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A New Generalization of the Inverted Gompertz Distribution with Different Methods of Estimation and Applications

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Abstract Designing appropriate models for analyzing data in various fields is essential as it helps professionals comprehend complex data patterns and their characteristics, leading to informed decision-making. Despite the diversity of probability distribution, the data may not conform to classical distributions in many instances. Consequently, there arises a need for a new distribution that can accommodate the intricacies of diverse data forms and enhance the goodness of fit. This article introduces a novel extended lifetime model called the new exponential exponentiated generalized inverted Gompertz based on the new exponential-X family of distributions. The article discusses some statistical properties associated with the proposed distribution. The parameters of the new distribution are estimated using multiple estimation techniques, and their performance is compared through Monte Carlo simulations. The demonstrated potential and effectiveness of the proposed distribution are exemplified by its application to three datasets within various fields.

Keywords T-X family, exponential-X family of distributions, inverted Gompertz distribution, maximum likelihood, maximum product of spacing, Ordinary and Weighted least squares estimators, Anderson–Darling estimators, Monte Carlo simulations.

AMS 2010 subject classifications 62E10 , 62N05

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1. Introduction

Professionals are always searching for statistical distributions that can accurately represent phenomena in various fields such as medicine, engineering, economics and other sciences. These distributions are crucial in helping specialists understand the behavior of different phenomena, estimate their features, and predict them correctly. Traditional statistical distributions are usually not appropriate in representing complex phenomena. Therefore, statisticians resort to creating new distributions to study various phenomena that current distributions cannot explain.

Statisticians use several methods to improve the functionality and adaptability of density and hazard rate functions for modeling data diversity. Compounding, adding parameters, composing, and transforming are some techniques for extending distributions. The beta-generated method by [18], the Kumaraswamy-generated method by [21] and the transformed-transformer approach by [7] are examples of some different generated methods.

These distributions have been developed and derived to enhance data fitting and analysis across various fields. [1] introduced an extended odd inverse Weibull generator family by incorporating an additional shape parameter into the inverse Weibull generator of distributions and applied it in engineering and chemistry. [24] merged the Odd

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Lomax-G family with Inverse Weibull to analyze Climate data. [6] introduced the exponential generalized inverse generalized Weibull distribution and utilized it in medical and engineering applications. Recently, [10] introduced a new distribution by combining features of the odd Weibull family and the inverse Gompertz distribution. This innovative family merged characteristics from both distributions and was subsequently applied in the field of psychology.

One of the effective transformation is the T-X method which proposed by [7] to generate families of continuous distributions and can be obtained by using any continuous random variable as a generator. Let $r(t)$ be the probability density function of a random variable $T \epsilon \left[a, b\right], -\infty < a < b < \infty$ and let $W\left[G(x)\right]$ be a function of the cumulative distribution function of a random variable X with the following conditions:

i. $W\left[G(x)\right] \epsilon \left[a, b\right]$.

ii. $W\left[G(x)\right]$ is differentiable and monotonically non-decreasing.

iii. $W\left[G(x)\right] \rightarrow a \; as \; x \rightarrow -\infty$ and $W\left[G(x)\right] \rightarrow b \; as \; x \rightarrow \infty$.

The cdf and the pdf of T-X family of distributions are defined respectively, by

$$F\left(x\right) = \int_{a}^{W\left[G(x)\right]} r\left(t\right) dt,$$

$$f\left(x\right) = \left\{ \frac{d}{dx} W\left[G(x)\right] \right\} r\left\{ W\left[G(x)\right] \right\} \tag{1}$$

where $W\left[G(x)\right]$ satisfies the condition from (i) to (iii).

[20] proposed the new lifetime exponential-X family (NLTE-X) based on $T - X$ transformation by setting $W[F(x)] = -log(\frac{(1-F(x))}{e^{\theta F(x)}})$ and put $T \sim Exp(1)$ with cdf given by $R(t) = 1 - e^{-t}, \quad t \geq 0$ and pdf given by $r(t) = e^{-t}, \; t > 0$.

Then the CDF and PDF of NLTE-X family corresponding to (1) are given by

$$G(x; \theta, \xi) = 1 - \left\{ \frac{1 - F(x; \Theta)}{e^{\theta F(x;\Theta)}} \right\}; \; \theta > 0, \; x > 0, \tag{2}$$

$$g(x; \theta, \xi) = f(x) \frac{\{1 + \theta \overline{F}(x; \Theta)\}}{e^{\theta F(x;\Theta)}}; \; \theta > 0, \; x > 0, \tag{3}$$

where Θ is the vector of distribution parameters and θ is an NLTE-X parameter.

The NLTE-X family was employed to produce various distributions, including the exponential Fréchet distribution [8], the exponential inverted Topp-Leone distribution [28], the exponentiated Weibull [9], and the exponential-X power family of distribution [19].

Recently, some generalizations of inverse distributions have been studied in the literature. These include the Kumaraswamy-inverse Weibull distribution proposed by [32], the Kumaraswamy inverse exponential distribution developed by [29], the alpha power inverse Rayleigh distribution developed by [27], the exponentiated inverse Rayleigh distribution introduced by [30], the Weibull inverted exponential distribution by [14], the Marshall-Olkin alpha power inverse Weibull distribution by [11], the alpha-power exponentiated inverse Rayleigh distribution by [4], and the odd Weibull inverse Topp-Leone distribution proposed by [5]. The Exponentiated Generalised Inverted Gompertz Distribution (EGIG) by [15] is a new lifetime model that applies exponentiated generalized (EG) family [7] to inverted Gompertz (IG) distributions [17]. The CDF and PDF of EGIG are given as

$$F(x)=\left[1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{\gamma}\right]^{\lambda};\ \alpha,\beta,\gamma,\lambda>0,\ x>0, \tag{4}$$

$$f(x)=\frac{\alpha\gamma\lambda}{x^2}e^{\frac{\beta}{x}}e^{-\frac{\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{(\gamma-1)}\left[1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{\gamma}\right]^{(\lambda-1)};\alpha,\beta,\gamma,\lambda>0,\ x>0, \tag{5}$$

where the scale parameter β and three shape parameters α, γ, and λ improve the EGIG's usefulness and adaptability. This distribution has proven its efficiency in representing multiple types of data.

The main objective of this article is to propose a new generalization of the EGIG based on the NLTE-X distribution family called the new exponential exponentiated generalized inverted Gompertz distribution (NEEGIG). The NEEGIG distribution offers several advantages:

- It enhances the flexibility of density and hazard rate functions, allowing for accurate modeling of various real-world applications.
- It provides improved flexibility over the EGIG distribution by introducing new generalizations and offering a better fit when compared to other distributions.
- The hazard rate function for NEEGIG can take on various forms, including symmetrical and asymmetrical shapes in its density function. This versatility allows NEEGIG to effectively model a diverse range of data from engineering, medicine, Climate, psychology and reliability fields.
- The NEEGIG distribution introduces new generalizations of the IG, EG, and EGIG distributions by incorporating new parameters, thereby improving their flexibility and ability to accurately characterize tail shapes.
- The NEEGIG distribution's CDF and hazard rate functions, moments, and entropy are expressed in closed forms, making it useful for analyzing complete and censored data.
- Furthermore, our analysis of medical,economic and engineering data demonstrates that the NEEGIG distribution outperforms other modern and popular lifetime models when applied to bladder cancer, athletes' sun skin folds, waiting times in a bank, gauge lengths and the losses from passenger automobile insurance policies.

This article is classified as follows: Section 2 describes the NEEGIG using graphical representations. Section 3 derived some of the NEEGIG properties. Section 4, six estimation methods are used to estimate the NEEGIG parameter: maximum likelihood (ML), maximum product of spacing (MPS), ordinary least squares (OLS), weighted least squares (WLS), Cramér-von Mises (CM), and Ander-son-Darling (AD). Section 5 shows extensive simulation studies to evaluate the performance of various estimators. Section 6 investigates three applications in different fileds data to assess the NEEGIG's modeling effectiveness. Section 7 concludes with some final remarks.

2. New Exponential Exponentiated Generalized Inverted Gompertz Distribution

The NEEGIG's CDF and PDF can be obtained by substituting (4) and (5) in (2) and (3), as follows:

$$G(x)=1-\left\{\frac{1-\left[1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{\gamma}\right]^{\lambda}}{e^{\theta\left[1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{\gamma}\right]^{\lambda}}}\right\};\ \theta,\alpha,\beta,\gamma,\lambda>0,\ x>0, \tag{6}$$

$$\begin{aligned} g(x) = {} & \frac{\alpha\gamma\lambda}{x^2} e^{\frac{\beta}{x}} e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)} \left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{(\gamma-1)} \left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{(\lambda-1)} \\ & \times e^{-\theta\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{\lambda}} \left[1+\theta\left\{1-\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{\lambda}\right\}\right]; \\ & \theta,\alpha,\beta,\gamma,\lambda > 0,\ x>0, \end{aligned} \tag{7}$$

The survival function, $S(x)$ is defined as

$$S(x) = \left\{ \frac{1-\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{\lambda}}{e^{\theta\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{\lambda}}} \right\}. \tag{8}$$

The hazard rate function (hrf), which is frequently used in lifetime modeling to denote the likelihood of failure, is described as

$$\begin{aligned} hrf = {} & \frac{\alpha\gamma\lambda}{x^2} e^{\frac{\beta}{x}} e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)} \left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{(\gamma-1)} \left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{(\lambda-1)} \\ & \times \left[\frac{1+\theta\left\{1-\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{\lambda}\right\}}{1-\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{\lambda}} \right]. \end{aligned} \tag{9}$$

