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A New Left Truncated Distribution for Modeling Failure time data: Estimation, Robustness study and Application

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Abstract Truncation arises in many practical situations such as Epidemiology, Material science, Psychology, Social Sciences and Statistics where one wants to study about data which lie above or below a certain threshold or with in a specified range. Left-truncation occurs when observations below a given threshold are not present in the sample. It usually arises in employment, engineering, hydrology, insurance, reliability studies, survival analysis etc. In this article, we develop and analyze a new left truncated distribution by truncating an asymmetric and heavy tailed distribution namely Esscher transformed Laplace distribution from the left so that the resulting distriution lies with in (b,∞). Various distributional and reliability properties of the proposed distribution are investigated. A real data analysis is done using failure time data.

Keywords Esscher transformed Laplace distribution, Estimation, Left truncation, Moments, Real data analysis

AMS 2010 subject classifications 62E10, 62G30

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1. Introduction

Truncation is the act or process of shortening something by removing a part of it. Truncated models are imperative to efficiently analyze the finite data that we observe in almost all the real life situations such as biology, hydrology, industry, reliability, medicine etc. Truncated distributions can be used to simplify the asymptotic theory of robust estimators of location and regression. The concepts of left, right and double truncation were common in statistical literature. Hence many researchers are interested in deriving truncated versions of standard probability distributions. Left-truncation arises due to early failures occuring before the data collection period. It occurs when observations below a threshold are not present in the sample. It finds applications in various fiels such as employment, engineering, hydrology, insurance, reliability studies, survival analysis etc. The basic concepts of left and right truncated distributions were introduced by Galton (1898) and later by Pearson and Lee (1908). Based on their studies, several truncated distributions were derived viz. truncated Gamma distribution, truncated exponential distribution, truncated t and F distributions, truncated Cauchy distribution, truncated Pareto distribution, truncated Weibull distribution, left truncated beta distribution, right and left truncated generalized Gaussian distribution, truncated Frechet-Weibull and Frechet distributions, Weibull truncated exponential distribution, right truncated mixture Topp-Leone with exonential distribution, [0,1] truncated Gompertz exponential distribution, truncated Gumbel distribution. simplex truncated multivariate normal distribution, truncated Cauchy power Weibull-G class of distributions, exponentiated left truncated power distribution, truncated bivariate Kumaraswamy exponential distribution, Weibull generalized truncated Poisson distribution, truncated exponentiated exponential distribution,

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truncated Weibull exponential distribution, left truncated Gumbel exponential distribution, left truncated Frechet distribution, truncated Pranav distribution. For details see, [11], [12] ,[24], [25], [5], [33], [32], [8], [9], [2]. [19], [21], [22], [26], [4], [6], [10], [13], [27], [29], [1], [3], [23], [31].

The article is organized as follows. In Section 2, Esscher transformed Laplace (ETL) distribution is reviewed. In Section 3, we derive the left truncated Esscher transformed Laplace (LTETL) distribution and studied its properties. Estimation of the parameter is done in Section 4. We conduct simulation study, checking robust nature and real applications of $LTETL(\eta, b)$ distribution for both $((b<0), b>0)$ cases were done in Sections 5 and 6 respectively. The article ended by Section 7.

2. Esscher Transformed Laplace (ETL) Distribution

Esscher transformed Laplace (ETL) distribution introduced by [16] is a new class of asymmetric and heavy-tailed distribution which is developed through Esscher transformation, a concept introduced by [14]. This distribution belongs to one parameter exponential family and is obtained from classical Laplace distribution through exponential tilting, (For details, see [16]).

The probability density function and distribution function of the one parameter Esscher transformed Laplace (ETL) distribution are

$$f(x,\eta)=\begin{cases}\frac{1-\eta^2}{2}e^{x(1+\eta)}; & x<0\\ \\ \frac{1-\eta^2}{2}e^{-x(1-\eta)}; & x\geq 0,\quad |\eta|<1\end{cases} \tag{1}$$

and

$$F(x)=\begin{cases}\frac{1-\eta}{2}e^{x(1+\eta)]}; & x<0\\ \\ \frac{1-\eta}{2}+\frac{1+\eta}{2}[1-e^{-x(1-\eta)]}]; & x\geq 0\end{cases}\quad . \tag{2}$$

This family of distributions is quite useful for modeling financial and flood level data which are asymmetric, heavy-tailed and having high skewness. For details (see, [18], [17] and [30].

3. Left Truncated Esscher Transformed Laplace (LTETL) distribution

In this section, we consider the left truncation of (1) beyond the interval (b,∞). Let X be a continuous random variable with probability density function f(x) and cumulative distribution function F(x). If values of X below a specified value b are excluded from the distribution, the remaining values have a distribution with pdf

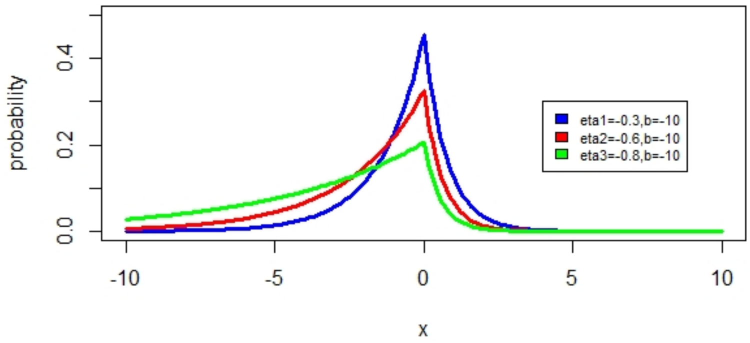
$$f(x,b)=\frac{f(x)}{1-F(b)};\quad b<x<\infty \tag{3}$$

and the distribution is said to be left truncated at b.

