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Distance k-domination and k-resolving domination of the corona product of graphs

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Abstract For two simple graphs G and H, the vertex corona product of G and H, denoted by $G \odot H$, is the graph obtained by adding a copy of H for each vertex of G and joining each vertex of G to all vertices in its corresponding copy of H. For $k \geq 1$, a set of vertices D in a graph G is a distance k-dominating set if any vertex in G is at a distance less or equal to k from a vertex in D. The minimum cardinality overall distance k-dominating sets of G is the distance k-domination number, denoted by $\gamma_k(G)$. The metric dimension of a graph is the smallest number of vertices required to distinguish all other vertices based on distances uniquely. The concept of distance k-resolving domination in graphs combines both distance k-domination and the metric dimension of graphs. In this paper, we investigate for all $k \geq 1$, the distance k-domination and the distance k-resolving domination in the vertex corona product of graphs. First, we show that for $k \geq 2$, the distance k-domination number of $G \odot H$ is equal to $\gamma_{k-1}(G)$ for any two graphs G and H. Then, we give the exact value of $\gamma_k(G \odot H)$ when G is a complete graph, complete m-partite graph, path and cycle. We also provide general bounds for $\gamma_k(G \odot H)$. Then, we examine the distance k-resolving domination number for $G \odot H$. For $k = 1$, we give bounds for $\gamma^r(G \odot H)$ the resolving domination number of $G \odot H$ and characterize the graphs achieving those bounds. Later, for $k \geq 2$, we establish bounds for $\gamma_k^r(G \odot H)$ the distance k-resolving domination number of $G \odot H$ and characterize the graphs achieving these bounds.

Keywords graph theory, domination, distance domination, metric dimension, distance resolving domination, vertex corona product

AMS 2010 subject classifications 05C12, 05C69, 05C76

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1. Introduction

In this paper, we consider only finite, undirected, and unweighted graphs. Graph terminology can be found in [6].

Distance in graphs is a fundamental concept in graph theory. Another well-known concept in graph theory is domination in graphs. A combination of distance and domination in graphs was introduced by Meir and Moon [21] in 1975. For $k \geq 1$, in a graph $G = (V, E)$, a set $D \subseteq V$ is a *distance k-dominating set*, if for any vertex $v \in V \setminus D$, there exists a vertex $u \in D$, such that $d_G(u, v) \leq k$, where $d_G(u, v)$ is the length of a shortest path joining u and v in G. The *distance k-domination number* is the minimum cardinality overall distance k-dominating sets of the graph G, it is denoted by $\gamma_k(G)$. Note that for $k = 1$, the distance 1-domination number is the domination number of the graph, here denoted simply by $\gamma(G)$. Distance k-domination in graphs finds applications to many problems, this includes geometric problems [20], communication networks [24], and facility location problems in operations research [16]. Precise descriptions and results on the distance k-domination in graphs can be found in [15].

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The metric dimension of a graph is the minimum number of vertices needed to identify all other vertices based on distances. The concept of metric dimension was introduced independently by Slater [23] in 1975, and by Harary and Melter [14] in 1976. Formally, for a graph $G = (V, E)$, let $W = \{w_1, w_2, ..., w_r\}$ be an ordered set of vertices in G. The *metric representation* of $v \in V$ with respect to W is the r-vector $c(v|W) = (d_G(v, w_1), d_G(v, w_2), ..., d_G(v, w_r))$. The set W is a *resolving set* of G, if for every two distinct vertices $v, u \in V$, $c(v|W) \neq c(u|W)$. The minimum cardinality of a resolving set of G is the *metric dimension* of G, and is denoted by $dim(G)$. Resolving sets have many applications to several problems involving graphs, like robot navigation in networks [19], pharmaceutical chemistry [5], coin weighing problems, strategies for Mastermind game [3, 18], and more. For a survey on the metric dimension of graphs, we refer the reader to [26]. Example of minimum resolving set of a path graph of order 8 is illustrated in Figure 1 (a).

A recently studied graph quantity closely related to both metric dimension and distance k-domination is the distance k-resolving domination in graphs [22, 29]. The concept of distance k-resolving domination combines together the concepts of resolvability and distance domination in graphs. A set $S \subseteq V$ in a graph G is a *distance k-resolving dominating set*, if S satisfies both conditions:

(i) For any $v \in V$, $d_G(v, S) \leq k$, where $d_G(v, S) = min\{d_G(v, x) : x \in S\}$.
(ii) For any two different vertices $u, v \in V$, there exists $x \in S$ such that $d_G(u, x) \neq d_G(v, x)$.

Notice that condition (i) means that S is a distance k-dominating set of G, and condition (ii) means that S is a resolving set of G. The *distance k-resolving domination number*, denoted by $\gamma_k^r(G)$, is the minimum cardinality of a distance k-resolving dominating set of G. When $k = 1$, the distance 1-resolving domination number is called the resolving domination number of the graph and is denoted simply by $\gamma^r(G)$. For studies on the resolving domination number of graphs, see for example [1, 2, 13, 17], and for results on the distance k-resolving domination number for $k \geq 2$, we refer the reader to [22, 29]. Examples of distance k-resolving dominating sets of path graph of order 8 for $k = 1$ and $k = 3$ are illustrated in Figure 1 (b) and (c), respectively.

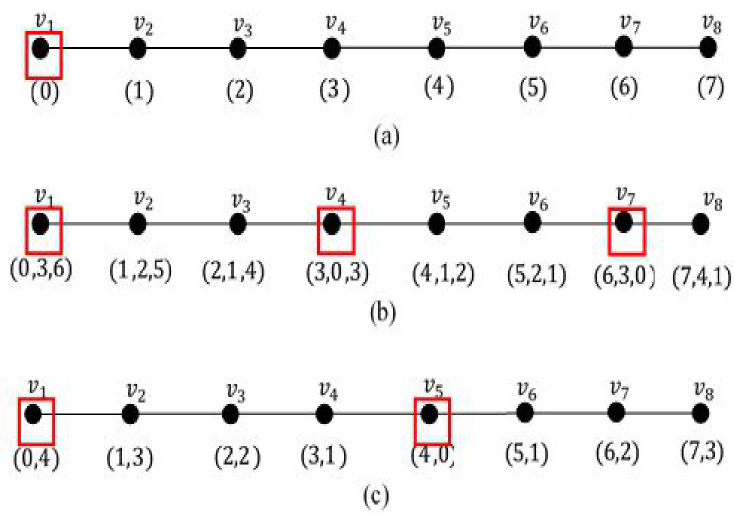


Figure 1. (a) Example of minimum resolving set of P_8 (b) Example of minimum resolving dominating set of P_8 (c) Example of minimum distance 3-resolving dominating set of P_8. Vertices are annotated with their metric representation which are all unique.

In this paper we investigate, for all $k \geq 1$, the distance k-domination and distance k-resolving domination of the vertex corona product of two graphs. The *vertex corona product* of two graphs G and H, denoted by $G \odot H$, is the graph obtained by adding a copy of H for each vertex v of G and joining v to every vertex of its corresponding copy of H. To avoid repetition, we call it simply the corona product of graphs.

Researchers have become increasingly interested in graph operations, such as their product. Researchers are drawn to the corona product among the many existing graph operations because of its intricate yet distinct structure, which creates copies of the original graph when one graph is multiplied on its own, creating a massive structure. Studying its uses and structural characteristics is therefore both necessary and fascinating. Since their introduction in 1970 by Frucht and Harary [11], many aspects and invariants of the corona product of graphs have been studied, see [8, 10, 25, 28, 31, 32]. Corona products are very important in data analytics, where large volumes of data must be quickly processed in order to make a conclusion. It may be applied in biotechnology to DNA sampling [30]. It may be used in chemistry [7] to understand chemical compound structures. It might also be useful in social science to understand a community's or group's behavioral patterns. In here we focus on the distance k-domination number and the distance k-resolving domination number of the corona product. For results on the metric dimension of the corona product of graphs, we refer to [10, 31, 32].

The paper is organized as follows. In Section 2, we give some preliminary results regarding the degree and the distance in the corona product of two graphs. We establish the maximum degree, minimum degree, diameter and radius for the corona product of any two graphs. Section 3 is dedicated to the domination number and the distance k-domination number of the corona product of two graphs. First, we provide another proof showing that $\gamma(G \odot H) = |G|$, where $|G|$ is the order of the graph G. Then we show that for $k \geq 2$, $\gamma_k(G \odot H) = \gamma_{k-1}(G)$ for any two graphs G and H. We use these results to give some bounds in general for $\gamma_k(G \odot H)$ in terms of k, the order of G, its diameter, and radius. We also give the exact value of $\gamma_k(G \odot H)$ for all $k \geq 1$, when H is any graph and G is a complete graph, complete m-partite graph, path, and cycle. In Section 4, we investigate the resolving domination number and the distance k-resolving domination number for $G \odot H$. We give bounds for $\gamma^r(G \odot H)$ in terms of the order of both G and H and characterize the graphs achieving both the upper bound and the lower bound. Afterward, for $k \geq 2$, we give both upper and lower bounds for $\gamma_k^r(G \odot H)$. Then we use the equivalence relationship between $dim(G \odot H)$ and $\gamma_k^r(G \odot H)$ to characterize the graphs achieving those bounds. Finally, we conclude the paper with several directions for future work.