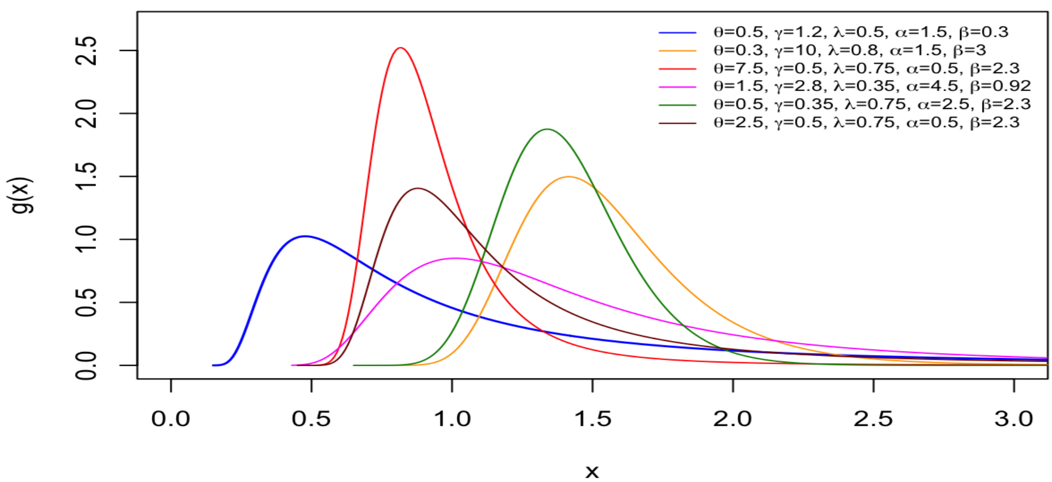


Figure 1. The NEEGIG density plots.

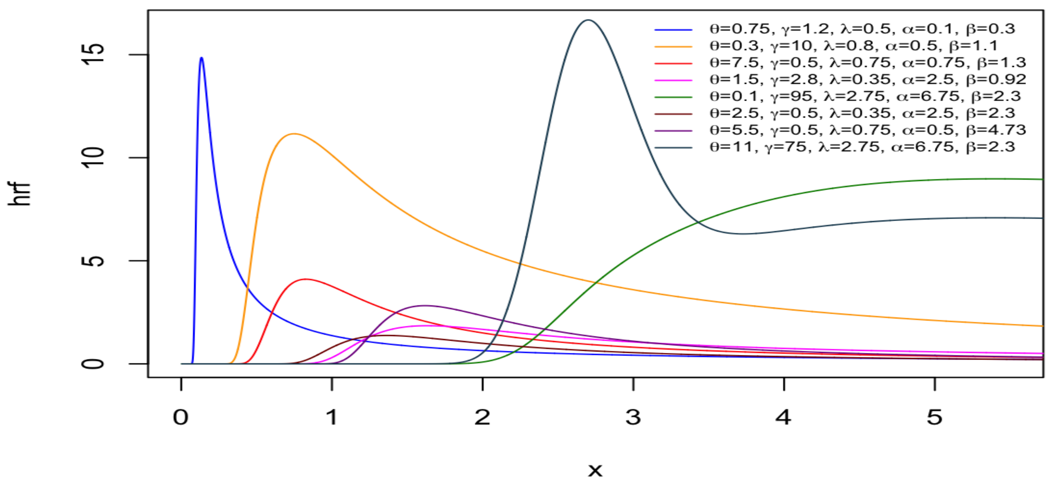


Figure 2. The NEEGIG hrf's plots.

Plots illustrating the PDF and hazard function (hrf) of the NEEGIG at various parameter values are shown. Figure (1) shows the PDF of the NEEGIG in various shapes, such as right-skewed, left-skewed, and decayed. Figure (2) shows different shapes of the hazard function, including upside-down, decreasing, and constant patterns. These diverse shapes demonstrate NEEGIG's significant adaptability in modeling real-world data.

2.1. Linear representation for the density of NEEGIG

Some mathematical expansions have been used to develop a linear representation of NEEGIG's PDF.

$$g(x) = \eta \frac{\alpha\gamma\lambda}{x^{(2+v_8)}} e^{-v_7 \frac{\beta}{x}}, \tag{10}$$

where

$$\begin{aligned}\eta = &\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{1}\sum_{v_3=0}^{v_2}\sum_{v_4=0}^{\infty}\sum_{v_5=0}^{\infty}\sum_{v_6=0}^{\infty}\sum_{v_7=0}^{v_6}\sum_{v_8=0}^{\infty} \frac{(-1)^{v_1+v_3+v_4+v_5+v_6+v_7}}{v_1! v_6! v_8!} \\ &\times \binom{1}{v_2}\binom{v_2}{v_3}\binom{\lambda(v_1+v_3+1)-1}{v_4}\binom{\gamma(v_4+1)-1}{v_5}\binom{v_6}{v_7}\theta^{v_1+v_2}\alpha^{v_6}\beta^{(v_8-v_6)}(v_5+1)^{v_6}(v_6+1)^{v_8}\end{aligned} \tag{11}$$

3. Properties of the NEEGIG

This section derives some of the NEEGIG's characteristic properties includes quantile function, moment, moment generating function, characteristic function, R ́enyi entropy and order statistics. The statistical properties are essential for describing and analyzing data from various perspectives. The quantile function is useful for determining the percentage of observations that are below a specific value. For instance, it can be used to calculate the percentage of individuals with low blood pressure in a given sample, or the proportion of students who fail using the quantile function. This function allows the data to be segmented based on the researcher's requirements. Moments, including mean and variance, are crucial for analyzing numerical data like financial data and for managing risks. Higher moments provide insight into the tails and extremities of a distribution. Moment generating functions simplify the process of deriving moments and analyzing complex distributions, which is essential for theoretical models and proofs. Characteristic functions provide a comprehensive description of distributions, helping in the study of limit theorems and complex random variables. Rényi entropy measures uncertainty and information content, impacting areas such as data compression, cryptography, and machine learning algorithms. Lastly, order statistics are essential for modeling extreme events, estimating product reliability, and analyzing survey data to identify outliers and understand distribution characteristics. All these properties improve the ability to model, predict, and make informed decisions in practical applications.

3.1. Quantile Function

To obtain the uth quantile function $(0 < u < 1)$ of $X \sim$ NEEGIG, invert (6) and solve the non-linear equation using the Lambert function $W[.]$.

$$x_u = \left\{ \frac{1}{\beta} \log \left\{ 1 - \frac{\beta}{\alpha} \log \left[1 - \left(1 - \left(1 - \frac{W(\theta e^{\theta}(1-u))}{\theta} \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{\gamma}} \right] \right\} \right\}^{-1}, 0 \leq u \leq 1. \tag{12}$$

The median of the NEEGIG is obtained by replacing $u = 0.5$ into equation (13).

$$Median(x) = \left\{ \frac{1}{\beta} \log \left\{ 1 - \frac{\beta}{\alpha} \log \left[1 - \left(1 - \left(1 - \frac{W(\theta e^{\theta}(0.5))}{\theta} \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{\gamma}} \right] \right\} \right\}^{-1}. \tag{13}$$

3.2. Moment

The rth moment of $X \sim$ NEEGIG is derived as follows:

$$\begin{aligned} \mu_r = E\left(x^r\right) &= \int_0^\infty x^r g(x) dx \\ &= \eta\alpha\gamma\lambda \int_0^\infty x^r\, x^{-(2+v_8)} e^{-v_7 \frac{\beta}{x}} dx. \end{aligned} \tag{14}$$

By substituting $y = v_7 \frac{\beta}{x}$, therefore the rth moment can be defined as

$$\mu_r = \eta\alpha\gamma\lambda(\beta v_7)^{r-v_8-1}\Gamma(v_8 + 1 - r) \tag{15}$$

where η is given by (11)

3.3. Moment Generating Function

The moment-generating function (MGF) is described as

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r. \tag{16}$$

Thus, by substituting (15) into (16), the MGF of the NEEGIG is obtained as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \eta\alpha\gamma\lambda(\beta v_7)^{r-v_8-1}\Gamma(v_8 + 1 - r) \tag{17}$$

where η is given by (11)

3.4. Characteristic Function

The NEEGIG characteristic function is as follows:

$$\phi_x(t) = E(e^{itx}) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \eta\alpha\gamma\lambda(\beta v_7)^{r-v_8-1}\Gamma(v_8+1-r), \tag{18}$$

where η is given by (11)

3.5. Rényi entropy

The Rényi entropy can be used to determine the uncertainty measurement of the random variable X. When the Rényi entropy value is high, the level of uncertainty in the data increases. According to [31], the Rényi entropy, $RE()$, can be expressed as

$$RE() = \frac{1}{1-} \log\left[\int_0^{\infty} [g(x)]^d x\right] \tag{19}$$

By substituting $g(x)$ given in (7) into the (19) and applying some expansions. $RE()$ is presented as

$$[g(x)]^{=} \eta^* \frac{(\alpha\gamma\lambda)}{x^{(2+v_8)}} e^{-v_7 \frac{\beta}{x}}, \tag{20}$$

where

$$\begin{aligned}\eta^* = &\sum_{v_1=0}^{\infty} \sum_{v_2=0} \sum_{v_3=0}^{v_2} \sum_{v_4=0}^{\infty} \sum_{v_5=0}^{\infty} \sum_{v_6=0}^{\infty} \sum_{v_7=0}^{v_6} \sum_{v_8=0}^{\infty} \frac{(-1)^{v_1+v_3+v_4+v_5+v_6+v_7}}{v_1! v_6! v_8!} \\ &\times \binom{}{v_2}\binom{v_2}{v_3}\binom{\lambda(v_1+v_3+)-}{v_4}\binom{\gamma(v_4+)-}{v_5}\binom{v_6}{v_7} \theta^{v_1+v_2}\alpha^{v_6}\beta^{v_8-v_6 v_1}(v_5+)^{v_6}(v_6+)^{v_8}\end{aligned} \tag{21}$$

By replacing (20) in (19), and calculating the integral, the Rényi entropy of the NEEGIG can be derived as

$$RE() = \frac{1}{1-} \log\left[\eta^*(\alpha\gamma\lambda)^(\beta\ v_7)^{-(2+v_8+3)}\Gamma(2+v_8+3)\right]. \tag{22}$$

3.6. Order statistics

Let $X_{i:n}$ represent the i^{th} order statistic for a random sample (RS) from the NEEGIG distribution, denoted by $X_1, X_2, \ldots, X_n$. Consequently, the probability density function (PDF) of the i^{th} order statistic, $f_{i:n}(x)$, is defined as:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) [F(x)]^{i-1} [1-F(x)]^{n-i}. \tag{23}$$

By utilizing binomial expansion, the PDF of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \frac{(-1)^k n!}{(i-1)!(n-i)!} \binom{n-i}{k} f(x) \left[1 - \left\{\frac{1-\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{\lambda}}{e^{\theta\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{\lambda}}}\right\}\right]^{k+i-1}. \tag{24}$$

where $f(x)$ given by (7).