The probability density function of left truncated Esscher transformed Laplace ($LTETL(\eta, b)$) distribution is given by

$$g(x,\eta,b)=\begin{cases}\frac{(1-\eta)(e^{-x(1-\eta)})}{e^{-b(1-\eta)})}; & b<x<\infty;\;\; b\geq 0\\ \\ \frac{(1-\eta^2)e^{\eta x-|x|}}{2-(1-\eta)e^{b(1+\eta)}}; & b<x<\infty;\;\; b<0\end{cases}\quad . \tag{4}$$

The pdf plots of $LTETL(\eta, b)$ for various values of η and b= -10 are given in Figure 1.



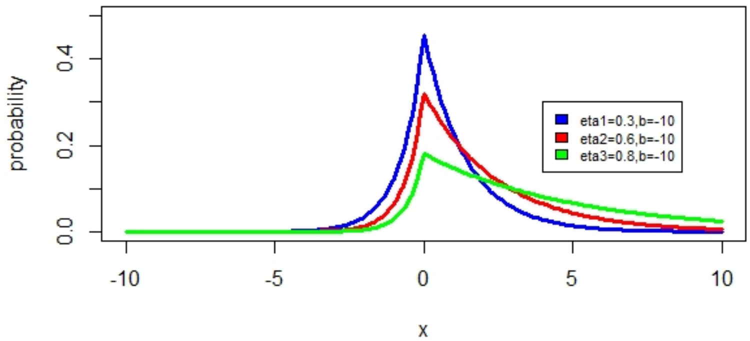


Figure 1. (a): Plots of $LTETL(\eta, b)$ distribution for $\eta \in (-1, 0)$, b=-10,(b): Plots of $LTETL(\eta, b)$ distribution for $\eta \in (0, 1)$, b=-10

The distribution function, characteristic function and moments of (4) are respectively

$$G(x,\eta) = \begin{cases} \frac{1-e^{-x(1-\eta)}}{e^{-b(1-\eta)}}; \quad b < x < \infty; \quad b \geq 0 \\ \\ \frac{2-(1+\eta)e^{-x((1-\eta)}-(1-\eta)e^{b(1+\eta)}}{2-(1-\eta)e^{b(1+\eta)}}; \quad b < x < \infty; \quad b < 0 \end{cases} , \tag{5}$$

$$\phi_X(t) = \begin{cases} \frac{(1-\eta)(e^{b(1+\eta)})}{1-\eta-it}; \quad b \geq 0, \quad t \in R \\ \\ \frac{1-\eta^2}{2-(1-\eta)e^{b(1+\eta)}} \left[\frac{2}{1-(\eta+it)^2} - \frac{e^{it+(1+\eta)b}}{1+(\eta+it)} \right]; \quad b < 0 \end{cases} \tag{6}$$

and

$$E(X^r) = \begin{cases} \frac{r!(e^{b(1-\eta)})}{(1-\eta)^r}; \quad b \geq 0 \\ \\ \frac{1-\eta^2}{2-(1-\eta)e^{b(1+\eta)}} \left[\frac{r!}{(1-\eta)^{r+1}} (\gamma(r+1, -b(1+\eta))) \right]; \quad b < 0 \end{cases} \tag{7}$$

where $\gamma(a,x)$ is the lower incomplete gamma function.
The mean of the distribution is

$$E(X)=\begin{cases}\frac{e^{b(1-\eta)}}{1-\eta}; \quad b\geq 0\\ \\ \frac{1-\eta^2}{2-(1-\eta)e^{b(1+\eta)}}\left[\frac{1}{(1-\eta)^2}(\gamma(2,-b(1+\eta)))\right]; \quad b<0\end{cases}$$

and

$$E(X^2)=\begin{cases}\frac{2e^{b(1-\eta)}}{(1-\eta)^2}; \quad b\geq 0\\ \\ \frac{1-\eta^2}{2-(1-\eta)e^{b(1+\eta)}}\left[\frac{2!}{(1-\eta)^3}(\gamma(3,-b(1+\eta)))\right]; \quad b<0\end{cases}$$

Therefore

$$V(X)=E(X^2)-[E(X)]^2.$$

3.1. Quantile function:

The α th quantile $q(\alpha)$ is

$$q(\alpha)=\begin{cases}\frac{-log\left[1-\alpha(e^{-b(1-\eta)})\right]}{1-\eta}; \quad b\geq 0\\ \\ \frac{-log\left[\frac{(1-\alpha)(2-(1-\eta)e^{b(1+\eta)})}{1+\eta}\right]}{1-\eta}; \quad b<0\end{cases}.$$