2. Preliminary results

In this section, we give some results on the degree and the distance in the corona product of two graphs, which will help us understand the structure of the corona product of graphs. First, we will give the formal definition of the corona product of two graphs G and H. The notations will be used throughout the remainder of the paper.

Let G be a graph of order n_1 with vertex set $V(G) = \{v_1, v_2, \ldots, v_{n_1}\}$ and let H be a graph of order n_2 with vertex set $V(H) = \{u_1, u_2, \ldots, u_{n_2}\}$. For $1 \leq i \leq n_1$, let H_i be the i-th copy of H and let $V(H_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,n_2}\}$. The corona graph $G \odot H$ is the graph with the vertex set $V(G \odot H) = V(G) \cup_{i=1}^{n_1} V(H_i)$, obtained from G and by joining the vertex v_i of G with an edge with every vertex from H_i the i-th copy of H.

The *open neighborhood* of a vertex v in a graph G, denoted by $N_G(v)$, is the set of vertices that are adjacent to v in G. The *degree* of a vertex v in G, denoted by $deg_G(v)$, is the number of vertices that are adjacent to v in G. From the definition of the corona product of two graphs, we get the following results.

Lemma 2.1
Let G and H be two graphs. Then the degree of a vertex x in $V(G \odot H)$ is

$$deg_{G \odot H}(x) = \begin{cases} deg_G(x) + n_2, & if\ x \in V(G); \\ deg_H(x) + 1, & if\ x \in V(H_i)\ for\ 1 \leq i \leq n_1. \end{cases}$$

The *maximum degree* of a graph G, denoted by $\Delta(G)$, is the maximum value of the degrees among all vertices of G. The *minimum degree* of a graph G is the minimum value of the degrees among all vertices of G and it is denoted by $\delta(G)$. Based on Lemma 2.1, we prove the following results regarding the maximum and minimum degrees of the corona product of any two graphs G and H.

Lemma 2.2
Let G and H be two graphs.

- We have $\Delta(G \odot H) = \Delta(G) + n_2$.
- If G is a trivial graph, then $\Delta(G \odot H) = n_2$, and $deg_{G \odot H}(v) = \Delta(G \odot H) = n_2$ if and only if $v \in V(G)$ or $deg_H(v) = n_2 - 1$.
- If G is not a trivial graph, we have $deg_{G \odot H}(v) = \Delta(G \odot H)$ if and only if $v \in V(G)$ and $deg_G(v) = \Delta(G)$.

Proof
Let v be a vertex in $V(G \odot H)$. If $v \in V(H_i)$ with $1 \leq i \leq n_1$, by Lemma 2.1 we have $deg_{G \odot H}(v) = deg_H(v) + 1$. Since $deg_H(v) \leq n_2 - 1$, we have $deg_{G \odot H}(v) \leq n_2$. If v is in $V(G)$, by Lemma 2.1 we have $deg_{G \odot H}(v) = deg_G(v) + n_2 \leq \Delta(G) + n_2$. If $deg_G(v) = \Delta(G)$, then $deg_{G \odot H}(v) = \Delta(G) + n_2$. Therefore $\Delta(G \odot H) = \Delta(G) + n_2$.

Let G be a trivial graph. Since $\Delta(G) = 0$, from the previous result we have $\Delta(G \odot H) = \Delta(G) + n_2 = n_2$. If $deg_{G \odot H}(v) = \Delta(G \odot H) = n_2$, suppose that $v \in V(H_i)$ with $1 \leq i \leq n_1$, and $deg_H(v) \leq n_2 - 2$. Based on Lemma 2.1 we have $deg_{G \odot H}(v) = deg_H(v) + 1 \leq n_2 - 1$, which is a contradiction. Therefore $v \in V(G)$ or $deg_H(v) = n_2 - 1$. Conversely, if $v \in V(G)$, then by Lemma 2.1, we have $deg_{G \odot H}(v) = n_2 = \Delta(G \odot H)$. Also, if $v \in V(H_i)$ and $deg_H(v) = n_2 - 1$, then by Lemma 2.1 we have $deg_{G \odot H}(v) = deg_H(v) + 1 = n_2 = \Delta(G \odot H)$.

If G is not trivial, let v be a vertex in $V(G \odot H)$ such that $deg_{G \odot H}(v) = \Delta(G \odot H)$. Suppose that $v \in V(H_i)$ with $1 \leq i \leq n_1$, or ($v \in V(G)$ and $deg_G(v) < \Delta(G)$). If $v \in V(H_i)$, based on Lemma 2.1 we have $\Delta(G \odot H) = deg_{G \odot H}(v) = deg_H(v) + 1 \leq n_2$. If u is a vertex in $V(G)$, by Lemma 2.1 we have $deg_{G \odot H}(u) = deg_G(u) + n_2$. Since G is not a trivial graph, we have $deg_G(u) \geq 1$. Therefore $deg_{G \odot H}(u) \geq n_2 + 1 > \Delta(G \odot H)$, which is a contradiction. Now, if $v \in V(G)$ and $deg_G(v) < \Delta(G)$, then by Lemma 2.1 we have $deg_{G \odot H}(v) = deg_G(v) + n_2 < \Delta(G) + n_2$, again a contradiction, since $\Delta(G \odot H) = \Delta(G) + n_2$. Conversely, if $v \in V(G)$ and $deg_G(v) = \Delta(G)$, then by Lemma 2.1 we have $deg_{G \odot H}(v) = deg_G(v) + n_2 = \Delta(G) + n_2 = \Delta(G \odot H)$. □

Lemma 2.3
Let G and H be two graphs.

- We have $\delta(G \odot H) = \delta(H) + 1$.
- For $v \in V(G)$, we have $deg_{G \odot H}(v) = \delta(G \odot H)$ if and only if G is a trivial graph and $H \cong K_{n_2}$.
- For $u_j \in V(H)$, we have for any $1 \leq i \leq n_1$, $deg_{G \odot H}(u_{i,j}) = \delta(G \odot H)$ if and only if $deg_H(u_j) = \delta(H)$.

Proof
Let $V(G) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(H) = \{u_1, u_2, \ldots, u_{n_2}\}$. For $1 \leq i \leq n_1$, in $G \odot H$ let us denote by H_i the i-th copy of H joined to the vertex v_i and let $V(H_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,n_2}\}$. For any $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$, from Lemma 2.1 we have $deg_{G \odot H}(v_i) = deg_G(v_i) + n_2 \geq n_2$ and $\delta(H) + 1 \leq deg_{G \odot H}(u_{i,j}) = deg_G(u_{i,j}) + 1 \leq n_2$. For some $1 \leq j \leq n_2$, if $deg_H(u_j) = \delta(H)$, then $deg_{G \odot H}(u_{i,j}) = \delta(H) + 1$. Therefore, $\delta(G \odot H) = \delta(H) + 1$.

Let $v \in V(G)$ be such that $deg_{G \odot H}(v) = \delta(G \odot H)$. From the previous result, we have $\delta(G \odot H) = \delta(H) + 1 \leq n_2$. Suppose that G is not a trivial graph or $H \ncong K_{n_2}$. If G is not a trivial graph, then for any $1 \leq i \leq n_1$ we have $deg_G(v_i) \geq 1$. By Lemma 2.1 we have $deg_{G \odot H}(v) = deg_G(v) + n_2 \geq n_2 + 1$, which is a contradiction. Now, if $H \ncong K_{n_2}$, then there exists a vertex u_j in $V(H)$ such that $deg_H(u_j) \leq n_2 - 2$. Based on Lemma 2.1 we have $deg_{G \odot H}(u_{i,j}) \leq n_2 - 1$. Since $deg_{G \odot H}(v) \geq n_2$ we have $deg_{G \odot H}(v) = \delta(G \odot H) > deg_{G \odot H}(u_{i,j})$, again a contradiction. Conversely, if G is a trivial graph and $H \cong K_{n_2}$, then from the definition of the corona product we have $G \odot H \cong K_{n_2+1}$ and $deg_{G \odot H}(v) = n_2 = \delta(G \odot H)$.