4. Estimation of Parameters

This section describes six methods for estimating the NEEGIG's parameters.

4.1. ML estimation method

If $x_1, ..., x_n$ are NEEGIG RS of size n, the log-likelihood function (ℓ) for $\Theta = (\theta, \gamma, \lambda, \alpha, \beta)$ is:

$$\begin{aligned}\ell(\Theta) =& n\log(\alpha) + n\log(\gamma) + n\log(\lambda) - 2\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\frac{\beta}{x_i} - \sum_{i=1}^{n}\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}} - 1)\\ &+ (\gamma-1)\sum_{i=1}^{n}\log\left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right) + (\lambda-1)\sum_{i=1}^{n}\log\left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]\\ &- \sum_{i=1}^{n}\theta\left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda} + \sum_{i=1}^{n}\log\left[1 + \theta\left\{1 - \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda}\right\}\right].\end{aligned} \tag{25}$$

The first derivatives of (25) with respect to $\Theta = (\theta, \gamma, \lambda, \alpha, \beta)$ are as follows:

$$\frac{\partial\ell}{\partial\theta} = -\sum_{i=1}^{n}\left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda} + \sum_{i=1}^{n}\frac{\left\{1 - \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda}\right\}}{\left[1 + \theta\left\{1 - \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda}\right\}\right]}, \tag{26}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\gamma} =& \frac{n}{\gamma} + \sum_{i=1}^{n}\log\left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)\\ &- (\lambda-1)\sum_{i=1}^{n}\frac{\left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\log\left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)}{\left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]}\\ &+ \sum_{i=1}^{n}\theta\lambda\left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda-1}\left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\log\left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)\\ &+ \sum_{i=1}^{n}\frac{\theta\lambda\left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda-1}\left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\log\left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)}{\left[1 + \theta\left\{1 - \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda}\right\}\right]},\end{aligned} \tag{27}$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log\left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right] \\
+ \sum_{i=1}^{n} \theta \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda} \log\left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right] \\
- \sum_{i=1}^{n} \frac{\theta \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda} \log\left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]}{\left[1 + \theta\left\{1 - \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda}\right\}\right]},
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{1}{\beta}(e^{\frac{\beta}{x_i}} - 1) + (\gamma - 1) \sum_{i=1}^{n} \frac{(e^{\frac{\beta}{x_i}} - 1) e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}}{\beta\left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)} \\
+ (\lambda - 1) \sum_{i=1}^{n} \frac{\gamma \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma-1} e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)} (e^{\frac{\beta}{x_i}} - 1)}{\beta\left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]} \\
+ (\lambda - 1) \sum_{i=1}^{n} \frac{\gamma \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma-1} e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)} (e^{\frac{\beta}{x_i}} - 1)}{\beta\left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]} \\
+ \sum_{i=1}^{n} \frac{\theta\lambda\gamma}{\beta} \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda-1} \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma-1} e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)} (e^{\frac{\beta}{x_i}} - 1) \\
+ \sum_{i=1}^{n} \frac{\theta\lambda\gamma \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda-1} \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma-1} e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)} (e^{\frac{\beta}{x_i}} - 1)}{\beta\left[1 + \theta\left\{1 - \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda}\right\}\right]},
\end{aligned} \tag{29}
$$

$$\frac{\partial \ell}{\partial \beta}=\sum_{i=1}^{n}\frac{1}{x_i}+\sum_{i=1}^{n}\frac{\alpha}{\beta^2}(e^{\frac{\beta}{x_i}}-1)-\sum_{i=1}^{n}\frac{\alpha}{\beta x_i}e^{\frac{\beta}{x_i}}+(\gamma-1)\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{\beta^2}(e^{\frac{\beta}{x_i}}-1)+\frac{\alpha}{\beta x_i}e^{\frac{\beta}{x_i}}\right)e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}}{\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)}$$

$$+\frac{\gamma(\lambda-1)}{\beta}\sum_{i=1}^{n}\frac{\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma-1}e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\left(\frac{\alpha}{\beta^2}(e^{\frac{\beta}{x_i}}-1)-\frac{\alpha}{\beta x_i}e^{\frac{\beta}{x_i}}\right)}{\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]} \tag{30}$$

$$+\sum_{i=1}^{n}\theta\lambda\gamma\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda-1}\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma-1}e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\left(\frac{\alpha}{\beta^2}(e^{\frac{\beta}{x_i}}-1)-\frac{\alpha}{\beta x_i}e^{\frac{\beta}{x_i}}\right)$$

$$-\sum_{i=1}^{n}\frac{\theta\lambda\gamma\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda-1}\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma-1}e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\left(\frac{\alpha}{\beta^2}(e^{\frac{\beta}{x_i}}-1)-\frac{\alpha}{\beta x_i}e^{\frac{\beta}{x_i}}\right)}{\left[1+\theta\left\{1-\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x_i}}-1)}\right)^{\gamma}\right]^{\lambda}\right\}\right]}.$$

Equations (26-30) can be solved numerically using an optimization approach, such as the Newton-Raphson method. The Newton-Raphson method is is familiar to users and known for its fast convergence and accuracy near the root, provided that the initial approximation is in proximity and the function exhibits good behavior. This method is readily accessible in the R programming environment. In contrast, methodologies such as the bisection and regula falsi exhibit comparatively slower convergence rates and diminished precision as compared to the Newton-Raphson method. Furthermore, these methods are not built-in the R program. While the fixed-point iteration method is straightforward to implement, it lacks guaranteed convergence and accuracy, with its efficacy heavily contingent upon the selection of the function transformation.

4.2. Maximum product of spacing estimation method

The MPS method was developed by [12]. The uniform spacings of a RS from NEEGIG of size n can be used to find the MPS such as

$$D_i=G(x_{i:n})-G(x_{i-1:n}),\quad i=1,2,...,n+1, \tag{31}$$

where $G(i)$ represents the CDF of the observation $x_{i:n}$ of the NEEGIG distribution. Then, the MPS estimates with respect to the parameters of NEEGIG can be developed by maximizing the log of the geometric mean of sample spacing given by

$$M=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i. \tag{32}$$

4.3. Ordinary and weighted least squares estimation methods

The OLS and WLS are suggested by [33]. Suppose that $x_{1:n}\le ...\le x_{i:n}\le ...\le x_{n:n}$ represent the order statistics of a RS from the NEEGIG. For the OLS, minimize the following expression with respect to the NEEGIG parameters:

$$O=\sum_{i=1}^{n}\left[1-\left\{\frac{1-\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{\lambda}}{e^{\theta\left[1-\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}\right)^{\gamma}\right]^{\lambda}}}\right\}-\mathcal{G}(i)\right]^2, \tag{33}$$

$\mathcal{G}(i)$ is the empirical CDF of the observation $x_{i:n}$ of the NEEGIG, that is usually estimated by $\mathcal{G}(i) = i/(n+1)$. Furthermore, WLS estimates with respect to the parameters of NEEGIG can be derived by minimizing the following expression:

$$W = \sum_{i=1}^{n} w_i \left[1 - \left\{ \frac{1 - \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}} - 1)} \right)^{\gamma} \right]^{\lambda}}{e^{\theta \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}} - 1)} \right)^{\gamma} \right]^{\lambda}}} \right\} - \mathcal{G}(i) \right]^2, \ w_i = \frac{(n+1)^2(n+2)}{i(n+1-i)}. \tag{34}$$

4.4. Cramér von Mises and Anderson Darling estimation methods

The CM and AD estimate methods are presented by [13] in the context of statistical testing. They are calculated using the difference between the CDF estimates and the empirical distribution function. The functions (35) and (36) are minimized with respect to the parameters of NEEGIG to find CM and AD estimates, respectively.

$$C = \frac{1}{12n} + \frac{1}{n} \sum_{i=1}^{n} \left[1 - \left\{ \frac{1 - \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}} - 1)} \right)^{\gamma} \right]^{\lambda}}{e^{\theta \left[1 - \left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}} - 1)} \right)^{\gamma} \right]^{\lambda}}} \right\} - \frac{2i-1}{2n} \right]^2, \tag{35}$$

$$A = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left[\log G(x_{i:n}) + \log \overline{G}(x_{n-i+1:n}) \right]. \tag{36}$$

5. Simulation Studies

In this section, numerical experiments are conducted to assess the effectiveness of diverse estimation techniques. Random sets of $N = 1000$ samples from the NEEGIG distribution are generated, with sample sizes $n = 30$, 100, 200, and 500. Four distinct sets of parameter values are considered:

- $SetI : \theta = 26, \gamma = 0.2, \lambda = 0.7, \alpha = 0.03, \beta = 0.18$
- $SetII : \theta = 1.2, \gamma = 0.73, \lambda = 0.6, \alpha = 1.3, \beta = 0.28$
- $SetIII : \theta = 17, \gamma = 0.5, \lambda = 1.2, \alpha = 0.04, \beta = 0.15$
- $SetIV : \theta = 2.84, \gamma = 0.92, \lambda = 2.46, \alpha = 0.19, \beta = 0.42$

Employing Monte Carlo simulation in the R programming language, we estimate NEEGIG parameters using six distinct methods include ML, MPS, OLS, WLS, CM and AD. For each parameter, we compute the mean estimates and the mean square error (MSE). The Monte Carlo simulation is conducted by following the steps below.

