		[7]	0.0033	0.0089	0.0222	0.0507	0.1064	0.2179	0.3978	0.6413	0.8726	0.9887
	10	[8]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8583	0.9828	0.9998
		[9]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3226	0.5900	0.8580	0.9828	0.9998
		[10]	0.0012	0.0047	0.0162	0.0521	0.1401	0.3227	0.5903	0.8584	0.9828	0.9998
	15	[11]	0.0003	0.0025	0.0119	0.0501	0.1650	0.4017	0.7176	0.9446	0.9975	1.0000
20	5	[12]	0.0033	0.0089	0.0222	0.0508	0.1066	0.2181	0.3983	0.6418	0.8730	0.9887
		[13]	0.0033	0.0089	0.0222	0.0508	0.1064	0.2179	0.3977	0.6412	0.8725	0.9887
		[14]	0.0033	0.0089	0.0222	0.0508	0.1064	0.2177	0.3979	0.6415	0.8730	0.9887
	10	[15]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8582	0.9828	0.9998
		[16]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3226	0.5901	0.8581	0.9828	0.9998
		[17]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8582	0.9828	0.9998
	15	[18]	0.0003	0.0025	0.0119	0.0501	0.1650	0.4016	0.7176	0.9446	0.9975	1.0000
		[19]	0.0003	0.0025	0.0119	0.0501	0.1651	0.4017	0.7177	0.9447	0.9975	1.0000
		[20]	0.0003	0.0025	0.0119	0.0502	0.1651	0.4017	0.7178	0.9449	0.9975	1.0000
	20	[21]	0.0001	0.0015	0.0109	0.0506	0.1862	0.4742	0.8143	0.9784	0.9998	1.0000
30	10	[22]	0.0013	0.0048	0.0162	0.0523	0.1403	0.3232	0.5909	0.8588	0.9829	0.9998
		[23]	0.0012	0.0047	0.0162	0.0522	0.1402	0.3229	0.5905	0.8584	0.9828	0.9998
		[24]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8582	0.9828	0.9998
	20	[25]	0.0001	0.0015	0.0109	0.0507	0.1863	0.4745	0.8144	0.9784	0.9998	1.0000
		[26]	0.0001	0.0015	0.0109	0.0507	0.1865	0.4747	0.8148	0.9785	0.9998	1.0000
		[27]	0.0001	0.0015	0.0109	0.0506	0.1862	0.4743	0.8143	0.9784	0.9998	1.0000
	25	[28]	0.0000	0.0007	0.0092	0.0527	0.2081	0.5472	0.8792	0.9917	1.0000	1.0000
		[29]	0.0000	0.0007	0.0092	0.0527	0.2081	0.5473	0.8792	0.9917	1.0000	1.0000
		[30]	0.0000	0.0007	0.0092	0.0527	0.2080	0.5472	0.8791	0.9917	1.0000	1.0000
	30	[31]	0.0000	0.0008	0.0078	0.0528	0.2306	0.6042	0.9203	0.9975	1.0000	1.0000
50	15	[32]	0.0003	0.0025	0.0118	0.0498	0.1656	0.4034	0.7177	0.9450	0.9975	1.0000
		[33]	0.0003	0.0025	0.0119	0.0501	0.1654	0.4020	0.7180	0.9450	0.9975	1.0000
		[34]	0.0003	0.0025	0.0119	0.0501	0.1650	0.4017	0.7176	0.9446	0.9975	1.0000
	30	[35]	0.0000	0.0008	0.0078	0.0528	0.2306	0.6044	0.9203	0.9975	1.0000	1.0000
		[36]	0.0000	0.0008	0.0078	0.0528	0.2306	0.6044	0.9203	0.9975	1.0000	1.0000
		[37]	0.0000	0.0007	0.0079	0.0529	0.2310	0.6049	0.9205	0.9975	1.0000	1.0000
	40	[38]	0.0001	0.0004	0.0057	0.0523	0.2730	0.7060	0.9656	0.9997	1.0000	1.0000
		[39]	0.0001	0.0004	0.0057	0.0523	0.2730	0.7059	0.9656	0.9997	1.0000	1.0000
		[40]	0.0001	0.0004	0.0057	0.0523	0.2730	0.7060	0.9656	0.9997	1.0000	1.0000
	50	[41]	0.0000	0.0001	0.0045	0.0514	0.3126	0.7824	0.9841	1.0000	1.0000	1.0000