We have $\delta(G \odot H) = \delta(H) + 1$. For $u_j \in V(H)$, if $deg_H(u_j) = \delta(H)$, then by Lemma 2.1 for $1 \leq i \leq n_1$ we have $deg_{G \odot H}(u_{i,j}) = deg_H(u_j) + 1 = \delta(H) + 1 = \delta(G \odot H)$. Now, for $1 \leq i \leq n_1$ we have $deg_{G \odot H}(u_{i,j}) = \delta(G \odot H)$. Since $deg_{G \odot H}(u_{i,j}) = deg_H(u_j) + 1$ we have $deg_H(u_j) + 1 = \delta(G \odot H) = \delta(H) + 1$. It follows that $deg_H(u_j) = \delta(H)$. □

The *distance* between two vertices u and v in a graph G, denoted by $d_G(u, v)$, is the minimum length overall paths joining u and v in G. If u and v lie in distinct components of G, we set $d_G(u, v) := \infty$. The following lemma is a straightforward consequence of the definition of the corona product of two graphs G and H.

Lemma 2.4
Let G and H be two graphs. Let H_i be the i-th copy of H in $G \odot H$ and let $V(G) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(H_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,n_2}\}$, for $1 \leq i \leq n_1$. Then

$$d_{G\odot H}(v_i, v_j) = d_G(v_i, v_j),\ for\ 1 \leq i, j \leq n_1,$$

$$d_{G\odot H}(v_i, u_{kl}) = d_G(v_i, v_k) + 1,\ for\ 1 \leq i, k \leq n_1,\ and\ 1 \leq l \leq n_2,$$

$$d_{G\odot H}(u_{ij}, u_{kl}) = \begin{cases} d_G(v_i, v_k) + 2, & if\ i \neq k; \\ 1, & if\ i = k\ and\ u_j u_k \in E(H); \\ 2, & if\ i = k\ and\ u_j u_k \notin E(H). \end{cases}$$

In a graph G the *eccentricity* of a vertex v, denoted by $ecc_G(v)$, is the distance between v and the farthest vertex from v in G. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum eccentricity among all the vertices of G. The diameter of G is also the greatest distance between any two pairs of vertices of G. The *radius* of a connected graph G, denoted by $rad(G)$, is the minimum eccentricity among the vertices of G. As a consequence of Lemma 2.4, we get the following results regarding the diameter and radius of the corona product of two graphs G and H.

Lemma 2.5
Let G and H be two graphs of order n_1 and n_2 respectively. Then the following statements hold.

- $diam(G \odot H) = 1$ if and only if G is a trivial graph and $H \cong K_{n_2}$, and we have $G \odot H \cong K_{n_2+1}$.
- $diam(G \odot H) = 2$ if and only if G is a trivial graph and $H \ncong K_{n_2}$.
- If G is not a trivial graph, then $diam(G \odot H) = diam(G) + 2 \geq 3$.

Proof
If G is a trivial graph and $H \cong K_{n_2}$, then from the definition of the corona product of G and H, we have $G \odot H$ is a complete graph of order $n_2 + 1$ and so $diam(G \odot H) = 1$. Conversely, if $diam(G \odot H) = 1$, we suppose that G is not a trivial graph or $H \ncong K_{n_2}$ with $n_2 \geq 1$. If G is not a trivial graph, then $n_1 \geq 2$. Let v_i and v_k be two distinct vertices in G. From the definition of the corona product of G and H, any two vertices $u_{i,j}$ and $u_{k,l}$ from the two copies H_i and H_k respectively, are not adjacent. It follows that $d_{G\odot H}(u_{i,j}, u_{k,l}) \geq 2$. Therefore, $diam(G \odot H) \geq 2$, which is a contradiction. On the other hand, if $H \ncong K_{n_2}$, let u_j and u_l be two vertices in H such that $u_i u_j \notin E(H)$. From Lemma 2.4, for some $1 \leq i \leq n_1$, we have $d_{G\odot H}(u_{i,j}, u_{i,l}) = 2$. Hence $diam(G \odot H) \geq 2$, again a contradiction.

If G is a trivial graph and $H \ncong K_n$, let $V(H) = \{u_1, u_2, \ldots, u_{n_2}\}$ and let u_i and u_j with $1 \leq i, j \leq n_2$ be such that $u_i u_j \notin E(H)$. Based on Lemma 2.4, we have $d_{G\odot H}(u_i, u_j) = 2$. Therefore, $diam(G \odot H) = 2$. Conversely, if $diam(G \odot H) = 2$, we suppose that G is not a trivial graph or $H \cong K_n$. If G is not a trivial graph, let v_i and v_k be two distinct vertices in G. Based on Lemma 2.4, $d_{G\odot H}(u_{ij}, u_{kl}) = d_G(v_i, v_k) + 2 \geq 3$ for $u_{ij} \in H_i$ and $u_{kl} \in H_l$. Therefore, $diam(G \odot H) \geq 3$, which is a contradiction. Now if $H \cong K_n$ and G is a trivial graph, then the previous case shows that $G \odot H \cong K_{n_2+1}$ and $diam(G \odot H) = 1$, again a contradiction.

If G is not a trivial graph, then $diam(G) \geq 1$. Let H_i be the i-th copy of H in $G \odot H$ and let $V(G) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(H_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,n_2}\}$ with $1 \leq i \leq n_1$. From Lemma 2.4 for any $1 \leq i, j \leq n_1$, we have $d_{G\odot H}(v_i, v_j) = d_G(v_i, v_j) \leq diam(G)$. For all $1 \leq i, k \leq n_1$ and $1 \leq l \leq n_2$, we have $d_{G\odot H}(v_i, u_{kl}) = d_G(v_i, v_k) + 1 \leq diam(G) + 1$. Also for all $1 \leq i, k \leq n_1$ and $1 \leq j, l \leq n_2$, we have $d_{G\odot H}(u_{ij}, u_{kl}) \leq d_G(v_i, v_k) + 2 \leq diam(G) + 2$. Hence $diam(G \odot H) \leq diam(G) + 2$. Conversely, let $v_i, v_k \in V(G)$ be such that $d_G(v_i, v_k) = diam(G)$. Based on Lemma 2.4, for some $1 \leq j, l \leq n_2$, we have $d_{G\odot H}(u_{ij}, u_{kl}) = d_G(v_i, v_k) + 2 = diam(G) + 2$. Therefore, $diam(G \odot H) \geq diam(G) + 2$. Thus, $diam(G \odot H) = diam(G) + 2 \geq 3$. □

Lemma 2.6
For two graphs G and H, the following statements hold.

- $rad(G \odot H) = 1$ if and only if G is a trivial graph, i.e., $G = K_1$.

- If G is not a trivial graph, then $rad(G \odot H) = rad(G) + 1$.

Proof
Let $V(G) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(H) = \{u_1, u_2, \ldots, u_{n_2}\}$. For $1 \le i \le n_1$, in $G \odot H$ denote by H_i the i-th copy of H joined to the vertex v_i and let $V(H_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,n_2}\}$.

- Let G be a trivial graph and let $V(G) = \{v\}$. We have the eccentricity of v in $G \odot H$ is 1. Therefore, $rad(G \odot H) = 1$.
  Conversely, if $rad(G \odot H) = 1$, suppose that G is not a trivial, graph which means that $n_1 \ge 2$. From Lemma 2.4, for any $v_i \in V(G)$, $1 \le i \le n_1$, for some $k \ne i$ we have $d_{G\odot H}(v_i, u_{kl}) = d_G(v_i, v_k) + 1 \ge 2$. Therefore, for any $v_i \in V(G)$, we have $ecc_{G\odot H}(v_i) \ge 2$. Also, for any $1 \le i, k \le n_1$ and $1 \le j, l \le n_2$, with $k \ne i$, we have $d_{G\odot H}(u_{ij}, u_{kl}) = d_G(v_i, v_k) + 2 \ge 3$. Therefore, for any $1 \le i \le n_1$ and $1 \le j \le n_2$, $ecc_{G\odot H}(u_{ij}) \ge 3$. It follows that, $rad(G \odot H) \ge 2$, which is a contradiction. Hence G is a trivial graph.
- If $n_1 \ge 2$, from Lemma 2.4, for any $v_i \in V(G)$, $1 \le i \le n_1$, for some $k \ne i$ we have $d_{G\odot H}(v_i, u_{kl}) = d_G(v_i, v_k) + 1$ and $d_{G\odot H}(v_i, v_k) = d_G(v_i, v_k)$. Therefore, for any $1 \le i \le n_1$, $ecc_{G\odot H}(v_i) \ge rad(G) + 1$. For any $1 \le i, k \le n_1$ and $1 \le j, l \le n_2$, with $k \ne i$, we have $d_{G\odot H}(u_{ij}, u_{kl}) = d_G(v_i, v_k) + 2$ and $d_{G\odot H}(v_i, u_{kl}) = d_G(v_i, v_k) + 1$. Therefore, for any $1 \le i \le n_1$ and $1 \le j \le n_2$, we have $ecc_{G\odot H}(u_{ij}) \ge rad(G) + 2$. Hence, $rad(G \odot H) \ge rad(G) + 1$.
  Let v_i be a central vertex of G, i.e., $ecc_G(v_i) = rad(G)$. From the above $ecc_{G\odot H}(v_i) \ge rad(G) + 1$. Based on Lemma 2.4, for any $1 \le i, k \le n_1$ and $1 \le l \le n_2$, we have $d_{G\odot H}(v_i, u_{kl}) = d_G(v_i, v_k) + 1 \le rad(G) + 1$ and $d_{G\odot H}(v_i, v_k) = d_G(v_i, v_k) \le rad(G)$. Therefore, $ecc_{G\odot H}(v_i) \le rad(G) + 1$. Hence, $ecc_{G\odot H}(v_i) = rad(G) + 1$. Thus, $rad(G \odot H) = rad(G) + 1$. □

Since the parameters of the corona product studied in the following sections have a close relationship with the neighborhood and distance in graphs, the results obtained in this preliminary section will be useful in the forthcoming sections.

