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A Connection between the Adjoint Variables and Value Function for Differential Games

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Abstract In this paper, we present a deterministic two-player nonzero-sum differential games (NZSDGs) in a finite horizon. The connection between the adjoint varaibles in the maximum principle (MP) and the value function in the dynamic programming principle (DPP) for differentail games is obtained in either case, whether the value function is smooth and nonsmooth. For the smooth case, the connection between the adjoint variables and the derivatives of the value function are equal to each other along optimal trajectories. Furthermore, for the nonsmooth case, this relation is represented in terms of the adjoint variables and the first-order super- and subdifferentials of the value function. We give an example to illustrate the theoretical results.

Keywords Nonzero-sum differential games, Maximum principle, Dynamic programming principle, Super- and subdifferentials

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1. Introduction

Differential games are a kind of dynamic game that evolves over time. The state of the game is represented by a system of differential equations involving multiple decision-makers, known as players. Each player aims to minimize or maximize his individual criteria ([14], [4]).

On the other hand, differential games are an extension of optimal control problems (OCPs). Due to their connection, some of the concepts and techniques used in the solution of OCPs can also be applied in the solution of differential game problems such as Pontryagin maximum principle (MP) and Bellman's dynamic programming principle (DPP) serve as the main significant approaches for differential games (see e.g.,[4]). The MP approach characterizes the open-loop Nash equilibrium (OLNE) solution of the differential games using Hamiltonian function and adjoint variables, while the DPP characterizes the feedback Nash equilibrium (FNE) using the value function solution to the Hamilton-Jacobi-Bellman (HJB) equations (see e.g., [4]) and there is a close relationship between them. The relation between MP and DPP can be regarded as the connection between adjoint variables and the value function, or the Hamiltonian systems and the HJB equations ([23]). There is a lot of research on the study of relationship between them in deterministic and stochastic optimal control problems (Single player differential games)(see [23, 24, 18, 19, 13, 7, 15, 22]).

The connection between MP and DPP for optimal control problems with a smooth value function was established by Fleming and Rishel [12], Yong and Zhou [23] and further investigated by Shi in [21] for (zero-sum) stochastic

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differential games with jump diffusions. However, even in very simple cases, the value function is not smooth and the HJB equations may not have a smooth solution at all, so this equations must be studied in viscosity solution (VS). This new notion is a kind of nonsmooth solutions was first proposed by Crandall and Lions [9] (see also Crandall et al.[10] and [2]) to overcome the difficulty that the value function of differential games or single player differential games (OCPs) is not smooth. The VS provides researchers to explore relationships between adjoint variables and value functions of deterministic and stochastic optimal control problems (see [3, 8, 24, 23, 17, 18, 19, 13, 7, 15, 22]). Barron [3], Zhou[24] and Yong and Zhou [23] established the relationship between MP and DPP for deterministic optimal control problems using viscosity solution. For stochastic optimal control problems, Nie et al.[18] examined the connection between MP and DPP for stochastic recursive optimal control problems using the viscosity solution framework in the general case. Hu et al.[13] explored this relationship specifically for fully coupled forward-backward stochastic control systems within the same viscosity solution framework. Chen [7] investigated the relationship between MP and DPP in infinite dimensional stochastic control systems. Li [15] studied the relationship between MP and DPP for stochastic recursive optimal control problems under volatility uncertainty. For the stochastic recursive optimal control problem with jumps, Wang [22] obtained the relationship between general MP and DPP.

The connection between the adjoint variables in MP and the value function in DPP for optimal control problems has important applications in mathematical economics and finance. Yong and Zhou [23] discusses the economic interpretations of the adjoint variable, also known as the shadow price, in both smooth and nonsmooth of the value function. For zero-sum stochastic differential games with jump diffusion, Shi [21] discusses a portfolio optimization problem under model uncertainty in an incomplete financial market in the smooth case, which motivated us to study this connection for nonzero-sum differential games. We provide an example of a producer-consumer game with sticky prices [6] to illustrate the connection between adjoint variables and value function in smooth and nonsmooth cases. This application illustrates how this connection can be applied to real-world situations, focusing on the economic interpretations of adjoint variables in differential games.

In this paper, we present a deterministic two-player nonzero-sum differential game on a finite horizon with a convex control domain. We use the two main approaches, both the MP and DPP, for differential games (e.g.,[4]) and obtain the connection between the adjoint variables in the MP and the value function in the DPP in either cases that value function is smooth and nonsmooth. The connection is established in terms of derivatives and super- and subdifferentials of the value function. We give an example of a producer-consumer game formulated as a two-player nonzero-sum differential game involving a producer and a consumer to illustrate the above theoretical results. This article represents a generalization of the results in [23] related to deterministic optimal control problems.

This paper is organized as follows. In Section 2, we give the problem formulation of nonzero-sum differential games (NZSDGs) and we recall the preliminaries results of the MP and DPP (e.g.,[4]). The Section 3 contain the main results of the connection between adjoint variables and the value function in both cases where the corresponding value function is smooth and nonsmooth. An example to illustrate the theoretical results is given in Section 4. Finally, we conclude our paper and give some future works.

2. Formulation of the game problem and preliminaries

In this section, we give the problem formulation of (NZSDGs) and we recall some preliminary of the MP and DPP (see e.g.,[4]) necessary for the main results. Let us consider a non-cooperative two-player nonzero-sum differential games on finite horizon and the dynamical system are described by (ODE)

$$\begin{cases} \dot{y}\left(s\right)=F\left(s,y\left(s\right),b_1\left(s\right),b_2\left(s\right)\right), \ s\in[0,T] \\ y(0)=y_0, \end{cases} \tag{1}$$

where $y(s)\in\mathbb{R}^n$ is the dynamic state of the game at time $s\in[0,T]$ that is influenced by both players and the control strategy for the i-th player $b_i:[0,T]\to B_i$, where B_i is convex and closed subset of $\mathbb{R}^{m_i}$, $(B_i=B_1\times B_2, i=1,2)$. $T>0$ is a fixed time horizon, and $\mathcal{B}_i$ is called admissible set of the control $b_i(\cdot)=(b_1(\cdot),b_2(\cdot))$ defined by the following:

$$\mathcal{B}_i([0,T])=\{b_i\left(.\right):[0,T]\to B_i|b_i\left(.\right)\in\mathbb{L}^2([0,T];\mathbb{R}^{m_i})\},\ i=1,2.$$

The cost functional for the two players is as follows:

$$J_i(s,y_0;b_1(\cdot),b_2(\cdot)) = \int_0^T G_i\left(s,y\left(s\right),b_1\left(s\right),b_2\left(s\right)\right)ds + h_i\left(y\left(T\right)\right),\ i=1,2. \tag{2}$$

We give the following assumptions for the coefficients of (1) and (2).

(DG1) The function $F:[0,T]\times\mathbb{R}^n\times B_1\times B_2\to\mathbb{R}^n$ is continuous and there exists a constant $M>0$ such that for every $s\in[0,T],y,\hat{y}\in\mathbb{R}^n,b,\hat{b}\in B_i$ with $b=(b_1,b_2)$, we have

$$\left|F\left(s,y,b\right)-F\left(s,\hat{y},\hat{b}\right)\right|\leq M\left(\left|y-\hat{y}\right|+\left|b-\hat{b}\right|\right),$$
$$\left|F\left(s,y,b\right)\right|\leq M\left(1+\left|y\right|+\left|b\right|\right),$$

(DG2) The functions $G_i:[0,T]\times\mathbb{R}^n\times B_1\times B_2\to\mathbb{R}$ and $h_i:\mathbb{R}^n\to\mathbb{R}$ are continuous, and there exists a constant $M>0$ such that

$$\left|G_i\left(s,y,b\right)-G_i\left(s,\hat{y},\hat{b}\right)\right|\leq M\left(\left|y-\hat{y}\right|+\left|b-\hat{b}\right|\right),$$
$$\left|h_i\left(y\right)-h_i\left(\hat{y}\right)\right|\leq M\left|y-\hat{y}\right|,$$
$$\left|G_i\left(s,y,b_1,b_2\right)\right|+\left|h_i\left(y\right)\right|\leq M\left(1+\left|y\right|\right),\forall s\in[0,T],y,\hat{y}\in\mathbb{R}^n,b,\hat{b}\in B_i,\ i=1,2.$$