1. Generate a random sample from the NEEGIG distribution with size n.
2. Calculate the ML, MPS, OLS, WLS, CM and AD estimations for each parameter $\Theta = (\theta, \gamma, \lambda, \alpha, \beta)$.
3. Repeat the steps from 1 to 2, N times.
4. For each parameter, calculate the average estimate,$(\hat{\Theta})$, and MSE, where the MSE is defined as

$$MSE = \text{var}(\widehat{\Theta}) + [\text{Bias}(\hat{\Theta})]^2 = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\Theta} - \Theta_{\text{true}})^2,$$

where $\Theta = (\theta, \gamma, \lambda, \alpha, \beta)$ and $Bias = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Theta} - \Theta_{\text{true}})$.

Table 1. Estimates and MSE of NEEGIG parameters for Set I.

Set I: $\theta = 26$, $\gamma = 0.2$, $\lambda = 0.7$, $\alpha = 0.03$ and $\beta = 0.18$.							
n		**ML**	**MPS**	**OLS**	**WLS**	**CM**	**AD**
30	$\hat{\theta}$	26.0102 (8.51e-04)	29.5436 (1.83e+02)	26.0139 (3.31e-02)	32.1310 (5.06e+02)	26.0071 (5.38e-03)	27.6106 (8.07e+01)
	$\hat{\gamma}$	0.3910 (0.2208)	2.9006 (3.93e+01)	1.2116 (1.81e+02)	1.8541 (1.03e+01)	0.6928 (2.26e+01)	0.7206 (2.2398)
	$\hat{\lambda}$	0.6531 (1.31e-02)	1.0841 (2.5157)	0.6336 (2.54e-02)	0.9380 (4.36e-01)	0.6438 (1.91e-02)	0.7568 (2.78e-01)
	$\hat{\alpha}$	0.0512 (3.13e-03)	0.1214 (2.31e-02)	0.0723 (1.19e-02)	0.0839 (1.01e-02)	0.0579 (7.71e-03)	0.0512 (3.46e-03)
	$\hat{\beta}$	0.1802 (1.97e-03)	0.1346 (8.46e-03)	0.1620 (1.44e-03)	0.1439 (7.04e-03)	0.1732 (1.28e-03)	0.1846 (5.91e-03)
100	$\hat{\theta}$	26.0033 (1.41e-04)	27.9965 (3.64e+01)	26.0015 (2.33e-04)	28.4298 (1.00e+02)	26.0005 (1.75e-05)	26.7209 (1.20e+01)
	$\hat{\gamma}$	0.2908 (6.32e-02)	0.6855 (1.0545)	0.3176 (1.0463)	0.5974 (9.82e-01)	0.2450 (7.72e-02)	0.3461 (2.18e-01)
	$\hat{\lambda}$	0.6632 (7.83e-03)	0.8584 (1.90e-01)	0.6756 (8.54e-03)	0.7687 (5.01e-02)	0.6784 (6.18e-03)	0.7261 (5.33e-02)
	$\hat{\alpha}$	0.0449 (1.64e-03)	0.0587 (3.87e-03)	0.0449 (2.74e-03)	0.0492 (1.97e-03)	0.0397 (1.78e-03)	0.0373 (8.48e-04)
	$\hat{\beta}$	0.1744 (8.80e-04)	0.1548 (2.90e-03)	0.1703 (5.06e-04)	0.1659 (2.64e-03)	0.1750 (3.82e-04)	0.1822 (2.33e-03)
200	$\hat{\theta}$	26.0008 (9.18e-06)	27.4120 (1.76e+01)	26.0005 (1.07e-05)	27.2349 (6.89e+01)	26.0002 (1.03e-05)	26.6329 (8.8307)
	$\hat{\gamma}$	0.2348 (1.33e-02)	0.3892 (1.62e-01)	0.2497 (8.74e-02)	0.3819 (1.91e-01)	0.2223 (2.87e-02)	0.2656 (6.79e-02)
	$\hat{\lambda}$	0.6827 (2.55e-03)	0.7982 (6.08e-02)	0.6844 (3.54e-03)	0.7557 (3.17e-02)	0.6881 (2.45e-03)	0.7220 (2.25e-02)
	$\hat{\alpha}$	0.0362 (4.59e-04)	0.0426 (9.17e-04)	0.0382 (9.63e-04)	0.0388 (5.59e-04)	0.0346 (4.97e-04)	0.0325 (3.06e-04)
	$\hat{\beta}$	0.1766 (3.94e-04)	0.1647 (1.33e-03)	0.1727 (2.63e-04)	0.1721 (1.25e-03)	0.1760 (1.69e-04)	0.1821 (1.23e-03)
500	$\hat{\theta}$	26.0001 (8.41e-07)	27.0709 (1.45e+01)	26.0001 (3.89e-07)	26.6801 (5.63e+01)	26.0000 (1.17e-07)	26.4008 (4.3260)
	$\hat{\gamma}$	0.2110 (2.47e-03)	0.2643 (2.31e-02)	0.2163 (3.07e-03)	0.2750 (3.11e-02)	0.2067 (1.02e-03)	0.2242 (1.63e-02)
	$\hat{\lambda}$	0.6944 (4.48e-04)	0.7569 (3.23e-02)	0.6926 (7.01e-04)	0.7497 (2.06e-02)	0.6952 (3.80e-04)	0.7137 (9.64e-03)
	$\hat{\alpha}$	0.0322 (8.92e-05)	0.0350 (2.06e-04)	0.0336 (1.40e-04)	0.0331 (2.35e-04)	0.0321 (6.12e-05)	0.0308 (1.03e-04)
	$\hat{\beta}$	0.1778 (1.26e-04)	0.1718 (4.50e-04)	0.1752 (1.15e-04)	0.1769 (5.29e-04)	0.1767 (7.67e-05)	0.1807 (5.07e-04)

Table 2. Estimates and MSE of NEEGIG parameters for Set II.

Set II: $\theta = 1.2$, $\gamma = 0.73$, $\lambda = 0.6$, $\alpha = 1.3$ and $\beta = 0.28$.							
n		**ML**	**MPS**	**OLS**	**WLS**	**CM**	**AD**
30	$\hat{\theta}$	0.0117 (2.19.e+02)	0.5284 (2.3039)	1.8596 (6.2053)	0.8838 (2.2675)	1.2885 (3.2694)	1.4944 (3.3430)
	$\hat{\gamma}$	1.4320 (5.05e+01)	1.6641 (3.1730)	1.5373 (1.25e+02)	1.3552 (1.6821)	2.4346 (6.36e+02)	0.8504 (6.69e-01)
	$\hat{\lambda}$	0.5502 (2.18e-01)	0.6754 (3.92e-01)	0.6432 (3.31e-01)	0.7393 (3.87e-01)	0.5854 (1.55e-01)	0.6324 (1.92e-01)
	$\hat{\alpha}$	1.6111 (9.7624)	1.7995 (2.2875)	2.0878 (4.2527)	1.4306 (9.32e-01)	2.0069 (4.6544)	1.5300 (1.0599)
	$\hat{\beta}$	0.3941 (2.63e+01)	0.2923 (1.21e-01)	0.4145 (2.12e-01)	0.2894 (1.53e-01)	0.4570 (2.36e-01)	0.3620 (1.21e-01)
100	$\hat{\theta}$	0.9118 (1.8520)	0.7816 (1.1219)	2.0901 (6.1966)	0.8888 (1.2538)	1.5890 (3.9488)	1.4656 (2.0380)
	$\hat{\gamma}$	0.8307 (1.26e-01)	0.9914 (2.16e-01)	0.6721 (3.20e-01)	0.9378 (2.07e-01)	0.7947 (9.95e-01)	0.7065 (1.23e-01)
	$\hat{\lambda}$	0.5897 (1.59e-01)	0.6069 (1.42e-01)	0.5569 (9.42e-02)	0.5866 (8.45e-02)	0.5724 (4.81e-02)	0.5915 (7.80e-02)
	$\hat{\alpha}$	1.3793 (6.27e-01)	1.4972 (6.08e-01)	1.7614 (1.3837)	1.3800 (3.91e-01)	1.5264 (9.39e-01)	1.4616 (5.58e-01)
	$\hat{\beta}$	0.4144 (6.58e-02)	0.3053 (4.162e-02)	0.3305 (6.403e-02)	0.3415 (4.50e-02)	0.3344 (6.22e-02)	0.3284 (3.81e-02)
200	$\hat{\theta}$	1.1051 (1.4607)	0.9330 (8.07e-01)	2.0233 (5.5211)	0.9829 (1.0586)	1.5837 (3.3968)	1.3795 (1.2492)
	$\hat{\gamma}$	0.7735 (7.815e-02)	0.8805 (9.38e-02)	0.6446 (1.99e-01)	0.8476 (1.00e-01)	0.7299 (1.38e-01)	0.7063 (8.15e-02)
	$\hat{\lambda}$	0.6034 (1.15e0-1)	0.5843 (8.39e-02)	0.5479 (5.98e-02)	0.5767 (5.61e-02)	0.5663 (2.37e-02)	0.5878 (4.90e-02)
	$\hat{\alpha}$	1.3446 (3.90e-01)	1.4635 (3.84e-01)	1.6146 (8.79e-01)	1.3515 (2.69e-01)	1.4519 (5.14e-01)	1.3888 (3.20e-01)
	$\hat{\beta}$	0.3655 (3.47546e-02)	0.2997 (2.29e-02)	0.3216 (3.81e-02)	0.3371 (2.97e-02)	0.3118 (3.30e-02)	0.3197 (2.20e-02)
500	$\hat{\theta}$	1.2295 (9.42e-01)	1.0790 (4.80e-01)	1.9129 (4.1275)	1.0631 (6.29e-01)	1.4169 (1.7265)	1.3145 (7.55e-01)
	$\hat{\gamma}$	0.7353 (4.14e-02)	0.7972 (3.76e-02)	0.6377 (1.35e-01)	0.7914 (4.98e-02)	0.7230 (5.92e-02)	0.7081 (4.64e-02)
	$\hat{\lambda}$	0.5946 (5.44e-02)	0.5737 (3.562e-02)	0.5638 (3.35e-02)	0.5689 (3.04e-02)	0.5838 (1.08e-02)	0.5901 (2.87e-02)
	$\hat{\alpha}$	1.3527 (2.18e-01)	1.4227 (2.09e-01)	1.4830 (4.97e-01)	1.3489 (1.40e-01)	1.3602 (1.83e-01)	1.3384 (1.60e-01)
	$\hat{\beta}$	0.3182 (1.265e-02)	0.2867 (9.06e-03)	0.3071 (1.98e-02)	0.3196 (1.46e-02)	0.2937 (1.48e-02)	0.3071 (1.18e-02)