3. Distance k-domination of the corona product of two graphs

In this section, for all $k \ge 1$, we will discuss the distance k-domination number of the corona product of two graphs.

First, we present the following relationship between the radius of the graph and the distance k-domination number, also known as the *Duality Lemma*.

Lemma 3.1
For $k \ge 1$, let G be a connected graph. Then we have $rad(G) \le k$ if and only if $\gamma_k(G) = 1$.

Corollary 3.1
For two graphs G and H and for all $k \ge 1$, we have $\gamma_k(G \odot H) = 1$ if and only if $rad(G) \le k - 1$.

Proof
Let G and H be two graphs. For $k \ge 1$, if $\gamma_k(G \odot H) = 1$, then by Lemma 3.1 we have $rad(G \odot H) \le k$. From Lemma 2.6, we have $rad(G \odot H) = rad(G) + 1$, so $rad(G) \le k - 1$.

Conversely, if $rad(G) \le k - 1$, then from Lemma 2.6 we have $rad(G \odot H) = rad(G) + 1 \le k$. From Lemma 3.1, it follows that $\gamma_k(G \odot H) = 1$. □

In the following, for $k \ge 1$, we prove that there exists always a minimum distance k-dominating set of $G \odot H$, which is a subset of the vertex set of G.

Lemma 3.2
For $k \ge 1$, there exists a γ_k-set D of the graph $G \odot H$ such that $D \subseteq V(G)$.

Proof
For $k \ge 1$, let $D \subset V(G \odot H)$ be a γ_k-set of $G \odot H$. Suppose that $u_{i,j} \in D \cap V(H_i)$, and let $v \in V(G \odot H)$ be

such that $d_{G\odot H}(u_{i,j}, v) \leq k$. Based on Lemma 2.4, if $v \in V(G)$ $d_{G\odot H}(u_{i,j}, v) = d_G(v_i, v) + 1 \geq d_{G\odot H}(v_i, v)$. If $v \in V(H_k)$ with $i \neq k$, we have $d_{G\odot H}(u_{i,j}, v) = d_G(v_i, v_k) + 2 \geq d_{G\odot H}(v_i, v) = d_G(v_i, v_k) + 1$. Therefore, $d_{G\odot H}(v_i, v) \leq d_{G\odot H}(u_{i,j}, v) \leq k$. Hence for any vertex $v \in V(G \odot H)$ such that $d_{G\odot H}(u_{i,j}, v) \leq k$, we have $d_{G\odot H}(v_i, v) \leq k$, which means that the open k-neighborhood, i.e., the set of vertices at distance less or equal to k, $N_k(u_{i,j}; G \odot H) \subseteq N_k(v_i; G \odot H)$. Since D is a distance k-dominating set containing $u_{i,j}$, it follows that replacing $u_{i,j}$ by v_i in D, produces the set $D' = D \cup \{v_i\} \setminus \{u_{i,j}\}$ which is also a distance k-dominating set and $|D'| = |D| = \gamma_k(G \odot H)$. Thus, for $k \geq 1$, there is always γ_k-set D of $G \odot H$ such that $D \subseteq V(G)$. □

The following result appeared in [12]. We give another proof by using Lemma 3.2.

Theorem 3.1 ([12])
For two graphs G and H of order n_1 and n_2 respectively, we have $\gamma(G \odot H) = n_1$.

Proof
Let G and H be two graphs of order n_1 and n_2 respectively. In $G \odot H$, since for $1 \leq i \leq n_1$ every vertex in H_i is adjacent to the vertex $v_i \in V(G)$. The set $D = V(G)$ is a dominating set of $G \odot H$. Therefore, $\gamma(G \odot H) \leq n_1$.

We prove equality by using contradiction. Suppose that $\gamma(G \odot H) < n_1$, from Lemma 3.2 there is a dominating set D of the graph $G \odot H$ of cardinality $|D| = \gamma(G \odot H)$ such that $D \subseteq V(G)$. From the supposition we have $|D| < n_1$, then there is at least one vertex $v_i \in V(G)$ for $1 \leq i \leq n_1$ and $v_i \notin D$. Let $v_i \in V(G) \setminus D$. We have the neighborhood $N_{G\odot H}(V(H_i)) = \{v_i\}$, which means any vertex from $V(H_i)$ is only adjacent to the vertex v_i from $V(G)$. Hence the vertices in $V(H_i)$ are not adjacent to any vertex in D, which means that D is not a dominating set of $G \odot H$, which is a contradiction. Therefore, $\gamma(G \odot H) \geq n_1$. Thus, $\gamma(G \odot H) = n_1$. □

Next, for $k \geq 2$, we show that the distance k-domination number of $G \odot H$ is equal to the distance $(k-1)$-domination number of the graph G.

Theorem 3.2
Let G and H be two graphs. For $k \geq 2$, we have $\gamma_k(G \odot H) = \gamma_{k-1}(G)$.

Proof
Let G and H be two graphs of order n_1 and n_2 respectively and let D be a γ_{k-1}-set of G. For any vertex $v_i \in V(G)$, there is a vertex $v_j \in D$ such that $d_G(v_i, v_j) \leq k-1$. In $G \odot H$, from Lemma 2.4, $d_{G\odot H}(v_i, v_j) = d_G(v_i, v_j) \leq k-1$. Also for any $u_{i,l} \in V(H_i)$, we have $d_{G\odot H}(u_{i,l}, v_j) = d_G(v_i, v_j) + 1 \leq k$. Hence D is a distance k-dominating set of $G \odot H$. Therefore, $\gamma_k(G \odot H) \leq |D| = \gamma_{k-1}(G)$.

Based on Lemma 3.2, let D be a γ_k-set of $G \odot H$ such that $D \subseteq V(G)$. For all $1 \leq i \leq n_1$, we have for any vertex $u_{i,l} \in V(H_i)$ there is a vertex $v_j \in D$ such that $d_{G\odot H}(u_{i,k}, v_j) \leq k$. Since $d_{G\odot H}(u_{i,k}, v_j) = d_G(v_i, v_j) + 1$, we have $d_G(v_i, v_j) \leq k-1$. Therefore, D is a distance $(k-1)$-dominating set of G. Hence $\gamma_{k-1}(G) \leq |D| = \gamma_k(G \odot H)$, and the proof is completed. □

For $k \geq 2$, Theorem 3.2, allow us to investigate $\gamma_k(G \odot H)$ through the study of $\gamma_{k-1}(G)$.

Lemma 3.3
Let G, H, and H' be three graphs. For $k \geq 1$, we have $\gamma_k(G \odot H) = \gamma_k(G \odot H')$.

Proof
For $k = 1$, by Theorem 3.1 we have $\gamma(G \odot H) = \gamma(G \odot H') = |V(G)|$.

Now, for $k \geq 2$, from Theorem 3.2 we have $\gamma_k(G \odot H) = \gamma_{k-1}(G)$, also $\gamma_k(G \odot H') = \gamma_{k-1}(G)$. It follows that $\gamma_k(G \odot H) = \gamma_k(G \odot H')$. □

Observation 3.1
For $k \geq 1$, if G' is a spanning subgraph of a graph G, then we have $\gamma_k(G) \leq \gamma_k(G')$.

Proposition 3.1
Let G and H be two graphs. If G' is a spanning subgraph of G, then we have $\gamma(G \odot H) = \gamma(G' \odot H)$. Moreover, for $k \geq 2$, we have $\gamma_k(G \odot H) \leq \gamma_k(G' \odot H)$.