3.2. Reliability Measurers

The survival function, failure rate, cumulative hazard rate and second rate of failure of $LTETL(\eta,b)$ distribution are respectively

$$S(x)=\frac{e^{-x(1-\eta)}+e^{-b(1-\eta)}-1}{e^{-b(1-\eta)}},$$

$$r(x)=\frac{(1-\eta)e^{-x(1-\eta)}}{e^{-x(1-\eta)}+e^{-b(1-\eta)}-1},$$

$$-logS(x)=-log\left[\frac{e^{-x(1-\eta)}+e^{-b(1-\eta)}-1}{e^{-b(1-\eta)}}\right],$$

and

$$r^*(x)=log\left[\frac{e^{-x(1-\eta)}+e^{-b(1-\eta)}-1}{e^{-(x+1)(1-\eta)}+e^{-b(1-\eta)-1}}\right].$$

3.3. Maximum Entropy Property

The Shannon's entropy is defined by

$$H(X)=E(-logf(x)).$$

and for $LTETL(\eta,b)$ distribution it is obtained as

$$H(X)=\begin{cases}e^{b(1-\eta)}+log\left[\frac{e^{-b(1-\eta)}}{1-\eta}\right]; \quad b\geq 0\\ \\ 1+\frac{(1-\eta^2)be^{b(1+\eta)}}{2-(1-\eta)e^{b(1+\eta)}}+log\left[\frac{2-(1-\eta)e^{b(1+\eta)}}{1-\eta^2}\right]; \quad b<0.\end{cases} \tag{8}$$

3.4. Order Statistics

Let $X_{(1:n)} \leq X_{(2:n)} \leq X_{(3:n)} \leq ... \leq X_{(n:n)}$ denote the order statistics obtained from a random sample $X_1, X_2, ...X_n$ taken independently from $LTETL(\eta, b)$ distribution. Then the probability density function of n^{th} order statistics is given by

$$f_{X_{(n:n)}}(x) = \begin{cases} n\frac{(1-\eta)(e^{-x(1-\eta)})}{e^{-b(1-\eta)})} \left[\frac{1-e^{-x(1-\eta)}}{e^{-b(1-\eta)}}\right]^{(n-1)} ; & b \geq 0 \\ n\frac{(1-\eta^2)e^{\eta x-|x|}}{2-(1-\eta)e^{b(1+\eta)}} \left[\frac{2-(1+\eta)e^{-x((1-\eta)}-(1-\eta)e^{b(1+\eta)}}{2-(1-\eta)e^{b(1+\eta)}}\right]^{(n-1)} ; & b < 0 \end{cases}$$

and the pdf of first order statistics is

$$f_{X_{(1:n)}}(x) = \begin{cases} n\frac{(1-\eta)(e^{-x(1-\eta)})}{e^{-b(1-\eta)})} \left[1 - \frac{1-e^{-x(1-\eta)}}{e^{-b(1-\eta)}}\right]^{(n-1)} ; & b \geq 0 \\ n\frac{(1-\eta^2)e^{\eta x-|x|}}{2-(1-\eta)e^{b(1+\eta)}} \left[1 - \frac{2-(1+\eta)e^{-x((1-\eta)}-(1-\eta)e^{b(1+\eta)}}{2-(1-\eta)e^{b(1+\eta)}}\right]^{(n-1)} ; & b < 0 \end{cases}$$

4. Estimation-$LTETL(\eta, b)$, $(b < 0)$

In this section, we estimate the parameter η of the $LTETL(\eta, b)$, $(b < 0)$ distribution using the method of maximum likelihood. We take a random sample $X_1, X_2...X_n$ of size n from the $LTETL(\eta, b)$, $(b < 0)$ distribution.
The log-likelihood function is

$$logL = nlog(1-\eta^2) + \sum(\eta x - |x|) - nlog\left[2 - (1-\eta)e^{b(1+\eta)}\right].$$

$$\frac{\partial logL}{\partial \eta} = \frac{-2\eta}{1-\eta^2} + [b(1-\eta) - 1]\frac{e^{b((1+\eta)}}{2-(1-\eta)e^{b((1+\eta)}} + \bar{X}.$$

The solution of $\frac{\partial logL}{\partial \eta} = 0$ will provide the MLE of η.
But, this equation cannot be solved analytically. So, we use the nlm function in R software for estimating the parameter η.

4.1. Estimation-$LTETL(\eta, b)$, $(b > 0)$

In this section, we estimate the parameter η of the $LTETL(\eta, b)$,$(b > 0)$ distribution using the maximum likelihood estimation method. We take a random sample $X_1, X_2...X_n$ of size n from the $LTETL(\eta, b)$, $(b > 0)$ distribution. Then the logarithm of likelihood function is

$$logL = nlog(1-\eta) - n\bar{X}(1-\eta) + nb(1-\eta)$$

Differentiating this equation partially with respect to η and equate it to zero,the maximum likelihood estimate of η is obtained.