Table 19. The power for $c_1 = 0(0.1)0.9$ with $k = 3, \lambda = 2.75, L_X = 1.25, c^* = 0.3$ and $\alpha = 0.05$

n	m	R	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
10	5	[1]	0.0033	0.0089	0.0222	0.0507	0.1063	0.2177	0.3977	0.6414	0.8728	0.9887
		[2]	0.0033	0.0089	0.0222	0.0507	0.1064	0.2179	0.3978	0.6413	0.8726	0.9887
		[3]	0.0033	0.0089	0.0222	0.0507	0.1063	0.2178	0.3978	0.6414	0.8727	0.9887
	10	[4]	0.0012	0.0047	0.0162	0.0521	0.1398	0.3225	0.5899	0.8578	0.9828	0.9998
15	5	[5]	0.0033	0.0089	0.0222	0.0508	0.1065	0.2180	0.3983	0.6417	0.8730	0.9887
		[6]	0.0033	0.0089	0.0222	0.0508	0.1065	0.2178	0.3980	0.6415	0.8730	0.9887
		[7]	0.0033	0.0089	0.0222	0.0507	0.1064	0.2179	0.3978	0.6413	0.8726	0.9887
	10	[8]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8583	0.9828	0.9998
		[9]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3226	0.5900	0.8580	0.9828	0.9998
		[10]	0.0012	0.0047	0.0162	0.0521	0.1401	0.3227	0.5903	0.8584	0.9828	0.9998
	15	[11]	0.0003	0.0025	0.0119	0.0501	0.1650	0.4017	0.7176	0.9446	0.9975	1.0000
20	5	[12]	0.0033	0.0089	0.0222	0.0508	0.1066	0.2181	0.3983	0.6418	0.8730	0.9887
		[13]	0.0033	0.0089	0.0222	0.0508	0.1064	0.2179	0.3977	0.6412	0.8725	0.9887
		[14]	0.0033	0.0089	0.0222	0.0508	0.1064	0.2177	0.3979	0.6415	0.8730	0.9887
	10	[15]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8582	0.9828	0.9998
		[16]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3226	0.5901	0.8581	0.9828	0.9998
		[17]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8582	0.9828	0.9998
	15	[18]	0.0003	0.0025	0.0119	0.0501	0.1650	0.4016	0.7176	0.9446	0.9975	1.0000
		[19]	0.0003	0.0025	0.0119	0.0501	0.1651	0.4017	0.7177	0.9447	0.9975	1.0000
		[20]	0.0003	0.0025	0.0119	0.0502	0.1651	0.4017	0.7178	0.9449	0.9975	1.0000
	20	[21]	0.0001	0.0015	0.0109	0.0506	0.1862	0.4742	0.8143	0.9784	0.9998	1.0000
30	10	[22]	0.0013	0.0048	0.0162	0.0523	0.1403	0.3232	0.5909	0.8588	0.9829	0.9998
		[23]	0.0012	0.0047	0.0162	0.0522	0.1402	0.3229	0.5905	0.8584	0.9828	0.9998
		[24]	0.0012	0.0047	0.0162	0.0521	0.1399	0.3227	0.5902	0.8582	0.9828	0.9998
	20	[25]	0.0001	0.0015	0.0109	0.0507	0.1863	0.4745	0.8144	0.9784	0.9998	1.0000
		[26]	0.0001	0.0015	0.0109	0.0507	0.1865	0.4747	0.8148	0.9785	0.9998	1.0000
		[27]	0.0001	0.0015	0.0109	0.0506	0.1862	0.4743	0.8143	0.9784	0.9998	1.0000
	25	[28]	0.0000	0.0007	0.0092	0.0527	0.2081	0.5472	0.8792	0.9917	1.0000	1.0000
		[29]	0.0000	0.0007	0.0092	0.0527	0.2081	0.5473	0.8792	0.9917	1.0000	1.0000
		[30]	0.0000	0.0007	0.0092	0.0527	0.2080	0.5472	0.8791	0.9917	1.0000	1.0000
	30	[31]	0.0000	0.0008	0.0078	0.0528	0.2306	0.6042	0.9203	0.9975	1.0000	1.0000
50	15	[32]	0.0003	0.0025	0.0118	0.0498	0.1656	0.4034	0.7177	0.9450	0.9975	1.0000
		[33]	0.0003	0.0025	0.0119	0.0501	0.1654	0.4020	0.7180	0.9450	0.9975	1.0000
		[34]	0.0003	0.0025	0.0119	0.0501	0.1650	0.4017	0.7176	0.9446	0.9975	1.0000
	30	[35]	0.0000	0.0008	0.0078	0.0528	0.2306	0.6044	0.9203	0.9975	1.0000	1.0000
		[36]	0.0000	0.0008	0.0078	0.0528	0.2306	0.6044	0.9203	0.9975	1.0000	1.0000
		[37]	0.0000	0.0007	0.0079	0.0529	0.2310	0.6049	0.9205	0.9975	1.0000	1.0000
	40	[38]	0.0001	0.0004	0.0057	0.0523	0.2730	0.7060	0.9656	0.9997	1.0000	1.0000
		[39]	0.0001	0.0004	0.0057	0.0523	0.2730	0.7059	0.9656	0.9997	1.0000	1.0000
		[40]	0.0001	0.0004	0.0057	0.0523	0.2730	0.7060	0.9656	0.9997	1.0000	1.0000
	50	[41]	0.0000	0.0001	0.0045	0.0514	0.3126	0.7824	0.9841	1.0000	1.0000	1.0000