Under assumption (DG1) for any $(s,y)\in[0,T]\times\mathbb{R}^n$ and the controls $b_i(.)\in\mathcal{B}_i[0,T]$, equation (1) admits a unique solution $y(\cdot)=y^{s,y_0,b_i(\cdot)}(\cdot)$ and under (DG2) the functional (2) is well-defined. (see Yong and Zhou ([23]). Consider the following nonzero-sum differential game roblem.
Problem (NZSDG). For given $(s,y)\in[0,T]\times\mathbb{R}^n$, find a $\bar{b}_i(\cdot)\in\mathcal{B}_i[0,T],i=1,2$, such that

$$J_i(s,y_0;\bar{b}_i(\cdot)) = \inf_{b_i(\cdot)\in\mathcal{B}_i[0,T]} J_i(s,y_0;b_i(\cdot)). \tag{3}$$

For $i=1,2$, here $\bar{b}_i(\cdot)\in\mathcal{B}_i[0,T]$ satisfying (3) is called a Nash equilibrium of Problem (NZSDG).

$$J_1(s,y_0;\bar{b}_1(\cdot),\bar{b}_2(\cdot))\leq J_1(s,y_0;b_1(\cdot),\bar{b}_2(\cdot)),\ \forall b_1(\cdot)\in\mathcal{B}_1[0,T],$$

$$J_2(s,y_0;\bar{b}_1(\cdot),\bar{b}_2(\cdot))\leq J_2(s,y_0;\bar{b}_1(\cdot),b_2(\cdot)),\ \forall b_2(\cdot)\in\ \mathcal{B}_2[0,T].$$

This implies that the controls $(\bar{b}_1(\cdot),\bar{b}_2(\cdot))$ represent a Nash equilibrium, indicating that neither player can benefit by changing their own control, making it the optimal choice for both [16].

Now we present here both approaches to find this equilibrium, based on the Pontryagin's maximum principle (MP) and dynamic programming principle (DPP) for differential games (e.g.,[4]). First of all, consider the MP for Problem (NZSDG), as published in multiple articles (see e.g.,[20], [23] and [4]), by using the necessary conditions (4), (5), (6) and (7)) for an open-loop Nash equilibrium (OLNE) $\bar{b}_i(\cdot)=(\bar{b}_1(\cdot),\bar{b}_2(\cdot))\in\mathcal{B}_1[0,T]\times\mathcal{B}_2[0,T]$ and the assumption is as follows:

(DG3) F is C^1 in (y,b) and its derivatives are bounded and uniformly Lipchitz in (y,b). In addition, G_i and h_i are C^1 in (y,b), and the partial derivatives G_y^i,G_b^i,h_y^i are uniformly Lipchitz and linear growth.

The Hamiltonian functions associated with this game $H_i:[0,T]\times\mathbb{R}^n\times B_i\times\mathbb{R}\to\mathbb{R}$ is defined by

$$H_i\left(s,y,b_1,b_2,p_i\right)=\left\langle F\left(s,y,b_1,b_2\right),p_i\right\rangle+G_i\left(s,y,b_1,b_2\right)\ ,\ i=1,2, \tag{4}$$

which the determination of Nash equilibrium is related to the minimization of the Hamiltonian.
Under the assumptions (DG1)-(DG3), let $(\bar{b}_1(\cdot),\bar{b}_2(\cdot))$ is an OLNE of Problem (NZSDG) and $\bar{y}(s)$ is the corresponding state trajectory, there exist a unique adjoint variables $(\bar{p}_i\left(.\right))\in\left(C\left([0,T];\mathbb{R}^n\right)\right)$ solution of the

adjoint equations

$$\begin{cases} \dot{\bar{p}}_i\left(s\right) = -H_y^i\left(s, \bar{y}\left(s\right), \bar{b}_i\left(s\right), \bar{p}_i\left(s\right)\right), \ s \in [0,T] \\ \bar{p}_i\left(T\right) = h_y^i\left(\bar{y}\left(T\right)\right) \end{cases}, \ i = 1, 2, \tag{5}$$

and the infimum condition

$$H_i^*\left(s, \bar{y}(s), \bar{p}_i(s)\right) = H_i\left(s, \bar{y}(s), \bar{b}_i(s), \bar{p}_i(s)\right) = \inf_{b_i(\cdot)\in\mathcal{B}_i[0,T]} H_i\left(s, \bar{y}(s), b_i(s), \bar{p}_i(s)\right). \tag{6}$$

Such that,

$$H_{b_i}^i\left(s, \bar{y}(s), \bar{b}_i(s), \bar{p}_i(s)\right) = 0, \ s \in [0,T]. \tag{7}$$

(DG4) $H_i, i = 1, 2$, is convex in (y, b_1, b_2) and $h_i, i = 1, 2$, is convex in y, $\forall s \in [0, T]$.

Under some appropriate convexity conditions (DG4) we can recall the sufficient maximum principle for an OLNE can be regarded as an extension of the MP for single player differential games in (see e.g.,[23], [24]) and for differential games in (e.g.,[4]), we introduce the following theorem (see e.g., [23],[4]).

Theorem 2.1
Let (DG1)-(DG4) hold. Suppose that $(\bar{b}_1(\cdot), \bar{b}_2(\cdot))$ admissible strategy with the corresponding state trajectory $\bar{y}(\cdot)$. Suppose there exist a solution $(\bar{p}_i\left(.\right)) \in \left(C\left([0,T]; \mathbb{R}^n\right), i = 1, 2\right)$ of the adjoint equations (5) such that the infimum conditions hold

$$H_i^*\left(s, \bar{y}(s), \bar{p}_i(s)\right) = \inf_{b_i(\cdot)\in\mathcal{B}_i[0,T]} H_i\left(s, \bar{y}(s), b_i(s), \bar{p}_i(s)\right), \ i = 1, 2.$$

Then, $(\bar{b}_1(\cdot), \bar{b}_2(\cdot))$ is an open-loop Nash equilibrium.

Next, we present the DPP (see e.g., [4]) for the (NZSDG) problem when the controls $\bar{b}_i(\cdot) = (\bar{b}_1(\cdot), \bar{b}_2(\cdot)) \in \mathcal{B}_1\left[s, T\right] \times \mathcal{B}_2\left[s, T\right]$ is feedback Nash equilibrium (FNE).
For $t \in [0, T]$ and $x \in \mathbb{R}^n$, we rewrite (1) and (2) as the following:

$$\begin{cases} \dot{y}\left(s\right) = F\left(s, y\left(s\right), b_1\left(s\right), b_2\left(s\right)\right), \ s \in [t, T] \\ y(t) = x, \end{cases} \tag{8}$$

The objective of the players is to minimize

$$J_i(t, x; b_1(\cdot), b_2(\cdot)) = \int_t^T G_i\left(s, y\left(s\right), b_1\left(s\right), b_2\left(s\right)\right) ds + h_i\left(y\left(T\right)\right), \ i = 1, 2, \tag{9}$$

and the value function

$$\begin{cases} W_i\left(t, x\right) = \inf_{b_i(\cdot)\in\mathcal{B}_i[t,T]} J_i(t, x; b_i(\cdot)) \\ W_i(T, x) = h_i(x), \ i = 1, 2, \end{cases} \tag{10}$$

represents the minimum cost that can be achieved starting at time t with state x under the optimal decision strategy $\bar{b}_i$.
We present the following Bellman's Principle of optimality [5] for the Problem (NZSDG).

$$W_i\left(t, x\right) = \inf_{b_i(\cdot)\in\mathcal{B}_i[t,T]} \left\{ \int_t^{\hat{t}} g_i\left(s, y\left(s\right), b_i\left(s\right)\right) ds + W_i\left(\hat{t}, y\left(\hat{t}\right)\right) \right\}, \forall \hat{t} \in [t, T], \ i = 1, 2. \tag{11}$$