Table 3. Estimates and MSE of NEEGIG parameters for Set III.

Set III: $\theta = 17$, $\gamma = 0.5$, $\lambda = 1.2$, $\alpha = 0.04$ and $\beta = 0.15$.							
n		**ML**	**MPS**	**OLS**	**WLS**	**CM**	**AD**
30	$\hat{\theta}$	17.0338 (1.00e-02)	23.6079 (2.55e+02)	17.1206 (1.0756)	24.7399 (7.35e+02)	17.0704 (2.35e-01)	17.8720 (2.67e+01)
	$\hat{\gamma}$	0.8038 (6.61e-01)	3.9916 (7.04e+01)	3.3370 (5.94e+02)	2.4541 (1.71e+01)	2.3923 (1.60e+02)	1.1967 (4.1552)
	$\hat{\lambda}$	1.2130 (1.93e-02)	2.1634 (7.84)	1.0544 (9.52e-02)	1.8745 (2.2360)	1.1021 (5.78e-02)	1.3380 (1.0613)
	$\hat{\alpha}$	0.0542 (2.30e-03)	0.1443 (3.74e-02)	0.1004 (2.38e-02)	0.1013 (1.27e-02)	0.0797 (1.58e-02)	0.0678 (5.90e-03)
	$\hat{\beta}$	0.1523 (2.75e-03)	0.0926 (1.23e-02)	0.1306 (2.23e-03)	0.1065 (1.01e-02)	0.1425 (2.06e-03)	0.1557 (8.11e-03)
100	$\hat{\theta}$	17.0081 (6.66e-04)	21.0752 (9.66e+01)	17.0065 (1.64e-03)	19.5427 (8.51e+01)	17.0042 (2.61e-04)	17.5761 (14.9560)
	$\hat{\gamma}$	0.6156 (1.19e-01)	1.0402 (1.6025)	0.7084 (1.1323)	0.9900 (1.1273)	0.6463 (2.81e-01)	0.6938 (4.87e-01)
	$\hat{\lambda}$	1.1890 (7.61e-04)	1.4385 (4.92e-01)	1.1400 (2.97e-02)	1.4536 (2.74e-01)	1.1479 (2.68e-02)	1.2796 (2.10e-01)
	$\hat{\alpha}$	0.0484 (7.91e-04)	0.0719 (4.51e-03)	0.0597 (3.58e-03)	0.0594 (2.07e-03)	0.0554 (2.75e-03)	0.0482 (1.27e-03)
	$\hat{\beta}$	0.1473 (1.02e-03)	0.1229 (3.30e-03)	0.1390 (6.55e-04)	0.1319 (3.06e-03)	0.1439 (6.23e-04)	0.1520 (2.96e-03)
200	$\hat{\theta}$	17.0028 (1.15e-04)	20.6256 (8.80e+01)	17.0025 (9.60e-05)	19.0608 (6.31e+01)	17.0016 (6.21e-05)	17.5286 (1.10e+01)
	$\hat{\gamma}$	0.5501 (3.66e-02)	0.7219 (3.30e-01)	0.5931 (1.22e-01)	0.7399 (3.86e-01)	0.5570 (7.57e-02)	0.5743 (1.40e-01)
	$\hat{\lambda}$	1.1917 (3.93e-04)	1.3193 (2.10e-01)	1.1622 (1.56e-02)	1.3427 (1.30e-01)	1.1782 (9.42e-03)	1.2251 (8.29e-02)
	$\hat{\alpha}$	0.0440 (3.02e-04)	0.0567 (1.21e-03)	0.0517 (1.44e-03)	0.0502 (8.54e-04)	0.0466 (8.97e-04)	0.0437 (4.73e-04)
	$\hat{\beta}$	0.1485 (4.69e-04)	0.1331 (1.55e-03)	0.1418 (3.87e-04)	0.1402 (1.60e-03)	0.1462 (2.95e-04)	0.1518 (1.41e-03)
500	$\hat{\theta}$	17.0008 (1.84e-05)	19.8045 (6.69e+01)	17.0010 (1.83e-05)	18.0569 (3.20e+01)	17.0004 (6.67e-06)	17.2623 (8.0426)
	$\hat{\gamma}$	0.5182 (8.80e-03)	0.5994 (1.44e-01)	0.5414 (2.63e-02)	0.6010 (8.65e-02)	0.5180 (8.50e-03)	0.5394 (5.15e-02)
	$\hat{\lambda}$	1.1959 (1.35e-04)	1.2704 (1.15e-01)	1.1827 (4.16e-03)	1.2594 (4.06e-02)	1.1919 (1.44e-03)	1.2186 (3.11e-02)
	$\hat{\alpha}$	0.0418 (8.58e-05)	0.0479 (4.01e-04)	0.0453 (3.51e-04)	0.0448 (3.25e-04)	0.0424 (1.17e-04)	0.0415 (1.84e-04)
	$\hat{\beta}$	0.1485 (1.50e-04)	0.1408 (5.94e-04)	0.1449 (1.67e-04)	0.1454 (6.48e-04)	0.1474 (1.11e-04)	0.1504 (6.00e-04)

Table 4. Estimates and MSE of NEEGIG parameters for Set IV.

Set IV: $\theta = 2.84$, $\gamma = 0.92$, $\lambda = 2.46$, $\alpha = 0.19$ and $\beta = 0.42$.							
n		**ML**	**MPS**	**OLS**	**WLS**	**CM**	**AD**
30	$\hat{\theta}$	2.9306 (8.36e-02)	3.0699 (1.55e-01)	3.6605 (16.8244)	4.6987 (20.8505)	3.7842 (20.1415)	3.5936 (6.9314)
	$\hat{\gamma}$	1.0507 (2.80e-01)	1.4046 (5.95e-01)	1.1607520 (1.2844)	1.5724 (3.8171)	1.1924 (1.7995)	1.1719 (1.9543)
	$\hat{\lambda}$	2.4655 (1.17e-03)	2.4894 (3.68e-03)	2.7173 (3.2422)	2.7452 (3.0357)	2.2915 (2.80e-01)	2.5090 (1.9084)
	$\hat{\alpha}$	0.2286 (3.47e-02)	0.3621 (7.96e-02)	0.2770 (1.25e-01)	0.3958 (2.33e-01)	0.3758 (3.44e-01)	0.2738 (1.02e-01)
	$\hat{\beta}$	0.4813 (8.52e-02)	0.2653 (9.42e-02)	0.4807 (2.13e-01)	0.3424 (1.60e-01)	0.4848 (1.67e-02)	0.4952 (1.59e-01)
100	$\hat{\theta}$	2.8646 (1.72e-02)	2.9080 (2.46e-02)	3.0896 (3.8377)	3.8037 (8.5145)	2.9480 (9.45e-01)	3.2401 (3.4160)
	$\hat{\gamma}$	0.9649 (7.08e-02)	1.1263 (1.39e-01)	1.0217 (3.30e-01)	1.0287 (2.64e-01)	0.9995 (1.59e-01)	0.9651 (1.84e-01)
	$\hat{\lambda}$	2.4615 (1.42e-04)	2.4689 (3.16e-04)	2.6102 (1.1357)	2.5937 (1.1670)	2.4293 (3.23e-02)	2.5541 (7.60e-01)
	$\hat{\alpha}$	0.2044 (9.65e-03)	0.2677 (2.00e-02)	0.2134 (1.90e-02)	0.2280 (1.63e-02)	0.2324 (3.05e-02)	0.2027 (1.12e-02)
	$\hat{\beta}$	0.4405 (2.93e-02)	0.3333 (3.46e-02)	0.4422 (7.04e-02)	0.4096 (4.16e-02)	0.4403 (6.58e-02)	0.4573 (4.45e-02)
200	$\hat{\theta}$	2.8476 (4.60e-03)	2.866 (9.50e-03)	3.0709 (2.6454)	3.5285 (5.2480)	2.8473 (1.29e-02)	3.2005 (2.5628)
	$\hat{\gamma}$	0.9361 (3.25e-02)	1.0467 (6.22e-02)	0.9936 (2.02e-01)	0.9623 (1.50e-01)	0.9628 (6.61e-02)	0.9341 (1.21e-01)
	$\hat{\lambda}$	2.4604 (4.42e-05)	2.4647 (1.02e-04)	2.5855 (6.91e-01)	2.5136 (5.85e-01)	2.4543 (2.08e-03)	2.5195 (4.92e-01)
	$\hat{\alpha}$	0.1962 (4.60e-03)	0.2381 (9.11e-03)	0.2007 (8.19e-03)	0.2050 (5.64e-03)	0.2079 (1.04e-02)	0.1915 (4.70e-03)
	$\hat{\beta}$	0.4308 (1.40e-02)	0.3601 (1.82e-02)	0.4306 (3.58e-02)	0.4236 (1.94e-02)	0.4302 (3.23e-02)	0.4521 (2.14e-02)
500	$\hat{\theta}$	2.8417 (4.33e-03)	2.8428 (5.04e-03)	3.0990 (1.5093)	3.2941 (2.8121)	2.8434 (3.37e-03)	3.2324 (1.9391)
	$\hat{\gamma}$	0.9323 (1.30e-02)	0.9844 (1.94e-02)	0.9328 (7.14e-02)	0.9331 (9.98e-02)	0.9363 (2.51e-02)	0.8980 (7.57e-02)
	$\hat{\lambda}$	2.4599 (1.68e-05)	2.4619 (2.63e-05)	2.5062 (2.78e-01)	2.4991 (3.12e-01)	2.4572 (7.66e-04)	2.4592 (2.48e-01)
	$\hat{\alpha}$	0.1938 (1.80e-03)	0.2144 (2.87e-03)	0.1935 (2.55e-03)	0.1927 (2.36e-03)	0.1978 (3.99e-03)	0.1886 (2.29e-03)
	$\hat{\beta}$	0.4222 (5.50e-03)	0.3860 (6.98e-03)	0.4252 (1.41e-02)	0.4281 (8.96e-03)	0.4228 (1.33e-02)	0.4469 (9.69e-03)