Proof
Let G and H be two graphs and let G' be a spanning subgraph of G. For $k = 1$, since G' is a spanning subgraph of G, by Theorem 3.1 we have $\gamma_k(G \odot H) = |V(G)|$ and $\gamma_k(G' \odot H) = |V(G')| = |V(G)|$. It follows that $\gamma_k(G \odot H) = \gamma_k(G' \odot H)$.

For $k \geq 2$, by Theorem 3.2 we have $\gamma_k(G \odot H) = \gamma_{k-1}(G)$ and $\gamma_k(G' \odot H') = \gamma_{k-1}(G')$. Also, from Observation 3.1 we have for $k \geq 2$, $\gamma_{k-1}(G) \leq \gamma_{k-1}(G')$. It follows that $\gamma_k(G \odot H) \leq \gamma_k(G' \odot H')$. □

In [21], Meir and Moon showed that $\gamma_k(G) \leq \dfrac{n}{k+1}$, for any connected graph G of order $n \geq k+1$. Topp and Volkmann [27] characterized the graphs achieving equality. Based on their results and Theorem 3.2, we have the following.

Theorem 3.3
For $k \geq 2$, let G be a connected graph of order $n \geq k$ and H any graph. Then we have

$$\gamma_k(G \odot H) \leq \frac{n}{k}.$$

There is equality $\gamma_k(G \odot H) = \dfrac{n}{k}$ if and only if at least one of the following conditions holds:

- $n = k$.
- $G \cong C_{2k}$.
- G is the graph obtained from a graph H' of order at least 2 by attaching a path of length $k-1$ to each vertex of H'.

Proof
For $k \geq 2$, let G be a connected graph of order $n \geq k$ and let H be any graph. Based on Theorem 3.2, we have $\gamma_k(G \odot H) = \gamma_{k-1}(G)$. From the upper bound in [21], we have for $k \geq 2$, $\gamma_{k-1}(G) \leq \dfrac{n}{k}$. It follows that $\gamma_k(G \odot H) \leq \dfrac{n}{k}$.

Based on the characterization in [27], for $k \geq 2$, we have $\gamma_{k-1}(G) = \dfrac{n}{k}$ if and only if $n = k$ or $G \cong C_{2k}$ or G is the graph obtained from a graph H' of order at least 2 by attaching a path of length $k-1$ to each vertex of H'. Since $\gamma_k(G \odot H) = \gamma_{k-1}(G)$, we get the results. □

In the following, we focus on the distance k-domination number of $G \odot H$ when G belongs to a certain family of graphs and H is any graph.

Let K_n be a *complete graph* of order n with $V(K_n) = \{v_i : 1 \leq i \leq n\}$. Every vertex of a complete graph is connected to the other vertices, so $E(K_n) = \{v_i v_j : 1 \leq i \leq n : 1 \leq j \leq n : i \neq j\}$. We have $diam(K_n) = rad(K_n) = 1$. It is easy to see that $\gamma_k(K_n) = 1$ for all $k \geq 1$.

Proposition 3.2
If G is a complete graph K_n and H is any graph, then

$$\gamma_k(K_n \odot H) = \begin{cases} n, & if\ k = 1, \\ 1, & if\ k \geq 2. \end{cases}$$

Proof
If $n = 1$, i.e., G is a trivial graph K_1, then by Lemma 2.6 for any graph H we have $rad(G \odot H) = 1$. It follows that for $k = 1$, we have $\gamma(K_1 \odot H) = n = 1$. Also for $k \geq 2$, we have $\gamma_k(K_1 \odot H) = 1$.

For $n \geq 2$, let K_n be a complete graph and H be any graph. We have the diameter and the radius of the complete graph are equal to 1 and the maximum and minimum degree is $\Delta(K_n) = \delta(K_n) = n - 1$. From Lemma 2.5 and 2.6 it follows that $diam(K_n \odot H) = 3$ and $rad(K_n \odot H) = 2$.

- If $k = 1$, then from Theorem 3.1, we have $\gamma_k(K_n \odot H) = n$.

- For $k \geq 2$, since the radius $rad(K_n \odot H) = 2$, we have $\gamma_k(K_n \odot H) = 1$ for $k \geq 2$. □

Let $K_{n_1,n_2,\ldots,n_m}$ be the complete m-partite graph of order $n = \sum_{i=1}^{m} n_i$, where $V_1, V_2, \ldots, V_m$ are its partite sets, with $|V_i| = n_i$ for $1 \leq i \leq m$. We have $diam(K_{n_1,n_2,\ldots,n_m}) = 2$ and $rad(K_{n_1,n_2,\ldots,n_m}) \leq 2$. If for all $1 \leq i \leq m$, we have $n_i \geq 2$, then we have $\gamma(K_{n_1,n_2,\ldots,n_m}) = 2$. Otherwise, $\gamma(K_{n_1,n_2,\ldots,n_m}) = 1$. For $k \geq 2$, we have $\gamma_k(K_{n_1,n_2,\ldots,n_m}) = 1$ for any complete m-partite graph.

Proposition 3.3
Let $K_{n_1,n_2,\ldots,n_m}$ be a complete m-partite graph of order n with $m \geq 2$ and H be any graph. Then we have

$$\gamma_k(K_{n_1,n_2,\ldots,n_m} \odot H) = \begin{cases} n, & if\ k = 1, \\ 1, & if\ k = 2 \text{ and there is } 1 \leq i \leq m, \text{ such that } n_i = 1, \\ 2, & if\ k = 2 \text{ and for all } 1 \leq i \leq m, \text{ we have } n_i \geq 2, \\ 1, & if\ k \geq 3. \end{cases}$$

Proof
Let $K_{n_1,n_2,\ldots,n_m}$ be a complete m-partite graph of order $n = \sum_{i=1}^{m} n_i$ with $m \geq 2$ and $n_i \geq 1$ for $1 \leq i \leq m$ and $V_1, V_2, \ldots, V_m$ are its partite sets and let H be any graph.

- For $k = 1$, from Theorem 3.1, we have $\gamma_k(K_{n_1,n_2,\ldots,n_m} \odot H) = n$.
- If $k = 2$ and there is $1 \leq i \leq m$ such that $n_i = 1$, let v_i be such that $V_i = \{v_i\}$. We have v_i is adjacent to all the other vertices in $K_{n_1,n_2,\ldots,n_m}$. Therefore $\gamma(K_{n_1,n_2,\ldots,n_m}) = 1$. From Theorem 3.2 we have $\gamma_2(K_{n_1,n_2,\ldots,n_m} \odot H) = \gamma(K_{n_1,n_2,\ldots,n_m}) = 1$.
- If $k = 2$ and for all $1 \leq i \leq m$ we have $n_i \geq 2$, then by Theorem 3.2 we have $\gamma_2(K_{n_1,n_2,\ldots,n_m} \odot H) = \gamma(K_{n_1,n_2,\ldots,n_m})$. Let $v_i \in V_i$ and $v_j \in V_j$ with $i \neq j$, we have v_i is adjacent to every vertex not in V_i and not adjacent to any vertex in V_i, which means a set consisting of a single vertex cannot be a dominating set of $K_{n_1,n_2,\ldots,n_m}$. Hence $\gamma(K_{n_1,n_2,\ldots,n_m}) \geq 2$. Also, v_j is adjacent to every vertex not in V_j. Therefore, the set $D = \{v_i, v_j\}$ is a dominating set of $K_{n_1,n_2,\ldots,n_m}$, which means that $\gamma(K_{n_1,n_2,\ldots,n_m}) \leq |D| = 2$. Thus, $\gamma(K_{n_1,n_2,\ldots,n_m}) = 2$. It follows that $\gamma_2(K_{n_1,n_2,\ldots,n_m} \odot H) = \gamma(K_{n_1,n_2,\ldots,n_m}) = 2$.
- If $k \geq 3$, we have $rad(K_{n_1,n_2,\ldots,n_m}) \leq 2$, then from Corollary 3.1 we have $\gamma_k(K_{n_1,n_2,\ldots,n_m} \odot H) = 1$. □

A special case of complete k-partite graphs is the star graph $K_{1,n}$ consisting of two partite sets, a single vertex adjacent to every vertex in an independent set of order n. The distance k-domination number of the corona product of star graph $K_{1,n}$ and any graph can be deducted from Proposition 3.3.

Corollary 3.2
Let $K_{1,n}$ be a star graph and H is any graph, then

$$\gamma_k(K_{1,n} \odot H) = \begin{cases} n+1, & if\ k = 1, \\ 1, & if\ k \geq 2. \end{cases}$$

Proof
If $k = 1$, from Proposition 3.3, we have $\gamma(K_{1,n} \odot H) = |K_{1,n}| = n + 1$. If $k \geq 2$, since $K_{1,n}$ is a complete bipartite graph with one partition consisting of a single vertex, from Proposition 3.3 it follows that $\gamma(K_{1,n} \odot H) = 1$ for $k \geq 2$. □

Let C_n be a *cycle graph* with vertex set $V(C_n) = \{v_i : 1 \leq i \leq n\}$, and edge set $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}$. For $0 \leq i, j \leq n-1$, with $i \neq j$, we have $d_{C_n}(v_i, v_j) = min\{|i-j|, n-|i-j|\}$. Thus, $diam(C_n) = rad(C_n) = \lfloor \frac{n}{2} \rfloor$.