5. Simulation Study-$LTETL(\eta, b)$, $(b < 0)$

In Statistics, the term robust or robustness refers to the strength of a statistical model, tests and procedures according to a specific conditions of the statistical analysis a study hopes to acheive. So in this Section, we study the performance of the estimator η as well as the robustness of $LTETL(\eta, b)$, $(b < 0)$ distribution with respect to b, using Monte Carlo simulation method. We generate 5000 random samples of sizes n= 20, 40, 75 and 100 from the $ETL(\eta)$ distribution for some true values of the parameter η = -0.3, 0.3 and 0.6. We select the left truncated data from each generated sample for different truncation points viz. b= -5, -20, and -50 and find out 5000 estimates of η for each sample sizes. The R code used for simulation is given below.

```
rm(list=ls(all=TRUE))
m<-5000
n<-100
theta=0.6
w1<-numeric()
w2<-numeric()
thetahat<- numeric(m)
b=-50
t=(1-theta)*(exp(b*(1+theta)))
for (j in 1: m){


for(i in 1:n){

u1=runif(n,0,1)
u2=runif(n,0,1)

x=log((u1^(1-theta))/(u2^(1+theta)))}


fn <- function(theta) {
-(n*log(1-theta^2)-(n*log(2-t))+sum((theta*x)-(abs(x))))}

result<-nlm(fn,0, hessian=TRUE)
result
mle<-result$estimate
thetahat[j]<-mle[1]
}
thetahat
meant<-mean(thetahat)
meant
var(thetahat)
bias_theta<-mean(thetahat)-theta
bias_theta

msetheta<-var(thetahat)+(bias_theta^2)
round(msetheta,4)
```

The estimates of the parameter, average bias and mean squared error of the estimates (MSE) are computed and they are given in Tables 1 to 3.

Table 1. Values of $\hat{\eta}$, average bias and average MSE for b= -5 and different values of n, η

sample size	η=-0.3			η=0.3			η=0.6		
n	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE
20	-0.26701	0.0329	0.0172	0.2673	-0.0326	0.0171	0.4562	-0.1437	0.0293
40	-0.2694	0.0305	0.0094	0.2695	-0.0304	0.0091	0.4601	-0.1398	0.0237
75	-0.2745	0.0254	0.0051	0.2734	-0.0265	0.0053	0.4658	-0.1341	0.0204
100	-0.2741	0.0254	0.0039	0.2748	-0.0251	0.0042	0.4655	-0.1345	0.0197

Table 2. Values of $\hat{\eta}$, average bias and average MSE for b=-20 and different values of n, η

sample size	η=-0.3			η=0.3			η=0.6		
n	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE
20	-0.26309	0.03690	0.0175	0.2653	-0.0165	0.0180	0.4554	-0.1443	0.0296
40	-0.2675	0.0324	0.0093	0.2704	-0.0295	0.0091	0.4620	-0.1379	0.0233
75	-0.2737	0.0262	0.0053	0.2729	-0.0270	0.0052	0.4650	-0.1349	0.0205
100	-0.2743	0.0256	0.0041	0.2731	-0.0268	0.0041	0.4663	-0.1336	0.0196

Table 3. Values of $\hat{\eta}$, average Bias and average MSE for b=-50 and different values of n, η

sample size	η=-0.3			η=0.3			η=0.6		
n	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE
20	-0.2648	0.0351	0.0174	0.2623	-0.0376	0.0181	0.4558	-0.1451	0.0297
40	-0.2705	0.0294	0.00946	0.2704	-0.0295	0.0093	0.4622	-0.1377	0.0232
75	-0.2721	0.0278	0.0052	0.2731	-0.0268	0.0053	0.4647	-0.1352	0.0206
100	-0.2743	0.0256	0.0041	0.2740	-0.0259	0.0041	0.4652	-0.1347	0.0199

From tables 1 to 3, it is clear that, for different values of the sample size (n), the average bias and average mean squared error (MSE) of the estimates seem to be reasonably small. But as the sample size increases, it is found that there is a slight decrease in both the measurers. Also, there is no considerable variation in the average bias and average MSE with respect to different values of the parameters and different choice of the truncation points. So $LTETL(\eta, b)$, $(b < 0)$ distribution is robust with respect to different truncation points.

5.1. Real Data Analysis

In this section, we assess the credibility of $LTETL(\eta, b)$, $(b < 0)$ distribution using the data set given in [7]. The data set represents 40 times to failure of turbocharger of one type of engine.

The data set is listed in Table 4.
We model this data using $LTETL(\eta, b)$,$(b < 0)$ distribution with b = -2.

Table 4. Observed values of times to failure of turbocharger

1.6, 2.0, 2.6, 3.0, 8.0, 8.1, 8.3, 8.4, 6.7, 6.5, 6.0, 6.3, 4.5, 3.9, 3.5, 5.0, 5.1, 7.1, 5.8 ,3.9, 4.8, 4.6, 5.4, 5.3, 5.6, 8.4, 8.5, 7.3, 7.9, 6.1, 7.8, 6.5, 7.0, 8.8, 7.7, 6.0, 7.3, 7.7, 8.7, 9.

Histogram and embedded pdf are given in Figures 2 and 3 respectively.