Similarly to the Pontryagin's MP approach, the search for the FNE is related to the minimization of the Hamiltonian (6). The development of the principle of optimality to equation (11), leads immediately to Hamilton-Jacobi-Bellman (HJB)

$$\begin{cases} \dfrac{\partial W_i}{\partial t}\left(t, x\right) + H_i^*\left(t, x, \dfrac{\partial W_i}{\partial x}\left(t, x\right)\right) = 0, \forall \left(t, x\right) \in [0, T] \times \mathbb{R}^n \\ W_i\left(T, x\right) = h_i\left(x\right), \end{cases} \quad i = 1, 2, \tag{12}$$

where $W_i(\cdot,\cdot) \in \left(C^{1,1}\left([0,T]\times\mathbb{R}^n\right);\mathbb{R}\right)$ and

$$H_i^*\left(t,x,\frac{\partial W_i}{\partial x}(t,x)\right) = H_i\left(t,x,\bar{b}_i,\frac{\partial W_i}{\partial x}(t,x)\right) = \inf_{b_i\in\mathcal{B}_i[t,T]} H_i\left(t,x,b_i,\frac{\partial W_i}{\partial x}(t,x)\right).\ i=1,2.$$

The following verification theorem is a generalization of similar results from (e.g., [23]) for a single player differential game that gives a sufficient condition for a FNE.

Theorem 2.2
(Verification Theorem). Let assumptions (DG1)-(DG2) hold. Assume that $W_i(\cdot,\cdot)\in C^{1,1}\left([0,T]\times\mathbb{R}^n\right)$ is a solution to equations (12). Then we have the following.
(i) $W_i(t,x) \le J_i(t,x;b_i(\cdot)), \forall (t,x)\in[0,T]\times\mathbb{R}^n, b_i(\cdot)\in\mathcal{B}_i[t,T]$.
(ii) Suppose

$$\frac{\partial W_i}{\partial t}(t,x) + H_i^*\left(t,x,\frac{\partial W_i}{\partial x}(t,x)\right) = 0, \forall (t,x)\in[0,T]\times\mathbb{R}^n,\ i=1,2,$$

and there exist an $(\bar{b}_1(\cdot),\bar{b}_2(\cdot))\in\mathcal{B}_1[t,T]\times\mathcal{B}_2[t,T]$ admissible strategy with the corresponding state trajectory $\bar{y}(\cdot)$ for Problem (NZSDG)

$$H_i^*\left(\hat{t},\bar{y}(\hat{t}),\frac{\partial W_i}{\partial x}\left(\hat{t},\bar{y}(\hat{t})\right)\right) = H_i\left(\hat{t},\bar{y}(\hat{t}),\bar{b}_1(\hat{t}),\bar{b}_2(\hat{t}),\frac{\partial W_i}{\partial x}\left(\hat{t},\bar{y}(\hat{t})\right)\right),\ \forall\hat{t}\in[t,T].$$

Then $\left(\bar{b}_1(\cdot),\bar{b}_2(\cdot)\right)$ is a feedback Nash equilibrium (FNE) with the optimal state $\bar{y}(\cdot)$ for Problem (NZSDG) in the point (t,x).

As the value function $W_i(\cdot,\cdot)$ is nonsmooth, it is crucial to recall the definition of viscosity solution (VS) (see [10] and [23]).

Definition 2.1. (Viscosity Solution) A continuous function w_i on $[0,T]\times\mathbb{R}^n$ is a viscosity subsolution (respectively,supersolution) of (12), if $w_i(T,y)\le(\ge)h_i(y)$ for all $y\in\mathbb{R}^n$ and

$$\phi_s^i(s,y) + H_i^*\left(s,y,\phi_y^i(s,y)\right) \ge (\le) 0,\ i=1,2,$$

whenever $w_i-\phi_i$ attains a local maximum (respectively, minimum) at $(s,y)\in[0,T]\times\mathbb{R}^n$ for $\phi_i\in C^{1,1}([0,T]\times\mathbb{R}^n)$. A function w_i is called a VS to (12) if it is both a viscosity subsolution and viscosity supersolution to (12).

Thus, the following result is the uniqueness of VS of the HJB equations (12) (see, Yong and Zhou[23]).

Proposition 2.1
Suppose (DG1)-(DG2) hold. Then, (10) satisfies

$$|W_i(t,x) - W_i(t,x^*)| \le M\left(|x-x^*| + |t-t^*|\right),\ \forall t,t^*\in[0,T], x,x^*\in\mathbb{R}^n,$$

and

$$|W_i(t,x)| \le M(1+|x|),\ \forall (t,x)\in[0,T]\times\mathbb{R}^n,\ i=1,2.$$

Furthermore, $W_i(\cdot,\cdot)$ is the viscosity solution to (12).

3. Main results

3.1. Smooth case

The following theorem states that the connection between the adjoint variables and the derivatives of the value function is equal along optimal trajectories.

Theorem 3.1
Assume (DG1)-(DG3) hold and $(t,x)\in[0,T)\times\mathbb{R}^n$ be fixed. Let $(\bar{b}_1(\cdot),\bar{b}_2(\cdot))$ is a Nash equilibrium with the optimal state $\bar{y}(\cdot)$ for Problem (NZSDG) and $\bar{p}_i$ be the corresponding solution of the adjoint equations (5). Assume that $W_i(\cdot,\cdot)\in\left(C^{1,1}([0,T]\times\mathbb{R}^n);\mathbb{R}\right)$, then

$$\begin{aligned}-\frac{\partial W_i}{\partial s}(s,\bar{y}(s)) =& H_i\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s),\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right)\\ &= \inf_{b_i\in\mathcal{B}_i[s,T]} H_i\left(s,\bar{y}(s),b_1(s),b_2(s),\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right),\ i=1,2,\end{aligned}\tag{13}$$

$\forall s\in[t,T]$. Further, if $W_i(\cdot,\cdot)\in\left(C^{1,2}([0,T]\times\mathbb{R}^n);\mathbb{R}\right)$ and W^i_{sx} is continuous,

$$\bar{p}_i(s)=\frac{\partial W_i}{\partial y}(s,\bar{y}(s)),\forall s\in[t,T],\ i=1,2.\tag{14}$$

Proof
By the optimality of $(\bar{y}(\cdot),\bar{b}_1(\cdot),\bar{b}_2(\cdot))$ for Problem (NZSDG)

$$\begin{cases}\dot{\bar{y}}(s)=F\left(s,\bar{y}(s),\bar{b_1}(s),\bar{b_2}(s)\right),\ s\in[t,T]\\ \bar{y}(t)=x,\end{cases}\tag{15}$$

and the cost functional:

$$W_i(t,x)=J_i(t,x;\bar{b_1}(\cdot),\bar{b_2}(\cdot))=\int_t^T G_i\left(s,\bar{y}(s),\bar{b_1}(s),\bar{b_2}(s)\right)ds+h_i(\bar{y}(T)),\ i=1,2,\forall t\in[s,T].\tag{16}$$

Differentiating both sides of the (16) with respect to s :

$$\frac{\partial W_i}{\partial s}(s,\bar{y}(s))+\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\,\dot{\bar{y}}(s)=-G_i\left(s,\bar{y}(s),\bar{b_1}(s),\bar{b}_2(s)\right),\ i=1,2$$

According to (15), we can deduce that

$$\frac{\partial W_i}{\partial s}(s,\bar{y}(s))+\left\langle F\left(s,\bar{y},\bar{b_1}(s),\bar{b}_2(s)\right),\frac{\partial W_i}{\partial y}(s,\bar{y}(t))\right\rangle=-G_i\left(s,\bar{y}(s),\bar{b_1}(s),\bar{b}_2(s)\right)$$

By (4), we get the first equality in (13)

$$-\frac{\partial W_i}{\partial s}(s,\bar{y}(s))=H_i\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s),\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right),\ i=1,2,$$