Proposition 3.4 ([9])
For $k \geq 1$ and $n \geq 3$, we have $\gamma_k(C_n) = \lceil \frac{n}{2k+1} \rceil$.

By using Theorem 3.1 and 3.2 together with the distance k-domination number of cycle graphs in Proposition 3.4, we get the distance k-domination number of the corona product of a cycle graph and any graph.

Proposition 3.5
If C_n is a cycle graph of order $n \geq 3$ and H is any graph, then

$$\gamma_k(C_n \odot H) = \begin{cases} n, & if\ k = 1, \\ \lceil \frac{n}{2k-1} \rceil, & if\ k \geq 2. \end{cases}$$

Proof
Let C_n be a cycle graph of order $n \geq 3$ and H a graph. If $k = 1$, then by Proposition 3.3, we have $\gamma(C_n \odot H) = |C_n| = n$.

For $k \geq 2$, from Theorem 3.2 we have $\gamma_k(C_n \odot H) = \gamma_{k-1}(C_n)$. Based on Proposition 3.4, we have $\gamma_{k-1}(C_n) = \lceil \frac{n}{2k-1} \rceil$. It follows that for $k \geq 2$, we have $\gamma_k(C_n \odot H) = \lceil \frac{n}{2k-1} \rceil$. For an illustration of Proposition 3.5, see Figure 2. □

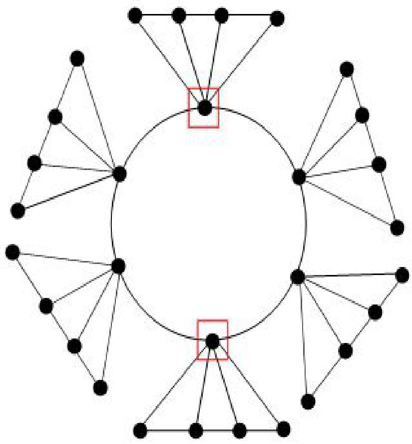


Figure 2. An illustration of distance k-dominating set with $k = 2$ of $C_6 \odot P_4$

Let P_n be the *path graph* of order $n \geq 3$ with vertex set $V(P_n) = \{v_i : 1 \leq i \leq n\}$, and edge set $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. For $1 \leq i, j \leq n$, with $i \neq j$, we have $d_{P_n}(v_i, v_j) = |i - j|$. Hence, the diameter of path is $diam(P_n) = n - 1$ and the radius $rad(P_n) = \lfloor \frac{n}{2} \rfloor$.

Proposition 3.6 ([9])
For $k \geq 1$ and $n \geq 3$, we have $\gamma_k(P_n) = \lceil \frac{n}{2k+1} \rceil$.

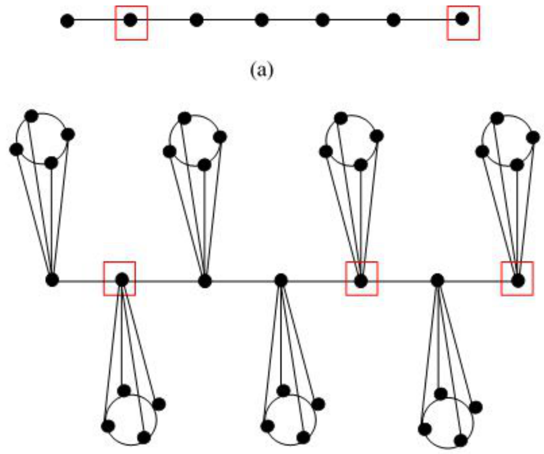
Based on Theorem 3.1 and 3.2 and the distance k-domination number of path graphs given in Proposition 3.6, we give the exact distance k-domination number of the corona product of a path graph and any graph.

Proposition 3.7
If P_n is a path graph of order $n \geq 3$ and H is any graph, then

$$\gamma_k(P_n \odot H) = \begin{cases} n, & if\ k = 1, \\ \lceil \frac{n}{2k-1} \rceil, & if\ k \geq 2. \end{cases}$$

Proof
Let P_n be a path graph of order $n \geq 3$ and H is any graph. For $k = 1$, from Proposition 3.3, we have $\gamma(P_n \odot H) = |P_n| = n$.

Now if $k \geq 2$, by Theorem 3.2 we have $\gamma_k(P_n \odot H) = \gamma_{k-1}(P_n)$. In Proposition 3.4, we have $\gamma_{k-1}(P_n) = \lceil \frac{n}{2k-1} \rceil$. It means that for $k \geq 2$, we have $\gamma_k(P_n \odot H) = \lceil \frac{n}{2k-1} \rceil$. For an illustration of Proposition 3.7, see Figure 3. □



(a)

(b)

Figure 3. (a) The illustration of distance 2-dominating set of P_7 (b) The illustration of distance 2-dominating set of $P_7 \odot C_4$

Davila et al. [9] proved the following lower bounds on the distance k-domination number of a graph in terms of its diameter and radius.

Proposition 3.8 ([9])
For $k \geq 1$, let G be a connected graph. Then we have

(a) $\gamma_k(G) \geq \frac{diam(G)+1}{2k+1}$;
(b) $\gamma_k(G) \geq \frac{2rad(G)}{2k+1}$.

By Lemma 2.5 and Proposition 3.8, if G is a nontrivial connected graph and H is any graph, we have $\gamma_k(G \odot H) \geq \frac{diam(G \odot H)+1}{2k+1} = \frac{diam(G)+3}{2k+1}$. By using the result in Theorem 3.2 and Proposition 3.8, we have been able to improve this lower bound for $\gamma_k(G \odot H)$.

Proposition 3.9
For $k \geq 2$, let G be a connected graph and H be any graph. Then we have $\gamma_k(G \odot H) \geq \frac{diam(G)+1}{2k-1}$, and this lower bound is sharp.

Proof
Let G be a connected graph and let H be any graph. For $k \geq 2$, by Theorem 3.2, we have $\gamma_k(G \odot H) = \gamma_{k-1}(G)$. In Proposition 3.8 (a) we have for $k \geq 2$, $\gamma_{k-1}(G) \geq \frac{diam(G)+1}{2k-1}$. It follows that $\gamma_k(G \odot H) \geq \frac{diam(G)+1}{2k-1}$.

For $k \geq 2$, if we consider the path graph P_n of order $n = p(2k-1)$ with $p \geq 1$, then by Proposition 3.7 for any graph H we have $\gamma_k(P_n \odot H) = \lceil \frac{n}{2k-1} \rceil = \frac{n}{2k-1}$. Since $diam(P_n) = n - 1$, we get that $\gamma_k(P_n \odot H) = \frac{diam(P_n)+1}{2k-1}$. Hence, for all $k \geq 2$ the path graph P_n of order $n = p(2k-1)$ with $p \geq 1$ achieves the lower bound $\frac{diam(G)+1}{2k-1}$. □

Also, by Lemma 2.6 and Proposition 3.8, if G is a nontrivial connected graph and H is any graph, we have $\gamma_k(G \odot H) \geq \frac{2rad(G \odot H)}{2k+1} = \frac{2rad(G)+2}{2k+1}$. From Theorem 3.2 and Proposition 3.8, we can improve this lower bound for $\gamma_k(G \odot H)$ in terms of the radius of G.

Proposition 3.10
For $k \geq 2$, let G be a connected graph and H be any graph. Then we have $\gamma_k(G \odot H) \geq \frac{2rad(G)}{2k-1}$, and this lower bound is sharp.

Proof
Let G be a connected graph and H be any graph. For $k \geq 2$, by Theorem 3.2, we have $\gamma_k(G \odot H) = \gamma_{k-1}(G)$. For $k \geq 2$, from Proposition 3.8 (b) we have, $\gamma_{k-1}(G) \geq \frac{2rad(G)}{2k-1}$. Thus, $\gamma_k(G \odot H) \geq \frac{2rad(G)}{2k-1}$.