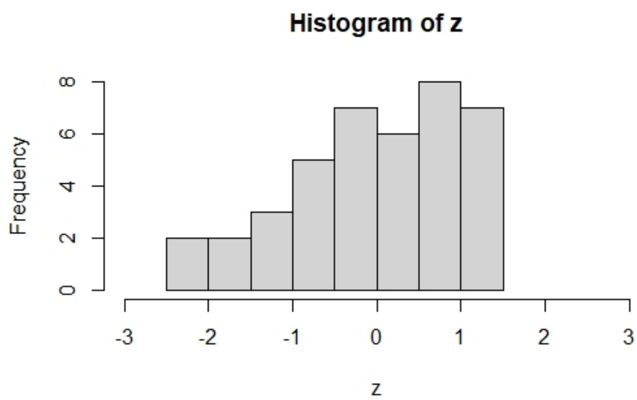


Figure 2. Histogram of Time to Failure of Turbocharger

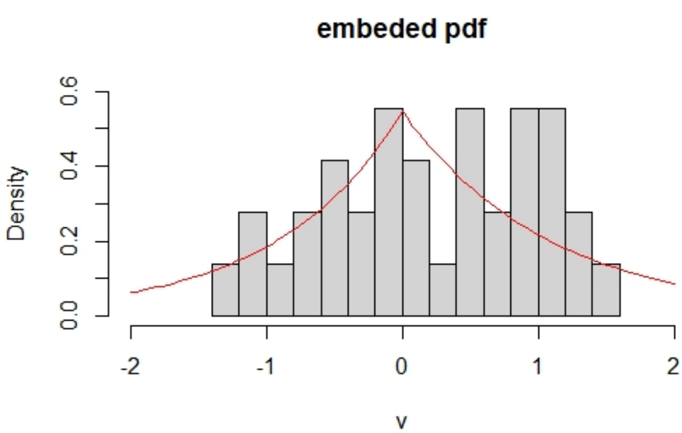


Figure 3. Embedded Pdf of LTETL (η, b) Distribution

We estimate the parameter and find out the values of log-likelihood, AIC, BIC and K-S distance for LTETL (η, b), $(b < 0)$ distribution using the R code given below.

```
library(MASS)
library(survival)
library(fitdistrplus)
x=c(   0.92433796,  0.97477932,-1.79949559, -1.59773014,
-1.14375788, -1.34552333,-1.14375788, 1.07566204,
 1.12610340,  0.26860025,  0.16771753, -0.08448928,
  0.06683480, -0.84110971,  -0.58890290, -0.53846154,
   0.47036570, -0.18537201,  -0.68978562, -0.79066835,
   -0.38713745, -0.43757881, -0.28625473,  1.12610340,
     1.17654477,  0.57124842,  0.87389660, -0.03404792,
0.82345523,  0.16771753,  0.41992434,  1.32786885 ,
0.77301387, -0.08448928, 0.57124842,  0.77301387,
 1.27742749,  1.42875158)

b= -2
dAltp<-function(x,a){  ((1-a^2)*(exp((a*x)-abs(x))))/
(2-((1-a)*exp(b*(1-a))))}# density
pAltp<- function(q,a){((2-((1+a)*exp(-q*(1-a)))-(1-a)*exp(b*(1+a)))/
(2-((1-a)*exp(b*(1+a)))))}#cdf
qAltp<- function(p,a){log(1+a)-log(1-p)-log(2-((1-a)*exp(b*(1+a))))/(1-a)}
falt<-fitdist(x,"Altp", method="mle",start=list(a=.1))

summary(falt)


b=-2

cdf<-function(x,a){((2-((1+a)*exp(-x*(1-a)))-(1-a)*exp(b*(1+a)))/
(2-((1-a)*exp(b*(1+a)))))}#cdf

ks.test(x,a<-c(0.0929),cdf)
```

Also we compare it with that of Laplace and left truncated Laplace distributions. The results are tabulated in Table 5.

Table 5. Values of MLE's, KS, Log likelihood, AIC, and BIC values of time to failure of turbocharger data

Distribution fitted	Estimates	KS	LL	AIC	BIC	P value
LTETL	$\hat{\eta} = 0.0929$	0.5263	-51.7	105.4	107.0	0.9501
Laplace	$\hat{\sigma} = 0.2684$	0.5500	-97.20	196.2	195.57	0.9294
Left truncated Laplace	$\hat{\sigma} = 0.6180$	0.6578	-53.19	108.38	107.95	0.7929

Table 5 shows that the left truncated Esscher transformed Laplace $(LTETL(\eta, b),\, b < 0)$ distribution is a better model when compared to the Laplace and left truncated Laplace distribution.

6. Simulation Study-$LTETL(\eta, b)$, $(b > 0)$

In this Section, we study the performance of the estimator η as well as the robustness of $LTETL(\eta, b)$, $(b > 0)$ distribution with respect to b, using Monte Carlo simulation method. The simulation algorithm is as follows.

Step 1: Generate 5000 random samples of sizes n= 400, 500, 750 and 1000 from the $LTETL(\eta, b)$, $(b > 0)$ distribution for some true values of the parameter η = -0.3, 0.3 and 0.6 for different truncation points viz. b= 0.001, 0.005 and 0.01

Step 2: Compute 5000 Maximum likelihood estimates of η.

Step 3: Compute average estimate, bias and MSE.

The R code used for simulation and robustness checking is

```
set.seed(1)
m=5000
n=1000
theta=numeric()
a1hat=rep(NA, times = m)
b=0.001
fn<-function(theta){L=(n*log(1-theta[1])+(n*b*(1-theta[1]))-sum(x)*(1-theta[1]))
return(-L)}
for(j in 1:m){
u=runif(n,0,1)
theta[1]=0.6
b=0.001
x=(-1/(1-theta[1]))*log(1-u*(exp(-b*(1-theta[1]))))
result<-nlm(fn,c(0.34))
ml<-result$estimate
a1hat[j]<-ml[1]

}
x
a1hat[j]
a1hat
esta1=mean(a1hat)
esta1

bias<-mean(a1hat)-theta[1]
bias
mse<-var(a1hat)+(bias)^2
mse
```

The results of the simulation study are given in Tables 6, 7 and 8.