Since $W_i\in C^{1,1}([0,T]\times\mathbb{R}^n)$ be a solution of the equations (12), we obtain that, for each $y\in\mathbb{R}^n$

$$\begin{aligned}&\frac{\partial W_i}{\partial s}(s,\bar{y}(s))+H_i\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s),\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right)\\ &=0\leq\frac{\partial W_i}{\partial s}(s,y)+H_i\left(s,y,\bar{b}_1(s),\bar{b}_2(s),\frac{\partial W_i}{\partial y}(s,y)\right)\end{aligned}$$

Thus we have the second equality in (13).
Therefore, if $W_i(\cdot,\cdot)\in\left(C^{1,2}([0,T]\times\mathbb{R}^n);\mathbb{R}\right)$ and W^i_{sy} is continuous, thus

$$\frac{\partial}{\partial y}\left\{\frac{\partial W_i}{\partial s}(s,y)+H_i\left(s,y,\bar{b}_1(s),\bar{b}_2(s),\frac{\partial W_i}{\partial y}(s,y)\right)\right\}|_{y=\bar{y}(s)}=0$$

This implies that

$$\frac{\partial}{\partial s}\left\{\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right\}+\frac{\partial^2 W_i}{\partial y^2}(s,\bar{y}(s))\,F\left(s,\bar{z}(s),\bar{b_1}(s),\bar{b}_2(s)\right)$$
$$+\frac{\partial W_i}{\partial y}(t,\bar{y}(s))\,F_y\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s)\right)+G_y^i\left(s,\bar{y}(s),\bar{b_1}(s),\bar{b}_2(s)\right)=0.\ i=1,2.$$

We have that

$$\frac{\partial}{\partial s}\frac{\partial W_i}{\partial y}(s,\bar{y}(s))=-H_y^i\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s),\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right).$$

Where,

$$H_y^i\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s),\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right)=\frac{\partial^2 W_i}{\partial y^2}(s,\bar{y}(s))\,F\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s)\right)$$
$$+\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\,F_y\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s)\right)+G_y^i\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s)\right).\ i=1,2.$$

Noting that $\frac{\partial W_i}{\partial y}(T,\bar{y}(T))=h_y^i(\bar{y}(T))$, and $\frac{\partial W_i}{\partial y}(s,\bar{y}(s))$ satisfies the equation (5). Then by the uniqueness of the solutions to the adjoint equation (5), we get (14). □

Remark 3.1. Note that the Theorem 3.1 is proved by Shi [21] in particular case of differential games (zero sum stochastic differential games) with jump diffusions.

3.2. Nonsmooth case

The relationship between the adjoint variables in MP and the value function in DPP is investigated in the framework of VS. We recall the notion of the first-order super- and subdifferentials (see [23]). For $w_i\in C([0,T]\times\mathbb{R}^n)$ and $(s,y)\in[0,T]\times\mathbb{R}^n$, we have as follows:

$$D_{s,y}^{1,+}w_i(s,y)=\left\{(q_i,p_i)\in\mathbb{R}\times\mathbb{R}^n\Big|\limsup_{t\to s,t\in[0,T),x\to y}\frac{w_i(t,x)-w_i(s,y)-q_i(t-s)-\langle p_i,x-y\rangle}{|t-s|+|x-y|}\le 0\right\}$$
$$D_{s,y}^{1,-}w_i(s,y)=\left\{(q_i,p_i)\in\mathbb{R}\times\mathbb{R}^n\Big|\liminf_{t\to s,t\in[0,T),x\to y}\frac{w_i(t,x)-w_i(s,y)-q_i(t-s)-\langle p_i,x-y\rangle}{|t-s|+|x-y|}\ge 0\right\}$$

Next, the viscosity solution to HJB equation (12) can be expressed equivalently in terms of super- and subdifferentials (see, [23]). For $w_i\in C([0,T]\times\mathbb{R}^n)$ is a VS of the equations (12) and for all $(s,y)\in[0,T]\times\mathbb{R}^n$,

$$\begin{cases} q_i+H_i^*(s,\bar{y},\bar{p}_i)\ge 0,\ \forall(q_i,\bar{p}_i)\in D_{s,y}^{1,+}w_i(s,y)\\ q_i+H_i^*(s,\bar{y},\bar{p}_i)\le 0,\ \forall(q_i,\bar{p}_i)\in D_{s,y}^{1,-}w_i(s,y),\quad i=1,2\\ w_i(T,y)=h_i(y).\end{cases}\tag{17}$$

Theorem 3.2
Assume (DG1)-(DG3) hold. Let $(t,x)\in[0,T)\times\mathbb{R}^n$ be fixed and $(\bar{b}_1(\cdot),\bar{b}_2(\cdot))$ is a Nash equilibrium with the optimal state trajectory $\bar{y}(\cdot)$ for Problem (NZSDG). Let $\bar{p}_i(\cdot)$ be the solution to equation (5). Suppose that the value function $W_i(\cdot,\cdot)\in(C([0,T]\times\mathbb{R}^n);\mathbb{R})$. Then

$$D_{s,y}^{1,-}W_i(s,\bar{y}(s))\subseteq\left\{\left(\mathcal{H}_i\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s),\bar{p}_i(s)\right),\bar{p}_i(s)\right)\right\}\subseteq D_{s,y}^{1,+}W_i(s,\bar{y}(s))\tag{18}$$

where

$$\mathcal{H}_i\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s),\bar{p}_i(s)\right)=-H_i\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s),\bar{p}_i(s)\right),\ i=1,2,$$

$$D_y^{1,-}W_i\left(s,\bar{y}(s)\right)\subseteq\{\bar{p}_i(s)\}\subseteq D_y^{1,+}W_i\left(s,\bar{y}(s)\right),\ i=1,2,\ \forall s\in[s,T], \tag{19}$$

and

$$\bar{q}_i=\mathcal{H}_i\left(s,\bar{y}(s),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right),\bar{p}_i(s)\right)=\inf_{b_i(\cdot)\in\mathcal{B}_i[0,T]}\mathcal{H}_i\left(s,\bar{y}(s),b_1(s),b_2\left(s\right),\bar{p}_i(s)\right),\ i=1,2, \tag{20}$$

$$\forall\left(\bar{q}_i,\bar{p}_i\right)\in D_{s,y}^{1,+}W_i\left(s,\bar{y}(s)\right)\cup D_{s,y}^{1,-}W_i\left(s,\bar{y}(s)\right),\ \forall s\in[s,T],$$

Proof
Note that

$$\lim_{h\to 0}\frac{1}{h}\int_s^{s+h}\psi(\vartheta)d\vartheta=\psi(s),\ a.e.s\in(t,T), \tag{21}$$

and $\psi(\vartheta)=F\left(\vartheta,\bar{y}\left(\vartheta\right),\bar{b}_1\left(\vartheta\right),\bar{b}_2\left(\vartheta\right)\right),G_i\left(\vartheta,\bar{y}\left(\vartheta\right),\bar{b}_1\left(\vartheta\right),\bar{b}_2\left(\vartheta\right)\right),i=1,2$. Fix $s\in(t,T)$ such that (21) holds. For any $\eta\in\mathbb{R}^n$ and $\tau\in[t,T]$, consider the following ODE:

$$\left\{\begin{array}{l}\dot{y}^{\tau,\eta}\left(\vartheta\right)=F\left(\vartheta,y^{\tau,\eta}\left(\vartheta\right),\bar{b}_1\left(\vartheta\right),\bar{b}_2\left(\vartheta\right)\right),\ \vartheta\in[\tau,T]\\ y^{\tau,\eta}(\tau)=\eta.\end{array}\right. \tag{22}$$