The ML, MPS, OLS, WLS, CM, and AD parameter estimates, along with their MSE, are detailed in Tables (1)-(4). The estimation is deemed optimal when it demonstrates the lowest MSE. Furthermore, it considered as consistent, when it converges to the true parameter value as the sample size increases. From the tables, generally, as the sample size increases, the MSE values tend to decrease and the estimate close the the true value of the parameter. The MSEs for the estimates MPS, OLS, WLS, CM, and AD are quite high when the sample size is small (n=30), but it decreases when the sample size increases. However, some parameter estimates still have high MSE even with larger sample sizes. The ML estimate consistently has the lowest MSE compared to the other estimators in all scenarios. ML is the most reliable choices for estimating NEEGIG parameters.

6. Applications

In this section, the NEEGIG model is employed to present five lifetime datasets in different field. The fitting of NEEGIG is compared with five competing models, which showing below The adequacy of the five data sets for the NEEGIG is determined by comparing its fit to five competing distributions with their CDFs defined as

- The Exponentiated Generalized Inverted Gompertz Distribution (EGIG) given in (4)
- The Inverse Gompertz Distribution (IG) [17]

  $F(x) = e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}, \alpha, \beta > 0,\ x > 0.$
- The Inverse Power Gompertz Distribution (IPG) Distribution [2]

  $F(x) = e^{-q(e^{\frac{\beta}{x^{\alpha}}}-1)}, q, \beta, p > 0,\ x > 0.$
- The Inverse Weibull (IW) Distribution [23]
  $F(x) = e^{-(c/x)^{d}}, c, d > 0,\ x > 0.$
- Inverted Exponential (IE) Distribution [26]
  $F(x) = e^{-(1/xv)}, v > 0,\ x > 0.$

Several goodness of fit (GOF) criteria are used to determine which distribution better fit the data, namely the corrected Akaike information criterion (CAIC), Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), and the Kolomogorov-Smirnov (K-S) test. The *p-value* corresponding to the K-S test is calculated. The formula of these measures are given by

$$\begin{aligned}
AIC &= -2\hat{\ell} + 2v \\
BIC &= -2\hat{\ell} + v\ log(n) \\
HQIC &= -2\hat{\ell} + 2v\ log(log(n)) \\
CAIC &= -2\hat{\ell} + 2kn/(n - v - 1) \\
AICC &= AIC + \frac{2v(v+1)}{n - v - 1}
\end{aligned}$$

where $\hat{\ell}$ refers to maximized log-likelihood function, v denotes the number of estimated parameters in the model and n is the the number of observations. The model that generates the smallest values for these indices largest value of *p-value* is regarded as the optimal model. Multiple criteria are used to get a comprehensive view of the model's performance and ensure robustness in the results.

Data 1: Remission Periods of Bladder Cancer Patients The data presents the remission periods in months of 128 bladder cancer patients, [16]:

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52
4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80
25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51	2.54
3.70	5.17	7.28	9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32
10.06	14.77	32.15	2.64	3.88	5.32	7.39	10.34	14.83	34.26	0.90
2.69	4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69	4.23	5.41
7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63	17.12	46.12
1.26	2.83	4.33	5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62
7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46
4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02
2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07
21.73	2.07	3.36	6.93	8.65	12.63	22.69				

Table 5. Measures of ML and GoF for the first data

Distributions	NEEGIG	EGIG	IG	IPG	IW	IE
Estimates	$\hat{\theta} = 158.2499$			$\hat{q} = 226.7999$	$\hat{c} = 3.2628$	$\hat{v} = 0.4024$
	$\hat{\gamma} = 0.1955$	$\hat{\gamma} = 0.7442$			$\hat{d} = 0.7519$	
	$\hat{\lambda} = 18.5990$	$\hat{\lambda} = 72.6584$		$\hat{a} = 0.7431$		
	$\hat{\alpha} = 0.0055$	$\hat{\alpha} = 0.0100$	$\hat{\alpha} = 2.434027$			
	$\hat{\beta} = 0.0340$	$\hat{\beta} = 0.0100$	$\hat{\beta} = 0.0100$	$\hat{\beta} = 0.0105$		
$-\ell$	-415.8215	-445.6837	-462.5061	-444.5020	-444.0008	-460.3823
CAIC	842.1348	899.6927	929.1081	895.1976	892.0976	922.7963
AIC	841.6430	899.3674	929.0121	895.0041	892.0016	922.7646
BIC	855.9032	910.7756	934.7162	903.5602	897.7057	925.6166
HQIC	847.4370	904.0026	931.3297	898.4804	894.3192	923.9234
K-S	0.0347	0.1412	0.2379	0.1431	0.1410	0.2315
p-value	9.97e-01	1.20e-02	1.01e-06	1.05e-02	1.23e-02	2.1848e-06

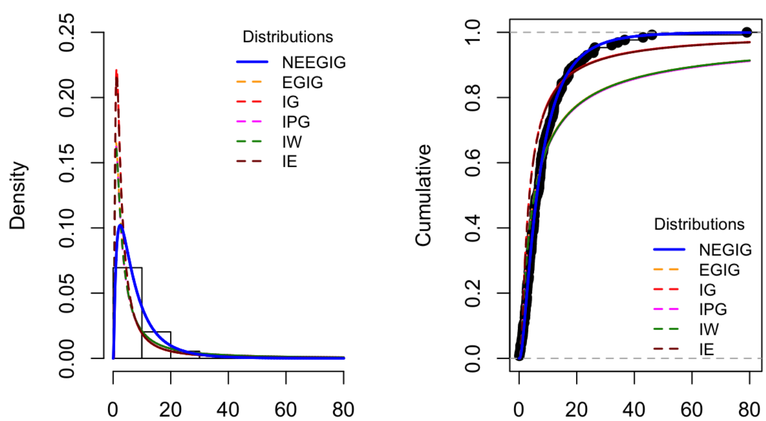


Figure 3. The NEEGIG is compared to other distributions for the first data. (Right): CDF for all distributions. (Left): observed and expected frequencies for all distributions.

Data 2: Australian Institute of Sports Athletes' Sun Skin Folds The data was collected at the Australian Institute of Sports and represents the sun of skin folds in 202 athletes, [34]:

28.0	98	89.0	68.9	69.9	109.0	52.3	52.8	46.7	82.7	42.3
109.1	96.8	98.3	103.6	110.2	98.1	57.0	43.1	71.1	29.7	96.3
102.8	80.3	122.1	71.3	200.8	80.6	65.3	78.0	65.9	38.9	56.5
104.6	74.9	90.4	54.6	131.9	68.3	52.0	40.8	34.3	44.8	105.7
126.4	83.0	106.9	88.2	33.8	47.6	42.7	41.5	34.6	30.9	100.7
80.3	91.0	156.6	95.4	43.5	61.9	35.2	50.9	31.8	44.0	56.8
75.2	76.2	101.1	47.5	46.2	38.2	49.2	49.6	34.5	37.5	75.9
87.2	52.6	126.4	55.6	73.9	43.5	61.8	88.9	31.0	37.6	52.8
97.9	111.1	114.0	62.9	36.8	56.8	46.5	48.3	32.6	31.7	47.8
75.1	110.7	70.0	52.5	67	41.6	34.8	61.8	31.5	36.6	76.0
65.1	74.7	77.0	62.6	41.1	58.9	60.2	43.0	32.6	48	61.2
171.1	113.5	148.9	49.9	59.4	44.5	48.1	61.1	31.0	41.9	75.6
76.8	99.8	80.1	57.9	48.4	41.8	44.5	43.8	33.7	30.9	43.3
117.8	80.3	156.6	109.6	50.0	33.7	54.0	54.2	30.3	52.8	49.5
90.2	109.5	115.9	98.5	54.6	50.9	44.7	41.8	38.0	43.2	70.0
97.2	123.6	181.7	136.3	42.3	40.5	64.9	34.1	55.7	113.5	75.7
99.9	91.2	71.6	103.6	46.1	51.2	43.8	30.5	37.5	96.9	57.7
125.9	49.0	143.5	102.8	46.3	54.4	58.3	34.0	112.5	49.3	67.2
56.5	47.6	60.4	34.9							