Table 6. Values of $\hat{\eta}$, average bias and average MSE for b= 0.001 and different values of n, η

sample size	η=-0.3			η=0.3			η=0.6		
n	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE
400	-0.3173	-0.0173	0.0044	0.2935	-0.0064	0.0012	0.5972	-0.0027	0.0004
500	-0.3163	-0.0163	0.0036	0.2941	-0.0058	0.0010	0.5975	-0.0024	0.0003
750	-0.3152	-0.0152	0.0024	0.2947	-0.0052	0.0006	0.5978	-0.0021	0.0002
1000	-0.3149	-0.0149	0.0018	0.2949	-0.0050	0.0005	0.5980	-0.0019	0.0001

Table 7. Values of $\hat{\eta}$, average bias and average MSE for b= 0.005 and different values of n, η

sample size	η=-0.3			η=0.3			η=0.6		
n	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE
400	-0.3574	-0.0574	0.0074	0.2810	-0.0189	0.0015	0.5928	-0.0071	0.0004
500	-0.3564	-0.0564	0.0065	0.2815	-0.0184	0.0013	0.5931	-0.0068	0.0003
750	-0.3554	-0.0554	0.0052	0.2821	-0.0178	0.0009	0.5934	-0.0065	0.0002
1000	-0.3550	-0.0550	0.0047	0.2823	-0.0176	0.0007	0.5935	-0.0064	0.0001

Table 8. Values of $\hat{\eta}$, average bias and average MSE for b= 0.01 and different values of n, η

sample size	η=-0.3			η=0.3			η=0.6		
n	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE
400	-0.4018	-0.1018	0.0146	0.2671	-0.0328	0.0022	0.5897	-0.0120	0.0005
500	-0.4008	-0.1008	0.0136	0.2677	-0.0322	0.0020	0.5882	-0.0117	0.0004
750	-0.3998	-0.0998	0.0122	0.2682	-0.0317	0.0016	0.5885	-0.0114	0.0003
1000	-0.3994	-0.0994	0.0116	0.2684	-0.0315	0.0014	0.5887	-0.0112	0.0002

From Tables 6, 7 and 8, it is clear that the estimated values of the parameter are close to the pre-specified values. Also, we can see that the MSE of the estimate decreases with an increase in the sample size. These results indicates that the maximum likelihood of the estimates of the parameter are consistent estimates.

6.1. Real Data Analysis

To express the acceptability of the $LTETL(\eta, b)$,$(b > 0)$ model, we use the data set used by [19]. He modeled Weibull truncated exponential distribution using this data set and showed that his proposed model is a better fit than many baseline distributions like Exponential, Gamma, Truncated exponential and Weibull distributions. [20] used Exponentiated Exponential distribution to model the same data. The data given below are the failure times of air conditioning system in an airplane. The data set consists of 30 observations and are recorded in Table 9.

Histogram of the data is given in Figure 4.

We fit the $LTETL(\eta, b)$,$(b > 0)$ distribution using the above data set with truncation point b =1. The embeded pdf is given in Figure 5.

We calculate the parameters, log ikelihood, AIC , BIC and K-S distance using the R code given below.

Table 9. Observed values of failure time

23, 261, 87, 7, 120, 14,
62, 47, 225, 71, 246, 21,
42, 20, 5, 12, 120, 11,
3, 14, 71, 11, 14, 11,
16, 90, 1, 16, 52, 95.