Denote by $y^{\tau,\eta}\left(\cdot\right)$ the solution of (22) starting from (τ,y) under the controls $\bar{b}_i\left(\cdot\right)=\left(\bar{b}_1\left(\cdot\right),\bar{b}_2\left(\cdot\right)\right)$, for $i=1,2$,

$$y^{\tau,\eta}\left(\vartheta\right)=\eta+\int_\tau^\vartheta F\left(\alpha,y^{\tau,\eta}\left(\alpha\right),\bar{b}_1\left(\alpha\right),\bar{b}_2\left(\alpha\right)\right)d\alpha,\ \vartheta\in\left[\tau,T\right],$$

and $\bar{y}\left(\cdot\right)$ the solution of ODE

$$\bar{y}\left(\vartheta\right)=\bar{y}\left(s\right)+\int_s^\vartheta F\left(\alpha,\bar{y}\left(\alpha\right),\bar{b}_1\left(\alpha\right),\bar{b}_2\left(\alpha\right)\right)d\alpha,\ \vartheta\in\left[\tau,T\right],$$

Then $\tau<s$ and for any $\vartheta\in[\tau,T]$, we have

$$\begin{aligned}y^{\tau,\eta}\left(\vartheta\right)-\bar{y}\left(\vartheta\right)=\eta-\bar{y}\left(s\right)-\int_s^\tau F\left(\alpha,\bar{y}\left(\alpha\right),\bar{b}_1\left(\alpha\right),\bar{b}_2\left(\alpha\right)\right)d\alpha\\ +\int_\tau^\vartheta\left[F\left(\alpha,y^{\tau,\eta}\left(\alpha\right),\bar{b}_1\left(\alpha\right),\bar{b}_2\left(\alpha\right)\right)-F\left(\alpha,\bar{y}\left(\alpha\right),\bar{b}_1\left(\alpha\right),\bar{b}_2\left(\alpha\right)\right)\right]d\alpha\\ =\eta-\bar{y}\left(s\right)-\int_s^\tau F\left(\alpha,\bar{y}\left(\alpha\right),\bar{b}_1\left(\alpha\right),\bar{b}_2\left(\alpha\right)\right)d\alpha\\ +\int_\tau^\vartheta F_y\left(\alpha,\bar{y}\left(\alpha\right),\bar{b}_1\left(\alpha\right),\bar{b}_2\left(\alpha\right)\right)\left(y^{\tau,\eta}\left(\alpha\right)-\bar{y}\left(\alpha\right)\right)d\alpha\\ +\int_\tau^\vartheta\epsilon_{\tau,\eta}\left(\alpha\right)\left(y^{\tau,\eta}\left(\alpha\right)-\bar{y}\left(\alpha\right)\right)d\alpha.\end{aligned} \tag{23}$$

We obtain the second equality of (23) by using the variational equation for $\xi\left(\vartheta\right)=y^{\tau,\eta}\left(\vartheta\right)-\bar{y}\left(\vartheta\right)$ given by

$$\left\{\begin{array}{l}\dot{\xi}\left(\vartheta\right)=F_y\left(\vartheta,\bar{y}\left(\vartheta\right),\bar{b}_1\left(\vartheta\right),\bar{b}_2\left(\vartheta\right)\right)\xi\left(\vartheta\right)+\epsilon_{\tau,\eta}\left(\vartheta\right)\xi\left(\vartheta\right),\ \vartheta\in[\tau,T]\\ \xi(\vartheta)=\eta-\bar{y}\left(s\right)-\int_s^\tau F\left(\vartheta,\bar{y}\left(\vartheta\right),\bar{b}_1\left(\vartheta\right),\bar{b}_2\left(\vartheta\right)\right)d\vartheta.\end{array}\right. \tag{24}$$

where,

$$\begin{cases} \epsilon_{\tau,\eta}\left(\alpha\right)=\int\limits_{0}^{1}\left\{F_{y}\left(\alpha,\bar{y}\left(\alpha\right)+\beta\left(y^{\tau,\eta}\left(\alpha\right)-\bar{y}\left(\alpha\right)\right),\bar{b}_{1}\left(\alpha\right),\bar{b}_{2}\left(\alpha\right)\right)\right. \\ \qquad\qquad\left.-F_{y}\left(\alpha,\bar{y}\left(\alpha\right),\bar{b}_{1}\left(\alpha\right),\bar{b}_{2}\left(\alpha\right)\right)\right\}d\beta \\ \lim\limits_{\tau\to s,\eta\to\bar{y}(s)}\epsilon_{\tau,\eta}\left(\alpha\right)=0,\quad\forall\alpha\in\left[0,T\right], \\ \sup\limits_{\alpha,\tau,\eta}\left|\epsilon_{\tau,\eta}\left(\alpha\right)\right|\leq K. \end{cases} \tag{25}$$

In this case, the assumption (DG3) was employed.
By the definition of $W_i\left(\tau,\eta\right)$

$$W_{i}\left(\tau,\eta\right)\leq\int\limits_{\tau}^{T}G_{i}\left(\vartheta,y^{\tau,\eta}\left(\vartheta\right),\bar{b}_{1}\left(\vartheta\right),\bar{b}_{2}\left(\vartheta\right)\right)d\vartheta+h_{i}\left(y^{\tau,\eta}\left(T\right)\right),\ i=1,2,$$

and the optimality of $\left(\bar{y}\left(\cdot\right),\bar{b}_{1}\left(\cdot\right),\bar{b}_{2}\left(\cdot\right)\right)$, we get

$$W_{i}\left(s,\bar{y}(t)\right)=\int\limits_{s}^{T}G_{i}\left(\vartheta,\bar{y}\left(\vartheta\right),\bar{b}_{1}\left(\vartheta\right),\bar{b}_{2}\left(\vartheta\right)\right)d\vartheta+h_{i}\left(\bar{y}\left(T\right)\right),\ i=1,2.$$

Then, compute $W_i\left(\tau,\eta\right)-W_i\left(s,\bar{y}\left(s\right)\right)$ we obtain

$$\begin{aligned} &W_{i}\left(\tau,\eta\right)-W_{i}\left(s,\bar{y}\left(s\right)\right) \\ &\leq\int\limits_{\tau}^{T}\left\{G_{i}\left(\vartheta,y^{\tau,\eta}\left(\vartheta\right),\bar{b}_{1}\left(\vartheta\right),\bar{b}_{2}\left(\vartheta\right)\right)-G_{i}\left(\vartheta,\bar{y}\left(\vartheta\right),\bar{b}_{1}\left(\vartheta\right),\bar{b}_{2}\left(\vartheta\right)\right)\right\}d\vartheta \\ &\qquad+\left\{h_{i}\left(y^{\tau,\eta}\left(T\right)\right)-h_{i}\left(\bar{y}\left(T\right)\right)\right\}-\int\limits_{t}^{\tau}G_{i}\left(\vartheta,\bar{y}\left(\vartheta\right),\bar{b}_{1}\left(\vartheta\right),\bar{b}_{2}\left(\vartheta\right)\right)d\vartheta \\ &=\int\limits_{\tau}^{T}\left\langle G_{y}^{i}\left(\vartheta,\bar{y}\left(\vartheta\right),\bar{b}_{1}\left(\vartheta\right),\bar{b}_{2}\left(\vartheta\right)\right),y^{\tau,\eta}\left(\vartheta\right)-\bar{y}\left(\vartheta\right)\right\rangle d\vartheta \\ &\qquad+\left\langle h_{y}^{i}\left(\bar{y}\left(T\right)\right),y^{\tau,\eta}\left(T\right)-\bar{y}\left(T\right)\right\rangle-\int\limits_{s}^{\tau}G_{i}\left(\vartheta,\bar{y}\left(\vartheta\right),\bar{b}_{1}\left(\vartheta\right),\bar{b}_{2}\left(\vartheta\right)\right)d\vartheta \\ &\qquad+\int\limits_{\tau}^{T}\tilde{\epsilon}_{\tau,\eta}\left(\vartheta\right)\left(y^{\tau,\eta}\left(\vartheta\right)-\bar{y}\left(\vartheta\right)\right)d\vartheta+o\left(\left|y^{\tau,\eta}\left(T\right)-\bar{y}\left(T\right)\right|\right),\ i=1,2. \end{aligned} \tag{26}$$