Table 6. Measures of ML and GoF for the second data

Distributions	NEEGIG	EGIG	IG	IPG	IW	IE
Estimates	$\hat{\theta} = 94.9774$			$\hat{q} = 227.0250$	$\hat{c} = 126.3550$	$\hat{v} = 0.0175$
	$\hat{\gamma} = 0.1822$	$\hat{\gamma} = 3.2545$			$\hat{d} = 0.9141$	
	$\hat{\lambda} = 4.7836$	$\hat{\lambda} = 22.6513$		$\hat{a} = 1.9688$		
	$\hat{\alpha} = 2.8200$	$\hat{\alpha} = 24.5336$	$\hat{\alpha} = 12.1376$			
	$\hat{\beta} = 74.0172$	$\hat{\beta} = 0.0100$	$\hat{\beta} = 112.3559$	$\hat{\beta} = 10.2374$		
$-\ell$	-952.2315	-954.9466	-969.5487	-967.7550	-1138.8340	-1055.7718
CAIC	1914.769	1918.096	1943.158	1941.631	2281.728	2113.564
AIC	1914.463	1917.893	1943.097	1941.510	2281.668	2113.544
BIC	1931.004	1931.126	1949.714	1951.435	2288.284	2116.852
HQIC	1921.156	1923.247	1945.774	1945.526	2284.345	2114.882
K-S	0.0701	0.0714	0.0988	0.1111	0.5874	0.3181
p-value	0.2731	0.2542	0.0387	0.0136	2.2e-16	2.2e-16

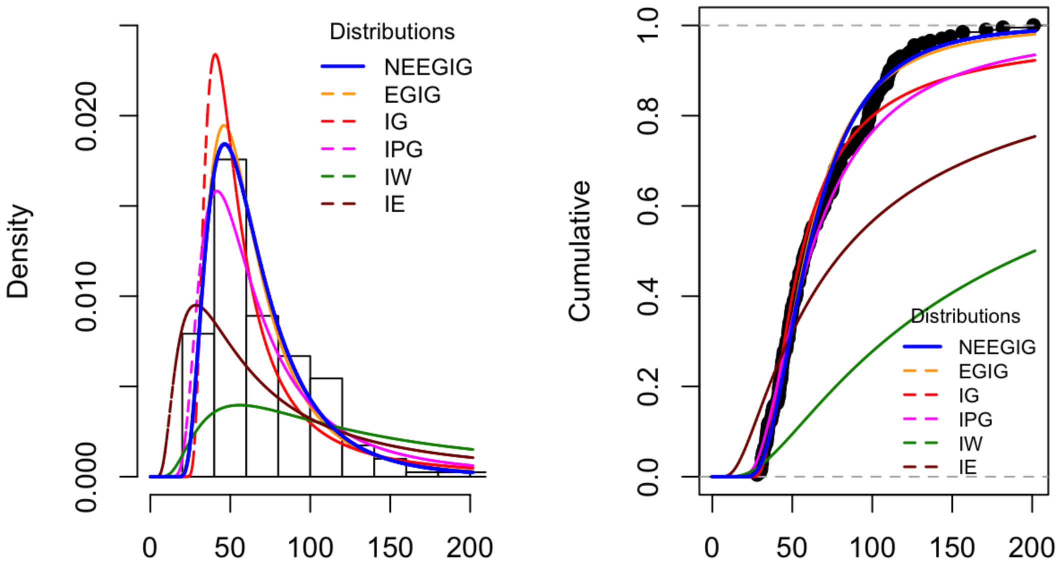


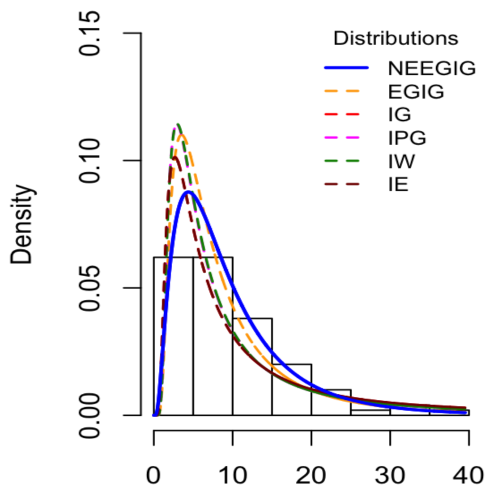
Figure 4. The NEEGIG is compared to other distributions for the second data. (Right): CDF for all distributions. (Left): observed and expected frequencies for all distributions.

Data 3: Waiting Times in a Bank The data was collected for 100 observations on waiting times (in minutes) before receiving service in a bank, [22]:

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7	2.9	3.1
3.2	3.3	3.5	3.6	4.0	4.1	4.2	4.2	4.3	4.3	4.4	4.4
4.6	4.7	4.7	4.8	4.9	4.9	5.0	5.3	5.5	5.7	5.7	6.1
6.2	6.2	6.2	6.3	6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6
7.7	8.0	8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5	11.9	12.4
12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9	14.1	15.4	15.4	17.3
17.3	18.1	18.2	18.4	18.9	19.0	19.9	20.6	21.3	21.4	21.9	23.0
27.0	31.6	33.1	38.5								

Table 7. Measures of ML and GoF for the third data

Distributions	NEEGIG	EGIG	IG	IPG	IW	IE
Estimates	$\hat{\theta} = 52.8236$			$\hat{q} = 226.7999$	$\hat{c}$ =5.0188	$\hat{v}$ =0.1870
	$\hat{\gamma} = 0.3257$	$\hat{\gamma} = 3.2857$			$\hat{d}$ =1.1639	
	$\hat{\lambda} = 30.9069$	$\hat{\lambda} = 0.3070$		$\hat{a} = 1.1520$		
	$\hat{\alpha} = 0.0151$	$\hat{\alpha} = 23.0302$	$\hat{\alpha} = 5.3366$			
	$\hat{\beta} = 0.0100$	$\hat{\beta} = 0.0100$	$\hat{\beta} = 0.0100$	$\hat{\beta} = 0.0282$		
$-\ell$	-318.0168	-324.8365	-336.5760	-334.5640	-334.3811	-336.5585
CAIC	646.6718	658.0940	677.2756	675.3781	672.8859	675.1578
AIC	646.0335	657.6730	677.1519	675.1281	672.7622	675.1170
BIC	659.0594	668.0936	682.3623	682.9436	677.9725	677.7222
HQIC	651.3053	661.8904	679.2606	678.2911	674.8709	676.1714
K-S	0.0495	0.1077	0.1671	0.1187	0.1163	0.1674
p-value	0.9667	0.1963	0.0075	0.1193	0.1333	0.0073



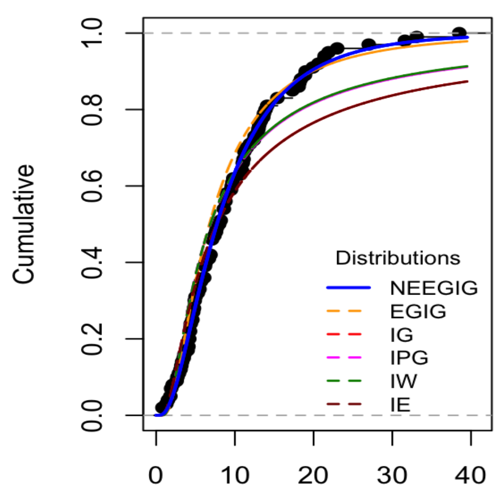


Figure 5. The NEEGIG is compared to other distributions for the third data. (Right): CDF for all distributions. (Left): observed and expected frequencies for all distributions.

Data 4: The observations of gauge lengths of 20 mm data,[25]:

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.426	2.434	2.435	2.382	2.478	2.554	2.514	2.511	2.490	2.535
2.566	2.570	2.586	2.629	2.800	2.773	2.770	2.809	3.585	2.818
2.642	2.726	2.697	2.684	2.648	2.633	3.128	3.090	3.096	3.233
2.821	2.880	2.848	2.818	3.067	2.821	2.954	2.809	3.585	3.084
3.012	2.880	2.848	3.433						

Table 8. Measures of ML and GoF for the fourth data

Distributions	NEEGIG	EGIG	IG	IPG	IW	IE
Estimates	$\hat{\theta} = 71.1386$			$\hat{q} = 226.7890$	$\hat{c}$ =2.1701	$\hat{v}$ =0.4219
	$\hat{\gamma} = 19.5286$	$\hat{\gamma} = 0.9832$			$\hat{d}$ =4.1081	
	$\hat{\lambda} = 0.4178$	$\hat{\lambda} = 0.5034$		$\hat{a} = 4.0869$		
	$\hat{\alpha} = 34.6795$	$\hat{\alpha} = 0.7159$	$\hat{\alpha} = 0.2331$			
	$\hat{\beta} = 0.0100$	$\hat{\beta} = 1.3310$	$\hat{\beta} = 7.3811$	$\hat{\beta} = 0.1039$		
$-\ell$	-51.8443	-56.8926	-82.5579	-69.1611	-69.0100	-141.3055
CAIC	114.5710	122.3650	169.2850	144.6652	142.1891	284.6666
AIC	113.6886	121.7853	169.1160	144.3224	142.0201	284.6111
BIC	125.2090	131.0015	173.7241	151.2346	146.6282	286.9152
HQIC	118.2842	125.4618	170.9542	147.0797	143.8583	285.5302
K-S	0.0578	0.0799	0.2071	0.1459	0.1455	0.4837
p-value	0.9654	0.7321	3.48e-03	8.55e-02	8.67e-02	1.77e-15