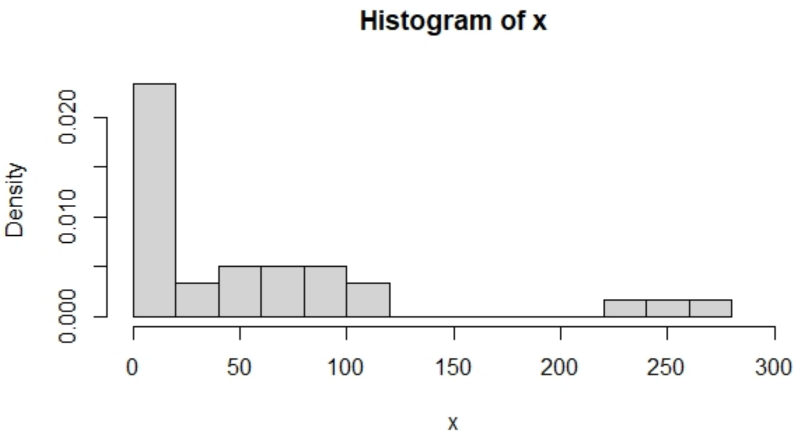


Figure 4. Histogram of failure time data

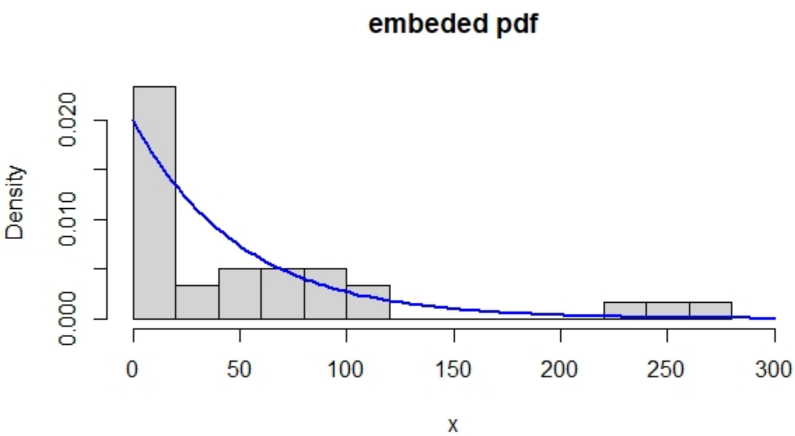


Figure 5. Embeded curve of LTETL distribution

Table 10. Values of MLE's, KS, Log likelihood, AIC, and BIC values of failure time data

Distribution fitted	estimate	KS	LL	AIC	BIC
LTETL	$\hat{\eta} = 0.98$	0.1612	-148.03	298.06	299.43
EXP	$\hat{\theta} = 0.02$	0.22	-156.89	315.78	317.22
Gamma	$\hat{\alpha} = 0.014$ $\hat{\beta} = 0.81$	0.18	-156.39	316.77	319.64
TEXP	$\hat{\theta} = 0.018$	0.228	-156.35	314.71	316.32

```
library(MASS)
library(survival)
library(fitdistrplus)
x=c(23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120,
 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52 ,95)

dAltp<-function(x,a){ (1-a)*exp(-x*(1-a))}# density
pAltp<- function(q,a){1-exp(-q*(1-a))}#cdf
qAltp<- function(p,a){-(log(1-p)/(1-a))}

falt<-fitdist(x,"Altp", method="mle",start=list(a=0.4))
summary(falt)
```

We made a comparison of LTETL$(b > 0)$ distribution with that of exponential(EXP), Gamma and truncated exponential(TEXP) distributions. The results are tabulated in Table 10.

Table 10 shows that the left truncated Esscher transformed Laplace ($LTETL(\eta, b)$,$b > 0$) distribution is a better model when compared to the exponential(EXP), Gamma and truncated exponential(TEXP) distributions.

7. Conclusion

Truncated distributions play a major role in practical statistics where the distributional values are limited to lie above or below a given threshold or with in a specified range. When there is no interest beyond the truncation point truncation happens. Here we develop and analyze a new left truncated distribution by truncating an asymmetric and heavy tailed distribution namely Esscher transformed Laplace distribution from the left so that the resulting distriution lies with in (b,∞). Various distributional and structural properties of the proposed distribution are studied. The parameter of the distribution is estimated using mle method. The accuracy of the estimator along with the robust nature of the proposed distribution with respect to the truncation points is evaluated using Monte Carlo simulation method. From this it is revealed that there is no appreciable difference in both average bias and average MSE for different choice of the truncation points and sample sizes. So, the LTETL distribution is robust with respect to the different truncation points. To illustrate the credibility of the proposed model a real data analysis is carried out using time to failure of turbocharger and failure time of air conditioning system in an airplane. It is found that the $LTETL(\eta, b)$, $(b < 0)$ distribution is more flexible than Laplace and left truncated Laplace distributions for failure of turbocharger data and $LTETL(\eta, b)$, $(b > 0)$ distribution is a better model than exponential(EXP), Gamma and truncated exponential(TEXP) distributions for failure time of air conditioning system in an airplane.