Table 20. The power for $c_1 = 0(0.1)0.9$ with $k = 5, \lambda = 2.75, L_X = 1.25, c^* = 0.3$ and $\alpha = 0.05$

n	m	R	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
10	5	[1]	0.0034	0.0093	0.0225	0.0512	0.1073	0.2192	0.3999	0.6443	0.8743	0.9890
		[2]	0.0034	0.0093	0.0226	0.0513	0.1075	0.2194	0.4002	0.6446	0.8743	0.9890
		[3]	0.0034	0.0093	0.0226	0.0513	0.1074	0.2194	0.4002	0.6445	0.8742	0.9890
	10	[4]	0.0014	0.0049	0.0163	0.0524	0.1408	0.3243	0.5925	0.8596	0.9833	0.9998
15	5	[5]	0.0033	0.0092	0.0224	0.0509	0.1072	0.2191	0.3999	0.6443	0.8744	0.9890
		[6]	0.0034	0.0093	0.0225	0.0511	0.1072	0.2194	0.4000	0.6445	0.8744	0.9890
		[7]	0.0034	0.0093	0.0226	0.0513	0.1075	0.2194	0.4002	0.6447	0.8743	0.9890
	10	[8]	0.0014	0.0049	0.0163	0.0524	0.1409	0.3243	0.5925	0.8596	0.9833	0.9998
		[9]	0.0014	0.0049	0.0163	0.0524	0.1408	0.3245	0.5926	0.8597	0.9833	0.9998
		[10]	0.0014	0.0049	0.0163	0.0524	0.1408	0.3243	0.5925	0.8597	0.9833	0.9998
	15	[11]	0.0003	0.0026	0.0120	0.0499	0.1653	0.4029	0.7188	0.9456	0.9976	1.0000
20	5	[12]	0.0034	0.0093	0.0225	0.0510	0.1071	0.2192	0.3997	0.6443	0.8743	0.9890
		[13]	0.0034	0.0093	0.0227	0.0514	0.1074	0.2193	0.4001	0.6445	0.8743	0.9890
		[14]	0.0034	0.0093	0.0225	0.0511	0.1072	0.2192	0.3998	0.6443	0.8743	0.9890
	10	[15]	0.0014	0.0049	0.0163	0.0524	0.1408	0.3244	0.5926	0.8596	0.9833	0.9998
		[16]	0.0014	0.0049	0.0163	0.0524	0.1408	0.3245	0.5926	0.8597	0.9833	0.9998
		[17]	0.0014	0.0049	0.0163	0.0524	0.1408	0.3244	0.5925	0.8596	0.9833	0.9998
	15	[18]	0.0003	0.0026	0.0120	0.0499	0.1653	0.4028	0.7188	0.9456	0.9976	1.0000
		[19]	0.0003	0.0026	0.0120	0.0498	0.1652	0.4028	0.7190	0.9457	0.9976	1.0000
		[20]	0.0003	0.0026	0.0120	0.0498	0.1653	0.4030	0.7190	0.9458	0.9976	1.0000
	20	[21]	0.0002	0.0016	0.0112	0.0506	0.1862	0.4757	0.8156	0.9789	0.9998	1.0000