Now, if we consider the path graph P_n of order $n = 2p(2k-1)$ with $p \geq 1$, then by Proposition 3.7 for any graph H we have $\gamma_k(P_n \odot H) = \frac{n}{2k-1}$. We have $rad(P_n) = \lfloor \frac{n}{2} \rfloor$, then if $n = 2p(2k-1)$, we have $2rad(P_n) = 2p(2k-1) = n$. It follows that $\gamma_k(P_n \odot H) = \frac{2rad(P_n)}{2k-1}$. Hence, the path graph P_n of order $n = 2p(2k-1)$ with $p \geq 1$ achieves the lower bound $\frac{2rad(G)}{2k-1}$ for all $k \geq 2$. □

4. Distance k-resolving domination of the corona product of two graphs

In this section, we will investigate the distance k-resolving domination, the concept combining both resolvability and distance k-domination.

First, we will prove the following useful lemma.

Lemma 4.1
Let G be a connected graph of order $n \geq 2$ and H be any nontrivial graph. If W is any resolving set of $G \odot H$, then for all $1 \leq i \leq n$, W contains at least one vertex from each copy $V(H_i)$.

Proof
If G is a connected graph of order $n \geq 2$ and H be any nontrivial graph. In $G \odot H$, for all $1 \leq i \leq n$, by Lemma 2.4, for any two distinct vertices $u_{i,l}$ and $u_{i,m}$ in the i-th copy H_i we have $d_{G \odot H}(u_{i,l}, x) = d_{G \odot H}(u_{i,m}, x)$ for every vertex $x \in V(G \odot H) \setminus V(H_i)$. Then for any W resolving set of $G \odot H$, W must contain at least one vertex of each $V(H_i)$ with $1 \leq i \leq n$. □

For a connected graph G of order n, in [1] we have the resolving domination number $1 \leq \gamma^r(G) \leq n-1$. Next, we give general bounds for $\gamma^r(G \odot H)$ for any connected graph G and any nontrivial graph H in terms of the orders of G and H.

Theorem 4.1
Let G be a connected graph of order $n_1 \geq 2$ and H be any nontrivial graph of order n_2. Then we have

$$n_1 \leq \gamma^r(G \odot H) \leq n_1 \cdot n_2.$$

Proof
If G is a connected graph of order $n_1 \geq 2$ and H is any graph of order $n_2 \geq 2$, based on Lemma 4.1, we have any resolving set of $G \odot H$ must contain at least one vertex of each $V(H_i)$ for all $1 \leq i \leq n_1$. Therefore, for all $1 \leq i \leq n_1$, any resolving dominating set must contain at least one vertex of each $V(H_i)$. It follows that $\gamma^r(G \odot H) \geq n_1$.

Let $S = \cup_{i=1}^{n_1} V(H_i)$. We have $V(G) = V(G \odot H) \setminus S$. For any two distinct vertices $v_i, v_j \in V(G)$, by Lemma 2.4 we have $d_{G \odot H}(v_i, u) = 1 \neq d_{G \odot H}(v_j, u) = d_G(v_i, v_j) + 1$ for $u \in V(H_i)$. Therefore, the set S is a resolving set of $G \odot H$. Also, any vertex v_i, with $1 \leq i \leq n_1$, is adjacent to a vertex u in $V(H_i)$. It follows that S is a dominating set of $G \odot H$. Therefore, S is a resolving dominating set of $G \odot H$. Thus, $\gamma^r(G \odot H) \leq |S| = n_1 \cdot n_2$. □

The *complement graph* of a graph G, denoted by $\overline{G}$, is the graph whose vertex set is $V(\overline{G}) = V(G)$ and where $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. We have the following characterization of graphs achieving the upper bound in Theorem 4.1.

Theorem 4.2
If G is a nontrivial connected graph of order n_1 and H a nontrivial graph of order n_2, then $\gamma^r(G \odot H) = n_1 \cdot n_2$ if and only if $H \cong \overline{K}_{n_2}$. Moreover, if $H \ncong \overline{K}_{n_2}$, then we have $\gamma^r(G \odot H) \leq n_1 \cdot (n_2 - 1)$.

Proof
If G is any connected graph of order $n_1 \geq 2$, we will show that $\gamma^r(G \odot \overline{K}_{n_2}) = n_1 \cdot n_2$ with $n_2 \geq 2$. Let H_i with $1 \leq i \leq n_1$ be the i-th copy of $\overline{K}_{n_2}$ in $G \odot \overline{K}_{n_2}$ where $V(H_i) = \{u_{i,l} : 1 \leq l \leq n_2\}$. For any two distinct vertices $u_{i,l}, u_{i,m} \in V(H_i)$, we have the neighborhood $N_{G\odot\overline{K}_{n_2}}(u_{i,l}) = N_{G\odot\overline{K}_{n_2}}(u_{i,m}) = \{v_i\}$, which means that any resolving set of $G \odot \overline{K}_{n_2}$ contains at least all but one of the vertices in $V(H_i)$ for all $1 \leq i \leq n_1$. Let S be a minimum resolving dominating set of $G \odot \overline{K}_{n_2}$. From above, S contains all but one of the vertices in $V(H_i)$ for all $1 \leq i \leq n_1$. Let $u_{i,l}$, with $1 \leq l \leq n_2$, be the vertex in $V(H_i)$ not in S. We have $N_{G\odot\overline{K}_{n_2}}(u_{i,l}) = \{v_i\}$ for all $1 \leq i \leq n_1$. If S is a dominating set of $G \odot \overline{K}_{n_2}$, then S must contain $u_{i,l}$ or v_i for all $1 \leq i \leq n_1$. Hence, $\gamma^r(G \odot \overline{K}_{n_2}) = |S| \geq n_1 \cdot n_2$. From the upper bound of Theorem 4.1, it follows that $\gamma^r(G \odot \overline{K}_{n_2}) = n_1 \cdot n_2$.

To show the converse, we suppose that H is not an empty graph $\overline{K}_{n_2}$, which means that H contains at least one edge. Let $u_l u_m \in E(H)$. Now, consider the set of vertices S in $G \odot H$, where $S = \cup_{i=1}^{n_1} V(H_i) \setminus \{u_{i,l}\}$. First, we show that S is a resolving set of $G \odot H$. For x and y two distinct vertices in $V(G \odot H) \setminus S$, we have the following.

- If $x = v_i \in V(G)$ and $y = v_j \in V(G)$ with $i \neq j$, then based on Lemma 2.4 for $u \in V(H_i) \cap S$, we have $d_{G\odot H}(x, u) = 1 < d_{G\odot H}(y, u) = d_G(x, y) + 1$.
- If $x = v_i \in V(G)$ and $y = u_{j,l} \in V(H_j) \setminus S$ with $i \neq j$, then from Lemma 2.4 for $u \in V(H_i) \cap S$, we have $d_{G\odot H}(x, u) = 1 < d_{G\odot H}(y, u) = d_G(v_j, v_i) + 2$.
- If $x = v_i \in V(G)$ and $y = u_{i,l} \in V(H_i) \setminus S$, for $u \in V(H_j) \cap S$ with $j \neq i$, from Lemma 2.4 we have $d_{G\odot H}(x, u) = d_G(v_i, v_j) + 1 < d_{G\odot H}(y, u) = d_G(v_i, v_j) + 2$.
- If $x = u_{i,l} \in V(H_i) \setminus S$ and $y = u_{j,l} \in V(H_j) \setminus S$ with $j \neq i$, then by Lemma 2.4 for $u \in V(H_i) \cap S$, we have $d_{G\odot H}(x, u) \leq 2$ and $d_{G\odot H}(y, u) = d_G(v_j, v_i) + 2 \geq 3$. Therefore, $d_{G\odot H}(x, u) \neq d_{G\odot H}(y, u)$.

It follows that S is a resolving set of $G \odot H$. Also, since $u_l u_m \in E(H)$, it is easy to see that S is a dominating set of $G \odot H$. Hence, S is a resolving dominating set of $G \odot H$. Thus, $\gamma^r(G \odot H) \leq |S| = n_1 \cdot (n_2 - 1)$. Therefore, the result follows. □

Next, we characterize the graph achieving the lower bound in Theorem 4.1.

Theorem 4.3
If G is a nontrivial connected graph of order n_1 and H a nontrivial graph of order n_2, then $\gamma^r(G \odot H) = n_1$ if and only if $H \cong K_2$.

Proof
If G is any connected graph of order $n_1 \geq 2$, we show first that $\gamma^r(G \odot K_2) = n_1$. Let H_i with $1 \leq i \leq n_1$ be the i-th copy of K_2 in $G \odot K_2$ where $V(H_i) = \{u_{i,1}, u_{i,2}\}$. From Theorem 4.2, we have $\gamma^r(G \odot K_2) \leq n_1 \cdot (n_2 - 1) = n_1$. Therefore, based on the lower bound in Theorem 4.1, we get that $\gamma^r(G \odot K_2) = n_1$.