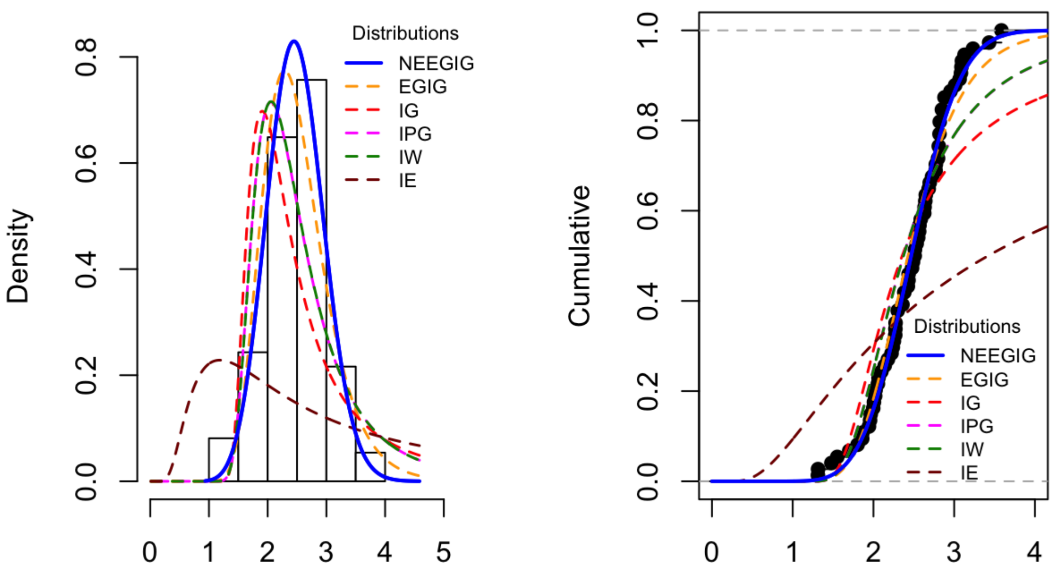


Figure 6. The NEEGIG is compared to other distributions for the third data. (Right): CDF for all distributions. (Left): observed and expected frequencies for all distributions.

Data 5: Passenger automobile insurance policies. The data set refers to losses from private passenger automobile insurance policies in United Kingdom. It consists of 32 observations and 4 variables. Variable number 4 which represents the number of claims was analyzed by [3] These data are available on R©software library:

21	40	23	5	63	171	92	44
140	343	318	129	123	448	361	169
151	479	381	166	245	970	719	304
266	859	504	162	260	578	312	96

Table 9. Measures of ML and GoF for the fifth data

Distributions	NEEGIG	EGIG	IG	IPG	IW	IE
Estimates	$\hat{\theta}$ =245.9760			$\hat{q}$ = 226.7997	$\hat{c}$ =40.6720	$\hat{v}$ =0.0147
	$\hat{\gamma}$ = 0.1660	$\hat{\gamma}$ = 0.7150			$\hat{d}$ =0.7217	
	$\hat{\lambda}$ = 27.0803	$\hat{\lambda}$ = 465.4793		$\hat{a}$ = 0.7112		
	$\hat{\alpha}$ = 0.0100	$\hat{\alpha}$ = 0.0172	$\hat{\alpha}$ = 67.9288			
	$\hat{\beta}$ = 0.0100	$\hat{\beta}$ = 0.0100	$\hat{\beta}$ = 0.0100	$\hat{\beta}$ = 0.1103		
$-\ell$	-213.3819	-222.2998	-227.4053	-222.3397	-226.9477	-227.3943
CAIC	439.0716	454.0811	459.2243	451.5366	458.3093	456.9218
AIC	436.7639	452.5996	458.8105	450.6795	457.8955	456.7885
BIC	444.0926	458.4626	461.7420	455.0767	460.8269	458.2543
HQIC	439.1931	454.5430	459.7822	452.1370	458.8672	457.2744
K-S	0.1002	0.1909	0.3256	0.1911	0.3876	0.3253
p-value	0.8730	0.1701	1.60e-03	0.1689	7.68e-05	1.62e-03

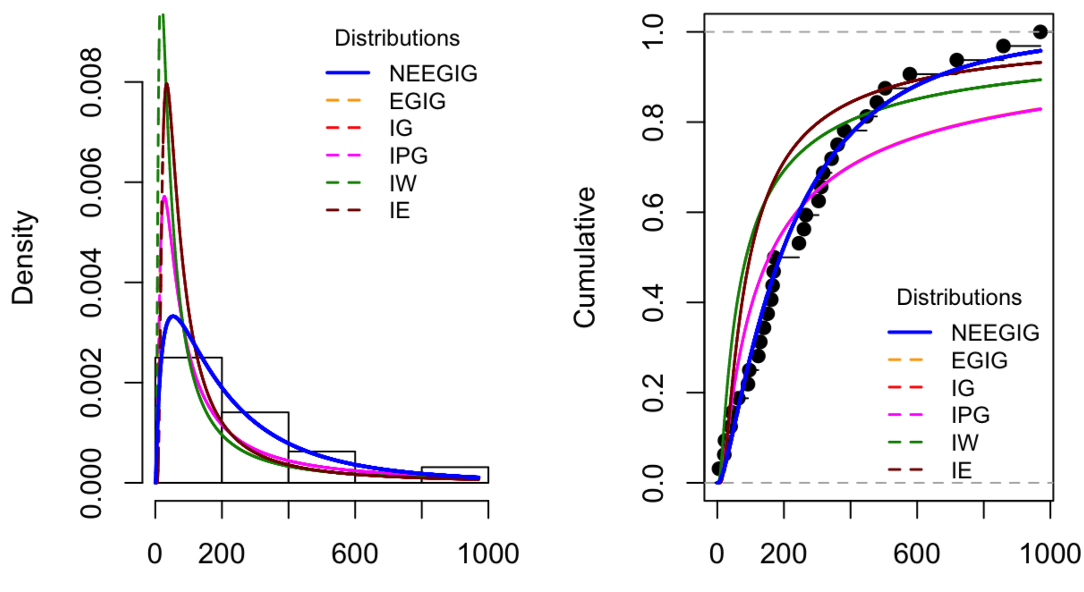


Figure 7. The NEEGIG is compared to other distributions for the fifth data. (Right): CDF for all distributions. (Left): observed and expected frequencies for all distributions.

Tables (5-9) provide a summary of each model's parameter MLs, log-likelihoods, and GoFs. The findings in Tables (5-9) indicate that the NEEGIG model exhibits the lowest values for CAIC, AIC, BIC, HQIC, and K-S measures, along with the highest p-values among all the fitted models. Additionally, Figures (3-7) illustrate that the NEEGIG model best aligns with the actual distribution of the examined datasets. Consequently, the NEEGIG model gave the best fit for data compared to competing distributions.

7. Concluding Remarks

The new distribution based on the NLTE-X family namely NEEGIG is presented in this article. The motivation behind the proposed NEEGIG is rooted in the idea that generalization offers increased flexibility for analyzing practical data. The hazard rate function of NEEGIG can take various forms, enabling it to model diverse hazards in real-world scenarios, including medical, biological, and other applications. Fundamental statistical properties are derived and obtained. The parameters of the NEEGIG distribution are estimated using six methods: ML, MPS, OLS, WLS, CM, and AD. A comprehensive simulation study is conducted to evaluate the performance of these methods, revealing that ML outperforms other methods in terms of mean square errors. The utility of NEEGIG is demonstrated through five applications in various fields. The conclusion is that the proposed NEEGIG model fits better than competing models. Anticipation arises that this generalization will pave the way for additional lifetime and reliability analysis applications.

Appendix

```
#NEEGIGW
#PDF

dNEEGIG=function(x,th,ga,lm,a,b ){
  p1=exp(b/x)
  p2=exp(-(a/b)*(p1-1))
  p3=(1-p2)^ga
  p33=(1-p2)^(ga-1)
  p4=(1-p3)^lm
  p44=(1-p3)^(lm-1)
  p5=exp(th*p4)
  p6=1+(th*(1-p4))
  pdf=a*ga*lm*x^(-2)*p1*p2*p33*p44*(1/p5)*p6
  return(pdf)}

#CDF
pNEEGIG=function(q,th,ga,lm,a,b ){
  p1=exp(b/q)
  p2=exp(-(a/b)*(p1-1))
  p3=(1-p2)^ga
  p4=(1-p3)^lm
  p5=exp(th*p4)
  cdf=1-((1-p4)/p5)
  return(cdf)
}
# Quantile
qNEEGIG=function(p,th,ga,lm,a,b ){
  u=runif(p,0,1)
  z=th*exp(th)*(1-u)
  y=(lambertW0(z)/th)
  f=(1-y)^(1/lm)
  k=(1-f)^(1/ga)
  g=1-((b/a)*(log(1-k)))
  x=(log(g)/b)^(-1)
```

```
  return(x)}
# Hazard rate function hrf(x)
hNEEGIG=function(x,th, ga,lm,a,b){
  h=dNEEGIG(x,th, ga,lm,a,b)/(1-pNEEGIG(x,th, ga,lm,a,b))
  return(h)
}
```

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