30	10	[22]	0.0014	0.0048	0.0163	0.0519	0.1405	0.3245	0.5921	0.8596	0.9834	0.9998
		[23]	0.0014	0.0049	0.0163	0.0524	0.1407	0.3242	0.5925	0.8597	0.9833	0.9998
		[24]	0.0014	0.0049	0.0163	0.0524	0.1408	0.3244	0.5926	0.8596	0.9833	0.9998
	20	[25]	0.0002	0.0016	0.0112	0.0507	0.1864	0.4759	0.8157	0.9790	0.9998	1.0000
		[26]	0.0002	0.0016	0.0112	0.0506	0.1863	0.4762	0.8155	0.9789	0.9998	1.0000
		[27]	0.0002	0.0016	0.0112	0.0506	0.1862	0.4756	0.8156	0.9789	0.9998	1.0000
	25	[28]	0.0000	0.0009	0.0096	0.0528	0.2074	0.5486	0.8800	0.9919	1.0000	1.0000
		[29]	0.0000	0.0009	0.0096	0.0529	0.2074	0.5487	0.8800	0.9919	1.0000	1.0000
		[30]	0.0000	0.0009	0.0096	0.0530	0.2076	0.5485	0.8798	0.9919	1.0000	1.0000
	30	[31]	0.0000	0.0007	0.0081	0.0527	0.2299	0.6050	0.9208	0.9976	1.0000	1.0000
50	15	[32]	0.0002	0.0026	0.0120	0.0500	0.1654	0.4028	0.7186	0.9459	0.9976	1.0000
		[33]	0.0003	0.0026	0.0120	0.0497	0.1654	0.4033	0.7187	0.9458	0.9976	1.0000
		[34]	0.0003	0.0026	0.0120	0.0499	0.1653	0.4029	0.7188	0.9456	0.9976	1.0000
	30	[35]	0.0000	0.0007	0.0081	0.0526	0.2298	0.6053	0.9209	0.9976	1.0000	1.0000
		[36]	0.0000	0.0007	0.0081	0.0527	0.2298	0.6053	0.9209	0.9976	1.0000	1.0000
		[37]	0.0000	0.0007	0.0082	0.0526	0.2298	0.6048	0.9207	0.9976	1.0000	1.0000
	40	[38]	0.0000	0.0003	0.0056	0.0525	0.2728	0.7058	0.9658	0.9997	1.0000	1.0000
		[39]	0.0000	0.0003	0.0056	0.0525	0.2724	0.7061	0.9658	0.9997	1.0000	1.0000
		[40]	0.0000	0.0003	0.0056	0.0525	0.2725	0.7060	0.9657	0.9997	1.0000	1.0000
	50	[41]	0.0000	0.0000	0.0045	0.0510	0.3117	0.7835	0.9842	1.0000	1.0000	1.0000

Data availability

The data and the statistical software R codes used in the simulation and examples are available upon readers' request to the authors.

Conflict of interest

The authors declare that they have no competing interests.

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