REFERENCES

1. S. Abbas, M. Farooq, J. D. Ahmed, S. S. Hanif and M. S. Qaiser, *Truncated Weibull-exponential distribution: methods and Applications*, Scientific reports, no.13, 2023.
2. S. Abid and R. Abdulrazak, *[0,1] Truncated Frechet-Weibull and Frechet distributions*, International Journal of Research in Industrial Engineering, vol. 7, no. 1, pp. 106–135, 2018.
3. B. R. Q. Abdul, and M. N. Wahhab, *A new left truncated Gumbel exponential distribution*, AIP Conf. Proc., vol. 2457, no. 1, pp. 1–16, 2023.
4. M. P. Adams, *Integral, Mean and covariance of the simplex-truncated multivariate normal distribution*, PLOS ONE, vol. 17, no. 7, pp. 1–31, 2022.
5. M. Ali and S Nadarajah, *truncated Pareto distribution*, Computer Communications, vol. 30, no. 1, pp. 1–4, 2006.
6. , N. Alotaibi, I. Elbatal, E. M. Almetwally, S. A. Alyami, A.S. Al-Moisheer, and M. Elgarhy, *Truncated Cauchy Power Weibull-G class of Distributions: bayesian and non-bayesian inference modelling for COVID-19 and carbon fiber data*, Mathematics, vol. 10, no. 9, pp. 1–25, 2022.
7. M. M. Alsobin, *The extended Weibull distribution with its properties, estimation and modelling skewed data*, Journal of King Saud Universituy-Science, vol. 34, no. 2, pp. 1–15, 2022.
8. K. Anithakumari, K. Srinivas Rao, and P. R. S. Reddy, *On a right truncated generalized Gaussian distribution*, Research Journal of Mathematical and Statistical Sciences, vol. 3, no. 4, pp. 1–9, 2015(a).
9. K. Anithakumari, K. Srinivas Rao, and P. R. S. Reddy, *On a left truncated generalized Gaussian distribution*, Journal of Advance Computing, vol. 4, no. 1, pp. 1–21, 2015(b).
10. M. Z. Arshad, O.S. Balogum, M.Z. Iqbal and P.E Oguntunde, *Exponentiated left truncated power distribution:On some properties of a new truncated model with applications to life time data* International Journal of Analysis and Applications, vol. 20, no. 23, pp. 20-23, 2022.
11. D. Chapman, *Estimating the parameters of a truncated Gamma distribution*, Annals of Mathematical Statistics, vol. 27, no. 2, pp. 498–506, 1956.
12. P. Consentino, V. Fiearra, and D. Luzio, *Truncated exponential frequency magnitude relationship in earthquake statistics*, Bulletin of the Seismological Society of America, vol. 67, no. 6, pp. 1615–1623, 1977.
13. H. H. El-Damrawy, A. A. M. Teamah and B. M. El-shiekh, *Truncated bivariate Kumaraswamy exponential distribution*, Journal of Statistics Applications and Probability, vol. 11, no. 2, pp. 461–4697, 2022.
14. F. Esscher, *On the probability function in the collective theory of risk*, Scandmaozan Actuarial Journal, vol. 15, pp. 175–195, 1932.
15. F. Galton, *An examination in the registered speeds of American trotting horses, with remarks on their value as hereditary data*, Proceedings of the Royal Society of London, vol.62, no. 1, pp. 310–315, 1898.
16. S. George and D. george, *Esscher transformed Laplace distribution and its applications*, Journal of Probability and Statistical Inference, vol. 10, no. 2, pp. 135–152, 2012.
17. D. George, S. George and H. Rimsha, *Esscher transformed and Geometric Esscher transformed Laplace processes*, Far East Journal of Theoretical Statistics, vol. 52, no. 6, pp. 455–483, 2016.
18. D. George and S. George, *Marshall-Olkin Esscher transformed Laplace distribution and processes*, Brazelian Journal of Probability and Statistics, vol. 27, no. 2, pp. 162–184, 2013.
19. A. Gul, M. Mohsin, M. Adil, M. Ali, *A modified truncated distribution for modeling the heavy tail, engineering and environmental science data*, PLOS ONE, vol. 16, no. 4, pp. 1-24, 2021.
20. R. D. Gupta and D. Kundu, *Exponentiated exponential family: an alternative to gamma and Weibull distributions*, Biometrical journal:Journal of Mathematical Methodsin Biosciences, vol. 43, no. 1, pp. 117–130, 2001.
21. H. A. A. Hashim and K. A Al-Kadim Al-Khafaji, *Right truncated of mixture Topp-Leone with exponential distribution*, Journal of Physics:Conference series, vol. 1804, no. 1, pp. 1-9, 2021.
22. H. M. Hussein and M. T. Ahmed, *Family of [0,1] truncated gompertz-exponential distribution with properties and application*, Turkish Journal of Computer and Mathematics, vol. 12, no. 14, pp. 1383–1399, 2021.
23. M. Irwan and A. Kurniawan, *Estimation of left truncated Frechet distribution parameters with maximum likelihood method*, AIP Conf. Proc., vol. 080012, 2023.
24. S. Kotz and S. Nadarajah, *Multivariate t distributions and their applications*, Cambridge University Press, New York, 2004.
25. S. Nadarajah and S. Kotz, *A truncated Cauchy distribution*, International Journal of Mathematical Education in Science and Technology, vol. 37, no. 5, pp. 605–608, 2006.
26. M. W. Naemah and B. A. Qasim, *A new truncated Gumbel distribution: Properties and Estimation* , Journal of Physics: Conference Series , vol. 1897, pp. 1-18, 2021.
27. A. Orabi, B. Ahmed, and D. Ziedan, *A new class of the Weibull-Generalized truncated Poisson distribution for reliability and life data analysis* , Applied Mathematical Sciences , vol. 16, no. 3, pp. 139–150, 2022.
28. K. Pearson and A. Lee, *On the generalized probable error in multiple normal correlation* , Biometrica , vol. 6, no. 1, pp. 59–68, 1908.
29. L. D. Ribeiro-Reis, *Truncated exponentiated exponential distribution: A class for unit interval* , Journal of Statistics and Management Systems, vol.25, pp. 2061-2072, 2022.
30. H. Rimsha and D. George, *q Esscher transformed Laplace distribution* , Communications in Statistics-Theory and Methods, vol. 8, no. 7, pp. 659–675, 2019.
31. K.K.Shukla, *Truncated Pranav distribution: Properties and Application* , RT and A , vol. 18, no. 1, pp. 179–192, 2023.
32. L. Zaninetti, *The initial mass function modeled by a left truncated beta distribution* , The Astro Physical Journal , vol. 765, no. 2, pp. 128–135, 2013.
33. T. Zhang and M. Xie, *On the upper truncated Weibull distribution and its reliability implications* , Reliability Engineering and System Safety, vol. 96, no. 1, pp. 194–200, 2011.