where $\tilde{\epsilon}_{\tau,\eta}\left(.\right)$ is defined similar to $\epsilon_{\tau,\eta}\left(.\right)$, with the substitution of F_y for G_y^i and has the same properties are present in (25)(see, [23]). Then, by the duality relation between the adjoint equation (5) $\bar{p}_i\left(.\right)$ and the variational equation (24) $y^{\tau,\eta}\left(.\right)-\bar{y}\left(.\right)$, we have

$$\begin{aligned} &\left\langle h_{y}^{i}\left(\bar{y}\left(T\right)\right),\xi\left(T\right)\right\rangle \\ &\quad=\left\langle\bar{p}_{i}\left(T\right),\xi\left(T\right)\right\rangle \\ &\quad=\left\langle\bar{p}_{i}\left(T\right),\xi\left(T\right)\right\rangle-\left\langle\bar{p}_{i}\left(\tau\right),\xi\left(\tau\right)\right\rangle+\left\langle\bar{p}_{i}\left(\tau\right),\xi\left(\tau\right)\right\rangle \\ &\quad=\int\limits_{\tau}^{T}\left\langle\dot{\bar{p}}_{i}\left(\vartheta\right),\xi\left(\vartheta\right)\right\rangle d\vartheta+\int\limits_{\tau}^{T}\left\langle\bar{p}_{i}\left(\vartheta\right),\dot{\xi}\left(\vartheta\right)\right\rangle d\vartheta+\left\langle\bar{p}_{i}\left(\tau\right),\xi\left(\tau\right)\right\rangle \\ &\quad=\left\langle\bar{p}_{i}\left(\tau\right),\xi\left(\tau\right)\right\rangle+\int\limits_{\tau}^{T}\left\langle\bar{p}_{i}\left(\vartheta\right),\epsilon_{\tau,\eta}\left(\vartheta\right)\xi\left(\vartheta\right)\right\rangle d\vartheta \\ &\quad-\int\limits_{\tau}^{T}\left\langle G_{y}^{i}\left(\vartheta,\bar{y}\left(\vartheta\right),\bar{b}_{1}\left(\vartheta\right),\bar{b}_{2}\left(\vartheta\right)\right),\xi\left(\vartheta\right)\right\rangle d\vartheta \end{aligned} \tag{27}$$

After that, with respect to the term on the right side of (27)

$$\begin{aligned}
&\langle \bar{p}_i(\tau), \xi(\tau)\rangle \\
&\quad= \left\langle \bar{p}_i(s), \eta - \bar{y}(s) - \int_s^\tau F\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \right\rangle \\
&\quad+ \left\langle \bar{p}_i(\tau) - \bar{p}_i(s), \eta - \bar{y}(s) - \int_s^\tau F\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \right\rangle \\
&\quad= \left\langle \bar{p}_i(s), \eta - \bar{y}(s) - \int_s^\tau F\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \right\rangle \\
&\quad+ \Bigg\langle \int_s^\tau \left[-F_y\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right)\bar{p}_i(\vartheta) - G_y^i\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right)\right] d\vartheta \\
&\qquad\quad, \eta - \bar{y}(s) - \int_s^\tau F\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \Bigg\rangle \\
&\quad= \langle \bar{p}_i(s), \eta - \bar{y}(s)\rangle - \left\langle \bar{p}_i(s), \int_s^\tau F\left(r, \bar{y}(r), \bar{b}_1(r), \bar{b}_2(r)\right) dr \right\rangle + o\left(|\tau - s| + |\eta - \bar{y}(s)|\right)
\end{aligned} \tag{28}$$

Here the properties presented in (25) was employed (see, [23]), for $\xi(\vartheta) = y^{\tau,\eta}(\vartheta) - \bar{y}(\vartheta)$, we have

$$\sup_{\tau \le \vartheta \le T} |\xi(\vartheta)| \le M\left[|\eta - \bar{y}(s)| + |\tau - s|\right],$$

and,

$$\int_\tau^T |\epsilon_{\tau,\eta}(\vartheta)\, \xi(\vartheta)| d\vartheta \le C\left[|\eta - \bar{y}(s)| + |\tau - s|\right]$$

Thus, by (26)-(28), we obtain

$$\begin{aligned}
&W_i(\tau, \eta) - W_i(s, \bar{y}(s)) \\
&\le \langle \bar{p}_i(s), \eta - \bar{y}(s)\rangle - \left\langle \bar{p}_i(s), \int_s^\tau F\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \right\rangle \\
&\qquad - \int_s^\tau G_i\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta + o\left(|\tau - s| + |\eta - \bar{y}(s)|\right) \\
&= \langle \bar{p}_i(s), \eta - \bar{y}(s)\rangle + (\tau - s)\,\mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \bar{p}_i(s)\right) \\
&\qquad + o\left(|\tau - s| + |\eta - \bar{y}(s)|\right),\ i = 1, 2,
\end{aligned} \tag{29}$$

which implies

$$\left(\mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \bar{p}_i(s)\right), \bar{p}_i(s)\right) \subseteq D_{s,y}^{1,+} W_i(s, \bar{y}(s)), i = 1, 2,\ \forall s \in [t, T],$$

by the definition of superdifferential and for such a s, $D_{s,y}^{1,+} W_i(s, \bar{y}(s))$ is nonempty.
Now we prove that

$$D_{s,y}^{1,-} W_i(s, \bar{y}(s)) \subseteq \left\{\left(\mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \bar{p}_i(s)\right), \bar{p}_i(s)\right)\right\},$$

with $s \in (t, T)$ such that (21) holds. For any $(\bar{q}_i, \bar{p}_i) \in D_{s,y}^{1,-} W_i(s, \bar{y}(s))$, by definition of subdifferential and (29), we have

$$\begin{aligned}
0 &\le \liminf_{\tau \uparrow s} \left\{\frac{W_i(\tau, \eta) - W_i(s, \bar{y}(s)) - \bar{q}_i(\tau - s) - \langle \bar{p}_i, \eta - \bar{y}(s)\rangle}{|\tau - s| + |\eta - \bar{y}(s)|}\right\} \\
&\le \liminf_{\tau \uparrow s} \left\{\frac{\left(\mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \bar{p}_i(s)\right) - \bar{q}_i\right)(\tau - s) + \langle \bar{p}_i(s) - \bar{p}_i, \eta - \bar{y}(s)\rangle}{|\tau - s| + |\eta - \bar{y}(s)|}\right\}
\end{aligned}$$

Thus, the first inclusion of (18) holds.
Let us show (19) by taking $\tau = s$ from the above proof of the inclusion in (18). Then we do not need s to satisfy (21). As a consequently, (19) holds for all $s \in [t, T]$.
Finally, we prove (20). Taking $s \in (t, T)$ such that (21) holds. If $\forall\left(\bar{q}_i, \bar{p}_i\right) \in D_{s,y}^{1,+} W_i\left(s, \bar{y}(s)\right)$, then by the definition of superdifferential and Bellman's Principle of optimality (11) we have