Conversely, if H has order $n_2 \geq 3$, we suppose that $\gamma^r(G \odot H) = n_1$. Now let S be a minimum resolving dominating set of $G \odot H$. By Lemma 4.1, S contains at least one vertex from each $V(H_i)$ with $1 \leq i \leq n_1$. Since $|S_i| = n_1$, S consists of the union of a single vertex from each $V(H_i)$. Now let $u_{i,l}, u_{i,m} \in V(H_i) \setminus S$. Since S is a dominating set of $G \odot H$, from Lemma 2.4, we have $u_{i,l}$ and $u_{i,m}$ are both adjacent in $G \odot H$ to $u \in V(H_i) \cap S$, which means that $d_{G\odot H}(u_{i,l}, u) = d_{G\odot H}(u_{i,m}, u) = 1$. Also, from Lemma 2.4, $d_{G\odot H}(u_{i,l}, u) = d_{G\odot H}(u_{i,m}, u)$ for $u \in V(H_j) \cap S$ for all $1 \leq j \leq n_1$ with $j \neq i$. It means that S is not a resolving set of $G \odot H$, which is a contradiction. Therefore, if $\gamma^r(G \odot H) = n_1$, then necessarily H has order $n_2 = 2$. The only graphs with order 2 are the graph K_2 and the graph $\overline{K}_2$. Based on Theorem 4.2, we have $\gamma^r(G \odot \overline{K}_2) = 2 \cdot n_1$. It follows that $\gamma^r(G \odot H) = n_1$ if and only if $H \cong K_2$. □

A comment without a proof in [32], suggested that the resolving domination number $\gamma^r(G \odot H)$, where it is called location domination number, is equal to $dim(G \odot H)$ for any two connected graphs G and H. However, this is not true. For example, if we consider G to be an edge K_2 and H to be the path P_3 of order 3, then we have $\gamma^r(K_2 \odot P_3) = 4$ and $dim(K_2 \odot P_3) = 2$. Besides the case where $k = 1$, the equality holds between the distance k-resolving domination number and the metric dimension of the corona product of two graphs.

Lemma 4.2
If G is a connected graph of order n_1 and H any nontrivial graph of order n_2, then for $k \geq 2$, $\gamma_k^r(G \odot H) = dim(G \odot H)$.

Proof
If $n_1 = 1$, i.e., G is a trivial graph K_1, then by Lemma 2.5 we have $diam(G \odot H) \leq 2$, which means that for $k \geq 2$, we have $\gamma_k^r(G \odot H) = dim(G \odot H)$. If $n_1 \geq 2$, let W be a metric basis of $G \odot H$. Based on Lemma 4.1, W contains at least one vertex from each copy $V(H_i)$ in $G \odot H$. For $v_i \in V(G)$, with $1 \leq i \leq n_1$, by Lemma 2.4 we have $d_{G \odot H}(v_i, u) = 1 \leq k$ for $u \in W \cap V(H_i)$. Also, for $u_{i,l} \in V(H_i) \setminus W$, by Lemma 2.4 we have $d_{G \odot H}(u_{i,l}, u) \leq 2 \leq k$ for $u \in W \cap V(H_i)$. Therefore, W is a distance k-dominating set of $G \odot H$. Hence $\gamma_k^r(G \odot H) \leq |W| = dim(G \odot H)$. In [22], we have $dim(G \odot H) \leq \gamma_k^r(G \odot H)$, it follows that for $k \geq 2$, $\gamma_k^r(G \odot H) = dim(G \odot H)$. □

Lemma 4.2, enables us to investigate $\gamma_k^r(G \odot H)$ throughout the study of $dim(G \odot H)$ and conversely $dim(G \odot H)$ can be explored throughout the study of $\gamma_k^r(G \odot H)$.

In [22], it is shown that $1 \leq \gamma_k^r(G) \leq n - 1$ for any connected graph G of order n. In the following we give bounds for $\gamma_k^r(G \odot H)$ in terms of the orders of G an H.

Theorem 4.4
Let G be a connected graph of order $n_1 \geq 2$ and H be any nontrivial graph of order n_2. Then for $k \geq 2$, we have

$$n_1 \leq \gamma_k^r(G \odot H) \leq n_1(n_2 - 1).$$

Proof
In [10], it is shown that $n_1 \leq dim(G \odot H) \leq n_1(n_2 - 1)$. Then by Lemma 4.2, for any connected graph G of order $n_1 \geq 2$ and any nontrivial graph H of order n_2, for $k \geq 2$, we have $n_1 \leq \gamma_k^r(G \odot H) \leq n_1(n_2 - 1)$. □

The graphs achieving the equality $dim(G \odot H) = n_1$ and $dim(G \odot H) = n_1(n_2 - 1)$ were given in [10] as follows.

Theorem 4.5 ([10])
If G is a nontrivial connected graph of order n_1 and H a nontrivial graph of order n_2, then the following statements hold.

- $dim(G \odot H) = n_1$ if and only if $H \in \{K_2, P_3, \overline{K}_2, \overline{P}_3\}$.
- $dim(G \odot H) = n_1(n_2 - 1)$ if and only if $H \in \{K_{n_2}, \overline{K}_{n_2}\}$.

From Lemma 4.2 and Theorem 4.5, we get the following characterizations.

Corollary 4.1
If G is a nontrivial connected graph of order n_1 and H is a nontrivial graph of order n_2, then for $k \geq 2$, the following statements hold.

- $\gamma_k^r(G \odot H) = n_1$ if and only if $H \in \{K_2, P_3, \overline{K}_2, \overline{P}_3\}$.
- $\gamma_k^r(G \odot H) = n_1(n_2 - 1)$ if and only if $H \in \{K_{n_2}, \overline{K}_{n_2}\}$.

Proposition 4.1 ([31])
If G is any connected graph of order $n_1 \geq 2$ and H is any graph of order $n_2 \geq 2$, then we have the following.

- $dim(G \odot H) = dim(G \odot \overline{H})$.
- If $diam(H) \leq 2$, then $dim(G \odot H) = n_1 \cdot dim(H)$.
- If $diam(H) \geq 6$ or H is a cycle of order $n_2 \geq 7$, then $dim(G \odot H) = n_1 \cdot dim(K_1 \odot H)$.

By combining Lemma 4.2 and Proposition 4.1, we get the following results for $\gamma_k^r(G \odot H)$ for $k \geq 2$.

Corollary 4.2
If G is any connected graph of order $n_1 \geq 2$ and H is any graph of order $n_2 \geq 2$, then for $k \geq 2$ we have the following.

- $\gamma_k^r(G \odot H) = \gamma_k^r(G \odot \overline{H})$.
- If $diam(H) \leq 2$, then $\gamma_k^r(G \odot H) = n_1 \cdot dim(H)$.
- If $diam(H) \geq 6$ or H is a cycle of order $n_2 \geq 7$, then $\gamma_k^r(G \odot H) = n_1 \cdot dim(K_1 \odot H)$.

5. Concluding remarks

In conclusion, this paper investigated the concepts of domination, distance domination, and distance resolving domination in the context of the vertex corona product of graphs. We have established exact values and bounds for the domination number, the distance k-domination number, and the distance k-resolving domination number of the corona product of two graphs G and H in general. We also explored these parameters for the corona product when G or H belong to some special classes of graphs such as paths, cycles, complete graphs, complete k-partite graphs, and diameter two graphs. There are still several avenues for further investigation in this area of study, such as studying these parameters for the corona products when G or H belong to some graph classes not studied here.

Since the lower bounds in Proposition 3.9 and 3.10 are attained and by Lemma 2.6 we have $\gamma_k(G \odot H) = \gamma_k(G \odot H')$ for any two graphs H and H', the following questions naturally arise.

- Can we characterize the graphs G having $\gamma_k(G \odot H) = \frac{diam(G)+1}{2k-1}$?
- Can we characterize the graphs G having $\gamma_k(G \odot H) = \frac{rad(G)+1}{2k-1}$?

Also, from Theorem 4.1, it would be interesting to investigate the following question regarding the resolving domination number of the corona product of two graphs.

- If G is a nontrivial connected graph of order n_1 and H a nontrivial graph of order n_2, let $n_1 + 1 \leq \gamma \leq n_1 \cdot (n_2 - 1)$. Is it possible to characterize for which graphs G and H we have $\gamma^r(G \odot H) = \gamma$?

From Lemma 4.2, for $k \geq 2$, finding the distance k-resolving domination number of $G \odot H$ is equivalent to finding $dim(G \odot H)$. In view of Theorem 4.4, a similar question as above could be proposed for the distance k-resolving domination number and the metric dimension of the corona product of two graphs.

- If G is a nontrivial connected graph of order n_1 and H a nontrivial graph of order n_2, let $n_1 + 1 \leq \gamma \leq n_1 \cdot (n_2 - 2)$. Is it possible to characterize for which graphs G and H we have $\gamma_k^r(G \odot H) = \gamma$?

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