$$
\begin{aligned}
0 \geq \limsup_{\vartheta \downarrow s} & \left\{ \frac{W_i\left(\vartheta, \bar{y}\left(\vartheta\right)\right) - W_i\left(s, \bar{y}\left(s\right)\right) - \bar{q}_i\left(\vartheta - s\right) - \left\langle \bar{p}_i, \bar{y}\left(\vartheta\right) - \bar{y}\left(s\right)\right\rangle}{\left|\vartheta - s\right| + \left|\bar{y}\left(\vartheta\right) - \bar{y}\left(s\right)\right|} \right\} \\
= \limsup_{\vartheta \downarrow s} & \left\{ \frac{1}{\left|\vartheta - s\right| + \left|\bar{y}\left(\vartheta\right) - \bar{y}\left(s\right)\right|} \left[- \int_s^{\vartheta} G_i\left(\alpha, \bar{y}\left(\alpha\right), \bar{b}_1\left(\alpha\right), \bar{b}_2\left(\alpha\right)\right) d\alpha \right. \right. \\
& \left. - \bar{q}_i\left(\vartheta - s\right) - \int_s^{\vartheta} \left\langle \bar{p}_i\left(\tau\right), F\left(\alpha, \bar{y}\left(\alpha\right), \bar{b}_1\left(\alpha\right), \bar{b}_2\left(\alpha\right)\right)\right\rangle d\alpha \right] \\
= \limsup_{\vartheta \downarrow s} & \left\{ \frac{1}{\left|\vartheta - s\right|} \left[- \int_s^{\vartheta} G_i\left(\alpha, \bar{y}\left(\alpha\right), \bar{b}_1\left(\alpha\right), \bar{b}_2\left(\alpha\right)\right) d\alpha \right. \right. \\
& \left. - \bar{q}_i\left(\vartheta - s\right) - \int_s^{\vartheta} \left\langle \bar{p}_i\left(\tau\right), F\left(\alpha, \bar{y}\left(\alpha\right), \bar{b}_1\left(\alpha\right), \bar{b}_2\left(\alpha\right)\right)\right\rangle d\alpha \right] \\
& \left. \frac{\left|\vartheta - s\right|}{\left|\vartheta - s\right| + \left|\bar{y}\left(\vartheta\right) - \bar{y}\left(s\right)\right|} \right\}
\end{aligned}
\tag{30}
$$

By using (21) and the limit of the first term on the right-hand side exists (constant). Because $\left|\bar{y}\left(\vartheta\right) - \bar{y}\left(s\right)\right| \leq C\left|\vartheta - s\right|$ for some constant $C > 0$, the inequality (30) yields

$$
\begin{aligned}
0 &\geq -G_i\left(s, \bar{y}\left(s\right), \bar{b}_1\left(s\right), \bar{b}_2\left(s\right)\right) - \bar{q}_i - \left\langle \bar{p}_i, F\left(s, \bar{y}\left(s\right), \bar{b}_1\left(s\right), \bar{b}_2\left(s\right)\right)\right\rangle \\
&= \mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1\left(s\right), \bar{b}_2\left(s\right), \bar{p}_i\right) - \bar{q}_i,
\end{aligned}
$$

Then

$$
\bar{q}_i \geq \mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1\left(s\right), \bar{b}_2\left(s\right), \bar{p}_i(s)\right). \tag{31}
$$

Similarly, letting $\vartheta \uparrow s$, we can conclude

$$
\bar{q}_i \leq \mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1\left(s\right), \bar{b}_2\left(s\right), \bar{p}_i(s)\right). \tag{32}
$$

Then, from (31) and (32) the first equality in (20) holds.
Next, since W_i is the viscosity solution (VS) of the equations (12) by (17) we have,

$$
\bar{q}_i - \inf_{b_i(\cdot) \in \mathcal{B}_i[0,T]} \mathcal{H}_i\left(s, \bar{y}(s), b_1(s), b_2\left(s\right), \bar{p}_i(s)\right) \geq 0, \ i = 1, 2,
$$

which yields the second equality of (20). □

Remark 3.2. We note that:
(i) When W_i is differentiable, the inclusions (18)-(19) is reduced to (13) and (14) in Theorem 3.1.
(ii) The principal results of this study might be considered as an extension of similar results in [23] related to deterministic optimal control problem.

4. Illustrative example

We provide the following example of a producer-consumer game with sticky prices, driven by ([6]) to illustrate our theoretical results (Theorem 3.1 and Theorem 3.2).

Consider a company manufacturing a good, let $y(s)$ denote the sale price of a good at time s, and this good is produced at rate $b_1(s)$ by the company and consumed at rate $b_2(s)$ by the consumer. The dynamical system (1) represent the variation of the price in time is given by the following (ODE)

$$\begin{cases} \dot{y}(s) = y(s)(b_2(s) - b_1(s)), \ s \in [0, T] \\ y(0) = y_0, \end{cases} \tag{33}$$

The controls $b_i(s) \geq 0, i = 1, 2$ represent the rate of production and consumption of a good at time s, respectively. According to (33), the price increases when the consumption is larger than the production of goods, and decreases otherwise.
To simplify, let $c_i(r), i = 1, 2$ denote the cost function of company i. We assume that

$$c_1(r) = \frac{r^2}{2}, \qquad c_2(r) = 2\sqrt{r}.$$

The producer's payoff is given by the profit generated from sales minus the cost of production $c_1(b_1(s))$, depending on the rate of production $b_1(s)$. The consumer's payoff is measured by a utility function $c_2(b_2(s))$, which represents the benefit obtained from consuming the goods minus the price paid to purchase the goods. Also, the payoff functional are

$$\begin{aligned} J_1(s, y_0; b_1(s), b_2(s)) &= \int_0^T [y(s) b_2(s) - c_1(b_1(s))] \, ds, \\ J_2(s, y_0; b_1(s), b_2(s)) &= \int_0^T [c_2(b_2(s)) - y(s) b_2(s)] \, ds, \end{aligned} \tag{34}$$

The problem is to maximize the payoffs for both the producer and the consumer (34), which can be rewritten as the minimization of

$$\begin{aligned} J_1(s, y_0; b_1(s), b_2(s)) &= -\int_0^T [y(s) b_2(s) - c_1(b_1(s))] \, ds, \\ J_2(s, y_0; b_1(s), b_2(s)) &= -\int_0^T [c_2(b_2(s)) - y(s) b_2(s)] \, ds. \end{aligned} \tag{35}$$

Where $W_i(s, y), i = 1, 2$ is the value function for the problem of minimizing (35), which represents the minimum achievable cost or loss that either the producer or the consumer can incur, given the initial time s and the state y.
The Pontryagin's maximum principle (MP) approach: Writing down the maximum principle for the above two-player nonzero-sum differential game (NZSDG), the Hamiltonian functions (4) has the form

$$H_1(s, y, b_1, b_2, p_1) = p_1 y (b_2 - b_1) - y b_2 + \frac{b_1^2}{2},$$

$$H_2(s, y, b_1, b_2, p_2) = p_2 y (b_2 - b_1) - 2\sqrt{b_2} + y b_2.$$

Using (7) we get the OLNE $\bar{b}_i(s) = \left(\bar{b}_1(s), \bar{b}_2(s)\right), i = 1, 2$ as follow

$$\bar{b_1}(s) = \bar{p}_1(s) \bar{y}(s),$$

$$\bar{b_2}(s) = \frac{1}{\bar{y}^2(s) (\bar{p}_2(s) + 1)^2},$$

here $\bar{y} > 0, \bar{p}_1 \geq 0$ and $\bar{p}_2 > -1$.
The adjoint equations (5) are

$$\begin{cases} \dot{\bar{p}}_1(s) = \bar{p}_1^2(s)\,\bar{y}(s) - \frac{\bar{p}_1(s)-1}{\bar{y}^2(s)(\bar{p}_2(s)+1)^2} \\ \bar{p}_1(T) = 0, \end{cases} \tag{36}$$

and

$$\begin{cases} \dot{\bar{p}}_2(s) = \bar{p}_1(s)\,\bar{p}_2(s)\,\bar{y}(s) - \frac{\bar{p}_2(s)+1}{\bar{y}^2(s)(\bar{p}_2(s)+1)^2} \\ \bar{p}_2(T) = 0. \end{cases} \tag{37}$$

The state equation (1) is given by the following (ODE)

$$\begin{cases} \dot{\bar{y}}(s) = \frac{1}{\bar{y}(s)(\bar{p}_2(s)+1)} - \bar{p}_1(s)\,\bar{y}^2(s) \\ \bar{y}(0) = y_0, \end{cases}$$

The dynamic programming approach: The value function W_i satisfies HJB equations (12), as established in Theorem 2.2, for the two players as follows

$$\begin{cases} \dfrac{\partial W_1}{\partial s}(s,y) + \inf_{b_1(s)\in\mathcal{B}_1[0,T]}\left\{ y(s)\,(b_2(s) - b_1(s))\,\dfrac{\partial W_1}{\partial y}(s,y) - y(s)\,b_2(s) + \frac{b_1^2(s)}{2}\right\} = 0, \\ W_1(T,x) = 0, \end{cases} \tag{38}$$

and

$$\begin{cases} \dfrac{\partial W_2}{\partial s}(s,y) + \inf_{b_2(s)\in\mathcal{B}_2[0,T]}\left\{ y(s)\,(b_2(s) - b_1(s))\,\dfrac{\partial W_2}{\partial y}(s,y) - 2\sqrt{b_2(s)} + y(s)\,b_2(s)\right\} = 0 \\ W_2(T,x) = 0, \end{cases} \tag{39}$$

Where the FNE for our problem, we can write as:

$$\bar{b_1}(s) = \bar{y}(s)\,.\frac{\partial W_1}{\partial y}(s,\bar{y}(s)),$$

$$\bar{b_2}(s) = \frac{1}{\bar{y}^2(s)\left(\dfrac{\partial W_2}{\partial y}(s,\bar{y}(s)) + 1\right)^2}.$$

In addition, the HJB equations (38) and (39) gets the following form:

$$\begin{cases} \dfrac{\partial W_1}{\partial s}(s,y) + \left\{ \bar{y}(s)\left(\frac{1}{\bar{y}^2(s)\left(\frac{\partial W_2}{\partial y}(s,\bar{y}(s))+1\right)^2} - \bar{y}(s)\,.\dfrac{\partial W_1}{\partial y}(s,\bar{y}(s))\right)\dfrac{\partial W_1}{\partial y}(s,y) \right. \\ \left. \quad -\bar{y}(s)\,\frac{1}{\bar{y}^2(s)\left(\frac{\partial W_2}{\partial y}(s,\bar{y}(s))+1\right)^2} + \frac{\left(\bar{y}(s).\frac{\partial W_1}{\partial y}(s,\bar{y}(s))\right)^2}{2}\right\} = 0, \\ W_1(T,x) = 0, \end{cases} \tag{40}$$

and

$$\begin{cases} \dfrac{\partial W_2}{\partial s}(s,y) + \Bigg\{ \bar{y}(s) \left(\dfrac{1}{\bar{y}^2(s)\left(\dfrac{\partial W_2}{\partial y}(s,\bar{y}(s))+1\right)^2} - \bar{y}(s) \,.\, \dfrac{\partial W_1}{\partial y}(s,\bar{y}(s)) \right) \dfrac{\partial W_2}{\partial y}(s,y) \\ \qquad -2\sqrt{\dfrac{1}{\bar{y}^2(s)\left(\dfrac{\partial W_2}{\partial y}(s,\bar{y}(s))+1\right)^2}} + \bar{y}(s)\, \dfrac{1}{\bar{y}^2(s)\left(\dfrac{\partial W_2}{\partial y}(s,\bar{y}(s))+1\right)^2} \Bigg\} = 0 \\ W_2(T,x) = 0, \end{cases} \tag{41}$$

4.1. The connection between PMP and DPP: Smooth case

In order to explain the results of Theorem 3.1, we can derive the equality (13) directly from equations (38)-(41). About equality (14), the adjoint variables $\bar{p}_i(s)\,, i = 1, 2$ represents the marginal value (also known as the shadow prices) of the sale price $\bar{y}(s)$. This provides an economic interpretation to the adjoint variables (see [11], [23], [1]). In addition, the change in the value of the sale price of the system from state $\bar{y}(s)$ to $\bar{y}(s) + \gamma y(s)$ is

$$W_i(s, \bar{y}(s) + \gamma y(s)) - W_i(s, \bar{y}(s)) \approx \bar{p}_i(s)\, \gamma y(s)\,.\; i = 1, 2. \tag{42}$$

This implies (Fréchet) differentiability of $W_i(s, \bar{y}(s))$ at $\bar{y}(s)$ (see e.g., [1]). Thus, $\bar{p}_i(s)\,, i = 1, 2$ represents the marginal value of the rate of change in the profit W_i for slight adjustments in the sale price $\bar{y}(s)$. As sale prices increase due to increased consumption, $\bar{p}_1(s)$ decreases for the producer while $\bar{p}_2(s)$ increases for the consumer. Furthermore, the marginal value for producer $\bar{p}_1(s)$ can be interpreted as the incremental profit of producing and selling another product, and for consumers, $\bar{p}_2(s)$ represents the maximum price they are actually willing to pay for the last thing they consume.

4.2. The connection between PMP and DPP: Nonsmooth case

Similar to the smooth case, we illustrate the result of the Theorem 3.2 when the value function $W_i(\cdot,\cdot)$ is nonsmooth, satisfying the viscosity solution (VS); see [23]. As we have seen in Subsection 4.1, since the second inclusion in (19) and when the increment $\gamma y(s)$ is small, the increase in the value of the system from state $\bar{y}(s)$ to $\bar{y}(s) + \gamma y(s)$ is defined as

$$W_i(s, \bar{y}(s) + \gamma y(s)) - W_i(s, \bar{y}(s)) \leq \bar{p}_i(s)\, \gamma y(s)\,. \tag{43}$$

Due to the positivity of both sides (43), we conclude that

$$|W_i(s, \bar{y}(s) + \gamma y(s)) - W_i(s, \bar{y}(s))| \leq \bar{p}_i(s)\, |\gamma y(s)|.$$

This indicates that the effect of slight changes $\gamma y(s)$ in the sale price on the producer's and the customer's payoffs is dependent on their individual marginal values. Then, as the sale price increases, the producer's marginal value $\bar{p}_1(s)$ decreases, suggesting that the rate of increase in the producer's reward per unit sold slows down. Meanwhile, the consumer's marginal value $\bar{p}_2(s)$ increases, suggesting that consumer are prepared to pay more for each unit they consume. The other side, the decrease in the value of the sale price state from state $\bar{y}(s)$ to $\bar{y}(s) - \gamma y(s)$, then

$$W_i(s, \bar{y}(s) - \gamma y(s)) - W_i(s, \bar{y}(s)) \leq -\bar{p}_i(s)\, \gamma y(s)\,. \tag{44}$$

Both sides of (44) are negative ($\gamma y(s) > 0$). So,

$$|W_i(s, \bar{y}(s) - \gamma y(s)) - W_i(s, \bar{y}(s))| \geq \bar{p}_i(s)\, |\gamma y(s)|.$$

When the sale prices decrease, the producer's marginal value $\bar{p}_1(s)$ increases, proving that the additional profit made from producing and selling more units rises as well. Conversely, the consumer's marginal value $\bar{p}_2(s)$

decreases, which indicates that consumers are less able to pay for every product that they consume as prices decrease. Similarly, we can also interpret the Hamiltonian $H_i\left(s, y, b_1, b_2, p_i\right), i=1,2$ as the rate of change for the maximum profit with respect to time using (18).

In a producer-consumer game, the adjoint variable $\bar{p}_i\left(s\right)$ illustrates how changes in the state $\bar{y}\left(s\right)$ (the sale prices) affect the optimal cost or payoff $W_i\left(s, \bar{y}\left(s\right)\right)$ (value function). To be more precise, the rate at which changes in the state variable affect the value function is indicated by the adjoint variable, often known as the marginal value. Essentially, $\bar{p}_i\left(s\right)$ expresses the sensitivity of optimal costs or payoffs to small state adjustments in the state.

5. Conclusion

In this paper, we have presented a deterministic two-player nonzero-sum differential game on a finite horizon. We have obtained the connection between the adjoint variables in the MP and the value function in the DPP in either cases that value function is smooth and nonsmooth. The connection is established in terms of derivatives and super- and subdifferentials of the value function. An example involving producer-consumer game is provided to illustrate our results. This article represents a generalization of the results in [23] related to deterministic optimal control problems. In the future, we will extend the theoretical results of this paper to stochastic nonzero-sum differential games to obtain the connection between the adjoint variables and the value function in both smooth and nonsmooth cases.

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